Identification of Transition Curves in Vehicular Roads and Railways

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In the paper attention is focused on the necessity to systematize the procedure for determining the shape of transition curves used in vehicular roads and railway routes. There has been presented a universal method of identifying curvature in transition curves by using differential equations. Curvature equations for such known forms of transition curves as clothoid, quartic parabola, the Bloss curve, cosinusoid and sinusoid, have been worked out and by the use these equations it was possible to determine some appropriate Cartesian coordinates. In addition some approximate solutions obtained in consequence of making certain simplifying assumptions orientated mainly towards railway routes, have been provided. Notice has been taken of limitations occurring in the application of smooth transition curves in railway systems, which can be caused by very small values of the horizontal ordinates in the initial region. This problem has provided an inspiration for finding a new family of the so-called parametric transition curves, being more advantageous not only over the clothoid but also over cubic parabola as far as dynamics is concerned.

Keywords: vehicular roads, railway routes, transition curves, identification method, a new solution.

1. INTRODUCTION

The problem of transition curves applies to both roads and railways. However, it could be seen that there is a clear disproportion in the interest of this problem. In case of roads there still can be observed an effort of searching for new solutions in the area (e.g. [1, 3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 16, 23, 25, 27, 29, 30, 31, 32, 33, 34, 36, 37, 42, 43]). In the case of railroads, the situation is completely different − the investigations of new transition curves are not as numerous, and, what is more, the works were worked out relatively long time ago (e.g. [2, 9, 10, 15, 17, 18, 19, 22, 24, 26, 28, 35, 38, 39, 40, 41]).

It is obvious that the use of transition curves is aimed at ensuring a continuous change of an unbalanced side acceleration between vehicular road lengths (or railway track sections) of diversified curvature, in a way that is advantageous for the dynamics of the road – vehicle system interactions. Such a requirement concerns all types of transition curves. In this situation it might seem that there is one defined algorithm of their formation, which is common for the whole family of curves under consideration. However, all the solutions that have been known so far, appear independently and use various names (sometimes originating from the name of their author). A knowledge of a general method of determining transition curve equations would make it possible to compare different forms of curves and to prepare an assessment of their usefulness for practical application.

As a matter of fact, this problem was to a large extent explained 20 years ago, mainly with respect to railway lines [24]. It was then that the method of identifying unbalanced accelerations, occurring on various types of transition curves, was worked out. The method was based on a comparative analysis of some selected transition curves provided with a dynamic model. Acceleration was in it a factor exciting transverse vibrations of the vehicle [20]. The basic conclusion resulting from the considerations given to the subject was to indicate the existing relation between the response of the system and the class of the exciting function. The dynamic effects were smaller (that is more advantageous), if the class of the function was higher. It appeared that the largest acceleration values were definitely connected with a third-order parabola (function class C^0), whereas in the case of the Bloss curve and the cosinusoid (class C^1) they were significantly smaller. The smallest values were noted on sinusoid (class C^2).

The transition curves are also defined in a different way. As regards vehicular roads they are often defined by function of angle $\theta(l)$ which is responsible for changing the direction of the longitudinal axis of a vehicle after it has travelled a length of a certain arc. The railway engineering is, traditionally dominated by the use of curvature $k(x)$ with a rectangular system of coordinates. The acceleration values over the length of the transition curve result from the curvature distribution. As can be expected it is the curvature distribution that should form the basis for the identification of the transition curves. In general it can be linear or nonlinear.

2. METHODOLOGY FOR IDENTIFYING TRANSITION CURVES

2.1. DETERMINATION OF THE CURVATURE EQUATIO

Let us make an attempt at generalizing the method presented in the paper [24] with a view to applying it also to typical vehicular roads. In the curvature distribution $k(l)$ one can make use of a similar procedure. The curvature should be described by a function of an appropriate class to produce a lesser (i.e. more advantageous) dynamic effect.

Function $k(l)$ should be sought among the differential equation solutions

$$k^{(m)}(l) = f\left[l, k(l), k'(l), \ldots, k^{(m-1)}(l)\right] \quad (1)$$

with conditions at first (for $l = 0$) and then (for $l = l_k$) of the transition curve

$$k^{(i)}(0^+) = 0 \qquad \text{for} \qquad i = 0, 1, 2, \ldots, n_1$$

$$k^{(j)}(l_k^-) = \begin{cases} \frac{1}{R} & \text{for} \quad j = 0 \\ 0 & \text{for} \quad j = 1, 2, \ldots, n_2 \end{cases} \quad (2)$$

The order of the differential equation (3) is $m = n_1 + n_2 + 2$, and the obtained function $k(l)$ is a function of class C^n within the interval $\langle 0, l_k \rangle$, where $n = \min(n_1, n_2)$.

