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On Some Gap Theorem for Euler's Method of Summation of Series

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三浦自治：級数のオイラーの総和法に対する或空隙定理に関して（英文）

Hardy and Littlewood (Hardy and Littlewood, Proceedings of the London Mathematical society, (2), vol. 25 (1926)) have proved the following theorem :

For a given series $\sum_{k=1}^{\infty} a_{n_k}$, ($a_{n_k} \neq 0$), let θ be a fixed constant such that

$$\frac{n_{k+1}}{n_k} \geq \theta > 1, \quad (k=1, 2, \dots).$$

If this series be summable by Abel's method of summation to the sum s , then this series is convergent and its sum is s .

Obreschkoff (Obreschkoff, Tôhoku Mathematical Journal, vol. 32 (1930)) obtained also a similar result for Cesàro's method.

Professor Okada (Y. Okada, Bull. of the American Math. Soc. (1937)) obtained the following result for Euler's method :

Theorem. Let $\sum_{n=0}^{\infty} a_n$ be a given series summable by Euler's method, that is, if

$$s_0 = 0, \quad s_n = a_0 + a_1 + \dots + a_{n-1}, \quad (n \geq 1),$$

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} \left\{ s_0 + ns_1 + \frac{n(n-1)}{2!} s_2 + \dots + s_n \right\} = s$$

exists; and for two given increasing sequences $\{n_k\}$, $\{n'_k\}$, ($n_k < n'_k$), of positive integers and for a given number a , ($1 \leq a < 2$), let

$$a_\nu = 0, \quad \text{for } n_k < \nu < n'_k, \quad (k=1, 2, \dots),$$

$$a_n = O(a^n).$$

If $n'_k/n_k \geq (1+\eta)/(1-\eta)$, ($k=1, 2, \dots$), for a positive number η such that

$$(1+\eta) \log(1+\eta) + (1-\eta) \log(1-\eta) - 2 \log a > 0,$$

then

$$\lim_{k \rightarrow \infty} \sum_{\nu=0}^{n'_k} a_\nu = s.$$

Professor Okada's result is about the special case, $p=1$ in Euler's method of summation to $\sum_{n=0}^{\infty} a_n$:

$$\lim_{n \rightarrow \infty} \frac{1}{(q+1)^n} \left\{ \binom{n}{0} q^n s_0 + \binom{n}{1} q^{n-1} s_1 + \dots + \binom{n}{n} s_n \right\} = s, \quad (q=2^p-1).$$

Here we will research in the same way about the general case, $p \geq 1$.

Theorem. Let $\sum_{n=0}^{\infty} a_n$ be a given series summable by Euler's method, that is, if $s_0=0, s_n=a_0+a_1+\dots+a_{n-1}$, ($n \geq 1$), $q=2^p-1$, ($p \geq 1$),

$$(1) \quad \lim_{n \rightarrow \infty} \frac{1}{(q+1)^n} \left\{ \binom{n}{0} q^n s_0 + \binom{n}{1} q^{n-1} s_1 + \dots + \binom{n}{n} s_n \right\} = s$$

exists; and for two given increasing sequences $\{n_k\}, \{n'_k\}$, ($n_k < n'_k$), of positive integers and for a given number α , let

$$(2) \quad a_\nu = 0, \text{ for } n_k < \nu < n'_k, \text{ (} k=1, 2, \dots \text{)}, \\ a_n = O(q^n),$$

where $1 \leq \alpha < \frac{q+1}{q}$.

If $n'_k/n_k \geq (1+\eta)/(1-\eta)$, ($k=1, 2, \dots$), for a positive number η such that

$$(1+\eta) \log(1+\eta) + (1-\eta) \log(1-\eta) - 2 \left(\log \left(\frac{2q}{q+1} \right) + \log \alpha \right) > 0,$$

then

$$(3) \quad \lim_{k \rightarrow \infty} \sum_{\nu=0}^{n_k} a_\nu = s.$$

Proof. To prove this, we can consider that all $n'_k - n_k - 1$ are even. Then putting

$$n_k + \frac{n'_k - n_k - 1}{2} + 1 = m,$$

from (2) we have

$$a_{m-1} = a_{m-2} = \dots = a_{n_k+1} = 0,$$

$$a_m = a_{m+1} = \dots = a_{n'_k-1} = 0.$$

Hence, if we put

$$s_n^{(p)} = \frac{1}{(q+1)^n} \left\{ \binom{n}{0} q^n s_0 + \binom{n}{1} q^{n-1} s_1 + \dots + \binom{n}{n} s_n \right\},$$

then we have

$$\begin{aligned} s_{2m}^{(p)} - s_m &= \frac{1}{(q+1)^{2m}} \left\{ q^{2m} s_0 + 2mq^{2m-1} s_1 + \frac{2m(2m-1)}{2!} q^{2m-2} s_2 + \dots + s_{2m} \right\} \\ &\quad - \frac{1}{(q+1)^{2m}} \left\{ q^{2m} s_m + 2mq^{2m-1} s_m + \frac{2m(2m-1)}{2!} q^{2m-2} s_m + \dots + s_m \right\} \\ &= \frac{1}{(q+1)^{2m}} \left\{ -(a_0 + \dots + a_{n_k}) q^{2m} - 2m(a_1 + \dots + a_{n_k}) q^{2m-1} - \frac{2m(2m-1)}{2!} \right. \\ &\quad \times (a_2 + \dots + a_{n_k}) q^{2m-2} - \dots - \frac{2m(2m-1) \dots (2m-n_k+1)}{n_k!} a_{n_k} q^{2m-n_k} \\ &\quad \left. + \frac{2m(2m-1) \dots (2m-n'_k)}{(n'_k+1)!} a_{n'_k} q^{2m-n'_k-1} + \dots + (a_{n'_k} + \dots + a_{2m-1}) \right\}. \end{aligned}$$

