Generalized Turán problems for disjoint copies of graphs

Dániel Gerbner * Abhishek Methuku † Máté Vizer ‡

September 10, 2018

Abstract

Given two graphs H and F, the maximum possible number of copies of H in an F-free graph on n vertices is denoted by $ex(n, H, F)$. We investigate the function $ex(n, H, kF)$, where kF denotes k vertex disjoint copies of a fixed graph F. Our results include cases when F is a complete graph, cycle or a complete bipartite graph.

Keywords: Turán numbers, disjoint copies, generalized Turán
AMS Subj. Class. (2010): 05C35, 05C38

1 Introduction

The vertex set of a graph G is denoted by $V(G)$ and its edge set is denoted by $E(G)$. The *disjoint union* $G \cup H$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ is the graph with the vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *join* $G + H$, of graphs G and H with disjoint vertex sets is the graph obtained by taking a copy of G and a copy of H on disjoint vertex sets and adding all the edges between them.

Given a positive integer k and a graph F, the vertex disjoint union of k copies of the graph F is denoted by kF. Let C_l denote a cycle of length l, $K_{s,t}$ denote the complete bipartite graph with parts of sizes s and t and let K_r denote the complete graph on r vertices.

For a set of graphs $\mathcal{F}$ the *Turán number* of $\mathcal{F}$, $ex(n, \mathcal{F})$, denotes the maximum number of edges of an n-vertex graph having no member of $\mathcal{F}$ as a subgraph. If $\mathcal{F}$ contains only a single graph F, we simply denote it by $ex(n, F)$. This function has been intensively studied, starting with the theorems of Mantel [15] and Turán [19] that determine $ex(n, K_{r+1})$ for $r \geq 3$. Turán also showed in [19] that a complete r-partite graph on n vertices with as equal parts as possible is the unique extremal graph. This extremal graph is called *Turán graph* and is denoted by $T_r(n)$. See, for example, [8, 18] for surveys on this topic.

*Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences. e-mail: gerbner@renyi.hu
†Central European University, Budapest. e-mail: abhishekmethuku@gmail.com
‡Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences. e-mail: vizermate@gmail.com.

Simonovits [17] and Moon [16] showed that if n is sufficiently large, then $K_{k-1} + T_r(n - k + 1)$ is the unique extremal graph for $\mathcal{F} = \{kK_{r+1}\}$. In [12] Gorgol initiated the systematic investigation of Turán numbers of disjoint copies of connected graphs and proved the following.

Theorem 1 (Gorgol)**.** *For every non-empty graph F and $k \geq 1$, we have*

$$ex(n, kF) = ex(n, F) + O(n).$$

In fact, Gorgol proved the following sharper upper bound: If F is an arbitrary connected graph and k is an arbitrary positive integer, then $ex(n, kF) \leq ex(n - (k-1)|V(F)|, F) + \binom{(k-1)|V(F)|}{2} + (k-1)|V(F)|(n - (k-1)|V(F)|)$ for $n \geq k|V(F)|$. For recent results about Turán numbers of disjoint copies of graphs see [4, 14, 20].

Given a graph H and a set of graphs $\mathcal{F}$, the maximum possible number of copies of H in an n-vertex graph that does not contain any copy of $F \in \mathcal{F}$ is denoted by $ex(n, H, \mathcal{F})$ and is called *Generalized Turán number*. If $\mathcal{F} = \{F\}$, we simply denote it by $ex(n, H, F)$. Note that $ex(n, K_2, F) = ex(n, F)$. Erdős [5] determined $ex(n, K_s, K_t)$ exactly. We will later use the following consequence of his result.

Proposition 2 (Erdős)**.** *For $s < t$ we have:*

$$ex(n, K_s, K_t) = (1 + o(1))\binom{t-1}{s}\left(\frac{n}{t-1}\right)^s.$$

Another notable result is that of Bollobás and Győri [2], who showed that $ex(n, K_3, C_5) = \Theta(n^{3/2})$. The systematic study of the function $ex(n, H, F)$ was initiated by Alon and Shikhelman in [1].

The function $ex(n, H, F)$ is closely related to the area of Berge hypergraphs. A *Berge cycle* of length k is an alternating sequence of distinct vertices and hyperedges of the form $v_1,h_1,v_2,h_2,\ldots,v_k,h_k,v_1$ where $v_i, v_{i+1} \in h_i$ for each $i \in \{1, 2, \ldots, k-1\}$ and $v_k, v_1 \in h_k$ and is denoted by Berge-C_k. Győri and Lemons [13] showed that any r-uniform hypergraph avoiding a Berge-C_{2l+1} contains $O(n^{1+1/l})$ hyperedges. They also showed that any r-uniform hypergraph avoiding a Berge-C_{2l} contains $O(n^{1+1/l})$ hyperedges. These results easily imply the following.

Theorem 3. *We have*

(a) For any $r \geq 3,\ l \geq 2$, we have

$$ex(n, K_r, C_{2l+1}) = O(n^{1+1/l}).$$

(b) For any $r, l \geq 2$, we have

$$ex(n, K_r, C_{2l}) = O(n^{1+1/l}).$$

Proof. We will prove (a) and (b) simultaneously. Let G be a C_k-free graph. Replace each clique of size r in it with a hyperedge on the same vertex set as the clique. It is easy to see that the resulting r-uniform hypergraph H does not contain a Berge-C_k, and in both cases $k = 2l$ and $k = 2l + 1$, H has at most $O(n^{1+1/l})$ hyperedges by the theorem of Győri and Lemons [13] mentioned before. This completes the proof as the number of cliques in G is equal to the number of hyperedges in H. □

Alon and Shikhelman [1] noted that while $ex(n, K_3, C_5) = \Theta(n^{3/2})$, we have $ex(n, K_3, 2C_5) = \Theta(n^2)$, showing that $ex(n, H, F)$ and $ex(n, H, kF)$ can have different order of magnitudes, unlike the graph case, where $ex(n, kF) = \Theta(ex(n, F))$ (see Theorem 1).

Our goal in this paper is to explore this phenomenon. Most of our theorems will relate $ex(n, H, kF)$ to $ex(n, H, F)$ for several graphs H and F.

General approach

The most typical example of a kF-free graph is obtained by taking an F-free graph G on $n-k+1$ vertices, and considering $K_{k-1}+G$. (We will sometimes refer to these $k-1$ vertices of degree $n-1$ in K_{k-1} as *universal* vertices of $K_{k-1}+G$.) Indeed, since any copy of F in $K_{k-1}+G$ must contain a vertex of K_{k-1} and as there are only $k-1$ vertices in K_{k-1}, it is impossible to find k vertex disjoint copies of F.

For example, let us take a C_5-free graph G on $n-1$ vertices, add a new vertex v and connect it to all the vertices of G. This graph shows $ex(n, K_3, 2C_5) = \Omega(n^2)$. In addition to the triangles in G (which are at most $O(n^{3/2})$ by the result of Bollobás and Győri [2] mentioned before), there are triangles which contain v and an edge of G. If G is the Turán graph $T_2(n-1)$, then there are $\Omega(n^2)$ many such triangles. What happens here is that instead of counting the copies of K_3 in a C_5-free graph, we count the copies of K_2 (which is a subgraph of K_3). As this happens to be of larger order of magnitude, we get more copies of K_3 in a $2C_5$-free graph than in a C_5-free graph.

