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On Finsler Spaces with (α, β)-metric. Regularity, Geodesics and Main Scalars.

(Dedicated to the memory of Professor Dr. Chōkō Shibata)

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(α, β)-計量をもつフィンスラー空間

―― 正則性，測地線，主スカラー ――

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Abstract

Ever since Professor Dr. C. Shibata started on the course of studying the differential geometry of Finsler spaces, he had been especially interested in Finsler spaces with (α, β)-metric, which have been paid special attention in the theoretical physics. We have a comprehensive survey [5] of the theory of Finsler spaces with (α, β)-metric written by the third author in 1991, in which ten papers of C. Shibata were quoted. We dedicate to his memory the present paper consisting of basic and important remarks on the theory of Finsler spaces with (α, β)-metric.

§ 1. Irregular (α, β)-metrics.

We shall consider an n-dimensional Finsler space $F^n=(M^n, L(\alpha, \beta))$ with (α, β)-metric $L(\alpha, \beta)$ ([3], §30; [1], 1.4.2). As usual, we suppose the following conditions :

(1) The associated Riemannian space $R^n=(M^n, \alpha)$ ([7]) is quasi-Riemannian. That is, the Riemannian metric

$$\alpha^2(x, dx)=a_{ij}(x)dx^i dx^j$$

is not always positive-definite, but is *regular* ($a=\det(a_{ij}) \neq 0$) ([1], 1.1.2).

(2) Throughout the paper we except the points (x, y) which $\beta(x, y)=b_i(x)y^i$ vanishes.

In the following, $(a^{ij}(x))=(a_{ij}(x))^{-1}$, and the raising and lowering of indices are done by means of a^{ij} and a_{ij} respectively.
For instance,

$$b^i=a^{ij}b_j, \qquad y_i=a_{ij}y^j.$$

Further, for a function $f(\alpha, \beta)$ of α and β such as $L(\alpha, \beta)$ and $F(\alpha, \beta)=L^2(\alpha, \beta)/2$, we use the following partial differential symbols :

$$f_1=\partial f/\partial\alpha, \qquad f_2=\partial f/\partial\beta.$$

Lemma 1. If we put $b^2=b_ib^i$ and $\gamma^2=b^2\alpha^2-\beta^2$, then γ^2, a quadratic form of y^i, does not vanish.

In fact, if $\gamma^2=(b^2a_{ij}-b_ib_j)y^iy^j$ vanishes, then we have $b^2a_{ij}=b_ib_j$. If $b^2\neq 0$ (resp. $b^2=0$), then we get the determinant $a=0$ (resp. $\beta=0$). Both of them lead us to a contradiction.

Now, as it was shown in ([3], §30) the fundamental tensor $g_{ij}(x, y)=\dot{\partial}_i\dot{\partial}_jF$ of F^n and its determinant $g=\det(g_{ij})$ are given by

$$g_{ij}=pa_{ij}+p_0b_ib_j+p_{-1}(b_iy_j+b_jy_i)+p_{-2}y_iy_j, \tag{1.1}$$

$$g=(p^{n-2}T)a, \tag{1.2}$$

where, using L or F, coefficients p, p_0, p_{-1}, p_{-2} are written as

$$\begin{cases} p=LL_1/\alpha=F_1/\alpha, \qquad p_0=LL_{22}+(L_2)^2=F_{22}, \\ p_{-1}=(LL_{12}+L_1L_2)/\alpha=F_{12}/\alpha, \\ p_{-2}=\{LL_{11}+(L_1)^2-LL_1/\alpha\}/\alpha^2=(F_{11}-F_1/\alpha)/\alpha^2. \end{cases} \tag{1.1a}$$

