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Asymptotic Power-law Index in Aggregation and Chipping Processes with Power-law Aggregation Kernel

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The asymptotic power-law behavior of a distribution function $P(X)$ for X clusters is analyzed for aggregation and chipping processes with power-law kernel of aggregation processes, $K(n, m \longrightarrow n+m) = n^{-\psi}m^{-\psi}/A^2$. An exact value of non-integer power-law index is obtained, that is, $P(X) \sim 1/X^{\alpha}$, $\alpha = \frac{n}{2} + 1 - \psi$ in the case $\psi < 0.5$. Numerical simulations agree well with this result.

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I. INTRODUCTION

Aggregation process can be found in various phenomena in both nature and social sciences[1–4]. These are nonequilibrium systems, thus not described by Gibbs distribution like thermal equilibrium system. In systems where only aggregation mechanism is operative, all the clusters in system eventually coagulates one condensed cluster. On the other hand, with some fragmentation mechanism the system evolves, in many cases, a power-law distribution[5]. We have already investigated the discrete aggregation-chipping processes with the constant kernels for both the aggregation and the chipping[6–8], the case where the chipping kernel has a power-law form[9], and the case with the constant chipping kernel[10]. In these papers, we have reported that the asymptotic power-law cluster size distribution $P(X) \sim 1/X^{\sigma}$ emerges when the weight of the chipping kernel has a power-law form $C(n) \sim 1/n^{\sigma}$ and the case $2 > \sigma > 1$. In the case $0 < \sigma \leq 1$, our analysis suggests that there is no asymptotic power-law solution, and in the case $\sigma = 0$ the chipping kernel becomes a constant, the exponential distribution appears. On the other hand, in the case $\sigma > 2$, still the power-law distribution holds, but also the condensed "big" cluster appears. These results suggest that even the chipping kernel has strong influences for the resultant cluster size distribution, and for the actual systems that mostly have the complex kernels of the aggregation and the chipping processes, further analysis is required to understand the resultant distribution.

In this paper, we consider the case where the aggregation kernel has a power-law form, and analyze the asymptotic power-law behavior of the cluster size distribution.

II. THEORY

Now we introduce a weighted-aggregation and chipping process, described as follows: initially N clusters are set where each cluster has a certain number of units, according to an initial distribution. The unit process consists of two parts, weighted-aggregation and chipping. First, two clusters are chosen according to a certain aggregation kernel $K(X, Y \longrightarrow X+Y)$ (X and Y clusters) are integrated and make one big cluster ($X+Y$ cluster). Secondly one unit is chipped off from a randomly chosen cluster which has more than 2 units ($Z(\geq 2)$ cluster) and become a cluster. In an one unit process, the total number of the clusters and the units are conserved (X, Y, Z clusters are changed into $(X+Y)$, $(Z-n)$ and n clusters). The whole process proceeds by repeating the unit process.

In the present paper we consider the case where the aggregation kernel $K(n, m \longrightarrow n+m)$ has a power-law form

$$K(n, m \longrightarrow n+m) = n^{-\psi} m^{-\psi} / A^2, \tag{1}$$

where $1/A = (\Sigma_{i=1}^{\infty} i^{-\psi})^{-1}$ is the normalization constant. In the limit $\psi \longrightarrow 0$, i.e. the aggregation kernel $K(n, m \longrightarrow n+m) = 1/A^2$ thus the two clusters for the aggregation are chosen freely, a resultant distribution is the power-law distribution $P(X) \sim 1/X^{2.5}$[7].

We analyze here the case ψ is a non-integer and $\psi < 0.5$. Note that if $\psi \leq 1$ A would be infinite, thus we need some high-cut for the system size. At a steady state, the basic equations of this system are

$$\frac{(1-P(1))(1-P(1))}{A^2}+\frac{P(2)}{1-P(1)}=\frac{P(1)^2}{A^2}\ \ (X=1), \tag{2}$$

$$\sum_{i+j=X}\frac{i^{-\psi}j^{-\psi}}{A^2}P(i)P(j)+\frac{P(X+1)}{1-P(1)}$$
$$=\left(\frac{2X^{-\psi}}{A}+\frac{1}{1-P(1)}\right)P(X)\ \ (X\geq 2). \tag{3}$$