To prove the upper bounds we will need the following operation: for an integer k, a graph F and a kF-free graph G, we consider the maximum number of disjoint copies of F in G. In the rest of the paper we denote the subgraph of G consisting of these copies by G_L, and the set of vertices spanned by G_L is denoted by $L(G)$. We denote by $R(G)$ the set $V(G) \setminus L(G)$ of the remaining vertices, and by G_R the subgraph of G induced by them. We call this partition of the vertices a *canonical* (k, F)*-partition* of G. (If it is clear from the context we simply write canonical partition.) Note that G_R is F-free.

Structure of the paper

The rest of this paper is divided into sections based on which graph is forbidden. In Section 2 we prove bounds on $ex(n, H, kF)$ for general F, while in Section 3 one of our main results is to determine the order of magnitude of $ex(n, K_s, kK_t)$ for all $s \geq t \geq 2$ and $k \geq 1$. In Section 4, we obtain bounds on $ex(n, K_r, kC_l)$. In Section 5 we study the case when F is a complete bipartite graph. We finish our article with some concluding remarks and open problems in Section 6.

2 Forbidding a general F

2.1 Counting arbitrary graphs

For a family of graphs $\mathcal{H}$, let us define $N(\mathcal{H}, G)$ as the number of copies of members of $\mathcal{H}$ in G. If $\mathcal{H} = \{H\}$, then we simply write $N(H, G)$ instead of $N(\mathcal{H}, G)$.

Let $\mathcal{H}^{ind}$ be the family of all induced subgraphs of a graph H. Let

$$\overline{ex}(n, H, F) := \max\{N(\mathcal{H}^{ind}, G) : G \text{ is an } F\text{-free graph on } n \text{ vertices}\}.$$

Remark 4. *Note that if F is a non-empty graph (i.e., contains at least one edge), then*

$$\overline{ex}(n, H, F) \geq \binom{n}{\alpha(H)}.$$

Indeed, let G be an F-free graph on n vertices and let $I \in \mathcal{H}$ be an induced subgraph spanned by a largest independent set of H. Then any set of $\alpha(H)$ vertices in G forms a copy of I.

Theorem 5. *For any $k \geq 2$ we have,*

$$ex(n, H, kF) = O(\overline{ex}(n, H, F)).$$

Moreover, if $k \geq |V(H)|$, then

$$ex(n, H, kF) = \Theta(\overline{ex}(n, H, F)).$$

Proof. For the lower bound, we take an F-free graph G on $n - k + 1$ vertices that contains $\overline{ex}(n-k+1, H, F)$ copies of induced subgraphs of H. Then $K_{k-1} + G$ is obviously kF-free. If $k \geq |V(H)|$, then every copy of an induced subgraph (having at least one vertex) of H in G can be extended to a copy of H in $K_{k-1} + G$, using the vertices of K_{k-1}. (A small technical issue is the following: Let Z be the induced subgraph of H with zero vertices. A copy of Z in G cannot be extended to a copy of H in $K_{k-1} + G$ if $k = |V(H)|$, but there is only one copy of Z in G.) Thus $K_{k-1} + G$ contains at least $\overline{ex}(n - k + 1, H, F) - 1$ copies of H. Now using the following standard argument, we conclude that $\overline{ex}(n-k+1, H, F) - 1 = \Omega(\overline{ex}(n, H, F))$: Consider a graph G on n vertices with $\overline{ex}(n, H, F)$ copies of induced subgraphs of H. Then a subgraph of G induced by a random subset of vertices of size $n - k + 1$, contains at least $(1 + o(1))\overline{ex}(n, H, F)$ copies of induced subgraphs of H. On the other hand, this subgraph contains at most $\overline{ex}(n - k + 1, H, F)$ copies of induced subgraphs of H. This finishes the proof of the lower bound.

Now we continue with the upper bound. Let us consider a kF-free graph G, and its canonical partition. Then every copy of H in G contains a subgraph in G_R, which contains an induced subgraph of H (note that this subgraph may have zero vertices). Moreover, each copy of an induced subgraph of H in G_R can be extended to a copy of H in G using vertices from $L(G)$ in at most $2^{|L(G)|} = O(1)$ ways. Therefore, the number of copies of H in G is at most $O(\overline{ex}(n, H, F))$, as desired. □

2.2 Counting triangles

Let $F_1, \ldots, F_k$ be graphs different from K_2 and let F be their vertex-disjoint union. (Note that the F_i's are not necessarily different.)

Theorem 6.

$$ex(n, K_3, F) = \Theta\left(\max_{1\le i\le k}\{ex(n, K_3, F_i)\} + \max_{1\le i<j\le k} ex(n, \{F_i, F_j\})\right).$$

Proof of Theorem 6. For the lower bound, consider the following two constructions.

1. Take an F_i-free graph containing the largest number of triangles. This graph is obviously F-free, showing that $ex(n, K_3, F) \ge \max_{1\le i\le k}\{ex(n, K_3, F_i)\}$.

2. Now let i, j be two integers with $1 \le i < j \le k$. Take an $\{F_i, F_j\}$-free graph G_0 on $n-1$ vertices, and add a universal vertex v. Then any copy of F_i and any copy of F_j in the resulting graph G must contain v, thus there are no vertex-disjoint copies of F_i and F_j in this graph. Therefore it does not contain F. Furthermore, the number of triangles in G is at least the number of edges in G_0, as these edges form a triangle with v. Thus we have $ex(n, K_3, F) \ge \max_{1\le i<j\le k} ex(n, \{F_i, F_j\})$.

For the upper bound, we use induction on k. The base case $k = 1$ is trivial. Let F' be the graph obtained by deleting F_i from F and F'' be the graph obtained by deleting F_j from F. Let us consider an F-free graph G on n vertices. If G is F'-free or F''-free, then we are done by induction. Thus we may assume G contains a copy of F' and a copy of F'', and let L be the union of their vertex sets. Note that these copies share at least one vertex, otherwise there would be a copy of F in G.

Let G' be the subgraph of G induced by $V(G)\setminus L$. Then G' is obviously both F_i-free and F_j-free, hence it contains at most $ex(n, \{F_i, F_j\})$ edges and at most $ex(n, K_3, F_i)$ triangles. Let T_s denote the set of triangles in G which contain exactly s vertices from L. Then we have $|T_3| = O(1)$, $|T_2| = O(n)$, $|T_1| = O(ex(n, \{F_i, F_j\}))$ and $|T_0| \le ex(n, K_3, F_i)$. Adding up these bounds, the proof is complete.

□

Remark 7.