The presented mathematical notation is an identification of the shape of the transition curves by using differential equations. It describes the way to find some solutions satisfying an arbitrary number of assumed conditions. The solutions can be of entirely different form with respect to the given conditions.

2.2. DETERMINATION OF CARTESIAN COORDINATES

Making use of the transition curves in field necessitates the determination of their coordinates in terms of the Cartesian system x, y. In order to this one should first determine function $\theta(l)$

$$\Theta(l) = \int k(l)\, dl \quad (3)$$

and next the transition curve equation expressed in a parametric form:

$$x(l) = \int \cos \Theta(l)\, dl \quad (4)$$

$$y(l) = \int \sin \Theta(l)\, dl \quad (5)$$

Here it is necessary to explain that the determination of $x(l)$ and $y(l)$ by using equations (4) and (5) will require expansion of the integrands, in a general way, into Taylor (or Maclaurin) series [21].

2.3. A SIMPLIFIED METHOD OF DETERMINING THE TRANSITION CURVE EQUATION

On railway routes, as well as vehicular roads of fast traffic, where note is taken of great circular arc radii and relatively long transition curves, use is made of a commonly simplified technique for determining the transition curve equation which consequently provides us with this formula in the form of explicit function $y(x)$. The simplification of the procedure is based on the assumption that the modeled curvature $k(l)$ is related to its projection on axis x, that is, $l = x$, and $l_k = x_k$. As a result of such assumptions we obtain an initial equation for curvature $k_0(x)$. The determination of function $y(x)$ in an accurate way by analytical approach is impossible, for the reason that it would require to solve the differential equation

$$k_0(x) = \frac{y''(x)}{\{1 + [y'(x)]^2\}^{\frac{3}{2}}}$$

Therefore $k_0(x)$ is also traditionally treated as an initial curvature, being an approximation of target curvature $k(x)$. The transition from $k_0(x)$ to $k(x)$ takes place in such a way that $k_0(x)$ is assumed as equation of the second derivative of function $y(x)$ being sought; thus,

$$y''(x) = k_0(x) \tag{6}$$

Then, the equation is integrated twice, which gives $y'(x)$ and $y(x)$; conditions $y(0) = 0$ and $y'(0) = 0$ are taken into account.

Curvature $k(x)$ of the transition curve obtained differs, of course, from the initial curvature $k_0(x)$. The difference depends on the tangent slope value $y'(x)$. With regard to the transition curves used for railway purposes (if advantage is taken of a system of coordinates where the outset of the curve is tangent to the x-axis) the value of $y'(x)$ along the length is small, therefore the difference between curvatures $k_0(x)$ and $k(x)$ is, in practice, insignificant.

3. ANALYSIS OF SOME KNOWN FORMS OF TRANSITION CURVES

3.1. CLOTHOID

The clothoid is a curve proposed in 1874 by French physicist Marie Cornu, in connection with his research in the field of optics (diffraction of light). The following basic boundary conditions

$$\begin{cases} k(0^+) = 0 \\ k(l_k^-) = \frac{1}{R} \end{cases} \tag{7}$$

and the differential equation

$$k''(l) = 0 \tag{8}$$

are used. After determining the constants, the solution of the differential problem (7), (8) is as follows:

$$k(l) = \frac{1}{R\, l_k} l \tag{9}$$

Thus, we are dealing here with a linear change of curvature. From expressions (3) and (9), by the use of integration, it is possible to find angle θ.

$$\Theta(l) = \frac{1}{2\, R\, l_k} l^2 \tag{10}$$

Having expanded function $\cos \Theta(l)$ and $\sin \Theta(l)$ into Maclaurin series [21], on the basis of equations (4) and (5) we have

$$x(l) = \int \cos \Theta(l)\, dl = l - \frac{1}{40\, R^2\, l_k^2} l^5 + \frac{1}{3456\, R^4\, l_k^4} l^9 - \frac{1}{599040\, R^6\, l_k^6} l^{13} + \ldots \tag{11}$$

$$y(l) = \int \sin \Theta(l)\, dl = \frac{1}{6\, R\, l_k} l^3 - \frac{1}{336\, R^3\, l_k^3} l^7 + \frac{1}{42240\, R^5\, l_k^5} l^{11} - \ldots \tag{12}$$

The simplified form of clothoid in terms of the x, y system is obtained on accepting the following assumptions: abscissa $x = l$, the final point abscissa $x_k = l_k$, the initial curvature $k_0(x) = \frac{1}{R\, l_k} x$.

$$y(x) = \frac{1}{6\, R\, l_k} x^3 \tag{13}$$

The simplified clothoid carries its own name, the cubic parabola, and it has been used for years as the basic type of transition curve in railway engineering. It does not mean at all that the solution is most advantageous.