Further T of (1.2) is written as

$$T=p(p+p_0b^2+p_{-1}\beta)+\{p_0p_{-2}-(p_{-1})^2\}\gamma^2. \tag{1.2a}$$

From the homogeneity property of F we have

$$F_1\alpha+F_2\beta=2F, \quad F_{11}\alpha+F_{12}\beta=F_1, \quad F_{21}\alpha+F_{22}\beta=F_2,$$

so that we obtain

$$\begin{aligned} F_{11}F_{22}-(F_{12})^2 &= \{2FF_{11}-(F_1)^2\}/\beta^2=L^3L_{11}/\beta^2 \\ &= -(2FF_{12}-F_1F_2)/\alpha\beta=-L^3L_{12}/\alpha\beta \\ &= \{2FF_{22}-(F_2)^2\}/\alpha^2=L^3L_{22}/\alpha^2. \end{aligned} \tag{1.3}$$

Consequently T is rewritten as

$$\begin{aligned} T &= 2FF_1/\alpha^3+\{F_{11}F_{22}-(F_1)^2\}\gamma^2/\alpha^2 \\ &= (L/\alpha)^3(L_1+L_{11}\alpha\gamma^2/\beta^2). \end{aligned} \tag{1.2'a}$$

Therefore this is the same T as ([1], 3.5.1).

Now, in the theory of Finsler spaces with (α, β)-metric, the regularity $(g\neq 0)$ of the metric is a very important assumption, but it has never been discussed. Accordingly, we shall find $L(\alpha, \beta)$ being irregular. By (1.2) and (1.2'a) it is $L(\alpha, \beta)$ which satisfies

$$L_1\beta^2+L_{11}\alpha(b^2\alpha^2-\beta^2)=0. \tag{1.4}$$

The coefficient $L_{11}\alpha^3$ of b^2 is zero whenever L_{11} vanishes, which leads us to a contradiction, because (1.4) gives $L_1=0$. Therefore from (1.4) b^2 is expressed as $b^2=f(\alpha, \beta)$ being a function of α and β. Since b^2 does not depend on y^i, from $\dot{\partial}_i\alpha=y_i/\alpha$ and $\dot{\partial}_i\beta=b_i$, differentiation this by y^i yields

$$f_1y_i/\alpha+f_2b_i=0.$$

Transvecting this by y^i and b^i, we have

$$f_1\alpha + f_2\beta = 0, \qquad f_1\beta/\alpha + f_2 b^2 = 0.$$

Thus from $b^2\alpha^2 - \beta^2 = \gamma^2 \neq 0$ (Lemma 1) it follows that $f_1 = f_2 = 0$, which implies b^2 being constant. So that (1.4) is rewritten as

$$\frac{L_{11}}{L_1} + \frac{b^2\alpha}{b^2\alpha^2 - \beta^2} - \frac{1}{\alpha} = 0,$$

and integrating this by α

$$L_1 = \frac{c_0\alpha}{\sqrt{|b^2\alpha^2 - \beta^2|}},$$

where c_0 being a function of β should be non-zero constant from the homogeneity of L_1. Further integrating this by α once more, we get

$$b^2 \neq 0 : \qquad L = c_1\sqrt{|b^2\alpha^2 - \beta^2|} + c(\beta),$$

$$b^2 = 0 : \qquad L = c_1\alpha^2/\beta + c(\beta),$$

where c_1 is non-zero constant, and $c(\beta)$ being a function of β must be $c(\beta) = c_2\beta$ (c_2 = constant), because L is (1) p-homogeneous with respect to α and β.

Summarizing up the above we have the following theorem.

Theorem 1. *An (α, β)-metric $L(\alpha, \beta)$ is irregular, if and only if b^2 is constant and L has the following form* :

(1) $b^2 \neq 0$: $\qquad L = c_1\sqrt{|b^2\alpha^2 - \beta^2|} + c_2\beta,$

(2) $b^2 = 0$: $\qquad L = c_1\alpha^2/\beta + c_2\beta,$

where c_1 ($\neq 0$) and c_2 are constants.