- *Note that by Theorem 6, we have $ex(n, K_3, kF) = \Theta(ex(n, K_3, F) + ex(n, F))$, for any integer $k \ge 2$. This shows that when k increases from 1 to 2, the order of magnitude of $ex(n, K_3, kF)$ can increase, but from then on (i.e., for $k \ge 2$), there is no further increase.*

- *The Compactness conjecture of Erdős and Simonovits [6] states that for any finite family $\mathcal{G}$ of graphs there is a $G \in \mathcal{G}$ such that $ex(n, \mathcal{G}) = \Theta(ex(n, G))$. It is known to be true for several classes of graphs, for example if $\mathcal{G}$ contains at most one bipartite graph.*

  Let $ex^{sec}(n, F)$ be the second largest of the Turán numbers $ex(n, F_i)$, $1 \le i \le k$. Note that if the Compactness conjecture is true (even if it is true only for families of two graphs), then $\max_{1\le i<j\le k} ex(n, \{F_i, F_j\}) = \Theta(ex^{sec}(n, F))$. (In particular, if F is non-bipartite, then this is the case.) Thus if the Compactness conjecture is true, then Theorem 6 can be stated as follows:

$$ex(n, K_3, F) = \Theta\left(\max_{1\le i\le k}\{ex(n, K_3, F_i)\} + ex^{sec}(n, F)\right).$$

Definition 8. *Let us define*

$$ex^*(n, F) := \max_G\{(k-1)\,|E(G)| + N(K_3, G) : G \text{ is an } n\text{-vertex } F\text{-free graph}\}.$$

Note that we have $ex(n, K_3, F) \le ex^*(n, F) \le (k-1)ex(n, F) + ex(n, K_3, F)$. Let us consider an arbitrary F, and let F_u be the graph we get by deleting the vertex u from F.

Theorem 9. *Let $|V(F)| \ge 4$. Then for every $u \in V(F)$ we have*

$$ex^*(n-k+1, F) \le ex(n, K_3, kF) \le ex^*(n, F) + (k-1)|V(F)|ex(n, F_u) + O(n).$$

Proof. For the lower bound of $ex(n, K_3, kF)$, take an F-free graph G on $n-k+1$ vertices for which $(k-1)\,|E(G)| + N(K_3, G)$ is maximum, and consider $K_{k-1} + G$. Then every edge of G, together with the $k-1$ universal vertices, gives $k-1$ triangles. This shows $ex(n, K_3, kF) \ge ex^*(n-k+1, F)$.

For the upper bound we use induction on k, the case $k = 1$ is trivial. If a kF-free graph G does not contain $(k-1)F$, then we are done by induction. So we may assume G contains a copy of $(k-1)F$.

Recall that F_u is the graph obtained by removing the vertex u from F. Let $X(u)$ be the set of neighbors of u in F. Let F^* be the graph we get by taking $(k-1)|V(F)|+1$ vertex disjoint copies of F_u and an additional vertex v (we call it the *center* of F^*), that is connected to all the vertices in $X(u)$ in each copy of F_u. In other words we take $(k-1)|V(F)|+1$ vertex-disjoint copies of F and identify the vertices u from each copy of F. So F^* is *created* by $(k-1)|V(F)|+1$ copies of F that intersect only in the center. Observe that if we are given a copy F_0 of F and a copy of F^*, such that F_0 does not contain the center of the F^*, then at least one of the copies of F which create F^* is disjoint from F_0.

Now let us assume G contains $k-1$ vertex-disjoint copies of F^* and let $v_1, \dots, v_{k-1}$ be their centers. In that case every copy F_0 of F contains at least one of the centers. Indeed, otherwise each of the $k-1$ copies of F^* contain a copy of F that is disjoint from F_0, and these copies are all disjoint from each other. Therefore they form a copy of kF, a contradiction. Thus by deleting $v_1, \dots, v_{k-1}$ we get an F-free graph G', and $ex(n, K_3, kF)$ is at most the number of triangles in G' plus the number of triangles t_i with exactly i vertices from $\{v_1, \dots, v_{k-1}\}$ for each $1 \le i \le 3$. t_1 is at most $k-1$ times the number of edges in G', whereas $t_2 + t_3 = O(n)$. Therefore, the total number of triangles in G is at most $ex^*(n-k+1, F) + O(n)$, as desired.

So we may assume G does not contain $(k-1)F^*$. Let us consider the canonical partition, i.e. a copy G_L of $(k-1)F$ and the graph G_R induced by the remaining vertices. Then there are $O(n)$ triangles containing 0 or 1 vertices from $R(G)$, and $N(K_3, G_R)$ triangles containing 3 vertices from $R(G)$. It remains to bound the number of triangles which contain exactly one vertex from the $(k-1)|V(F)|$ vertices not in $R(G)$.

Let us consider the subgraph H which consists of the largest number (which is at most $(k-2)$) of vertex-disjoint copies of F^* where each copy has its center in $L(G)$ and all its other vertices in $R(G)$. Let x be a vertex in $L(G)$ that does not belong to H and consider the graph H_0 induced by its neighborhood in $R(G) \setminus V(H)$. Observe that H_0 cannot contain more than $(k-1)|V(F)|+1$ vertex-disjoint copies of F_u. Thus the number of edges in H_0 is at most

$$ex(n, ((k-1)|V(F)|+1)F_u) = ex(n, F_u) + O(n)$$

using Theorem 1. In addition the number of edges incident to $R(G) \cap V(H)$ is at most

$$(k-2)((k-1)|V(F)|+1)|V(F)|n = O(n).$$

Thus x is in $ex(n, F_u) + O(n)$ triangles. There are at most $(k-1)|V(F)|$ vertices in $L(G)$. Moreover, there are at most $k-2$ vertices in $L(G)$ that belong to H, and each of them is in at most $|E(G_R)|$ triangles. Therefore, the number of triangles which contain exactly one vertex from the $(k-1)|V(F)|$ vertices not in $R(G)$ is at most $(k-2)|E(G_R)|+(k-1)|V(F)|ex(n, F_u) + O(n)$.

Therefore, the total number of triangles in G is at most

$$N(K_3, G_R) + (k-2)|E(G_R)|+(k-1)|V(F)|ex(n, F_u) + O(n)$$

$$\le ex^*(n, F) + (k-1)|V(F)|ex(n, F_u) + O(n),$$

completing the proof.

□

Remark 10. *Note that the proof shows a stronger upper bound. In one of the two cases we get an upper bound matching the lower bound. In the other case we obtain an upper bound of the form $(k-2)\,|E(G)|+N(K_3, G)$ rather than $(k-1)\,|E(G)|+N(K_3, G)$ as in the definition of $ex^*(n, F)$. In case $ex(n, F_u)$ has smaller order of magnitude than $ex(n, F)$ and n is large enough, this implies that $ex(n, K_3, kF) = ex^*(n-k+1, F)$.*

Note that if F_u is not a forest, we also know $ex(n, |V(F)|F_u) = (1+o(1))ex(n, F_u)$ by Theorem 1. Theorem 9 shows that if $ex(n, F_u) = o(ex^*(n, F))$, then we have

$$ex(n, K_3, kF) = (1+o(1))ex^*(n, F). \tag{1}$$

This implies the following.