3.2. QUARTIC PARABOLA

Now the number of conditions has been increased. At the same time the conditions are being differentiated for the first and the second half of the transition curve. An appropriate differential equation is used.

$$k'''(l) = 0 \tag{14}$$

- For the first half of the transition curve (i.e. for $l \in \langle 0, \frac{l_k}{2} \rangle$) we have the conditions:

$$\begin{cases} k(0^+) = k'(0^+) = 0 \\ k\left(\frac{1}{2} l_k^-\right) = \frac{1}{2\, R} \end{cases} \tag{15}$$

Solution of differential problem of (14), (15) is as follows:

$$k(l) = \frac{2}{R\, l_k^2} l^2 \tag{16}$$

$$\Theta(l) = \frac{2}{3\, R\, l_k^2} l^3 \tag{17}$$

$$x(l) = \int \cos\Theta(l)\,dl = l - \frac{2}{63\,R^2\,l_k^4} l^7 + \frac{2}{3159\,R^4\,l_k^8} l^{13} - \ldots \tag{18}$$

$$y(l) = \int \sin\Theta(l)\,dl = \frac{1}{6\,R\,l_k^2} l^4 - \frac{2}{405\,R^3\,l_k^6} l^{10} + \frac{1}{14580\,R^5\,l_k^{10}} l^{16} - \ldots \tag{19}$$

A simplified form of the quartic parabola is obtained for $k_0(x) = \frac{2}{R\,l_k^2} x^2$.

$$y(x) = \frac{1}{6\,R\,l_k^2} x^4 \tag{20}$$

- For the second half of the transition curve (i.e. for $l \in \langle \frac{l_k}{2}, l_k \rangle$) the conditions are as follows:

$$\begin{cases} k\left(\frac{1}{2} l_k^+\right) = \frac{1}{2\,R} \\ k(l_k^-) = \frac{1}{R} \\ k'(l_k^-) = 0 \end{cases} \tag{21}$$

Now we have

$$k(l) = -\frac{1}{R} + \frac{4}{R\,l_k} l - \frac{2}{R\,l_k^2} l^2 \tag{22}$$

$$\Theta(l) = \frac{l_k}{6\,R} - \frac{1}{R} l + \frac{2}{R\,l_k} l^2 - \frac{2}{3\,R\,l_k^2} l^3 \tag{23}$$

After using the equations (18) and (19) for estimating the values $x\left(\frac{l_k}{2}\right)$ and $y\left(\frac{l_k}{2}\right)$, we obtain

$$x(l) = \int \cos\Theta(l)\,dl = x\left(\frac{l_k}{2}\right) + \cos\left(\frac{l_k}{12\,R}\right)\left(l - \frac{l_k}{2}\right) - \frac{1}{4\,R}\sin\left(\frac{l_k}{12\,R}\right)\left(l - \frac{l_k}{2}\right)^2 - \left[\frac{1}{24\,R^2}\cos\left(\frac{l_k}{12\,R}\right) + \frac{1}{3\,R\,l_k}\sin\left(\frac{l_k}{12\,R}\right)\right]\left(l - \frac{l_k}{2}\right)^3 + \left[\frac{1}{192\,R^3}\sin\left(\frac{l_k}{12\,R}\right) - \frac{1}{8\,R^2\,l_k}\cos\left(\frac{l_k}{12\,R}\right) + \frac{2}{12\,R\,l_k^2}\sin\left(\frac{l_k}{12\,R}\right)\right]\left(l - \frac{l_k}{2}\right)^4 + \ldots \tag{24}$$

$$y(l) = \int \sin\Theta(l)\,dl = y\left(\frac{l_k}{2}\right) + \sin\left(\frac{l_k}{12\,R}\right)\left(l - \frac{l_k}{2}\right) + \frac{1}{4\,R}\cos\left(\frac{l_k}{12\,R}\right)\left(l - \frac{l_k}{2}\right)^2 - \left[\frac{1}{24\,R^2}\sin\left(\frac{l_k}{12\,R}\right) - \frac{1}{3\,R\,l_k}\cos\left(\frac{l_k}{12\,R}\right)\right]\left(l - \frac{l_k}{2}\right)^3 - \left[\frac{1}{192\,R^3}\cos\left(\frac{l_k}{12\,R}\right) + \frac{1}{8\,R^2\,l_k}\sin\left(\frac{l_k}{12\,R}\right) + \frac{2}{12\,R\,l_k^2}\cos\left(\frac{l_k}{12\,R}\right)\right]\left(l - \frac{l_k}{2}\right)^4 + \ldots \tag{25}$$