where α_n is a normalization constant,

$$\alpha_n=\left(\sum_{s=1}^{n-1}\frac{1}{s^{\sigma}}\right)^{-1}. \tag{4}$$

We now assume that the system has an asymptotic power-law solution, $P(X)\sim 1/X^{\alpha}$. In this case, the first term of the left hand side of (3) can be written as

$$\sum_{i+j=X}\frac{i^{-\psi}j^{-\psi}}{A^2}P(i)P(j)$$
$$\sim\sum_{i+j=X}\frac{i^{-\psi}j^{-\psi}}{A^2}\frac{1}{i^{\alpha}}\frac{1}{j^{\alpha}}=\sum_{i+j=X}\frac{1}{A^2}\frac{1}{i^{\alpha+\psi}}\frac{1}{j^{\alpha+\psi}}. \tag{5}$$

Also, the first term of the right hand side of (3) is,

$$\frac{2X^{-\phi}}{A}P(X)\sim\frac{1}{A}\frac{1}{X^{\alpha+\psi}}. \tag{6}$$

Therefore, using the z-transform $\phi(z)=\sum_{X=0}^{\infty}P(X)z^{-X}$, the basic equation in z-space has the form near $z \sim 1$

$$
\begin{aligned}
&C_1\{\phi_r^{\psi}(z)+B(1-z)^{-(1-\alpha-\psi)}\}^2 \\
&\quad + f(z)\{\phi_r(z)+B(1-z)^{-(1-\alpha)}\} \\
&\quad\quad = C_2\{\phi_r^{\psi}(z)+B(1-z)^{-(1-\alpha-\psi)}\} \\
&+\frac{1}{1-P(1)}\{\phi_r(z)+B(1-z)^{-(1-\alpha)}\}+g(z),
\end{aligned} \tag{7}
$$

where $\phi_r^{\psi}(z)$ is a regular term of the z-transform for $P(X) \sim 1/X^{\alpha+\psi}$, $\phi_r^{\psi}(z)$ is a regular term of the z-transform for $P(X) \sim 1/X^{\alpha}$, B, C_1 and C_2 are certain constants, and $f(z)$ and $g(z)$ are certain regular functions of z. Here we use the singularity of $\phi(z) = \sum_{X=0}^{\infty} P(X)z^{-X}$ near $z \to 1$ is $\phi_s(z) \sim (1-1/z)^{\beta-1}$ for $P(X) \sim 1/X^{\beta}$[11],

If the equation (7) holds, then the singular terms in (7) must satisfy

(1) $(1-z)^{-\{2(1-\alpha-\psi)\}} = (1-z)^{-(1-\alpha)}$,

or

(2) $(1-z)^{-\{2(1-\alpha-\psi)\}} = (1-z)^{n}$ where n is an integer, thus this term becomes a regular term.

The condition (1) leads to an equation $\alpha = 1-2\psi$. When $\psi > 1$, $\alpha < 1$ thus the condition (1) is irrelevant. Therefore, we have the asymptotic power-law solution,

$$
P(X) \sim 1/X^{\alpha}, \quad \alpha = \frac{n}{2}+1-\psi. \tag{8}
$$

When $\psi < 0.5$ and $n = 2$, $\frac{n}{2}+1-\psi > 2.0$, thus it is clear that the condensed cluster will appear in those cases.

III. NUMERICAL RESULTS

Figures 1 and 2 show the cumulative distributions $C_m(X) = \Sigma_{x \leq X} P(x)$ of the simulation results with $\psi = 0.02$ and $\psi = 0.03$ respectively. The total number of the clusters N is 1000 and the mean value of the cluster size $\langle X \rangle =$ 1000.

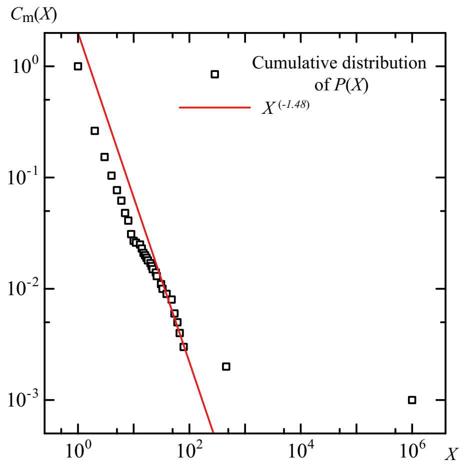

FIG. 1: Cumulative distribution of the simulation result for weighted-aggregation and chipping process with the number of the clusters $N = 1000$, $\psi = 0.02$ and $\langle X \rangle = 1000$.