Remark. Thus a Kropina metric $L = \alpha^2/\beta$ is irregular when b^2 vanishes. In the celebrated paper [7] of C. Shibata concerned with Kropina metrics, we have ρ ($= b^2$) in the denominator of (2.4) giving g^{ij}.

Example 1 : For a Randers metric $L = \alpha + \beta$, from (1.2'a) we get $T = (L/\alpha)^3 \neq 0$, so that it is regular. As to a *generalized Kropina metric* $L = \alpha^t\beta^{1-t}$ ($t \neq 0,1$) ([8], (3.19)1)), we obtain

$$T = t\alpha^{4t-4}\beta^{2-4t}\{(2-t)\beta^2 + (t-1)b^2\alpha^2\}.$$

Hence $T = 0$ gives

$$(2-t)b_ib_j + (t-1)b^2a_{ij} = 0.$$

Consequently we obtain $b^2 = 0$ and $t = 2$, so that this case reduces to the above remark. Therefore a generalized Kropina metric which is not a Kropina metric is all regular.

§ 2 . Geodesic equations in the Riemannian parameter.

By the arc-length s (*Finslerian parameter*) the equations of a geodesic of $F^n = (M^n, L(\alpha, \beta))$ is written in the well-known form

$$\text{(2.1)} \qquad \frac{d^2x^i}{ds^2} + 2G^i\left(x, \frac{dx}{ds}\right) = 0,$$

where functions $G^i(x, y)$ are given by

$$2G^i = g^{ir}(y^j \dot{\partial}_r \partial_j F - \partial_r F), \qquad F = \frac{L^2}{2}.$$

Especially for an (α, β)-metric, the results of [4] may be applied to G^i. In case of the Randers metric $L = \alpha + \beta$, however, using the arc-length τ (*Riemannian parameter*) in the associated Riemannian space $R^n = (M^n, \alpha)$, the equations of a geodesic have been simply written by Randers himself ([1], (1.3.2. 2)). So, in this section, using the parameter τ, we shall give the equations of a geodesic of F^n.

Using the parameter τ, (2.1) is written as

$$\text{(2.1')} \qquad \frac{d^2x^i}{d\tau^2} + 2G^i\left(x, \frac{dx}{d\tau}\right) = -\frac{\tau''}{(\tau')^2}\frac{dx^i}{d\tau},$$

where $\tau' = d\tau/ds$. Since $d\tau = \alpha(x, dx)$ and $ds = L(x, dx)$, we obtain.

$$\frac{d\tau}{ds} = \frac{1}{L(x, \dot{x})}, \qquad \frac{d^2\tau}{ds^2} = -\frac{1}{(L)^3}\left(L_1\frac{d\alpha}{d\tau} + L_2\frac{d\beta}{d\tau}\right),$$

where $\dot{x}^i = dx^i/d\tau$. Paying attention to $\alpha(x, \dot{x}) = 1$ along a geodesic, we get

$$\frac{d^2\tau}{ds^2} = \frac{-L_2}{(L)^3}\frac{d\beta}{d\tau} = -\frac{L^2}{(L)^3}\{b_i\ddot{x}^i + (b_{i;j} + b_r\gamma_i{}^r{}_j)\dot{x}^i\dot{x}^j\},$$

where $b_{i;j}$ are the covariant derivatives of b_i with respect to Levi-Civita connection $\{\gamma_j{}^i{}_k(x)\}$ of the associated Riemannian space.

In the following we shall employ the symbols used in [4].