The approximate solution is obtained for $k_0(x) = -\frac{1}{R} + \frac{4}{R\,l_k} x - \frac{2}{R\,l_k^2} x^2$; the following conditions are valid: $y'\left(\frac{l_k}{2}\right) = \frac{l_k}{12\,R}$, $y\left(\frac{l_k}{2}\right) = \frac{l_k^2}{96\,R}$.

$$y(x) = -\frac{l_k^2}{48\,R} + \frac{l_k}{6\,R} x - \frac{1}{2\,R} x^2 + \frac{2}{3\,R\,l_k} x^3 - \frac{1}{6\,R\,l_k^2} x^4 \tag{26}$$

Over the whole length of the curve there occurs a nonlinear change of the curvature. A similar situation will take place with respect to other smooth transition curves under consideration for which the number of the boundary conditions $n_1 = n_2$ (for $n_1 \neq n_2$ the curvature distribution becomes asymmetrical) .

3.3. BLOSS CURVE

In 1936, German engineer A. E. Bloss proposed a spiral transition for railways, in which as a curvature he used a simple polynomial of 3rd degree in relation to the length of the arc. In vehicular roads this curve is very often called Göldner curve. Further on the number of conditions is increased

$$\begin{cases} k(0^+) = k'(0^+) = 0 \\ k(l_k^-) = \frac{1}{R} \\ k'(l_k^-) = 0 \end{cases} \tag{27}$$

and use is made of differential equation

$$k^{(4)}(l) = 0 \tag{28}$$

The solution of problem (27), (28) provides the curvature equation

$$k(l) = \frac{3}{R\,l_k^2} l^2 - \frac{2}{R\,l_k^3} l^3 \tag{29}$$

$$\Theta(l) = \frac{1}{R\,l_k^2} l^3 - \frac{1}{2\,R\,l_k^3} l^4 \tag{30}$$

$$x(l) = \int \cos\Theta(l)\,dl = l - \frac{1}{14\,R^2\,l_k^4} l^7 + \frac{1}{16\,R^2\,l_k^5} l^8 - \frac{1}{72\,R^2\,l_k^6} l^9 + \ldots \tag{31}$$

$$y(l) = \int \sin \Theta(l)\, dl = \frac{1}{4\, R\, l_k^2} l^4 - \frac{1}{10\, R\, l_k^3} l^5 - \frac{1}{60\, R^3\, l_k^6} l^{10} + \frac{1}{44\, R^3\, l_k^7} l^{11} - \ldots \quad (32)$$

The simplified form of Bloss curve, using the x, y system, is obtained on the assumption $k_0(x) = \frac{3}{R\, l_k^2} x^2 - \frac{2}{R\, l_k^3} x^3$.

$$y(x) = \frac{1}{4\, R\, l_k^2} x^4 - \frac{1}{10\, R\, l_k^3} x^5 \quad (33)$$

3.4. COSINUSOID

Satisfying conditions (27) the curvature is identified by means of another differential equation.

$$k^{(4)}(l) + \frac{\pi^2}{l_k^2} k''(l) = 0 \quad (34)$$

The curvature equation is as follows

$$k(l) = \frac{1}{2\, R} - \frac{1}{2\, R} \cos\left(\frac{\pi}{l_k} l\right) \quad (35)$$

$$\Theta(l) = \frac{1}{2\, R} l - \frac{l_k}{2\pi\, R} \sin\left(\frac{\pi}{l_k} l\right) \quad (36)$$

$$x(l) = \int \cos \Theta(l)\, dl = l - \frac{\pi^4}{2016\, R^2\, l_k^4} l^7 + \frac{\pi^6}{25920\, R^2\, l_k^6} l^9 - \ldots \quad (37)$$

$$y(l) = \int \sin \Theta(l)\, dl = \frac{\pi^2}{48\, R\, l_k^2} l^4 - \frac{\pi^4}{1440\, R\, l_k^4} l^6 + \frac{\pi^6}{80640\, R\, l_k^6} l^8 - \ldots \quad (38)$$

A simplified form of the cosine curve is secured on the assumption that $k_0(x) = \frac{1}{2\, R} - \frac{1}{2\, R} \cos\left(\frac{\pi}{l_k} x\right)$.

$$y(x) = \frac{1}{4\, R} x^2 + \frac{l_k^2}{2\pi^2\, R} \left[\cos\left(\frac{\pi}{l_k} x\right) - 1\right] \quad (39)$$

