THE KNESER–POULSEN CONJECTURE FOR SPECIAL CONTRACTIONS

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ABSTRACT. The Kneser–Poulsen Conjecture states that if the centers of a family of N unit balls in $\mathbb{E}^d$ is contracted, then the volume of the union (resp., intersection) does not increase (resp., decrease). We consider two types of special contractions.

First, a *uniform contraction* is a contraction where all the pairwise distances in the first set of centers are larger than all the pairwise distances in the second set of centers. We obtain that a uniform contraction of the centers does not decrease the volume of the intersection of the balls, provided that $N \geq (1+\sqrt{2})^d$. Our result extends to intrinsic volumes. We prove a similar result concerning the volume of the union.

Second, a *strong contraction* is a contraction in each coordinate. We show that the conjecture holds for strong contractions. In fact, the result extends to arbitrary unconditional bodies in the place of balls.

1. Introduction

We denote the Euclidean norm of a vector p in the d-dimensional Euclidean space $\mathbb{E}^d$ by $|p| := \sqrt{\langle p, p\rangle}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product. For a positive integer N, we use $[N] = \{1, 2, \ldots, N\}$. Let $A \subset \mathbb{E}^d$ be a set, and $k \in [d]$. We denote the $k-$th intrinsic volume of A by $\mathrm{V}_k(A)$; in particular, $\mathrm{V}_d(A)$ is the $d-$dimensional volume. The closed Euclidean ball of radius ρ centered at $p \in \mathbb{E}^d$ is denoted by $\mathbf{B}[p, \rho] := \{q \in \mathbb{E}^d \ : \ |p - q| \leq \rho\}$, its volume is $\rho^d \kappa_d$, where $\kappa_d := \mathrm{V}_d(\mathbf{B}[o, 1])$. For a set $X \subset \mathbb{E}^d$, the intersection of balls of radius ρ around the points in X is $\mathbf{B}[X, \rho] := \cap_{x\in X}\mathbf{B}[x, \rho]$; when ρ is omitted, then $\rho = 1$. The *circumradius* $\mathrm{cr}(X)$ of X is the radius of the smallest ball containing X. Clearly, $\mathbf{B}[X, \rho]$ is empty, if, and only if, $\mathrm{cr}(X) > \rho$. We denote the unit sphere centered at the origin $o \in \mathbb{E}^d$ by $\mathbb{S}^{d-1} := \{u \in \mathbb{E}^d \ : \ |u| = 1\}$.

It is convenient to denote the (finite) point configuration consisting of N points $p_1, p_2, \ldots, p_N$ in $\mathbb{E}^d$ by $\mathbf{p} = (p_1, \ldots, p_N)$, also considered as a point in $\mathbb{E}^{d\times N}$. Now, if $\mathbf{p} = (p_1, \ldots, p_N)$ and $\mathbf{q} = (q_1, \ldots, q_N)$ are two configurations of N points in $\mathbb{E}^d$ such that for all $1 \leq i < j \leq N$ the inequality $|q_i - q_j| \leq |p_i - p_j|$ holds, then we say that $\mathbf{q}$ is a *contraction* of $\mathbf{p}$. If $\mathbf{q}$ is a contraction of $\mathbf{p}$, then there may or may not be a continuous motion $\mathbf{p}(t) = (p_1(t), \ldots, p_N(t))$, with $p_i(t) \in \mathbb{E}^d$ for all $0 \leq t \leq 1$ and $1 \leq i \leq N$ such that $\mathbf{p}(0) = \mathbf{p}$ and $\mathbf{p}(1) = \mathbf{q}$, and $|p_i(t) - p_j(t)|$ is monotone decreasing for all $1 \leq i < j \leq N$. When there is such a motion, we say that $\mathbf{q}$ is a *continuous contraction* of $\mathbf{p}$.

In 1954 Poulsen [Pou54] and in 1955 Kneser [Kne55] independently conjectured the following.

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Conjecture 1. *If* $\mathbf{q} = (q_1, \ldots, q_N)$ *is a contraction of* $\mathbf{p} = (p_1, \ldots, p_N)$ *in* $\mathbb{E}^d$*, then*

$$\mathrm{V}_d\left(\bigcup_{i=1}^{N} \mathbf{B}[p_i]\right) \geq \mathrm{V}_d\left(\bigcup_{i=1}^{N} \mathbf{B}[q_i]\right).$$

A similar conjecture was proposed by Gromov [Gro87] and also by Klee and Wagon [KW91].

