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On Non-degenerate Nuclear Statistical Models

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中野嘉弘： 非縮退原子核統計モデルに就て

Some notes on non-degenerate nuclear statistical model and the Brueckner's theory of nuclear structure are described. Four methods to estimate the order of magnitude of the non-degeneracy of nucleus or the epuivalent temperature kT are given. The beginning three methods considering only Coulomb effects give altogether the high degree of non degeneracy such as $kT \gtrsim 20 MeV$. The fourth method taking further nuclear symmetry energy into account makes easily kT lower down to e.g. 5 MeV, which in turn shows the significant importance of the symmetry energyin the theory of nuclear stability condition.

§ 1. Introduction

Atomic nuclei are not always treated as to be completely degenerate in the variety of nuclear models, for example, Euler [1] and Watanabe [2] showed that the momentum distribution of nucleons in real atomic nuclei should be given by the non-degenerate Fermi gas model with its equivalent temperature as $kT \sim 7\ MeV$.

This paper is a continuance of the paper titled "On the Nuclear Stability Curve"[3], and gives four methods to estimate the order of magnitude of non-degeneracy of nuclei or the equivalent temperature kT ; the first is based on the Heisenberg curve, the second on the neutron or proton numbers in nuclei not in their ratio but individually, and the third by the improved form of the second method taking the nuclear surface and/or curvature effects.

Some notes on the Brueckner's [4] theory of nuclear structure are also given. The fourth method taking the nuclear symmetry energy effect brings the equivalent temperature kT much lower, which in turn reflects the importance of symmetry energy in the theory of atomic nuclei.

§ 2. Estimation of kT from nuclear stability curve

As is already discussed in the paper "On the Nuclear Stability Curve", the most fundamental character of atomic nuclei is the so called nuclear stability —— Heisenberg curve —— $Z = A/(2 + 0.0146\ A^{\frac{2}{3}})$, which is given in the form of particle density as $\rho_p = \rho/\ (1+\Delta)$, where $\Delta = 1+0.0146\ \ A^{\frac{2}{3}}$, ρ_p and ρ are the proton and neucleon number density in the nucleus.

If we assume that the nucleus is a mixture of the two kinds of Fermions which are different in their possession of electric charge or not, and apply the Fermi statistics, then neutral nucleon (neutron) number density is given by

$$\rho_N = 4\pi h^{-3} (2MkT)^{\frac{3}{2}} I_{\frac{1}{2}}(\eta), \qquad (1)$$

where M : nucleon mass and

$$I_{\frac{1}{2}}(\eta) = \int_0^{\infty} \frac{y^{\frac{1}{2}}}{exp(y-\eta)+1} dy$$ (tabulated by McDougal and Stoner[5]).

In the charged nucleon (proton) number density ρ_p, η is replaced by $\eta - \frac{eV}{kT}$, where ηkT is the Fermi level, eV is the electrostatic energy per proton and is given as $Z/A^{\frac{1}{3}}$ (MeV) for nuclear radius constant $r_0 = 1.40 \times 10^{-13}$ cm.

From the relation $\rho = \rho_N + \rho_p$, we get $\rho_p = \rho/(1+\delta)$, where $\delta = I_{\frac{1}{2}}(\eta)/I_{\frac{1}{2}}\left(\eta - \frac{eV}{kT}\right)$. In the demand that the theoretical δ should be conistent with the Δ of real nuclei in the nuclear stability relation, it can be found for all nuclei that the respectively definite value of η is established at the every assumed kT value as are shown in Table 1.

Table 1.

kT (MeV)	7	15	25	23.58
η	7.5	3.2	0.6	1

The Fermi level ~ 24 MeV[6] of real nuclei gives reasonably the choice of parameter pair as $\eta = 1$, $kT = 23.5$ MeV.

§ 3. The second estimation of kT from neutron number density

In the preceding section, we discussed from the view point of proton-neutron number ratio. If we consider only neutron number per nucleus, then we can transform Eq. (1) to the form as follows,

$$N/A = 0.6\, (r_0 \zeta \sqrt{\beta}/\lambda_N)^3 \cdot I_{\frac{1}{2}}(\eta), \qquad (2)$$

where λ_N : Nucleon Compton wave length divided by 2π. $\beta = kT/Mc^2$,

ζ : nuclear density parameter defined as nuclear radius $\zeta r_0 = 1.40 \times 10^{-13}$ cm.

Evaluation with the Fermi-Dirac function tabulated by McDougal and Stoner[5] gives the result that $kT \gtrsim 23.58 MeV$ in order not to lead to such a contradiction as $N/A > 1$ in the case of the ordinarily accepted nuclear radius ($\zeta \sim 1$) and the normal nucleon mass. Otherwise, we must allow an unusually small nuclear radius and/or such a small effective nucleon mass as proposed by Brueckner.[4]

This is the second estimation of the non-degeneracy of atomic nuclei.