Here the resultant distributions are the power-law $\sim 1/X^{\alpha}$, and the power-law index α is practically the same as $\frac{5}{2} - \psi$. This is reasonable because when $\psi \longrightarrow 0$, α would be $5/2$. Therefore we conclude that there is an asymptotic solution $P(X) \sim 1/X^{\frac{5}{2}-\psi}$ in the case $\psi < 0.5$. Also we can see the condensed clusters appear at the right side of those pictures. Apparently, It is required to do simulations with $\psi > 0.5$, however due to the problem of calculation time, it

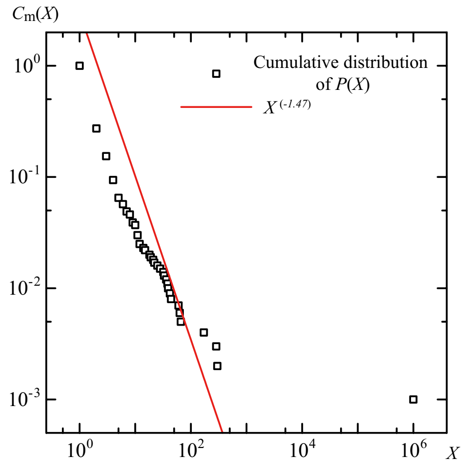

FIG. 2: Cumulative distribution of the simulation result for weighted-aggregation and chipping process with the number of the clusters $N = 1000$, $\psi = 0.03$ and $\langle X \rangle = 1000$.

is difficult to make such simulations with the simulation circumstance we have now.

IV. DISCUSSION AND SUMMARY

These results show that the power-law aggregation kernel influences directly to the power-law index of the resultant distribution. It is, in some point of view, obvious, however it also casts another questions.

When ψ becomes bigger than 0.5, there is no reason to break this asymptotic solution until $\psi \leq 1.0$. Thus we believe the solution $P(X) \sim 1/X^{\frac{5}{2}-\psi}$ holds $0.5 < \psi \leq 1.0$. This means that the condensed cluster would disappear when $0.5 < \psi \leq 1.0$, because the power-law index of the resultant distribution α

would be $\alpha < 2.0$.

What, then, will happen if ψ is larger than 1.0? The simulation results show that the power-law index α of the distribution is $\alpha = \frac{5}{2} - \psi$ when $\psi < 0.5$. However, our analysis shows only that $-2(1-\alpha-\psi)$ must be integer, so it is also possible, from the theory, that $\alpha = \frac{3}{2} - \psi$, $\alpha = 7/2 - \psi$ or another solutions. If, when $\psi > 1.0$, will the power-law index of the distribution α jump from $\frac{5}{2} - \psi$ to $\frac{7}{2} - \psi$ for example, or still hold the solution $\frac{5}{2} - \psi$? When $\psi > 2.5$, will the power-law distribution still hold, or break into other distribution like an exponential one? We cannot answer those questions now. However these results at least show that the power-law index ψ of the aggregation kernel could be another type of the control parameter for the resultant distribution $P(X)$ than the chipping kernel we have reported before, and also could affect the emergence of the power-law distribution or the condensed cluster. It expands the possibility of finding the corresponding phenomena in the real world.

In summary, we introduce the weighted-aggregation and chipping model, and analyze the case that the aggregation kernel $K(n, m \longrightarrow n+m) = n^{-\psi}m^{-\psi}/A^2$. When $\psi < 0.5$, our analysis shows there is an asymptotic solution of the probability distribution function $P(X) \sim 1/X^{\frac{5}{2}-\psi}$. The simulation results agree with this analysis, but we need to reduce the simulation time for getting more information when ψ is larger than 0.5. These results also show the strong possibility of the singularity analysis, and we believe that this analysis will show more profound features of the aggregation systems.

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