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On Motions in an Areal Space

Masao GAMA
Department of Mathematics, Asahigawa Branch Hokkaido University of Education

蒲　雅夫：面積空間における運動について

§ 0. Introduction

We consider an areal space $A_n^{(m)}$ of the submetric class with the fundamental function $F(x^i, p_\alpha^i)$,

where

$$F^2 = \det|g_{ij}p_\alpha^i p_\beta^j|,$$

$$g_{ij} = \Big(\frac{1}{m}L_{ij}^{\alpha\beta} + p_i^\alpha p_j^\beta\Big) g_{\alpha\beta},$$

$$L_{ij}^{\alpha\beta} = F^{-1}(FF_{;i}^{\ \alpha}{}_{;j}^{\ \beta} - F_{;i}^{\ \alpha}F_{;j}^{\ \beta} + F_{;j}^{\ \alpha}F_{;i}^{\ \beta}),$$

$$p_i^\alpha = F^{-1}F_{;i}^{\ \alpha}, \qquad {}_{;i}^{\ \alpha} = \partial/\partial p_\alpha^i.$$

The purpose of this paper is to define a motion in our space by the use of Lie derivatives and show some results concerning it. Throughout the paper Latin indices run from 1 to n and Greek from 1 to m.

§ 1. Lie derivatives in an areal space

Consider an infinitesimal transformation

(1.1) $\quad \bar{x}^i = x^i + \xi^i(x^j)d\tau,$

where $\xi^i(x^j)$ is a contravariant vector field of class C^2 and $d\tau$ an infinitesimal, which carries a point (x^i) of a surface $A_{(m)}: x^i = x^i(u^\alpha)$ to a point $(\bar{x}^i)$ of another surface $\bar{A}_{(m)}: \bar{x}^i = \bar{x}^i(u^\alpha)$. By this transformation p_α^i undergoes this transformation

(1.2) $\quad \bar{p}_\alpha^i = p_\alpha^i + \xi^i{}_{,j}p_\alpha^j d\tau,$

where, ${}_{,j} = \partial/\partial x^j$.

According to [2][1] and [3] the Lie derivative of a geometric object $\Omega(x^i, p_\alpha^i)$ (e.g. a tensor, connection parameters etc.) with respect to the vector field ξ^i may be defined thus:

(1.3) $\quad \underset{\xi}{\pounds}\Omega = (\overset{v}{d}\Omega - \overset{m}{d}\Omega)/d\tau,$

where

$$\overset{v}{d}\Omega = \Omega(\bar{x}^i, \bar{p}_\alpha^i) - \Omega(x^i, p_\alpha^i),$$

$$\overset{m}{d}\Omega = \bar{\Omega}(\bar{x}^i, \bar{p}_\alpha^i) - \Omega(x^i, p_\alpha^i),$$

1) Numbers in brackets refer to the references at the end of the paper.

and $\bar{\Omega}(\bar{x}^l, \bar{p}^l_\alpha)$ denotes the component of $\Omega(x^i, p^i_\alpha)$ in the coordinate system $(\bar{x}^i)$, (1.1) being interpreted as a coodrinate transformation from (x^i) to $(\bar{x}^i)$.

By definition the Lie derivative of g_{ij} with respect to ξ^i is

(1.4) $$\underset{\xi}{\pounds} g_{ij} = 2\{\xi_{(i|j)} + C_{(i|j),h}{}^{\alpha}\xi^h_{|k} p^k_\alpha\}^{2)},$$

where

$$\xi_i = g_{ij}\xi^j,$$
$$C_{lj,h}{}^{\alpha} = g_{lj}C^l_{i,h}{}^{\alpha},$$
$$C^l_{i,h}{}^{\alpha} = \tfrac{1}{2} g^{lk}\{g_{ik;h}{}^{\alpha} + 2g_{rs;h}{}^{\alpha}\gamma^r_{[i}\beta^s_{k]}\}^{3)},$$
$$\gamma^r_i = \delta^r_i - \beta^r_i, \qquad \beta^r_i = p^r_\alpha p^\alpha_i,$$

symbol $|j$ denoting the covariant derivative with respect to x^j[4] and δ^r_i the Kronecker delta.

Lie derivatives of p^l_α, p^α_l and the fundamental function F with respect to ξ^i are as follows:

(1.5) $$\underset{\xi}{\pounds} p^i_\alpha = 0,$$

(1.6) $$\underset{\xi}{\pounds} p^\alpha_i = \beta^h_j{}_{;i}{}^{\alpha}\xi^j_{|h} = (\gamma^h_i p^\alpha_j + p^h_\beta L^{\alpha\beta}_{ij})\xi^j_{|h},$$

(1.7) $$\underset{\xi}{\pounds} F = F\beta^i_j\xi^j_{|i}$$

respectively.