Corollary 11. *(a) If F has chromatic number $r > 3$, then we have*

$$ex(n, K_3, kF) = (1 + o(1))\binom{r-1}{3}\left(\frac{n}{r-1}\right)^3.$$

(b) For $k \geq 1$, we have

$$ex(n, K_3, kK_{2,t}) = (1 + o(1))\left(\frac{(k-1)(t-1)^{1/2}}{2} + \frac{(t-1)^{3/2}}{6}\right)n^{3/2}.$$

Proof. First let us prove (a). For any vertex $u \in V(F)$, trivially $ex(n, F_u) = O(n^2)$. Alon and Shikhelman [1] showed that $ex(n, K_3, F) = (1 + o(1))\binom{r-1}{3}\left(\frac{n}{r-1}\right)^3$ which is equal to $ex^*(n, F)$ asymptotically. Thus $ex(n, F_u) = o(ex^*(n, F))$, so by (1), we are done.

Now we prove (b). Alon and Shikhelman [1] showed that

$$ex(n, K_3, K_{2,t}) = (1 + o(1))\frac{(t-1)^{3/2}}{6}n^{3/2}.$$

In fact, they establish the lower bound $ex(n, K_3, K_{2,t}) \geq (1+o(1))\frac{(t-1)^{3/2}}{6}n^{3/2}$ by considering the $K_{2,t}$-free graph constructed by Füredi [7] and counting the number of triangles in it. This graph contains $\frac{(t-1)^{1/2}}{2}n^{3/2}(1 + o(1)) = ex(n, K_{2,t})$ edges, so it follows that

$$ex^*(n, K_{2,t}) \geq (1 + o(1))\left(\frac{(k-1)(t-1)^{1/2}}{2} + \frac{(t-1)^{3/2}}{6}\right)n^{3/2}$$

and this bound is sharp since by definition, $ex^*(n, K_{2,t}) \leq ex(n, K_3, K_{2,t}) + (k-1)ex(n, K_{2,t})$. Now it can be easily seen that $ex(n, F_u) = O(n) = o(ex^*(n, F))$ if $F = K_{2,t}$ for some u. So, by (1) again, the proof is complete.

□

2.3 Counting complete graphs

Theorem 12. *Let F be a graph and k, r be integers. Let $\max_{m \leq r}(ex(n, K_m, F)) = ex(n, K_{m_0}, F)$. Then we have*

$$ex(n, K_r, kF) = O(ex(n, K_{m_0}, F)).$$

Moreover, if $k > r - m_0$, then

$$ex(n, K_r, kF) = \Theta(ex(n, K_{m_0}, F)).$$

Proof. Let us consider a kF-free graph G, and its canonical partition. Then every copy of K_r consists of m vertices in $R(G)$ and $r - m$ vertices in $L(G)$ for some integer $m \leq k$. These latter ones can be chosen in at most $\binom{k|V(F)|}{r-m} = O(1)$ ways, thus we have

$$ex(n, K_r, kF) = O(\sum_{m \le r} ex(n, K_m, F)) \le O(ex(n, K_{m_0}, F)).$$

Let us now consider the graph on $n - r + m_0$ vertices that contains the most copies of K_{m_0}, and add $r - m_0$ universal vertices. The resulting graph contains $\Omega(ex(n, K_{m_0}, F))$ copies of K_r. On the other hand, every copy of F contains at least one of the additional vertices, thus there are at most $r - m_0 < k$ pairwise vertex-disjoint copies of F in it.

□

3 Forbidding complete graphs

As we mentioned in the introduction, Erdős [5] determined the exact value of $ex(n, K_s, K_t)$ for $s < t$ and Simonovits [17] determined the exact value of $ex(n, kK_t)$ for sufficiently large n.

In this section we investigate the function $ex(n, K_s, kK_t)$. First we present our main result of this section, which determines the order of magnitude for every s, t and k. Then we show two asymptotic results (for special values of s and t).

3.1 $ex(n, K_s, kK_t)$

The following theorem determines the order of magnitude of $ex(n, K_s, kK_t)$ for all s, t and k (as n tends to infinity). Note that if $s < t$, then the Turán graph shows $ex(n, K_s, kK_t) = \Theta(n^s)$.

Theorem 13. *Let $s \ge t \ge 2$ and $k \ge 1$ be arbitrary integers and let $x := \lceil \frac{kt-s}{k-1} \rceil - 1$. Then we have*

$$ex(n, K_s, kK_t) = \Theta(n^x).$$

Proof. For the lower bound, consider the Turán graph $K_{s-x} + T_x(n-s+x)$. This graph is kK_t-free as k vertex-disjoint copies of K_t together contain at most kx vertices from $T_x(n-s+x)$ and at most $s-x$ vertices from K_{s-x}. Thus they together contain at most $s + (k-1)x < kt$ vertices. On the other hand, the Turán-graph $T_x(n-s+x)$ contains $\Omega(n^x)$ copies of K_x, and they all can be extended to different copies of K_s.

To prove the upper bound we will repeatedly apply the canonical partition operation.

Step 1:

(1.1) Consider a kK_t-free graph G_1 and its canonical (k, K_t)-partition.

(1.2) Let us fix an arbitrary nonempty $X_1 \subset L(G_1)$ (of size x_1) and let

$$\mathcal{A}(X_1) := \{A : A \text{ is a copy of } K_s \text{ in } G_1 \text{ with } |V(A) \cap L(G_1)| = X_1\}.$$

Note that $V(A) \cap R(G_1)$ spans a (copy of) K_{s-x_1} for all $A \in \mathcal{A}(X_1)$ and let G_2 be the subgraph of G_1 spanned by the union of $\{V(A) \cap R(G_1) : A \in \mathcal{A}(X_1)\}$. So G_2 is a graph on the vertex set $R(G_1)$. We consider two cases:

Case 1: G_2 contains k disjoint copies of K_{s-x_1}. Observe that $s - x_1 < t$ and let us denote the corresponding copies of K_{s-x_1} by $A_1, \dots, A_k$. We claim that in this case we have

$$x_1 + k(s - x_1) < kt.$$

Otherwise we could complete $A_1, \dots, A_k$ from X_1 into k disjoint copies of K_t as every vertex of G_2 is connected to every vertex of X_1 and that would be a contradiction. This inequality implies $s - x_1 \le x$, and as obviously there are $O(n^{s-x_1})$ copies of K_s in $\mathcal{A}(X_1)$ we stop the application of canonical partitions here, and we are done.

Case 2: G_2 does not contain k disjoint copies of K_{s-x_1}. In this case we jump to Step 2.

Now we describe the ith step for $i \ge 2$:

Step i: We have from the $(i-1)$th step:

1) a sequence of subsets $X_1, L(G_1), \dots, X_{i-1}, L(G_{i-1})$ of the vertex set of our initial graph G_1, where:

1.1) $X_j \subset L(G_j)$ for all $j \le i - 1$, and

1.2) $L(G_j)$ $(j \le i - 1)$ are pairwise disjoint.