Conjecture 2. *If* $\mathbf{q} = (q_1, \ldots, q_N)$ *is a contraction of* $\mathbf{p} = (p_1, \ldots, p_N)$ *in* $\mathbb{E}^d$*, then*

$$\mathrm{V}_d\left(\bigcap_{i=1}^{N} \mathbf{B}[p_i]\right) \leq \mathrm{V}_d\left(\bigcap_{i=1}^{N} \mathbf{B}[q_i]\right).$$

In fact, both conjectures have been stated for the case of non-congruent balls.

Conjecture 1 is false in dimension $d = 2$ when the volume V_2 is replaced by V_1: Habicht and Kneser gave an example (see details in [BC02]) where the centers of a finite family of unit disks on the plane is contracted, and the union of the second family is of larger perimeter than the union of the first. On the other hand, Alexander [Ale85] conjectured that under any contraction of the center points of a finite family of unit disks in the plane, the perimeter of the intersection does not decrease. We pose the following more general problem.

Problem 1. *Is it true that whenever* $\mathbf{q} = (q_1, \ldots, q_N)$ *is a contraction of* $\mathbf{p} = (p_1, \ldots, p_N)$ *in* $\mathbb{E}^d$*, then*

$$\mathrm{V}_k\left(\bigcap_{i=1}^{N} \mathbf{B}[p_i]\right) \leq \mathrm{V}_k\left(\bigcap_{i=1}^{N} \mathbf{B}[q_i]\right)$$

holds for any $k \in [d]$*?*

For a recent comprehensive overview on the status of Conjectures 1 and 2, which are often called the Kneser–Poulsen conjecture in short, we refer the interested reader to [Bez13]. Here, we mention the following two results only, which briefly summarize the status of the Kneser–Poulsen conjecture. In [Csi98], Csikós proved Conjectures 1 and 2 for *continuous* contractions in all dimensions. On the other hand, in [BC02] the first named author jointly with Connelly proved Conjectures 1 and 2 for *all* contractions in the Euclidean plane. However, the Kneser–Poulsen conjecture remains open in dimensions three and higher.

1.1. **The Kneser-Poulsen conjecture for uniform contractions.** We will investigate Conjectures 1 and 2 and Problem 1 for special contractions of the following type. We say that $\mathbf{q} \in \mathbb{E}^{d\times N}$ is a *uniform contraction of* $\mathbf{p} \in \mathbb{E}^{d\times N}$ *with separating value* $\lambda > 0$, if

$$\text{(UC)} \qquad |q_i - q_j| \leq \lambda \leq |p_i - p_j| \text{ for all } i, j \in [N], i \neq j.$$

Our first main result is the following.

Theorem 1.1. *Let* $d, N \in \mathbb{Z}^+, k \in [d]$*, and let* $\mathbf{q} \in \mathbb{E}^{d\times N}$ *be a uniform contraction of* $\mathbf{p} \in \mathbb{E}^{d\times N}$ *with some separating value* $\lambda \in (0, 2]$*. If* $N \geq \left(1 + \sqrt{2}\right)^d$ *then*

$$\mathrm{V}_k\left(\bigcap_{i=1}^{N} \mathbf{B}[p_i]\right) \leq \mathrm{V}_k\left(\bigcap_{i=1}^{N} \mathbf{B}[q_i]\right). \tag{1}$$

The strength of this result is its independence of the separating value λ.

The idea of considering uniform contractions came from a conversation with Peter Pivovarov, who pointed out that such conditions arise naturally when sampling the point-sets $\mathbf{p}$ and $\mathbf{q}$ randomly. If one could find distributions for $\mathbf{p}$ and $\mathbf{q}$ that satisfy the reversal of (1) for $k = d$, while simultaneously satisfying (UC) (with some positive probability), it would lead to a counter-example to Conjecture 2. Related problems, isoperimetric inequalities for the volume of random ball polyhedra, were studied in [PP16].

Our second main result is the proof of Conjecture 1 under conditions analogous to those in Theorem 1.1.

Theorem 1.2. *Let $d, N \in \mathbb{Z}^+$, and let $\mathbf{q} \in \mathbb{E}^{d\times N}$ be a uniform contraction of $\mathbf{p} \in \mathbb{E}^{d\times N}$ with some separating value $\lambda \in (0, 2]$. If $N \geq (1 + 2d^3)^d$ then*

$$\mathrm{V}_d\left(\bigcup_{i=1}^{N} \mathbf{B}[p_i]\right) \geq \mathrm{V}_d\left(\bigcup_{i=1}^{N} \mathbf{B}[q_i]\right). \tag{2}$$

Again, the strength of this result is its independence of the separating value λ. Most likely, a more careful computation than the one presented here will give a condition $N \geq d^{cd}$ with a universal constant c below 3. It would be very interesting to see an exponential condition, that is, one of the form $N \geq e^{cd}$.