3.5. SINE CURVE

In addition to the assumptions made earlier more conditions are set out.

$$\begin{cases} k(0^+) = k'(0^+) = k''(0^+) = 0 \\ k(l_k^-) = \frac{1}{R} \\ k'(l_k^-) = k''(l_k^-) = 0 \end{cases} \quad (40)$$

Conditions (40) determine the order of the differential equation. They are expressed in the following form:

$$k^{(6)}(l) + \frac{4\, \pi^2}{l_k^2} k^{(4)} l = 0 \quad (41)$$

Having solved the differential problem (40), (41) it is possible to obtain a formula for the curvature equation:

$$k(l) = \frac{1}{R\, l_k} l - \frac{1}{2\pi\, R} \sin\left(\frac{2\pi}{l_k} l\right) \quad (42)$$

$$\Theta(l) = -\frac{l_k}{4\pi^2\, R} + \frac{1}{2\, R\, l_k} l^2 + \frac{l_k}{4\pi^2\, R} \cos\left(\frac{2\pi}{l_k} l\right) \quad (43)$$

$$x(l) = \int \cos \Theta(l)\, dl = l - \frac{\pi^4}{648\, R^2\, {l_k}^6}\, l^9 + \ldots \quad (44)$$

$$y(l) = \int \sin \Theta(l)\, dl = \frac{\pi^2}{30\, R\, {l_k}^3}\, l^5 - \frac{\pi^4}{315\, R\, {l_k}^5}\, l^7 - \frac{\pi^6}{5670\, R\, {l_k}^7}\, l^9 + \ldots \quad (45)$$

A simplified form of the sinusoid is obtained on making the assumptions $x = l$, $x_k = l_k$ and $k_0(x) = \frac{1}{R\, l_k} x - \frac{1}{2\pi\, R} \sin\left(\frac{2\pi}{l_k} x\right)$.

$$y(x) = -\frac{l_k}{4\pi^2\, R} x + \frac{1}{6\, R\, l_k} x^3 + \frac{l_k^2}{8\pi^3\, R} \sin\left(\frac{2\pi}{l_k} x\right) \quad (46)$$

4. LIMITATIONS NOTED IN THE USE OF SMOOTH TRANSITION CURVES IN RAILWAY ROUTES

A comparison of transition curves related to railway routes requires that certain assumptions should be made with respect to the permissible values of kinematic parameters being in force, acceleration growth ψ_{per} and the speed of lifting the railway rolling stock wheel at the superelevation ramp f_{per} . The assumption of equal values of ψ_{per} and f_{per} calls for the elongation of particular smooth transition curves in relation to the cubic parabola (by the use of an appropriate coefficient A).

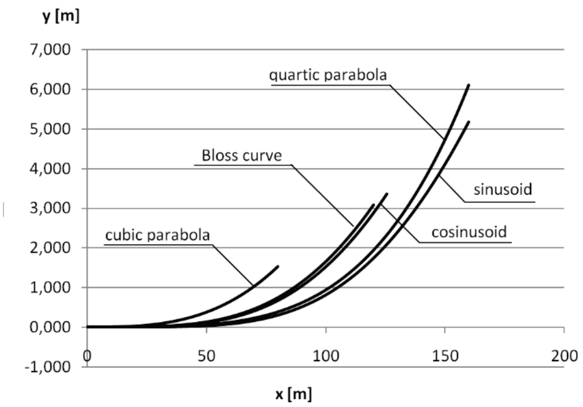

Fig. 1. The shaping of the horizontal ordinates along the length of the cubic parabola and the smooth transition curves.

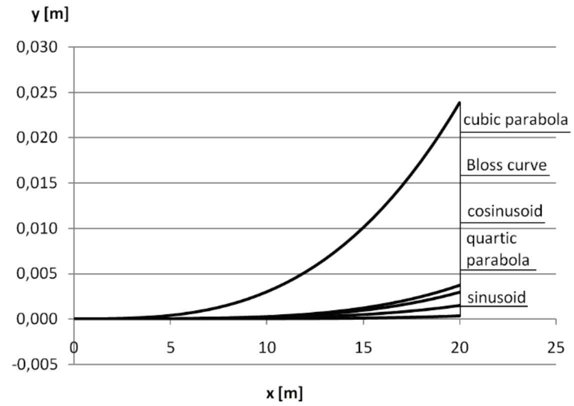

Fig. 2. The shaping of horizontal ordinates in the outset region of transition curves under consideration.