Therefore

$$\begin{aligned} |s_{2m}^{(p)} - s_m| &\leq \left(\frac{q}{q+1} \right)^{2m} \frac{2m(2m-1) \dots (2m-n_k+1)}{n_k!} \{ (|a_0| + \dots + |a_{n_k}|) \\ &\quad + (|a_1| + \dots + |a_{n_k}|) + \dots + |a_{n_k}| \} + \{ |a_{n'_k}| + (|a_{n'_k}| + |a_{n'_k+1}|) + \dots \\ &\quad + (|a_{n'_k}| + \dots + |a_{2m-1}|) \}. \end{aligned}$$

Since from (2) we can find a positive constant M such that $|a_n| < Mq^n$, ($n=0, 1, 2, \dots$); we get

$$|s_{2m}^{(p)} - s_m| < 2M \left(\frac{q\alpha}{q+1} \right)^{2m} (n_k+1)^2 \frac{2m(2m-1) \dots (2m-n_k+1)}{n_k!}$$

On Some Gap Theorem for Euler's Method of Summation of Series

$$< 4M \frac{e^{2m} \log a'}{2^{2m}} \frac{2m\Gamma(2m)}{\Gamma(m+\lambda)\Gamma(m-\lambda)},$$

where $a' = \frac{2q\alpha}{q+1}$, $\lambda = m - n_k$.

Let us now put

$$f(m) = \frac{e^{2m} \log a'}{2^{2m}} \frac{2m\Gamma(2m)}{\Gamma(m+\lambda)\Gamma(m-\lambda)} = \frac{1}{2^{2m}} \left(\frac{2q}{q+1} \alpha \right)^{2m} \frac{2m\Gamma(2m)}{\Gamma(m+\lambda)\Gamma(m-\lambda)}$$

or

$$f(m) = \frac{e^{2m} \log a'}{2^{2m}} \frac{2m\Gamma(2m)}{\Gamma(m(1-\delta))\Gamma(m(1+\delta))},$$

where $\lambda = m\delta$, $(0 < \delta = (n'_k - n_k + 1)/(n'_k + n_k + 1) < 1)$.

Then

$$\begin{aligned} \log f(m) &= 2m \log a' - 2m \log 2 + \log(2m) \\ &+ \left(2m - \frac{1}{2}\right) \log(2m) - 2m + O(1) \\ &- \left\{m(1-\delta) - \frac{1}{2}\right\} \log((1-\delta)m) + (1-\delta)m + O(1) \\ &- \left\{m(1+\delta) - \frac{1}{2}\right\} \log((1+\delta)m) + (1+\delta)m + O(1) \\ &= -m\psi(\delta) + \frac{3}{2} \log m + O(1), \end{aligned}$$

where

$$\psi(\delta) = (1+\delta)\log(1+\delta) + (1-\delta)\log(1-\delta) - 2\log a'.$$

Since from our assumption we have $1 \leq \alpha < \frac{q+1}{q}$, that is,

$$1 \leq \frac{2q}{q+1} \leq \frac{2q}{q+1} \alpha < 2, \text{ we get } 1 \leq a' < 2.$$

A fixed number η_0 , ($1 > \eta_0 \geq 0$) such that $\psi(\eta_0) = 0$ for the above a' exists necessarily.

Any number η such that $\eta_0 < \eta < 1$ for this η_0 gives $\psi(\eta) > 0$.

When η is so fixed, it follows from (1) that

$$\lim s_m = \lim s_m^{(p)} = s \text{ for } 1 > \delta > \eta.$$

From the other assumption $n'_k/n_k \geq (1+\eta)/(1-\eta)$ to the above η , we have

$$\frac{n'_k - n_k + 1}{n'_k + n_k + 1} > \eta.$$

Consequently $1 > \delta > \eta$ since

$$\delta = (n'_k - n_k + 1)/(n'_k + n_k + 1).$$

Therefore

$$\lim_{k \rightarrow \infty} S_{n_k} = \lim_{m \rightarrow \infty} s_m = s.$$

Thus our theorem is proved.

Remark. From Theorem and Knopp's theorem (Knopp, Math. Zeitschrift, vol. 15 (1922)) follows immediately Ostrowski's theorem :

Let $f(Z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series whose radius of convergence is 1.

If $a_\nu = 0$ for $n_k < \nu < n'_k$,

and

$$\frac{n'_k}{n_k} < 1 + \theta, \quad (k=1, 2, \dots),$$

Shiroji Miura

θ being a positive constant, then the partial sums s_{n_k} of this series converge uniformly in a full neighbourhood of every regular point of the function, $f(z)$ on the unit circle.