We note that if $d, N \in \mathbb{Z}^+$, and $\mathbf{q} \in \mathbb{E}^{d\times N}$ is a uniform contraction of $\mathbf{p} \in \mathbb{E}^{d\times N}$ with some separating value $\lambda \in [2, +\infty)$, then (2) holds trivially.

1.2. **The Kneser-Poulsen conjecture for strong contractions.** Let us refer to the coordinates of the point $x \in \mathbb{E}^d$ by writing $x = (x^{(1)}, \dots, x^{(d)})$. Now, if $\mathbf{p} = (p_1, \dots, p_N)$ and $\mathbf{q} = (q_1, \dots, q_N)$ are two configurations of N points in $\mathbb{E}^d$ such that for all $1 \leq k \leq d$ and $1 \leq i < j \leq N$ the inequality $|q_i^{(k)} - q_j^{(k)}| \leq |p_i^{(k)} - p_j^{(k)}|$ holds, then we say that $\mathbf{q}$ is a *strong contraction* of $\mathbf{p}$. Clearly, if $\mathbf{q}$ is a strong contraction of $\mathbf{p}$, then $\mathbf{q}$ is a contraction of $\mathbf{p}$ as well.

We describe a non-trivial example of strong contractions. Let $H := \{x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{E}^d \ : \ x^{(i)} = h\}$ be a hyperplane of $\mathbb{E}^d$ orthogonal to the ith coordinate axis in $\mathbb{E}^d$. Moreover, let $R_H : \mathbb{E}^d \to \mathbb{E}^d$ denote the reflection about H in $\mathbb{E}^d$. Furthermore, let $H^+ := \{x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{E}^d \ : \ x^{(i)} > h\}$ and $H^- := \{x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{E}^d \ : \ x^{(i)} < h\}$ be the two open halfspaces bounded by H in $\mathbb{E}^d$. Now, let us introduce the *one-sided reflection about* H^+ as the mapping $C_{H^+} : \mathbb{E}^d \to \mathbb{E}^d$ defined as follows: If $x \in H \cup H_-$, then let $C_{H^+}(x) := x$, and if $x \in H^+$, then let $C_{H^+}(x) := R_H(x)$. Clearly, for any point configuration $\mathbf{p} = (p_1, \dots, p_N)$ of N points in $\mathbb{E}^d$ the point configuration $\mathbf{q} := (C_{H^+}(p_1), \dots, C_{H^+}(p_N))$ is a strong contraction of $\mathbf{p}$ in $\mathbb{E}^d$.

Clearly, if $H_1, \dots, H_k$ is a sequence of hyperplanes in $\mathbb{E}^d$ each being orthogonal to some of the d coordinate axis of $\mathbb{E}^d$, then the composite mapping $C_{H_k^+} \circ \dots \circ C_{H_2^+} \circ C_{H_1^+}$ is a strong contraction of $\mathbb{E}^d$.

We note that the converse of this statement does not hold. Indeed, $\mathbf{q} = (-100, -1, 0, 99)$ is a strong contraction of the point configuration $\mathbf{p} = (-100, -1, 1, 100)$ in $\mathbb{E}^1$, which cannot be obtained in the form $C_{H_k^+} \circ \dots \circ C_{H_2^+} \circ C_{H_1^+}$ in $\mathbb{E}^1$.

The question whether Conjectures 1 and 2 hold for strong contractions, is a natural one. In what follows we give an affirmative answer to that question. We do a bit more. Recall that a *convex body* in $\mathbb{E}^d$ is a compact convex set with non-empty interior. A convex body K is called an *unconditional* (or, 1-*unconditional*) convex body if for any $x = (x^{(1)}, \dots, x^{(d)}) \in K$ also $(\pm x^{(1)}, \dots, \pm x^{(d)}) \in K$ holds. Clearly, if K is an unconditional convex body in $\mathbb{E}^d$, then K is symmetric about the origin o of $\mathbb{E}^d$. Our third main result is a generalization of the Kneser–Poulsen-type results published in [Bou28] and [Reh80].