Thus it is possible to make a comparison of the horizontal ordinates (Fig. 1). In the given case the curve in the form of cubic parabola is a reference adopted for the following geometric system:

- maximal speed of trains $v = 100$ km/h,
- radius of circular arc $R = 700$ m,
- value of superelevation along arc $h_0 = 80$ mm,
- length of transition curve (with a rectilinear superelevation ramp) $l_k = 80$ m.

The lengths of the smooth transition curves in Fig. 1 are as follows: for sinusoid and quartic parabola ($A = 2$) – $l_k = 160$ m, for cosinusoid ($A = \pi/2$) – $l_k = 125{,}664$ m, and for Bloss curve ($A = 1{,}5$) – $l_k = 120$ m.

The theoretical analyses performer and the experimental works carried out unambiguously indicate a lesser (that is, more advantageous) dynamic interactions while travelling on smooth transition curves. As already mentioned a dominant role here is played by the class of function describing the curvature. However, in spite of their indisputable advantages the range of application of smooth transition curves In railway tracks under exploitation is very limited. The basic reason for the existing skepticism about this question seems to be the very low value of the horizontal ordinates In the initial region of the curves analyzed. It is often difficult to set them out correctly in the field and in practice, leads to shortcuts of the transition curve made in comparison with the brief foredesign.

Table 1. Selected values of horizontal ordinates y [mm].

Transition curve	$x = 5$ m	$x = 10$ m	$x = 15$ m	$x = 20$ m
Cubic parabola	0.37202	2.97619	10.04464	2.80952
Quartic parabola	0.00581	0.09301	0.47084	1.48810
Bloss curve	0.01524	0.23975	1.19280	3.70370
Cosinusoid	0.01162	0.18562	0.93728	2.95150
Sinusoid	0.00036	0.01143	0.08642	0.36183

Let us, now, take a closer look at the shaping of the horizontal ordinates of the transition curves of Fig. 1 along the length of the first 20 m. This is illustrated in Fig. 2. Table 1 provides particular numerical values. When analyzing the data of Table 1, it is surprising that the usefulness of smooth transition curves causes such big doubts. The horizontal ordinates of the outset region are very small with respect to these curves. They are many times smaller than the ordinates of the cubic parabola. The Bloss curve takes relatively the best place among the smooth curves. The most advantageous sinusoid with respect to dynamics, in fact, seems to be impossible to be carried out, in a given case its ordinates along the distance of the first 20 m do not reach even 1 mm.

The presented considerations relating to the railway routes can also be referred to vehicular roads. It all leads to the conclusion that the major cause of the difficulties encountered, lies in excessive smoothing of the curvature near the original smooth transition curve. In order to take preventive measures, it is necessary to resign from the zeroing condition of the curvature derivative at the outset point, and assume a certain numerical value instead, smaller, however, than it occurs in the case of clothoid, or a cubic parabola. In this way an idea to find a new family, called parametric transition curves, has emerged [18].

5. PARAMETRIC TRANSITION CURVE

Advantage is taken of differential equation (27) with the following conditions (where constant $C > 0$):

$$\begin{cases} k(0^+) = 0 & k(l_k^-) = \frac{1}{R} \\ k'(0^+) = \frac{C}{R\, l_k} & k'(l_k^-) = 0 \end{cases} \tag{47}$$

The solution of the differential problem (27), (47) leads to the determination of the curvature equation of a new transition curve which on account of the occurring parameter *C*, will be referred to as parametric curve. The equation has the form

$$k(l) = \frac{C}{R\, l_k} l + \frac{3-2\,C}{R\, l_k^2} l^2 - \frac{2-C}{R\, l_k^3} l^3 \tag{48}$$

Function (48) describing the curvature is a function of class C^0. However, due to the fact that the transition near the region of the circular arc is mild, and grows milder in the initial area (than in the case of the clothoid and the cubic parabola) the obtained curve can still be included among smooth transition curves. Moreover, the curve, in view of dynamics, is more advantageous than both the clothoid and the cubic parabola.

From equation (48) it follows that

$$\Theta(l) = \frac{C}{2\, R l_k} l^2 + \frac{3-2\,C}{3\, R l_k^2} l^3 - \frac{2-C}{4\, R l_k^3} l^4 \tag{49}$$

The parametric transition curve equations are as follows:

$$x(l) = \int \cos\Theta(l)\, dl = l - \frac{C^2}{40\, R^2\, l_k^2} l^5 - \frac{C(3-2\,C)}{36\, R^2\, l_k^3} l^6 + \frac{9\,C(2-C) - 4(3-2\,C)^2}{504\, R^2\, l_k^4} l^7 + \frac{(3-2\,C)(2-C)}{96\, R^2\, l_k^5} l^8 + \ldots \tag{50}$$

$$y(l) = \int \sin\Theta(l)\, dl = \frac{C}{6\, R\, l_k} l^3 + \frac{3-2\,C}{12\, R\, l_k^2} l^4 - \frac{2-C}{20\, R\, l_k^3} l^5 - \frac{C^3}{336\, R^3\, l_k^3} l^7 - \frac{C^2\,(3-2\,C)}{192\, R^3\, l_k^4} l^8 + \ldots \tag{51}$$