§ 4. The third estimation of kT from the nuclear surface effect

In the preceding § 3. the ratio N/A is constant independently of A because of the constant η for a given kT value. In the proton case, however, the ratio Z/A is dependent on mass number A through the Coulomb effect eV in the argument $\eta-\frac{eV}{kT}$ instead of η in the neutron case. This Z/A, however, has a tendency to contradict with the behavior of real nuclei, especially in the region of lighter elements.

Now, we take the surface correction as done by Brueckner[4] and other corrections in the integration of momentum space into account according to Hill and Wheeler[7]. The surface correction gives the result for neutrons as follows,

$$N=2V\frac{4\pi}{h^3}\int_0^\infty \frac{p^2\left(1-\frac{p_0}{p}\right)dp}{exp\left[\frac{p^2}{2MkT}-\eta\right]+1}. \tag{3}$$

In the proton case, N and η must be replaced by Z and $\eta-\frac{eV}{kT}$ respectively. In the above Eq. (3), if we introduce only the surface correction, it holds that

$p_0/\hbar\equiv k_0=3\pi/4R.$ Considering further the effect of curvature correction, we accepted as $p_0=0.839\cdot\frac{3h}{8R}$.

Then, after integraing Eq. (3), we get the relation

$$N/A=X^2\{0.6\cdot X\cdot I_{\frac{1}{2}}(\eta)-0.839\ A^{-\frac{1}{3}}\ ln\,(e^\eta+1)\}, \tag{4}$$

where $X=(r_0\zeta)\cdot\lambda_N^{-1}\ (M^*M^{-1}\beta)^{\frac{1}{2}}$ and M^* is a effective nucleon mass.

Similar relation holds for the proton number Z/A. If we accept the value $\eta=1$ ($kT=23.58MeV$) according to the preceding sections, and $X=0.98$, then we can reproduce the ratios N/A and Z/A consistently with the behavior of real nuclei (see Fig. 1).

Disagreemnt in the lighter elements ($A\lesssim 50$) may be inherited from the statistical model itself, but some improvement can be obtained if we take further the full correction of curvature into account. Then it holds

$$N/A=X^2\ \{0.6\cdot X\cdot I_{\frac{1}{2}}(\eta)-A^{-\frac{1}{3}}\ ln\,(e^\eta+1)+0.225\cdot X^{-\frac{3}{2}}\cdot A^{-\frac{2}{3}}\}. \tag{5}$$

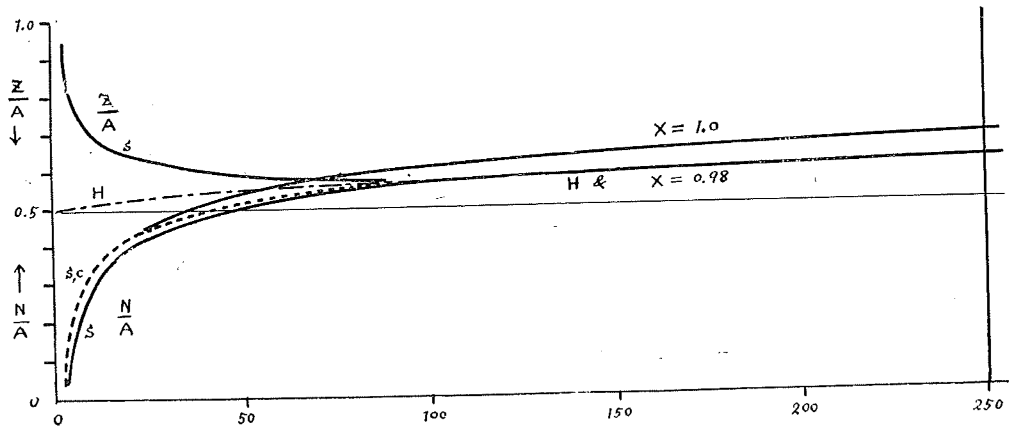

Fig. 1. Mass number vs. the ratios N/A and Z/A. H : Heisenberg curve, X : the parameter in Eqs. (4) and, (5). s : with surface correction using Eq. (4) and the corresponding for Z/A. s, c : sith surface and curvature corrections using Eq. (5). The parameter X=0.98 reproduce the N/A and Z/A consistently with Heisenberg curve.

The result is shown by the broken line in Fig. 1. All these show remarkable significance of finite nucleus.

The accepted value $X=0.98$ means $M^*=0.91\ M$ for the $r_0=1.40\times10^{-13}cm$, and if we take $M'=M$ then we get $r_0\zeta=1.37\times10^{-13}cm$.