2) A set of copies of K_s in G_1 parametrized by $X_1, \dots, X_{i-1}$:

$$\mathcal{A}(X_1, \dots, X_{i-1}) :=$$

$$\{A : A \text{ is a copy of } K_s \text{ in } G_1 \text{ with } |V(A) \cap L(G_1)| = X_1, \dots, |V(A) \cap L(G_{i-1})| = X_{i-1}\}.$$

3) G_i, that is the subgraph of G_{i-1} spanned by the (union of the) edges of the copies in $\mathcal{A}(X_1, \dots, X_{i-1})$ on the vertex set $R(G_{i-1})$, and G_i is a $kK_{s-x_1-\dots-x_{i-1}}$-free graph.

We do the following in Step i:

(i.1) We consider the canonical $(k, K_{s-x_1-\dots-x_{i-1}})$-partition of G_i.

(i.2) We fix an arbitrary nonempty $X_i \subset L(G_i)$ (of size x_i) and let

$$\mathcal{A}(X_1, \dots, X_i) :=$$

$$\{A : A \text{ is a copy of } K_s \text{ in } G_1 \text{ with } |V(A) \cap L(G_1)| = X_1, \dots, |V(A) \cap L(G_i)| = X_i\}.$$

Note that $V(A) \cap R(G_i)$ spans a (copy of) $K_{s-x_1-\ldots-x_i}$ for all $A \in \mathcal{A}(X_1, \ldots, X_i)$ and let G_{i+1} be the subgraph of G_i spanned by the union of the edges of the elements of $\{V(A) \cap R(G_i) : A \in \mathcal{A}(X_1, \ldots, X_i)\}$. So G_{i+1} is a graph on the vertex set $R(G_i)$. We consider two cases:

Case 1: G_{i+1} contains k disjoint copies of $K_{s-x_1-\ldots-x_i}$. Let us denote the corresponding copies of $K_{s-x_1-\ldots-x_i}$ by $A_1, \ldots, A_k$. We claim that in this case we have

$$x_1 + \ldots + x_i + k(s - x_1 - \ldots - x_i) < kt.$$

Otherwise we could complete the sets $V(A_1) \cap R(G_1), \ldots, V(A_k) \cap R(G_2)$ from $X_1 \cup \ldots \cup X_i$ into k disjoint copies of K_t as all the vertices of G_{i+1} are connected to all vertices in $X_1 \cup \ldots \cup X_i$ and that would be a contradiction. This inequality implies $s - x_1 - \ldots - x_i \leq x$, and as obviously there are $O(n^{s-x_1-\ldots-x_i})$ copies of K_s in $\mathcal{A}(X_1, \ldots, X_i)$ we stop the application of canonical partitions here.

Case 2: G_{i+1} does not contain k disjoint copies of $K_{s-x_1-\ldots-x_i}$. In this case we jump to Step $(i+1)$.

Note that our algorithm finishes in at most kt steps as we always choose nonempty subsets. There are at most $O(1)$ ways to pick $X_1, \ldots, X_s$ and every copy of a K_s will be an element of some $\mathcal{A}(X_1, \ldots, X_j)$ for some $j \leq s$ (and we stop the algorithm in Step j), so we are done with the proof.

□

Theorem 14. *If $t > s$, then we have*

$$ex(n, K_s, kK_t) = (1 + o(1)) \binom{t-1}{s} \left(\frac{n}{t-1}\right)^s.$$

Proof. To prove the lower bound one just considers the Turán graph $T_{t-1}(n)$.

For the upper bound consider a kK_t-free graph G and its canonical partition. First let us count the copies of K_s that have a common vertex with $L(G)$. There are $O(1)$ ways to pick the vertices from $L(G)$ and $O(n^{s-1})$ ways to pick the remaining vertices from $R(G)$.

Now let us count those copies that are in G_R. As G_R is a K_t-free graph on at most n vertices, by Proposition 2 there are at most

$$(1 + o(1)) \binom{t-1}{s} \left(\frac{n}{t-1}\right)^s$$

copies of K_s in it. Adding these bounds up, the proof is complete. □

Theorem 15. *If $s \geq t \geq s - k + 2$ then we have*

$$ex(n, K_s, kK_t) = (1 + o(1)) \binom{k-1}{s-t+1} \left(\frac{n}{t-1}\right)^{t-1}.$$

Proof. For the lower bound consider the graph $K_{k-1}+T_{t-1}(n-k+1)$. Counting the number of K_s's that contain exactly $s-t+1$ vertices from K_{k-1} gives the desired lower bound.

For the upper bound consider a kK_t-free graph G and its canonical partition. Then any copy of K_s contains at most $t-1$ vertices from $R(G)$. The number of those copies that contain exactly i vertices from $R(G)$ is at most $\binom{t(k-1)}{s-i}\binom{n}{i} = O(n^i)$, thus it is enough to only take care of those copies that contain exactly $t-1$ vertices from $R(G)$.

Let

$$\mathcal{A} := \{A : A \text{ is a copy of } K_s \text{ in } G \text{ with } |V(A) \cap R(G)| = t-1\}.$$

Then we want to upper bound the cardinality of $\mathcal{A}$. Note that any element of $\mathcal{A}$ intersects $R(G)$ in $t-1$ vertices spanning a complete graph in G. Let $B_1, \dots, B_r$ be the copies of K_t defining G_L with for some $r < k$.

Fix an arbitrary $A_1 \in \mathcal{A}$ that intersects B_1. One can easily see that the cardinality of

$$\mathcal{A}_1' := \{A \in \mathcal{A} : V(A) \cap V(B_1) \cap R(G) \neq \emptyset\}$$

is $O(n^{t-2})$. Consider any $A_1' \in \mathcal{A} \backslash \mathcal{A}_1'$ such that $A_{1,R} = V(A) \cap R(G)$ and $A_{1,R}' = V(A_1') \cap R(G)$ are disjoint and $A_1' \cap B_1 \neq \emptyset$. The existence of such A_1' implies that $A_1 \cap B_1$ and $A_1' \cap B_1$ are the same one element since otherwise B_1 could be replaced by two disjoint copies of K_t in G (namely those spanned by $V(A_R) \cup \{x\}$ and $V(A_R') \cup \{y\}$ for some $x \neq y \in B_1$), contradicting the definition of canonical partition.

In the same way for every $1 < i \leq r$ we can fix $A_i \in \mathcal{A}$, that intersects B_i, and (except a set $\mathcal{A}_i' \subset \mathcal{A}$, whose cardinality is $O(n^{t-2})$) we get that for every $A \in \mathcal{A} \setminus \mathcal{A}_i'$ we have

$$A \cap B_i \subset B_i \cap A_i$$

where $A_i \cap B_i$ contains either 0 or 1 vertex. Altogether we have that for every $A \in \mathcal{A} \backslash \cup_{i=1}^r \mathcal{A}_i'$

$$V(A) \cap L(G) \subset \cup_{i=1}^r (V(B_i) \cap V(A_i)).$$

By Proposition 2 there are at most $(1+o(1))\left(\frac{n}{t-1}\right)^{t-1}$ copies of a K_{t-1} in a K_t-free graph. The statement easily follows.