We put

$$\text{(2.2)} \qquad r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}), \qquad s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}), \qquad s^i{}_j = a^{ir}s_{rj}, \qquad s_i = b_r s^r{}_i.$$

Then we obtain

$$\frac{d^2\tau}{ds^2} = -\frac{L_2}{(L)^3}(G + r_{00} + b_r\gamma_0{}^r{}_0), \qquad G = b_i\ddot{x}^i.$$

Thus, as it was shown in [4], we put

$$2G^i = \gamma_0{}^i{}_0 + 2B^i,$$

and (2.1') becomes

$$\text{(2.1'')} \qquad \frac{d^2x^i}{d\tau^2} + \gamma_0{}^i{}_0 + 2B^i = \frac{L_2}{L}(G + r_{00} + b_r\gamma_0{}^r{}_0)\frac{dx^i}{d\tau}.$$

Now, it follows from [4] that B^i are

$$\text{(2.3)} \qquad B^i = \frac{E}{\alpha}y^i + \frac{\alpha L_2}{L_1}s^i{}_0 - \frac{\alpha L_{11}}{L_1}\left(C + \frac{\alpha}{2\beta}r_{00}\right)c^i,$$

where C, E and c^i are given by

$$\text{(2.4)} \qquad C = -\frac{\alpha^2 L_2}{\beta L_1}s_0 - \frac{\alpha L_{11}}{\beta^2 L_1}\gamma^2\left(C + \frac{\alpha}{2\beta}r_{00}\right),$$

$$\text{(2.5)} \qquad \frac{2L}{\alpha}E = L_2\left(r_{00} + \frac{2\beta}{\alpha}C\right),$$

and

$$c_i = \frac{1}{\alpha}y_i - \frac{\alpha}{\beta}b_i.$$

We now obtain

$$B^i b_i = \frac{\beta}{\alpha}(E - C),$$

because of $C=B^i c_i$ and $B^i b_i = \frac{\beta}{\alpha}B^i(\frac{1}{\alpha}y_i - c_i)$.

On eliminating C from this and (2.5), we get

$$2B^i b_i = r_{00} - \frac{2L_1}{L_2}E. \tag{2.6}$$

Consequently, transvecting (2.1″) by b_i and substituting (2.6) we have

$$G + b_r \gamma_0{}^r{}_0 + r_{00} = \frac{2LE}{\alpha L_2}. \tag{2.7}$$

Also, eliminating C by means of (2.3) and (2.5) we have

$$B^i = \frac{E}{\alpha}y^i + \frac{\alpha L_2}{L_1}s^i{}_0 - \frac{\alpha L L_{11} E}{\beta L_1 L_2}c^i.$$

Thus, putting $p^i = b^i - \frac{\beta}{\alpha^2}y^i$, we have

$$B^i = \frac{E}{\alpha}y^i + \frac{\alpha L_2}{L_1}s^i{}_0 + \frac{L L_{11} E \alpha^2}{L_1 L_2 \beta^2}p^i. \tag{2.8}$$

Therefore, by means of (2.7) and (2.8), (2.1″) is rewritten as

$$\frac{d^2x^i}{d\tau^2} + \gamma_0{}^i{}_0 + \frac{2L_2}{L_1}s^i{}_0 + \frac{2LL_{11}E}{L_1 L_2 \beta^2}p^i = 0. \tag{2.9}$$

Theorem 2. *In terms of arc-length τ in the associated Riemannian space $R^n = (M^n, \alpha)$, the equations of a geodesic of a Finsler space $F^n = (M^n, L(\alpha, \beta))$ with (α, β)-metric are written as (2.9), where $\gamma_j{}^i{}_k(x)$ are the Christoffel symbols of R^n, $p^i = b^i - \beta\dot{x}^i$, E is given by (2.4) and (2.5), and y^i of each points (x, y) is $\dot{x}^i = dx^i/d\tau$.*

Remark. From (2.4) C is uniquely determined and E from (2.5), provided that $1 + (\alpha L_{11}\gamma^2/\beta^2 L_1)$ does not vanish. When this is zero, $L(\alpha, \beta)$ is irregular from (1.4).
The vector p^i was often used by C. Shibata : [7], (2.7) and [8] (2.4)1).