Theorem 1.3. *Let $K_1, \dots, K_N$ be (not necessarily distinct) unconditional convex bodies in $\mathbb{E}^d$, $d \geq 2$. If $\mathbf{q} = (q_1, \dots, q_N)$ is a strong contraction of $\mathbf{p} = (p_1, \dots, p_N)$ in $\mathbb{E}^d$, then*

$$\mathrm{V}_d\left(\bigcup_{i=1}^{N}(p_i + K_i)\right) \geq \mathrm{V}_d\left(\bigcup_{i=1}^{N}(q_i + K_i)\right), \tag{3}$$

and

$$\mathrm{V}_d\left(\bigcap_{i=1}^{N}(p_i + K_i)\right) \leq \mathrm{V}_d\left(\bigcap_{i=1}^{N}(q_i + K_i)\right). \tag{4}$$

We note that the assumption that the bodies are unconditional cannot be dropped in Theorem 1.3. Indeed, Figure 1 shows two families of translates of a triangle. Both configurations of the three translation vectors are a strong contraction of the other configuration. The intersection of the first family is a small triangle, while the intersection of the second is a point. Additionally, the union of the first family is of larger area (resp., perimeter) than the union of the second.

Also note that in Theorem 1.3 we cannot replace volume by surface area. Indeed, Figure 2 shows two families of translates of unconditional planar convex bodies. The second family is a contraction of the first, while the union of the second family is of larger perimeter than the union of the first.

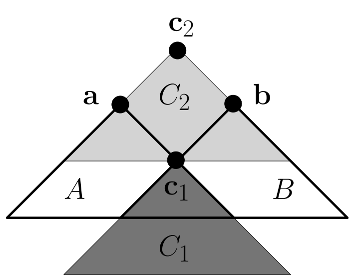

FIGURE 1. First family of translates of a triangle: A, B, C_1; second family: A, B, C_2, where, for the translation vectors, we have $b = -a$, and $c_2 = -c_1$. Both configurations of the three translation vectors are a strong contraction of the other configuration.

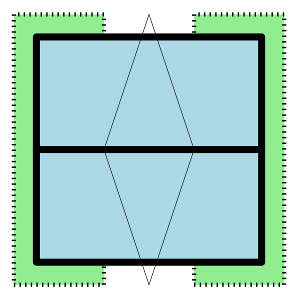

FIGURE 2. First family of unconditional sets: The two vertical rectangles, the two horizontal rectangles and the diamond in the middle; second family: The two vertical rectangles, the upper horizontal rectangle taken twice (once as itself, and once as a translate of the lower horizontal rectangle) and the diamond in the middle.

We prove Theorem 1.1 in Section 2, Theorem 1.2 in Section 3, and finally, Theorem 1.3 in Section 4.

2. Proof of Theorem 1.1

Theorem 1.1 clearly follows from the following

Theorem 2.1. *Let* $d, N \in \mathbb{Z}^+, k \in [d]$*, and let* $\mathbf{q} \in \mathbb{E}^{d\times N}$ *be a uniform contraction of* $\mathbf{p} \in \mathbb{E}^{d\times N}$ *with some separating value* $\lambda \in (0, 2]$*. If*

(a) $N \geq \left(1 + \frac{2}{\lambda}\right)^d$,

or

(b) $\lambda \leq \sqrt{2}$ *and* $N \geq \left(1 + \sqrt{\frac{2d}{d+1}}\right)^d$,

then (1) *holds.*

In this section, we prove Theorem 2.1. We may consider a point configuration $\mathbf{p} \in \mathbb{E}^{d\times N}$ as a subset of $\mathbb{E}^d$, and thus, we may use the notation $\mathbf{B}[\mathbf{p}] = \bigcap_{i\in N} \mathbf{B}[p_i]$. We define two quantities that arise naturally. For $d, N \in \mathbb{Z}^+, k \in [d]$ and $\lambda \in (0, 2]$, let

$$f_k(d, N, \lambda) := \min \left\{ \mathrm{V}_k \left(\mathbf{B}[\mathbf{q}]\right) \ : \ \mathbf{q} \in \mathbb{E}^{d\times N}, |q_i - q_j| \leq \lambda \text{ for all } i, j \in [N], i \neq j \right\},$$

and

$$g_k(d, N, \lambda) := \max \left\{ \mathrm{V}_k \left(\mathbf{B}[\mathbf{p}]\right) \ : \ \mathbf{p} \in \mathbb{E}^{d\times N}, |p_i - p_j| \geq \lambda \text{ for all } i, j \in [N], i \neq j \right\}.$$

In this paper, for simplicity, the maximum of the empty set is zero.