□

4 Forbidding cycles

The systematic study of counting substructures in $2k$-cycle-free graphs was initiated independently in [9, 10] and [11].

4.1 Counting complete graphs

For odd cycles, we have the following interesting phenomenon, depending on whether the size of the clique is bigger than k or not.

Theorem 16. *(a) If $r \leq k$, then $ex(n, K_r, kC_{2l+1}) = \Theta(n^2)$.*

(b) If $r > k+1$, then $ex(n, K_r, kC_{2l+1}) = O(n^{1+1/l})$.

Proof. First, let us prove (a). The lower bound follows from Theorem 12, as

$$\max_{m \leq r}(ex(n, K_m, C_{2l+1})) \geq ex(n, K_2, C_{2l+1}) = \Theta(n^2).$$

The quadratic upper bound similarly follows from Theorem 12, using that for any $r \geq 2$, $ex(n, K_r, C_{2l+1}) = O(n^2)$ (we used Theorem 3 for $r \geq 3$).

Now we prove (b). Consider a kC_{2l+1}-free graph G, and its canonical partition. Then every copy of K_r consists of m vertices in $R(G)$ and $r-m$ vertices in $L(G)$ for some m. If $m > 2$, then by Theorem 3, there are $O(n^{1+1/l})$ copies of a K_m in G_R, as it is C_{2l+1}-free. Moreover, there are $O(1)$ ways to choose the $r-m$ vertices in $L(G)$, so there are at most $O(n^{1+1/l})$ copies of K_r in G such that $m > 2$. Note that the case $m \leq 1$ only gives linearly many copies of K_r. Thus we only have to deal with the case $m = 2$. In other words, it only remains to show that the number of copies of K_r in G which contain exactly two vertices from $R(G)$ is $O(n^{1+1/l})$.

Let G'_R be the subgraph of G_R consisting of only those edges $xy \in E(G_R)$ such that x, y and some $r-2$ vertices from $L(G)$ form a copy of K_r in G. Clearly, the number of copies of K_r in G which contain exactly two vertices from $R(G)$ is $O(E(G'_R))$ because each edge of G'_R can be extended to such a copy of K_r in at most $\binom{|L|}{r-2} = O(1)$ ways. Since an edge xy of G'_R appears in a copy of K_r that contains $r-2$ vertices in $L(G)$, x and y have at least $r-2 \geq k$ common neighbors in $L(G)$. If G'_R contains k vertex-disjoint copies of C_{2l}, then we pick an arbitrary edge from each of these k copies. For these k edges $e_1, \dots, e_k$ we can greedily pick k vertices $v_1, \dots, v_k$ in $L(G)$ such that v_i is adjacent to both endpoints of $e_i = x_i y_i$ for every i. Then we replace e_i with $v_i x_i$ and $v_i y_i$, and this way we get k vertex disjoint copies of C_{2l+1} in G, a contradiction. Thus G'_R does not contain k vertex-disjoint copies of C_{2l}, hence

$$|E(G'_R)| \leq ex(n, kC_{2l}) = ex(n, C_{2l}) + O(n),$$

by Theorem 1. A well-known theorem of Bondy and Simonovits [3] states that $ex(n, C_{2l}) = O(n^{1+1/l})$, so

$$|E(G'_R)| \leq ex(n, kC_{2l}) = O(n^{1+1/l}).$$

Therefore, the number copies of K_r in G which contain exactly two vertices from $R(G)$ is also $O(n^{1+1/l})$, as desired. □

For even cycles, we have the following.

Proposition 17. *For any $r \geq 2$, $l \geq 2$, we have $ex(n, K_r, kC_{2l}) = O(n^{1+1/l})$.*

Proof. Using Theorem 5, we get $ex(n, K_r, kC_{2l}) = O(\overline{ex}(n, K_r, C_{2l})) = O(\sum_{i=1}^{r} ex(n, K_i, C_{2l}))$, as any induced subgraph of K_r is also a complete graph. By Theorem 3, we have $ex(n, K_i, C_{2l}) = O(n^{1+1/l})$ for any $i \geq 1$, so $ex(n, K_r, kC_{2l}) = O(n^{1+1/l})$, as required. □

5 Forbidding bipartite graphs

Let $K_{a,b}$ denote a complete bipartite graph with color classes of sizes a and b with $a \leq b$. Alon and Shikhelman proved the following.

Proposition 18 ([1], Proposition 4.10)**.** *If $s \leq t$ and $a \leq b < s$ then $ex(n, K_{a,b}, K_{s,t}) = O(n^{a+b-ab/s})$.*

We will prove that the same upper bound holds for $ex(n, K_{a,b}, kK_{s,t})$. Note that they also gave a constant factor in their proof; our proof would give a worse constant factor for the case $k > 1$. They also showed that in some range of a and b this order of magnitude is sharp, this immediately implies the same for the case $k > 1$.

Proposition 19. *If $s \leq t$ and $a \leq b < s$ then $ex(n, K_{a,b}, kK_{s,t}) = O(n^{a+b-ab/s})$.*

Proof. Let G be a $kK_{s,t}$-free graph and consider its canonical partition. A copy of $K_{a,b}$ intersects G_R in a copy of $K_{a',b'}$ for some $0 \leq a' \leq a$ and $0 \leq b' \leq b$. For fixed a' and b', there are $O(n^{a'+b'-a'b'/s})$ such copies by Proposition 18 since G_R is $K_{s,t}$-free. Increasing a' by 1, increases $a' + b' - a'b'/s$ by $1 - b'/s$ which is positive since $b' < s$. Applying a similar argument for a', we get that $a' + b' - a'b'/s$ is maximized when $a' = a$ and $b' = b$. Therefore, $O(n^{a'+b'-a'b'/s}) \leq O(n^{a+b-ab/s})$.

Moreover, the number of ways to extend a copy of $K_{a',b'}$ to a copy of $K_{a,b}$ by adding vertices from $L(G)$ is at most a constant (where this constant depends on a, b, s, t,but not on n). Now as $0 \leq a' \leq a$ and $0 \leq b' \leq b$, there are only at most $(a + 1)(b + 1)$ ways to pick a' and b', finishing the proof. □

Note that by Theorem 5 and Remark 4 we have the following general lower bound: $ex(n, K_{a,b}, kK_{s,t}) \geq \Omega(n^{\alpha(K_{a,b})}) = \Omega(n^b)$ if $k \geq a+b$ and $a \leq b$. If $b > s$, then $a+b-ab/s < b$, so the upper bound in the above proposition cannot hold in this case. Instead, we have the following.

Proposition 20. *If $a \leq b$, $b \geq s$, $s \leq t$, then $ex(n, K_{a,b}, kK_{s,t}) = O(n^b)$. Moreover, if $k > a$, then we have $ex(n, K_{a,b}, kK_{s,t}) = \Theta(n^b)$.*

Proof. Let G be a $kK_{s,t}$-free graph and let us consider its canonical partition. A copy of $K_{a,b}$ intersects G_R in a copy of $K_{a',b'}$ for some $0 \leq a' \leq a$ and $0 \leq b' \leq b$ with $a' \leq b'$. Let us fix a' and b' and consider two cases.