Example 2 : We shall consider a Randers metric $L = \alpha + \beta$.
Because of $L_{11} = 0$, the equations of a geodesic are given by

$$\frac{d^2x^i}{d\tau^2} + \gamma_0{}^i{}_0 + 2s^i{}_0 = 0.$$

This coincides with the result of G. Randers ([1], (1.3.2, 2)).

Example 3 : Secondly we shall consider a generalized Kropina metric $L = \alpha^t\beta^{1-t}$ $(t \neq 0, 1)$.
From (2.4) we have

$$r_{00} + \frac{2\beta}{\alpha}C = \frac{r_{00}\beta^2 + 2s_0\alpha^2\beta(t-1)/t}{\beta^2 + (t-1)\gamma^2},$$

and (2.5) is written as

$$E = \frac{\alpha}{2\beta}(1-t)\left(r_{00} + \frac{2\beta}{\alpha}C\right).$$

Therefore the equations of a geodesic are given by

$$\frac{d^2x^i}{d\tau^2} + \gamma_0{}^i{}_0 + \frac{2(1-t)}{\beta t}s^i{}_0 + \frac{(t-1)\{r_{00}\beta + 2s_0(t-1)/t\}}{\beta\{\beta^2 + (t-1)\gamma^2\}}p^i = 0$$

Especially, in case of a Kropina metric $(t = 2)$, we obtain

$$\frac{d^2x^i}{d\tau^2} + \gamma_0{}^i{}_0 - \frac{1}{\beta}s^i{}_0 + \frac{r_{00}\beta + s_0}{\beta b^2}p^i = 0.$$

§ 3. Berwald frames.

In the theory of Finsler spaces the so-called C-tensor, defined by

$$C_{ijk} = \dot{\partial}_k(g_{ij}/2) = \dot{\partial}_i\dot{\partial}_j\dot{\partial}_k(L^2/4),$$

plays an important role : For instance, a Finsler space is a Riemannian space, a Berwald space and a Landsberg space, if and only if $C_{ijk} = 0$, $C_{ijk|h} = 0$ and $C_{ijk|0} = 0$ respectively ([1], [3]).

Throughout the present and the next sections we restrict our consideration to two-dimensional Finsler spaces. Then the C-tensor can be written as

(3.1) $$L\,C_{ijk} = I\,m_im_jm_k,$$

in the Berwald frame (l, m), where $I=I(x, y)$ is a scalar field, called the main scalar. The formula of I for a Finsler space with (α, β)-metric has been given in the paper [2]. It is, however, written in an isothermal coordinate system of the associated Riemannian space for the sake of computation and, as a consequence, it is not easy to put in practice.

The purpose of the present section is to establish the relation between the Berwald frames of a Finsler space $F^2 = (M^2, L(\alpha, \beta))$ with (α, β)-metric and its associated Riemannian space $R^2 = (M^2, \alpha)$, and, in the next section, we shall find the formula of the main scalar of F^2 as an application of the relation.

Now the *Berwald frame* (l, m) has been defined in ([3], § 28) but we shall here consider it without the assumption of positive-definiteness of the metric.

The contravariant components l^i and the covariant components l_i of the first vector l of (l, m) are given by

(3.2) (1) $l^i = y^i/L(x, y)$, (2) $l_i = \dot{\partial}_iL(x, y)$.

Then, in terms of l_i and the angular metric tensor $h_{ij} = L(\dot{\partial}_i\dot{\partial}_jL)$ the fundamental tensor $g_{ij} = \dot{\partial}_i\dot{\partial}_j(L^2/2)$ is written as

(3.3) $$g_{ij} = l_il_j + h_{ij}.$$

Since the rank of the symmetric matrix (h_{ij}) of order two is equal to one, we get a sign $\varepsilon = \pm 1$ and (m_1, m_2) satisfying

(3.4) $$h_{ij} = \varepsilon m_im_j, \qquad i, j = 1, 2.$$

It is noted that the covariant vector m_i is determined by (3.4) up to the orientation, while ε is uniquely determined. ε is called the *signature* of the space. Thus (3.3) can be rewritten as

