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Infinitesimal Projective Transformations on the Tangent Bundles with the Horizontal Lift Connection

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水平リフト接続を持つ接バンドル上の無限小射影変換

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Abstract

Let (M, g) be a complete Riemannian manifold and TM its tangent bundle with the horizontal lift connection. If TM admits a non-affine infinitesimal projective transformation, then M and TM are locally flat.

1. Introduction

In the present paper everything will be always discussed in the C^{∞}-category, and manifolds will be assumed to be connected and dimension $n > 1$.

Let M be a manifold and TM its tangent bundle. We denote by $\Im^r_s(M)$ the set of all tensor fields of type (r, s) on M. Similarly, we denote by $\Im^r_s(TM)$ the corresponding set on TM.

Let ∇ be an affine connection on M. A vector field V on M is called an infinitesimal projective transformation if there exists a 1-form Ω such that

$$(L_V \nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X$$

for any $X, Y \in \Im^1_0(M)$, where L_V is the Lie derivation with respect to V. In this case Ω is called the associated 1-form of V. Especially, if $\Omega = 0$, then V is called an infinitesimal affine transformation.

Next, let g be a Riemannian metric on M and ∇ the Levi-Civita connection of g. Denote by V a vector field on M. V is called an infinitesimal conformal transformation if there exists a function f on M satisfying $L_V g = fg$. Especially, if f is constant, then V is called an infinitesimal homothetic transformation. Furthermore, if $f = 0$, then V is called an infinitesimal isometry.

In this paper, we prove the following

Theorem. *Let (M, g) be a complete Riemannian manifold and TM its tangent bundle with the horizontal lift connection. If TM admits a non-affine infinitesimal projective transformation, then M and TM* are *locally flat.*

2. Preliminaries

Let (M, g) be a Riemannian manifold, ∇ the Riemannian connection of g and $\Gamma_{ji}{}^h$ the coefficients of ∇, i.e., $\Gamma_{ji}{}^a \partial_a := \nabla_{\partial_j}\partial_i$, where $\partial_h = \frac{\partial}{\partial x^h}$ and (x^h) is the local coordinates of M. We define a local frame $\{E_i, E_{\bar{i}}\}$ of TM as follows:

$$E_i := \partial_i - y^b \Gamma_{ib}{}^a \partial_{\bar{a}} \text{ and } E_{\bar{i}} := \partial_{\bar{i}},$$

where (x^h, y^h) is the induced coordinates of TM and $\partial_{\bar{i}} := \frac{\partial}{\partial y^i}$, and call this frame $\{E_i, E_{\bar{i}}\}$ the adapted frame of TM. Then $\{dx^h, \delta y^h\}$ is the dual frame of $\{E_i, E_{\bar{i}}\}$, where $\delta y^h := dy^h + y^b \Gamma_{ab}{}^h dx^a$.

By the definition of the adapted frame, we have the following

Lemma 1. *The Lie brackets of the adapted frame of TM satisfy the following identities:*

(1) $[E_j, E_i] = y^b K_{ijb}{}^a E_{\bar{a}}$

(2) $[E_j, E_{\bar{i}}] = \Gamma_{ji}{}^a E_{\bar{a}}$,

(3) $[E_{\bar{j}}, E_{\bar{i}}] = 0$,

where $K = (K_{kji}{}^h)$ denotes the Riemannian curvature tensor of (M, g) defined by $K_{kji}{}^h := \partial_k \Gamma_{ji}{}^h - \partial_j \Gamma_{ki}{}^h + \Gamma_{ji}{}^a \Gamma_{ka}{}^h - \Gamma_{ki}{}^a \Gamma_{ja}{}^h$.

Let $\tilde{\nabla}$ be the horizontal lift connection on TM defined as follows:

$$\tilde{\nabla}_{E_j} E_i = \Gamma_{ji}{}^a E_a,\ \tilde{\nabla}_{E_j} E_{\bar{i}} = \Gamma_{ji}{}^a E_{\bar{a}},$$
$$\tilde{\nabla}_{E_{\bar{j}}} E_i = 0,\ \tilde{\nabla}_{E_{\bar{j}}} E_{\bar{i}} = 0.$$

This connection is the metric connection of the comlete lift metric $\tilde{g} = 2g_{ba} dx^b \delta y^a$ or the lift metric $\tilde{g} = g_{ba} dx^b dx^a + 2g_{ba} dx^b \delta x^a$. But it is not necessary for the present paper to use the lift metric itself.

We need the following well known lemma to prove Theorem.