The latter radius parameter is the conclusion from the analysis of high energy neutron scatterings by nuclei[8]. Hence, it seems to be unnecessary to introduce the small effective nucleon mass M^* as half as normal one which has been proposed by Brueckner et al.[4]

Thus, we get thrice the reasonable estimation of equivalent temperature kT.

§ 5. Evaluation of nuclear energies

We now evaluate the nuclear energies as an extension of the work of Brueckner[4] under the assumption of the degree of non-degeneracy estimated as $kT=23.58\ MeV$ $(\eta=1)$ in the Fermi type momentum distribution.

Then, the total kinetic energy of neutrons is given, taking the surface effect into account, as follows

$$T_N=V\cdot h^{-3}.\ 4\pi(2MkT)^{\frac{5}{2}}/2M\cdot\{I_{\frac{3}{2}}(\eta)-\sqrt{y_0}\cdot I_1(\eta)\}, \qquad (6)$$

where $$I_{\frac{3}{2}}(\eta)=\int_0^\infty dy\cdot y^{\frac{3}{2}}/[exp\ (y-\eta)+1], \qquad (7a)$$

$$I_1(\eta)=\int_0^\infty dy\cdot y/[exp(y-\eta)+1], \qquad (7b)$$

and $$\sqrt{y_0}=p_0/\sqrt{2MkT}=0.839\cdot\frac{3}{8R}\cdot\frac{h}{\sqrt{2MkT}}=1.4\cdot X^{-1}.\ A^{-\frac{1}{3}}. \qquad (7c)$$

Thus, the kinetic energy per nucleon $\overline{T}=(T_N+T_P)/A$ is given by

$$T=0.6X^{-3}kT\left\{I_{\frac{3}{2}}(\eta)+I_{\frac{3}{2}}\left(\eta-\frac{eV}{kT}\right)-\sqrt{y_0}\left[I_1(\eta)+I_1\left(\eta-\frac{eV}{kT}\right)\right]\right\}, \qquad (8)$$

Substituting the values $kT=23.58\ MeV$, $\eta=1$ and $X=0.98$ according to the estimations of the preceding sections, using the Mc Dougal and Stoner's table[5] for $I_{\frac{3}{2}}$ and calculating numerically the I_1 function, we get the result as shown in Table 2.

Table 2.

A	50	700	200	300	∞
T (MeV)	50.6	50.2	48	48	35.5

This is similar with the result of Brueckner[4], but we now find the tendency that the $\overline{T}_N$ increases and T_p decreases with increasing mass number A.

To find the potential energy we must evaluate the integral of the type as follows

$$\int\int^\infty N(k_1)f\left(\frac{1}{2}(\boldsymbol{K}_1-\boldsymbol{K}_2)\right)N(k_2)\ d\boldsymbol{K}_1\ d\boldsymbol{K}_2, \qquad (9)$$

where $f\left(\frac{1}{2}(\boldsymbol{K}_1-\boldsymbol{K}_2)\right)$ is the Fourier transform of nuclear two body potential, and

$$N(k_i)d\boldsymbol{K}_i=V(2\pi)^{-3}d\boldsymbol{K}_i(1-k_0/k_i)/(exp[\hbar^2k_i^2/2MkT-\eta]+1). \qquad (10)$$

Defining the relative momentum $\boldsymbol{K}\equiv\frac{1}{2}(\boldsymbol{K}_1-\boldsymbol{K}_2)$ and transforming into,

$$\int_0^\infty f(k)\,kdk\int_k^\infty N(k_1)\,k_1dk_1\int_{|2k-k_1|}^{k_1} N(k_2)\,k_2\,d\,k_2 \tag{11}$$
$$\equiv\int_0^\infty dk\,f(k)\,P(k)$$

where $P(k)$ is an extension of the function defined by Brueckner in his paper[4].

According to the non-degeneracy, the function $N(k_i)$ and then $P(k)$ trails into the region of high k value, hence we may expect the possibility to improve the defect of too small equilibrium radius of nucleus in the argument of saturation problem given by Brueckner[4]. The calculation of Eq. (11) is much more complicated than in the degenerate case of Brueckner[4].

§ 6. Symmetry energy effect

Symmetry energy effects in the non-degenerate statistical nuclear theory will be considered in this section. Eq.(3) can be rewritten as follows

$$N=\frac{V}{\pi^2}\int_0^\infty\frac{k^2\,dk}{exp\,[(\varepsilon-\varepsilon_n)/\tau]+1}\,, \tag{3a}$$

where $\tau\equiv kT$, $\varepsilon=p^2/2M$, $\varepsilon_n\equiv\eta kT$: Fermi level for neutron, and

$k\ (=\hbar p)$: wave number vector.