(3.3′) $$g_{ij} = l_il_j + \varepsilon m_im_j.$$

The equation $h_{ij}y^j = 0$ yields $m_jl^j = 0$ from (3.4). The contravariant components $m^i = g^{ij}m_j$ satisfies

$$m_i = g_{ij}m^j = (l_il_j + \varepsilon m_im_j)m^j = (\varepsilon m_jm^j)m_i,$$

which implies $m_jm^j = \varepsilon$. Thus m^i are given by

(3.5) (1) $m^il_i = 0$, (2) $m^im_i = \varepsilon$.

It follows from (3.3′) that we have

(3.6) $$g = \det(g_{ij}) = \varepsilon(l_1m_2 - l_2m_1)^2.$$

Hence ε is the sign of g.

We consider the derivatives of (l, m) by y^i. From (3.2) we get immediately

(3.7) (1) $L\dot{\partial}_j l^i = \varepsilon m^i m_j$, (2) $L\dot{\partial}_j l_i = \varepsilon m_i m_j$.

Next we differentiate $m_i l^i = 0$ by y^k and get $(L\dot{\partial}_k m_i)l^i = -m_k$ from (3.7). From (3.3′) we get similarly

$$2LC_{ijk} = \varepsilon(m_i l_j + l_i m_j)m_k + \varepsilon(\dot{\partial}_k m_i)m_j + \varepsilon m_i(\dot{\partial}_k m_j).$$

Transvecting this by $m^i m^j$ and using (3.1) and (3.5), we have $(L\dot{\partial}_k m_i)m^i = Im_k$. Consequently we obtain (3.8)(2). Further we differentiate $m_i = g_{ij}m^j$ by y^k and using (3.8)(2), we get (3.8)(1). Therefore we obtain

(3.8) (1) $L\dot{\partial}_j m^i = -(l^i + \varepsilon I m^i)m_j$, (2) $L\dot{\partial}_j m_i = -(l_i - \varepsilon m_i I)m_j$.

Proposition 1. *The contravariant and covariant components of the Bervald frame (l, m) are given by (3.2), (3.4) and (3.5), where the signature ε is the sign of g : (3.6). Their derivatives by y^j are given by (3.7) and (3.8), where I is the main scalar.*

Now we are concerned with a two-dimensional Finsler space $F^2 = (M^2, L(\alpha, \beta))$ with (α, β)-metric and its associated Riemannian space $R^2 = (M^2, \alpha)$. We shall denote by (l, m) and (u, v) the Berwald frames of F^2 and R^2 respectively. (l, m) is found as above. As for (u, v) we have first from (3.2).

(3.9) (1) $u^i = y^i/\alpha(x, y)$, (2) $u_i = \dot{\partial}_i \alpha(x, y)$.

Next, if we denote by k_{ij} the angular metric tensor of R^2, then we have $k_{ij} = \alpha(\dot{\partial}_i \dot{\partial}_j \alpha)$ and, similarly to (3.3′), (3.4), (3.5) and (3.6)

(3.10)
(1) $a_{ij} = u_i u_j + e v_i v_j$, $e = \pm 1$,
(2) $k_{ij} = e v_i v_j$,
(3) $v^i u_i = 0$, $v^i v_i = e$,
(4) $a = \det(a_{ij}) = e(u_1 v_2 - u_2 v_1)^2$.

Further, similarly to (3.7) and (3.8), we have

(3.11)
(1) $\alpha\dot{\partial}_j u^i = e v^i v_j$, $\alpha\dot{\partial}_j u_i = e v_i v_j$,
(2) $\alpha\dot{\partial}_j v^i = -u^i v_j$, $\alpha\dot{\partial}_j v_i = -u_i v_j$,

where (2) are obtained from I=0 of R^2.