Lemma 2 ([K1]). *If a complete Riemannian manifold M admits a non-isometric infinitesimal homothetic transformation, then M is locally flat.*

3. Infinitesimal projective transformation on *TM*

Proposition 1. *Let (M, g) be a Riemannian manifold and TM its tangent bundle with the horizontal lift connection. $\tilde{V}$ is an infinitesimal projective transformation with the associated 1-form $\tilde{\Omega}$ on TM if and only if there exist $\varphi, \psi \in \Im_0^0(M)$, $B = (B^h)$, $D = (D^h) \in \Im_0^1(M)$, $A = (A_i^h)$, $C = (C_i^h) \in \Im_1^1(M)$ satisfying*

(1) $(\tilde V^h,\ \tilde V^{\bar h}) = (B^h + y^a A_a^h,\ D^h + y^a C_a^h + y^a y^h \Phi_a)$,

(2) $(\tilde\Omega_i,\ \tilde\Omega_{\bar i}) = (\partial_i \psi,\ \partial_i \varphi) = (\Psi_i\ ,\ \Phi_i\ )$,

(3) $\nabla_j \Phi_i = 0\ ,\ \nabla_j \Psi_i = 0$,

(4) $\nabla_j A_i^h = \Phi_i\, \delta_j^h$,

(5) $\nabla_j C_i^h = \Psi_j\, \delta_i^h - K_{aji}{}^h B^a$,

(6) $L_B \Gamma_{ji}{}^h = \nabla_j \nabla_i B^h + K_{aji}{}^h B^a = \Psi_j\, \delta_i^h + \Psi_i\, \delta_j^h$,

(7) $\nabla_j \nabla_i D^h = 0$.

(8) $K_{kja}{}^h A_i^a = 0$,

where $(\tilde V^h,\ \tilde V^{\bar h}) := \tilde V^a E_a + \tilde V^{\bar a} E_{\bar a} = \tilde V,\ (\tilde\Omega_i,\ \tilde\Omega_{\bar i}) := \tilde\Omega_a dx^a + \tilde\Omega_{\bar a}\delta y^a = \tilde\Omega$.

Proof. Here we prove only the necessary condition because it is easy to prove the sufficient condition. Let $\tilde V$ be an infinitesimal projective transformation with the associated 1-form $\tilde\Omega$ on TM.

From $(L_{\tilde V}\tilde\nabla)(E_{\bar j},\ E_{\bar i}) = \tilde\Omega_{\bar j} E_{\bar i} + \tilde\Omega_{\bar i} E_{\bar j}$, we obtain

(3.1) $$\partial_{\bar j}\partial_{\bar i}\tilde V^h = 0$$

and

(3.2) $$\partial_{\bar j}\partial_{\bar i}\tilde V^{\bar h} = \tilde\Omega_{\bar j}\delta_i^h + \tilde\Omega_{\bar i}\delta_j^h .$$

From (3.1), we have

$$\partial_{\bar i}\tilde V^h = A_i^h$$

and

(3.3) $$\tilde V^h = B^h + y^a A_a^h ,$$

where B^h and A_i^h are certain functions which depend only on x^h. The coordinate transformation rule implies that $B = (B^h) \in \Im_0^1(M)$ and $A = (A_i^h) \in \Im_1^1(M)$.

From $(L_{\tilde V}\tilde\nabla)(E_{\bar j},\ E_i) = \tilde\Omega_{\bar j} E_i + \tilde\Omega_i E_{\bar j}$, using (3.3), we obtain

(3.4) $$\tilde\Omega_{\bar j}\delta_i^h = \nabla_i A_j^h$$

and

(3.5) $$\tilde\Omega_j \delta_i^h = K_{aij}{}^h B^a + y^a\left(K_{bij}{}^h A_a^b + K_{bia}{}^h A_j^b\right) + \Gamma_{ai}{}^h \partial_{\bar j}\tilde V^{\bar a}\partial_{\bar j}\left(E_i \tilde V^{\bar h}\right).$$

Contracting j and h in (3.4), we have $\tilde\Omega_{\bar j} = \nabla_i A_a^a$ · Therefore we have

(3.6) $$\tilde\Omega_{\bar i} = \Phi_i = \partial_i \varphi,$$

where $\varphi_i := A_a^a$ and $\Phi_i := \partial_i \varphi$. From (3.4) and (3.6), we get

(3.7) $$\nabla_j A_i^h = \Phi_i\, \delta_j^h .$$

From (3.2) and (3.6), we have

$$\partial_{\bar i}\tilde V^{\bar h} = C_i^h + y^h \Phi_i + y^a \Phi_a \delta_i^h$$

and

(3.8) $$\tilde V^{\bar h} = D^h + y^a C_a^h + y^a y^h \Phi_a ,$$

where D^h and $C_i{}^h$ are certain functions which depend only on x^h. We can see that $D = (D^h) \in \Im_0^1(M)$ and $C = (C_i{}^h) \in \Im_1^1(M)$.