The largest values of the kinematic parameters are noted at point $l_0 = \frac{3-2\,C}{3\,(2-C)}\, l_k$, where also the curvature derivative is the maximum

$$\max k'(l) = k'(l_0) = \left[C + \frac{(3-2\,C)^2}{3\,(2-C)}\right] \frac{1}{R\, l_k} \tag{52}$$

Expression $A = \left[C + \frac{(3-2\,C)^2}{3\,(2-C)}\right]$ determines the relation between the length of the parametric transition curve and the cubic parabola (for which in the calculation formulae $A = 1$). This coefficient assumes the smallest value for $C = 1$ and amounts to $A = \frac{4}{3} = 1.3333$.

A simplified form of the parametric curve is obtained on the assumption that $x = l$, $l_k = x_k$, $k_0(x) = \frac{C}{R\, l_k} x + \frac{3-2\,C}{R\, l_k^2} x^2 - \frac{2-C}{R\, l_k^3} x^3$.

$$y(x) = \frac{C}{6\, R\, l_k}\, x^3 + \frac{3-2\,C}{12\, R\, l_k^2}\, x^4 - \frac{2-C}{20\, R\, l_k^3}\, x^5 \tag{53}$$

The parametric transition curve was determined because of the limitations indicated at an earlier point regarding the use of smooth transition curves on railway routes. Therefore it will be legitimate to consider its properties while working on the application of the simplified solution and the utilization of equation (53).

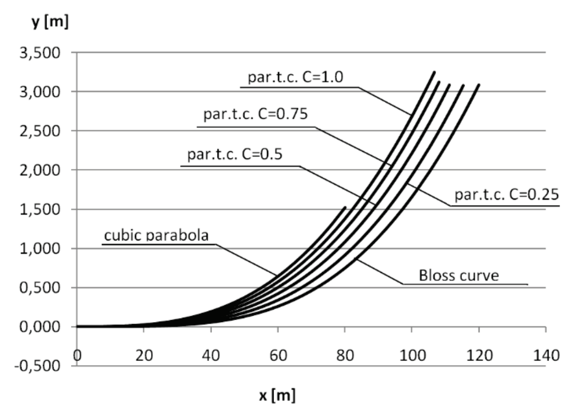

Fig. 3. The shaping of the horizontal ordinates along the length of the parametric transition curve (C = 0,25, C= 0,5, C=0,75, and C = 1,0), the cubic parabola, and the Bloss curve.

Fig. 3 shows the shaping of the horizontal ordinates of the obtained parametric curve for $C \in \langle 0, 1 \rangle$ in the background of appropriate diagrams for the cubic parabola and the Bloss curve. As can be seen in this figure, for $C \in \langle 0, 1 \rangle$ ordinates $y(x)$ differ evidently along the whole length of the transition curve. The parametric curve $C = 1$ has horizontal ordinates which are an approximation to the cubic parabolic ordinates. The ordinates of the remaining curves are fund between the cubic parabolic ordinates and the Bloss curve (i.e. curve $C = 0$).

For $C \geq 1$ ordinates $y(x)$ are similar along the entire length, and also share similarity with ordinates of the cubic parabola (particular curves are different only in length). All these factors create a situation where the use of transition curves which have $C \geq 1$, would not be advantageous; they make the process of entering the circular arc easier. However, the initial region is still affected by violent disturbances in the curvature, which are characteristic of the cubic parabola.

Thus, solutions that may prove useful for applications in practice should be sought among parametric curves of $C \in (0,1)$. The selection criterion should, of course, be based on values of the horizontal ordinates in the outset region. Fig. 4 illustrates the shaping of the horizontal ordinates in the outset region of some selected parametric curves.

The choice of the form of the parametric curve (i.e. the acceptance of the required value of parameter C) will depend on a particular geometric situation. The problem of fundamental significance will become the value analysis of the horizontal ordinates of the outset region. Table 2 illustrates the magnitudes of these ordinates along the length of the first 20 m for various values of parameter C (for the numerical data used in the paper). The ordinates are much greater than those for the smooth transition curves given in Table 1.