If $b' < s$, then by Proposition 18 there are $O(n^{a'+b'-a'b'/s})$ copies of $K_{a',b'}$ in G_R as it is $K_{s,t}$-free. By the same argument as in the proof of Proposition 19, it is easy to see that $a'+b'-a'b'/s$ increases, if we increase a' as long as $b' < s$. So $a'+b'-a'b'/s < s+b'-sb'/s = s \leq b$. Thus there are at most $O(n^b)$ copies of $K_{a',b'}$ in G_R.

If $b' \geq s$, then we first claim that there are at most $O(n^b)$ copies of $K_{a',b'}$ in G_R. Indeed, there are at most $O(n^{b'})$ ways to pick b' vertices, and they have at most $t - 1$ common neighbors (otherwise we can find a $K_{s,t}$ in G_R). Thus there are at most $\binom{t-1}{a'} = O(1)$ ways to pick the other class of $K_{a',b'}$, so there are at most $O(n^{b'}) \leq O(n^b)$ copies of $K_{a',b'}$ in G_R again.

Each copy of $K_{a',b'}$ in G_R can be extended to a copy of $K_{a,b}$ in G by adding vertices from $L(G)$ in $O(1)$ ways. Thus for any fixed a' and b', there are at most $O(n^b)$ copies of $K_{a,b}$ in G and as there are only at most $(a+1)(b+1)$ choices for a' and b', the proof is complete.

For the moreover part, take a $K_{s,t}$-free graph G on $n-k+1$ vertices and consider $K_{k-1}+G$. It is clearly $kK_{s,t}$-free. Any set of a vertices from K_{k-1} together with any set of b vertices from G forms a copy of $K_{a,b}$, which finishes the proof. □

Let us focus now on the case $k=1$ and $b \geq s$, as this case was not examined in [1].

Proposition 21. *(a) If $s \leq a \leq b \leq t$, then $ex(n, K_{a,b}, K_{s,t}) = O(n^s)$.*

(b) If $a < s \leq b \leq t$, then $ex(n, K_{a,b}, K_{s,t}) = \Theta(n^b)$.

Proof. For (a) let us consider a $K_{s,t}$-free graph G and an arbitrary set S of s vertices. We claim that S is contained in the larger color class (i.e., with size b) of at most $O(1)$ copies of $K_{a,b}$. Indeed, the vertices of S have at most $t-1$ common neighbors, thus there are $\binom{t-1}{a}$ ways to pick the other side, and they have at most $t-1$ common neighbors, thus there are at most $\binom{t-1-s}{b-s} = O(1)$ copies of $K_{a,b}$ such that their larger color class contains S. So there are at most $O(n^s)$ copies of $K_{a,b}$, as desired.

For (b) let us consider the graph $K_{s-1,n-s+1}$. It is easy to see that it contains $\Omega(n^b)$ copies of $K_{a,b}$. The upper bound $O(n^b)$ follows from Proposition 20.

□

6 Concluding remarks and open problems

• Our conclusion is that the significant difference between $ex(n, H, F)$ and $ex(n, H, kF)$ mostly comes from the ability to count subgraphs of H due to the universal vertices (vertices of degree $n-1$). A particular example when this is the case is $ex(n, F, kF)$. We did not deal with this especially interesting case, but for complete graphs Theorem 15 implies

$$ex(n, K_t, kK_t) = (k-1+o(1))\left(\frac{n}{t-1}\right)^{t-1}.$$

• Another natural direction is to count lH instead of H. Here we present two results of this type.

Proposition 22.

$$ex(n, lK_2, K_3) = \frac{1}{l!}\left\lfloor\frac{n^2}{4}\right\rfloor\left\lfloor\frac{(n-2)^2}{4}\right\rfloor\dots\left\lfloor\frac{(n-2l+2)^2}{4}\right\rfloor.$$

Proof. The lower bound is given by the complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$.

Let G be an triangle free graph on n vertices. To prove the upper bound, we first select an edge e_1 from G and then we select an edge e_2 disjoint from e_1, and then an edge e_3 disjoint from both e_1 and e_2 and so on. By Mantel's theorem, the maximum number of edges in a triangle-free graph is at most $\lfloor n^2/4 \rfloor$ so we can pick $e_1 = u_1v_1$ in at most $\lfloor n^2/4 \rfloor$ ways. Since the subgraph of G induced by $V(G) \setminus \{u_1, v_1\}$ is also triangle-free, we can pick e_2 in at most $\lfloor (n-2)^2/4 \rfloor$ ways and e_3 in at most $\lfloor (n-4)^2/4 \rfloor$ ways (by Mantel's theorem again) and so on, giving a total of $\lfloor n^2/4 \rfloor \lfloor (n-2)^2/4 \rfloor \dots \lfloor (n-2l+2)^2/4 \rfloor$ ordered tuples of l independent edges $(e_1, e_2, \dots, e_l)$. Since each copy of lK_2 is counted $l!$ times, this implies the desired upper bound.

□

Using Proposition 22, we prove the following asymptotic result.

Theorem 23. *Let $l < k$. Then*

$$ex(n, lK_3, kK_3) = (1 + o(1)) \binom{k-1}{l} \left(\frac{n^2}{4}\right)^l .$$

Proof. The lower bound is given by the graph $K_{k-1} + K_{\lfloor (n-k+1)/2 \rfloor, \lceil (n-k+1)/2 \rceil}$.

For the upper bound, let G be a kK_3-free graph and consider its canonical partition. We say that a triangle is *good* if it has exactly one vertex in $L(G)$.

We claim that it suffices to only count those copies of lK_3 in which every triangle is good. Indeed, no triangle of lK_3 has three vertices in $R(G)$, and if any of them has two vertices in $L(G)$, then we can pick at least $l+1$ vertices from $L(G)$ in $O(1)$ ways, and at most $2l-1$ vertices from $R(G)$ in $O(n^{2l-1}) = o(n^{2l})$ ways, which is covered by the error term in the theorem. So from now on we count the number of copies of lK_3 in which every triangle is good; we will refer to such a copy of lK_3 as a good copy.

We know that the subgraph of G induced by $L(G)$ consists of (maximum possible number of) vertex-disjoint triangles $A_1, \dots, A_r$ for some $r \le k-1$. Let a_i, b_i, c_i be the vertices of A_i for each $1 \le i \le r$.

A good copy of lK_3 can contain only one of the vertices a_i, b_i, c_i for any $1 \le i \le r$, because otherwise we can find more than r vertex-disjoint triangles in G, a contradiction. So in order to count the number of good copies of lK_3 in G, we first pick l of the $r \le k-1$ triangles -say $A_1, A_2, \dots, A_l$ without loss of generality - from G_L in $\binom{r}{l} \le \binom{k-1}{l}$ ways, and then count the number of good copies of lK_3 in which every triangle has a vertex in one of the triangles $A_1, A_2, \dots, A_l$. Now we bound this latter number.