We consider b_i of $\beta = b_i(x)y^i$ in the frame (u, v). Putting $b_i = B_1 u_i + B v_i$, we transvect it by $y^i = \alpha u^i$ and $b^i = a^{ij}b_j$. Then we get $\beta = B_1\alpha$ and $b^2 = (B_1)^2 + e(B)^2$. Hence we obtain

(3.12) $$b_i = \frac{\beta}{\alpha}u_i + Bv_i,$$

(3.12a) $$(B)^2 = e\left(b^2 - \frac{\beta^2}{\alpha^2}\right) = e\frac{\gamma^2}{\alpha^2}.$$

Now we deal with the main purpose of the present section, finding the expression of (l, m) in (u, v). First (3.2) and (3.9) give $Ll^i = \alpha u^i$ and $l_i = L_1 u_i + L_2 b_i$. Hence (3.12) leads to

(3.13) (1) $l^i = \frac{\alpha}{L}u^i$, (2) $l_i = \frac{L}{\alpha}u_i + L_2 B v_i$.

Next (3.11)(1) gives

$$h_{ij} = L\dot{\partial}_j(L_1 u_i + L_2 b_i)$$
$$= LL_{11}u_iu_j + LL_{12}(u_ib_j + u_jb_i) + LL_{22}b_ib_j + e\frac{LL_1}{\alpha}v_iv_j.$$

Since we have $L_{11}\alpha + L_{12}\beta = L_{21}\alpha + L_{22}\beta = 0$ from the homogeneity, the *Weierstrass invariant* $w(\alpha, \beta)$ of the space, similarly to the case of a two-dimensional Finsler space ([1], 1.1.3), can be defined by

(3.14) $$\frac{L_{11}}{\beta^2} = \frac{-L_{12}}{\alpha\beta} = \frac{L_{22}}{\alpha^2} = w(\alpha, \beta).$$

Then we have $h_{ij} = eL(L_1/\alpha + w\gamma^2)v_iv_j$ by virtue of (3.12). Thus (3.4) leads to

(3.15) $$m_i = mv_i,$$

(3.15a) $$(m)^2 = e\varepsilon L\left(\frac{L_1}{\alpha} + w\gamma^2\right).$$

Finally we put $m^i = ru^i + sv^i$. Transvecting this by l_i and m_i, (3.13) and (3.15) lead to $Lr/\alpha + eL_2Bs = 0$ and $ems = \varepsilon$. Hence we get

(3.16) $$m^i = -\frac{\varepsilon}{Lm}(L_2\alpha Bu^i - eLv^i).$$

Proposition 2. *In the Berwald frame (u, v) of its associated Riemannian space R^2 of a two-dimensional Finsler space F^2 with (α, β)-metric, the Berwald frame (l, m) of F^2 is written as (3.13), (3.15) and (3.16), where B and m are given by (3.12a) and (3.15a) respectively, w by (3.14), and ε and e are signatures of F^2 and R^2 respectively. The vector b_i is written as (3.12).*

Example 4 : For a Kropina metric $L = \alpha^2/\beta$ we have

$$l^i = \frac{\beta}{\alpha}u^i, \qquad l_i = \frac{\alpha}{\beta}u_i - \left(\frac{\alpha}{\beta}\right)^2 Bv_i,$$

$$m^i = \varepsilon\frac{B\alpha}{m\beta}u^i + \frac{e\varepsilon}{m}v^i, \qquad m_i = mv_i,$$

$$b_i = \frac{\beta}{\alpha}u_i + Bv_i, \qquad (B)^2 = e\frac{\gamma^2}{\alpha^2}, \qquad (m)^2 = e\varepsilon\frac{2\alpha^4}{\beta^4}b^2.$$

C. Shibata gives in [7]

(2.2) $$h_{ij} = \left(\frac{\alpha}{\beta}\right)^2(2a_{ij} - l_ib_j - l_jb_i).$$

It is observed that this h_{ij} is written as $h_{ij} = e(2b^2\alpha^4/\beta^4)v_iv_j$, which is equal to $h_{ij} = \varepsilon m_im_j$.