Substituting (3.8) into (3.5), we obtain

(3.9) $$\tilde\Omega_i \delta_j^h = \nabla_i C_j{}^h + K_{aij}{}^h B^a + y^a\left(K_{bij}{}^h A_a^b + K_{bia}{}^h A_j^b + \delta_a^h \nabla_i \Phi_j + \delta_j^h \nabla_i \Phi_a\right).$$

On the other hand, from $(L_{\tilde V}\tilde\nabla)(E_j, E_{\bar i}) = \tilde\Omega_j E_{\bar i} + \tilde\Omega_{\bar i} E_j$, we get

(3.10) $$\tilde\Omega_j \delta_i^h = \nabla_i C_j^h + K_{aji}{}^h B^a + y^a\left(K_{bji}{}^h A_a^b + \delta_a^h \nabla_j \Phi_i + \delta_i^h \nabla_j \Phi_a\right).$$

Comparing (3.9) with (3.10), we obtain

(3.11) $$K_{kja}{}^{h} A_i^a = 0$$

and

(3.12) $$\tilde{\Omega}_j \delta_i^h = \nabla_j C_i^h + K_{aji}{}^{h} B^a + y^a \left(\delta_a^h \nabla_j \Phi_i + \delta_i^h \nabla_j \Phi_a \right).$$

Contracting i and h in (3.12), we get

(3.13) $$n\tilde{\Omega}_j = \nabla_j C_a^a + (n+1) y^a \nabla_j \Phi_a .$$

Here, we put $\psi := \frac{1}{n} C_a^a$ and $\Psi_i := \nabla_i \psi = \partial_i \psi$. Then, from (3.12) and (3.13), we obtain

(3.14) $$\tilde{\Omega}_i = \Psi_i + \frac{n+1}{n} y^a \nabla_i \Phi_a ,$$

(3.15) $$\nabla_j C_i{}^{h} = \Psi_j \delta_i^h - K_{aji}{}^{h} B^a ,$$

and

(3.16) $$n \delta_k^h \nabla_j \Phi_i - \delta_i^h \nabla_j \Phi_k = 0.$$

Contracting k and h in (3.16), we have

(3.17) $$\nabla_j \Phi_i = 0.$$

From (3.14) and (3.17), we get

(3.18) $$\tilde{\Omega}_i = \Psi_i .$$

From $(L_{\tilde{V}} \tilde{\nabla})(E_j, E_i) = \tilde{\Omega}_j E_i + \tilde{\Omega}_i E_j$, we obtain

(3.19) $$L_B \Gamma_{ji}{}^{h} = \Psi_j \delta_i^h + \Psi_i \delta_j^h ,$$

(3.20) $$\nabla_j \nabla_i D^h = 0$$

and

(3.21) $$\nabla_k \nabla_j C_i{}^{h} + \nabla_k \left(K_{aji}{}^{h} B^a \right) = 0.$$

Substituting (3.15) into (3.21), we have

(3.22) $$\nabla_j \Psi_i = 0.$$

Q.E.D.

Proof of Theorem.

We put $X^h := A_a{}^{h} \Phi^a$. Then, using (3.7) and (3.17), we have

(3.23) $$\begin{aligned} L_X g_{ji} &= \nabla_j X_i + \nabla_i X_j \\ &= (\nabla_j A_{ai}) \Phi^a + (\nabla_i A_{aj}) \Phi^a \\ &= 2(\Phi_a \Phi^a) g_{ji} . \end{aligned}$$

Similarly we put $Y^h := \left(\nabla_a B^h - C_a{}^{h} \right) \Psi^a$. Then, using (3.15), (3.19) and (3.22),

(3.24) $$\begin{aligned} L_Y g_{ji} &= (\nabla_j \nabla_a B_i - \nabla_j C_{ai}) \Psi^a + (\nabla_i \nabla_a B_j - \nabla_i C_{aj}) \Psi^a \\ &= \left\{ \left(-K_{bjai} B^b + \Psi_j g_{ai} + \Psi_a g_{ji} \right) - \left(\Psi_j g_{ai} - K_{bjai} B^b \right) \right\} \Psi^a + (\Psi_a g_{ji}) \Psi^a \\ &= 2(\Psi_a \Psi^a) g_{ji} . \end{aligned}$$

Therefore X and Y are infinitesimal homothetic transformations. If M is not locally flat, then $\Phi = \Psi = 0$ by virtue of Lemma 2, and consequently $\tilde{\Omega} = 0$. This is a contradiction. Therefore M is locally flat. In this case TM is also locally flat.

Q.E.D.

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