First let us assume that there are two good copies of lK_3 in G which use two different vertices of the triangle $A_i = a_ib_ic_i$, say a_i and b_i for some $1 \le i \le l$. Let the corresponding good triangles of these lK_3's be a_ixy and b_ipq. Then the edges xy and pq must share a vertex, because otherwise we can replace A_i with the triangles a_ixy and b_ipq to produce more than r vertex-disjoint triangles in G, a contradiction. Thus it is easy to see that number of good triangles containing a_i is at most $2n$ since there are at most $2n$ edges that share a vertex with pq. Similarly, the number of good triangles containing b_i or c_i is also at most $2n$ each. Therefore, the total number of good triangles which have a vertex in A_i is at most $6n = O(n)$ in this case. This implies that the number of good copies of lK_3 in which every triangle has a vertex in one of the triangles $A_1, A_2, \dots, A_l$ is at most $O(n) \cdot O(n^{2l-2}) = o(n^{2l})$, which is covered by the error term of the theorem again.

So we can assume that every such good copy of lK_3 contains only one of the vertices a_i, b_i, c_i for each $1 \leq i \leq l$, say $u_1, u_2, \dots, u_l$. Thus we can count the number of those copies by picking a copy of lK_2 from G_R in at most $ex(n, lK_2, K_3) = (1 + o(1))\frac{1}{l!}\left(\frac{n^2}{4}\right)$ ways by Proposition 22 (recall that G_R is triangle-free), and then pairing the l edges of lK_2 with $u_1, u_2, \dots, u_l$ in at most $l!$ ways.

Therefore, the total number of good copies of lK_3 is at most $(1 + o(1))\binom{k-1}{l}\left(\frac{n^2}{4}\right)^l$, as required.

□

• We mention some more specific open problems.

◦ A lower bound of $\Omega(n^s)$ is trivial in Proposition 21 for $s = 1$. However it would be appealing to prove it for all s or even in case $s = 2$.

◦ It would be also interesting to improve Theorem 13 and prove an asymptotic result.

◦ In this article our results mostly obtain the order of magnitude or asymptotics of various quantities. It would be interesting to prove exact results corresponding to them.

• Finally, let us mention that the Turán number of the disjoint union of graphs $F_1, F_2, ..., F_k$, has not been investigated when the F_i's can be different. (See Theorem 1 and the comment after it, for the case when all the F_i's are the same.) It is not hard to prove the following proposition. However, it would be interesting to prove a sharper result in this case.

Proposition 24. *Let us suppose that we have graphs $F_1, ..., F_k$ and let $F = \cup_{1\leq i\leq k} F_i$. Then we have*

$$ex(n, F) = \max\{ex(n, F_i) : i \leq l\} + O(n).$$

Proof of Proposition 24. Let j be an integer such that $ex(n, F_j) = \max\{ex(n, F_i) : i \leq l\}$. Then the lower bound follows by taking an F_j-free graph with maximum possible number of edges.

For the upper bound, consider an F-free graph G. Let F' be a subgraph of G consisting of vertex disjoint copies of $F_1, \dots, F_j$ where j is an integer which is chosen as large as possible. Clearly $j < l$ as G is F-free. Then, of course, the subgraph of G induced by $V(G) \setminus V(F')$ is F_{j+1}-free, so it contains at most $ex(n, F_{j+1}) \leq \max\{ex(n, F_i) : i \leq l\}$ edges. Moreover, there are at most $O(n)$ edges incident to the vertices of F'. Adding these bounds up, the proof is complete. □

Acknowledgements

We are grateful to an anonymous reviewer for very carefully reading our paper and for many helpful comments.

Research of Gerbner was supported by the János Bolyai Research Fellowship of the Hungarian Academy of Sciences and by the National Research, Development and Innovation Office – NKFIH, grant K 116769.

Research of Methuku was supported by the National Research, Development and Innovation Office – NKFIH, grant K 116769.
Research of Vizer was supported by the National Research, Development and Innovation Office – NKFIH, grant SNN 116095.

References

[1] N. Alon, C. Shikhelman. Many T copies in H-free graphs. *Journal of Combinatorial Theory*, Series B, **121**, 146–172, 2016.

[2] B. Bollobás, E. Győri. Pentagons vs. triangles. *Discrete Mathematics*, **308**(19), 4332–4336, 2008.

[3] J. A. Bondy, M. Simonovits. Cycles of even length in graphs. *Journal of Combinatorial Theory, Series B*, **16**(2), 97–105, 1974.

[4] N. Bushaw, N. Kettle. Turán numbers of multiple paths and equibipartite forests. Combinatorics, Probability and Computing, **20**(6), 837–853, 2011.

[5] P. Erdős. On the number of complete subgraphs contained in certain graphs. *Magyar Tud. Akad. Mat. Kut. Int. Közl.*, **7**, 459–474, 1962.

[6] P. Erdős, M. Simonovits. Compactness results in extremal graph theory. *Combinatorica*, **2**(3), 275–288. 1982.

[7] Z. Füredi. New asymptotics for bipartite Turan numbers. *J. Combin. Theory Ser. A*, 75(1):141–144, 1996.

[8] Z. Füredi, M. Simonovits. The history of degenerate (bipartite) extremal graph problems. In: *Erdős Centennial*. Springer Berlin Heidelberg, 169–264, 2013.

[9] D. Gerbner, E. Győri, A. Methuku, M. Vizer. Generalized Turán numbers for even cycles. *arXiv:* 1712.07079, 2017.

[10] D. Gerbner, C. Palmer. Counting copies of a fixed subgraph in F-free graphs. *arXiv:1805.07520*, 2018.

[11] L. Gishboliner, A. Shapira. A Generalized Turán Problem and its Applications. *arXiv*: 1712.00831

[12] I. Gorgol. Turán numbers for disjoint copies of graphs. *Graphs and Combinatorics*, **27**(5), 661–667, 2011.

[13] E. Győri, N. Lemons. Hypergraphs with no cycle of a given length. *Combinatorics, Probability and Computing*, **21**(1-2), 193–201, 2012.

[14] H. Liu, B. Lidicky, C. Palmer. On the Turán number of forests. *The Electronic Journal of Combinatorics*, **20**(2), P62, 2013.

[15] W. Mantel. Problem 28. *Wiskundige Opgaven* **10** 60–61, 1907.

[16] J. W. Moon. On independent complete subgraphs in a graph. *Canad. J. Math* **20** 95–102, 1968.

[17] M. Simonovits. A method for solving extremal problems in graph theory, stability problems. in: *Theory of Graphs (Proc. Colloq., Tihany)*, 279–319, 1966.

[18] M. Simonovits. Paul Erdős influence on extremal graph theory. in: *The mathematics of Paul Erdős*, II, 148-192, Algorithms Combin., 14, Springer, Berlin, 1997.

[19] P. Turán. On an extremal problem in graph theory (in Hungarian), *Mat. Fiz. Lapok* **48**, 436–452, 1941.

[20] L. T. Yuan, X. D. Zhang. The Turán number of disjoint copies of paths. *Discrete Mathematics*, **340**(2), 132–139, 2017.