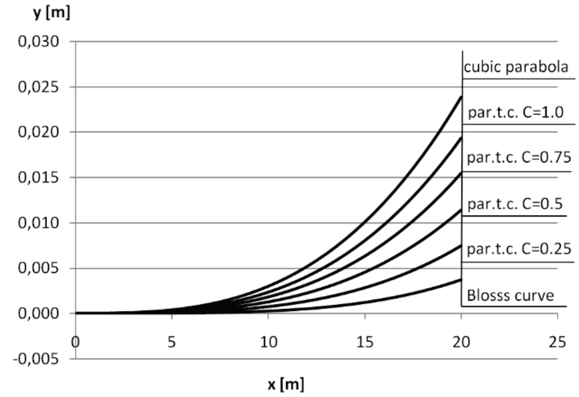

Fig. 4. The shaping of the horizontal ordinates of some selected parametric curves (and the cubic parabola, and the Bloss curve) in the outset region.

In the case analyzed it might appear that the best solution would be to apply parametric curve $C = 0{,}5$. It indicates quite clearly the horizontal ordinates in the initial region (though twice smaller than the cubic parabola). In addition it is characterized by relatively mild transfer from a straight to a transition curve; the curvature derivative at the initial point presents only 36 % of the value which appears on the cubic parabola.

6. CONCLUSIONS

- The requirements imposed upon transition curves relating both to vehicular roads and railway routes are clearly defined and for this reason there should be one common algorithm to create them. Meanwhile all the solutions that have been known so far, are still used independently and bear various names.

Table 2. Selected values of horizontal ordinates y [mm] for various parametric curves.

Curve	A	l_k [m]	x = 5 m	x = 10 m	x = 15 m	x = 20 m
C = 0.1	1.47544	118.035	0.03991	0.43272	1.82934	5.17768
C = 0.2	1.45185	116.148	0.06533	0.63122	2.48293	6.68835
C = 0.3	1.42941	114.353	0.09148	0.83501	3.15259	8.23309
C = 0.4	1.40833	112.667	0.11831	1.04364	3.83676	9.80796
C = 0.5	1.38889	111.111	0.14574	1.25648	4.53310	11.40720
C = 0.6	1.37143	109.714	0.17365	1.47253	5.23825	13.02267
C = 0.7	1.35641	108.513	0.20187	1.69041	5.94746	14.64299
C = 0.8	1.34444	107.556	0.23016	1.90814	6.65403	16.25237
C = 0.9	1.33636	106.909	0.25816	2.12293	7.34869	17.82905
C = 1.0	1.33333	106.667	0.28537	2.33089	8.01849	19.34291

Moreover, the transition curves are also defined in different ways, with respect to vehicular roads they are often denoted by angle function $\theta(l)$ which is responsible for changing the direction of the longitudinal axis of the vehicle after travelling along a certain arc, whereas in railway routes the major functions are related to the use of curvature $k(x)$ with a system of rectangular coordinates.

- As can be expected the basis for the identification of the transition curves should be the curvature distribution along their length deciding about the occurring unbalanced accelerations. The dynamic interactions are smaller (i.e. more advantageous), if the function class describing curvature is higher. The paper presents a method identifying curvature on transition curves by differential equations. The method makes a reference to the approach used in [24] related to identification of accelerations. The curvature equations have been determined for some known forms of transition curves, such as, the clothoid, the quartic parabola, the Bloss curve, the cosinusoid and the sinusoid. Taking advantage of these equations the Cartesian coordinates were found. Approximate solutions were also provided after some simplified Assumption had been made, orientated to a large extent towards railway routes.
- Smooth transition curves, i.e. curves of nonlinear distribution of curvature, have been known for a long time and possess a number of unquestionable advantages. First of all they are characterized by minor values of dynamic interactions. The range of their applications in railway routes has been limited so far. Unfortunately, the curve have one main drawback, namely, a very small value of the horizontal ordinates in the outset region, in practice often impossible for execution and hard to be maintained.
- The basic reason for the difficulties occurring in some known forms of smooth transition curves is connected with curvature being too much mitigated in the outset region. Thus it is necessary to find a new form of the transition curve, and abandon the condition of zeroing the curvature derivative at the outset point. For this purpose use has been made of the curvature identification method described in the paper, contributing simultaneously to the formation of a family of parametric transition curves.
- The parametric transition curve recommended for practical application is characterized by a mild proceeding of the curvature In the region of entering the circular arc, and is disturbance at the starting point (though smaller than in the case of the clothoid, Or the cubic parabola as well). The acceptance of an appropriate value of parameter $C \in (0,1)$ depends on a particular geometric situation and occurs as a result of the value analysis of horizontal ordinates in the outset region of the transition curve.

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