§4. Main scalar.

We shall consider the main scalar I of a two-dimensional Finsler space with (α, β)-metric. If we write (3.6) in the Berwald frame (u, v), then we have from (3.10) (4)

$$\frac{\varepsilon g}{ea} = \left(\frac{Lm}{\alpha}\right)^2 = e\varepsilon\left(\frac{L}{\alpha}\right)^3(L_1 + w\alpha\gamma^2).$$

Since (1.2) reduces to $g/a = T$ in the two-dimensional case, the above can be written as

(4.1) $$\frac{g}{a} = T = e\varepsilon\left(\frac{Lm}{\alpha}\right)^2 = \left(\frac{L}{\alpha}\right)^3(L_1 + w\alpha\gamma^2),$$

which yields $T_2 = \partial T/\partial\beta$ of the form

$$T_2 = 2e\varepsilon\frac{Lm}{\alpha^2}(Lm)_2. \tag{4.2}$$

Now (3.15) and (3.11) give

$$\dot{\partial}_j m_i = (m_1 u_j + m_2 b_j)v_i - \frac{m}{\alpha}u_i v_j.$$

Substituting (3.12), we have

$$\dot{\partial}_j m_i = \frac{1}{\alpha}(m_1\alpha + m_2\beta)v_i u_j - \left(\frac{m}{\alpha}u_i - m_2 B v_i\right)v_j.$$

Since $m(\alpha, \beta)$ is positively homogeneous of degree zero in (α, β), the first term of the right-hand side of the above vanishes. On the other hand, (3.8) together with (3.13) and (3.15) yields

$$\dot{\partial}_j m_i = -\frac{m}{\alpha}u_i v_j + \frac{m}{L}(\varepsilon Im - L_2 B)v_i v_j.$$

Comparing these two expressions of $\dot{\partial}_j m_i$, we obtain $I=\varepsilon B(Lm)_2/(m)^2$ and (4.2) yields $I=eBT_2\alpha^2/2L(m)^3$. Since the sign of I depends on the orientation of m_i, it is enough to deal with I^2. Thus (3.12a) and (4.1) lead to

$$\varepsilon I^2 = \frac{L^4\gamma^2}{4\alpha^4}\frac{(T_2)^2}{(T)^3}. \tag{4.3}$$

Theorem 3. *The main scalar I of a two-dimensional Finsler space with (α, β)-metric is given by (4.3), where T is given by (4.1) and $T_2 = \partial T/\partial\beta$.*

Example 5 : (1) We consider a two-dimensional Randers metric $L=\alpha+\beta$. Then we have $w=0$ and $T=(L/\alpha)^3$, so that

$$\varepsilon I^2 = \frac{9\gamma^2}{4(\alpha+\beta)\alpha}, \qquad \textit{Cf.}\ [1],\ p.127\ ;\ [2].$$

(2) We deal with a two-dimensional generalized Kropina metric $L = \alpha^r\beta^s$, $r + s = 1$. We have

$$w = -rs\alpha^{r-2}\beta^{s-2}, \qquad T=r\alpha^{4r-4}\beta^{4s-2}\{(1 + s)\beta^2 - sb^2\alpha^2\}.$$

Hence we obtain

$$\varepsilon I^2 = \frac{s^2\{2(1 + s)\beta^2 + (1 - 2s)b^2\alpha^2\}^2\gamma^2}{r\{(1 + s)\beta^2 - sb^2\alpha^2\}^3}, \qquad \textit{Cf.}\ [2].$$

In particular, for a Kropina metric ($r = 2, s = -1$) we have

$$\varepsilon I^2 = \frac{9\gamma^2}{2b^2\alpha^2}.$$

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