Now, if we introduce the quantity α in the meaning of symmetry energy, neutron excess ends ε in Eq. (3a) to $\varepsilon+\alpha$ ($\alpha>0$). Then, it holds

$$N=\frac{V}{\pi^2}\int_0^\infty\frac{k^2dk}{exp[(\varepsilon+\alpha-\varepsilon_n)/\tau]+1} \tag{12}$$

Because of the Coulomb energy and proton defect, the ε in the proton case tends to $\varepsilon+eV-\alpha$, then

$$Z=\frac{V}{\pi^2}\int_0^\infty\frac{k^2dk}{exp\,[\varepsilon+eV-\alpha-\varepsilon_p)/\tau]+1}\,, \tag{13}$$

where ε_p ($\equiv\eta kT$) is the Fermi level for proton and of course equal to ε_n.

Putting $\varepsilon_n-\alpha=\mu_n$, $\varepsilon_p-\overline{eV-\alpha}=\mu_p$, then we get

$$N\sim\frac{2}{3}\mu_n^{\frac{3}{2}}+\frac{\pi^2}{12}\tau^2\cdot\mu_n^{-\frac{1}{2}}+\cdots\cdots\cdots\cdots,$$

$$Z\sim\frac{2}{3}\mu_p^{\frac{3}{2}}+\frac{\pi^2}{12}\tau^2\cdot\mu_p^{-\frac{1}{2}}+\cdots\cdots\cdots\cdots. \tag{14}$$

In the first order approximation, neutron excess ratio is

$$\frac{N-Z}{N+Z}=\frac{\mu_n^{\frac{3}{2}}-\mu_p^{\frac{3}{2}}}{\mu_n^{\frac{3}{2}}+\mu_p^{\frac{3}{2}}}\equiv\gamma,\quad\text{hence}\ \frac{\mu_n^{\frac{3}{2}}}{\mu_p^{\frac{3}{2}}}=\frac{1+\gamma}{1-\gamma}. \tag{15}$$

$$\text{Putting}\ \frac{\mu_N}{(1+\gamma)^{\frac{2}{3}}}=\frac{\mu_p}{(1-\gamma)^{\frac{2}{3}}}\equiv\mu \tag{16}$$

then,

$$\mu_N-\mu_p=\mu[(1+\gamma)^{\frac{2}{3}}-(1-\gamma)^{\frac{2}{3}}]. \tag{17}$$

Hence,

$$Z = \frac{V}{\pi^2} \int_0^\infty \frac{k^2 dk}{exp[(\varepsilon - \varepsilon_p - \mu[(1+\gamma)^{\frac{2}{3}} - (1-\gamma)^{\frac{2}{3}}])/\tau] + 1} \tag{18}$$

Now, if we describe the Heisenberg curve $Z/A = 1/(1+\delta)$, then

$$\delta = I_{\frac{1}{2}}(\eta)/I_{\frac{1}{2}}(\eta - \frac{\mu}{kT}\{(1+\gamma)^{\frac{2}{3}} - (1-\gamma)^{\frac{2}{3}}\}), \tag{19}$$

where $\eta kT \sim 24MeV$ as is well known.[6)]

The best fit with the Heisenberg curve is given by the set of parameter of $\eta = 4$ or $kT = 5 \sim 6MeV$ (see Fig. 2).

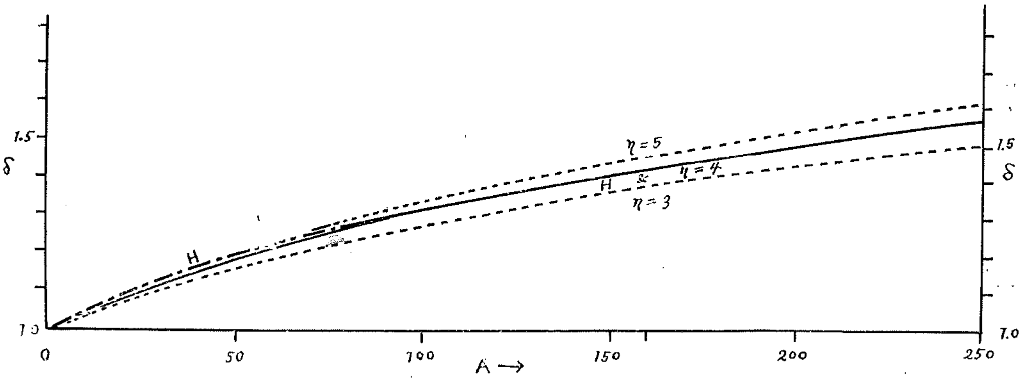

Fig. 2. δ defined by Eq. (19) vs. mass number A. H : Heisenberg curve. ηkT means the Fermi energy. The parametr $\eta = 4$ gives good agreement with Heisenberg curve. This means $kT = 5 \sim 6$ Mev.

This kT is much lower than the 23.58 MeV estimated before and is the same order of magnitude as was given by Euler[1)] and Watanabe[2)]. This result shows the important significance of the symmetry energy effect in the theory of atomic nuclei.

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