From (1.6) and (1.7) we have

(1.8) $$(\underset{\xi}{\pounds} F)_{;i}{}^{\alpha} = p^\alpha_i \underset{\xi}{\pounds} F + F\underset{\xi}{\pounds} p^\alpha_i$$

on taking account of the relation

$$\beta^h_j \Gamma^{*j}_{hk}{}_{;i}{}^{\alpha} = 0.$$

As to Lie derivatives of the connection parameters Γ^{*i}_{jk} and $C^i_{j,r}{}^{\alpha}$ with respect to ξ^i, we find

(1.9) $$\underset{\xi}{\pounds} \Gamma^{*l}_{jh} = R^l_{jhk}\xi^k + \Gamma^{*i}_{jh}{}_{;k}{}^{\alpha}\xi^k_{|l} p^l_\alpha + \xi^l_{|j|h}$$

where

$$R^l_{jhk} = 2\{\Gamma^{*l}_{j[h,k]} - \Gamma^{*i}_{j[h}{}_{;|l|}{}^{\alpha}\Gamma^{*l}_{|\alpha|k]} + \Gamma^{*l}_{j[h}\Gamma^{*i}_{|l|k]}\},$$
$$\Gamma^{*l}_{\alpha k} = \Gamma^{*l}_{rk} p^r_\alpha,$$

(1.10) $$\underset{\xi}{\pounds} C^i_{j,r}{}^{\alpha} = \tfrac{1}{2} g^{il}\{(\delta^h_j\delta^k_l + 2\gamma^h_{[j}\beta^k_{l]})(\underset{\xi}{\pounds} g_{hk})_{;r}{}^{\alpha} + 4C_{(h[j),|r|}{}^{\alpha}\underset{\xi}{\pounds} p^\beta_{l]} p^h_\beta - 2C^h_{j,r}{}^{\alpha}\underset{\xi}{\pounds} g_{hl}\}$$

respectively.

From (1.4) and (1.9) we get

(1.11) $$(\underset{\xi}{\pounds} g_{ij})_{|k} = (g_{ij}{}_{;h}{}^{\delta} p^l_\delta + \delta^l_i g_{hj} + \delta^l_j g_{ih})\underset{\xi}{\pounds} \Gamma^{*h}_{lk}$$

on taking account of the relation

$$R_{ijkh} + R_{jikh} + g_{ij}{}_{;l}{}^{\beta} R^l_{\beta kh} = 0^{5)}.$$

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2) The round brackets denote the symmetric part, e.g. $2\xi_{(i|j)} = \xi_{i|j} + \xi_{j|i}$.

3) The square brackets denote the alternating part, e.g. $2\gamma^r_{[i}\beta^s_{k]} = \gamma^r_i\beta^s_k - \gamma^r_k\beta^s_i$.

4) See [1].

5) See [1].

§ 2. Motions in an areal space

When the transformation (1.1) does not change the fundamental function (Fx^i, p^i_α) of an areal space $A^{(m)}_n$, we call this transformation a *motion* in the areal space.

By this definition we have

Theorem 1. *In order that an infinitesimal transformation* (1.1) *be a motion in an areal space, it is necessary and sufficient that the Lie derivative of the fundamental function with respect to* ξ^i *vanish, that is,*

(2.1) $\underset{\xi}{\pounds} F=0$

from which, taking account of (1.7) we have

Corollary. *In order that an infinitesimal transformation* (1.1) *be a motion in an areal space, it is necessary and sufficient that*

(2.2) $\beta^i_j \xi^j_{|i}=0.$

By virtue of (1.4), (2.2) is equivalent to

(2.3) $g^{\alpha\beta} \underset{\xi}{\pounds} g_{\alpha\beta}=0.$

Corollary. *If the Lie derivative of* g_{ij} *with respect to* ξ^i *vanishes, then the transformation* (1.1) *is a motion.*

The Lie derivative of $L^{\alpha\beta}_{ij}$ with respect to ξ^i is expressed as follows:

(2.4) $\underset{\xi}{\pounds} L^{\alpha\beta}_{ij}=p^\alpha_j \underset{\xi}{\pounds} p^\beta_i + p^\beta_i \underset{\xi}{\pounds} p^\alpha_j + (\underset{\xi}{\pounds} p^\alpha_i);^\beta_j$

Consequently by virtue of Theorem 1, (1.8) and (2.4) we have

Theorem 2. *If the transformation* (1.1) *is a motion, then we have*

(2.5) $\underset{\xi}{\pounds} p^\alpha_i=0, \qquad \underset{\xi}{\pounds} L^{\alpha\beta}_{ij}=0.$

By virtue of corollary, Theorem 2 and (1.10), we have

Theorem 3. *If the Lie derivative of* g_{ij} *with respect to* ξ^i *vanishes, then we have* (2.5) *and*

(2.6) $\underset{\xi}{\pounds} C^{i}_{j,}{}^{\alpha}_{r}=0.$

From (1.11) we get

(2.7) $(g^{\alpha\beta} \underset{\xi}{\pounds} g_{\alpha\beta})_{|k}=2\beta^l_h \underset{\xi}{\pounds} \Gamma^{*h}_{lk}.$

Thus by virtue of Corollary and (2.7) we have

Theorem 4. *If the transformation* (1.1) *is a motion, then we have*

(2.8) $\beta^l_h \underset{\xi}{\pounds} \Gamma^{*h}_{lk}=0.$

REFERENCES

[1] Gama, M. (1965), On areal spaces of the submetric class. Tensor, N. S., Vol. 16, p. 262–268.

[2] Rund, H. (1959), The Differential Geometry of Finsler spaces. Springer-Verlag.

[3] Yano, K. (1955), The theory of Lie derivatives and its applications. North-Holland.