Clearly, to establish Theorem 2.1, it will be sufficient to show that $f_k \geq g_k$ with the parameters satisfying the assumption of the theorem.

2.1. Some easy estimates.

We call the following estimate Jung's bound on f_k.

Lemma 2.2. *Let* $d, N \in \mathbb{Z}^+, k \in [d]$ *and* $\lambda \in (0, \sqrt{2}]$*. Then*

$$f_k(d, N, \lambda) \geq \left(1 - \sqrt{\frac{2d}{d+1}} \frac{\lambda}{2}\right)^k \mathrm{V}_k \left(\mathbf{B}[o]\right). \tag{5}$$

Proof of Lemma 2.2. Let $\mathbf{q} \in \mathbb{E}^{d\times N}$ be a point configuration in the definition of f_k. Then Jung's theorem [Jun01, DGK63] implies that the circumradius of the set $\{q_i\}$ in $\mathbb{E}^d$ is at most $\sqrt{\frac{2d}{d+1}} \frac{\lambda}{2}$. It follows that $\mathbf{B}[\mathbf{q}]$ contains a ball of radius $1 - \sqrt{\frac{2d}{d+1}} \frac{\lambda}{2}$. By the monotonicity (with respect to containment) and the degree-k homogeneity of V_k, the proof of the Lemma is complete. □

The following is a (trivial) packing bound on g_k.

Lemma 2.3. *Let* $d, N \in \mathbb{Z}^+, k \in [d]$ *and* $\lambda > 0$.

$$\text{If } N \left(\frac{\lambda}{2}\right)^d \geq \left(1 + \frac{\lambda}{2}\right)^d, \text{ then } g_k(d, N, \lambda) = 0. \tag{6}$$

Proof of Lemma 2.3. Let $\mathbf{p} \in \mathbb{E}^{d\times N}$ be such that $|p_i - p_j| \geq \lambda$ for all $i, j \in [N], i \neq j$. The balls of radius $\lambda/2$ centered at the points $\{p_i\}$ form a packing. By the assumption, taking volume yields that the circumradius of the set $\{p_i\}$ is at least one. Hence, $\mathbf{B}[\mathbf{p}]$ is a singleton or empty. □

We note that we could have a somewhat better estimate in Lemma 2.3 if we had a good upper bound on the maximum density of a packing of balls of radius $\frac{\lambda}{2}$ in a ball of radius $1 + \frac{\lambda}{2}$.

2.2. **Intersections of balls — An additive Blaschke–Santalo type inequality.** Let X be a non-empty subset of $\mathbb{E}^d$ with $\mathrm{cr}(X) \le \rho$. For $\rho > 0$, the *ρ-spindle convex hull* of X is defined as

$$\mathrm{conv}_\rho(X) := \mathbf{B}[\mathbf{B}[X, \rho], \rho].$$

It is not hard to see that

$$\mathbf{B}[X, \rho] = \mathbf{B}[\mathrm{conv}_\rho(X), \rho]. \tag{7}$$

We say that X is *ρ-spindle convex*, if $X = \mathrm{conv}_\rho(X)$.

Fodor, Kurusa and Vígh [FKV16, Theorem 1.1]. proved a Blaschke–Santalo-type inequality for the volume of spindle convex sets. The main result of this section is an additive version of this inequality, which covers all intrinsic volumes.

Theorem 2.4. *Let $Y \subset \mathbb{E}^d$ be a ρ-spindle convex set with $\rho > 0$, and , $k \in [d]$. Then*

$$\mathrm{V}_k\left(Y\right)^{1/k} + \mathrm{V}_k\left(\mathbf{B}[Y, \rho]\right)^{1/k} \le \rho \mathrm{V}_k\left(\mathbf{B}[o]\right)^{1/k}. \tag{8}$$

Motivated by [FKV16] we observe that Theorem 2.4 clearly follows from the following proposition combined with the Brunn–Minkowski theorem for intrinsic volumes, cf. [Gar02, equation (74)].

Proposition 2.5. *Let $Y \subset \mathbb{E}^d$ be a ρ-spindle convex set with $\rho > 0$. Then*

$$Y - \mathbf{B}[Y, \rho] = \mathbf{B}[o, \rho].$$

Proposition 2.5 has been known (cf. [FKV16, equation (7)]), but only with some hint on its proof. For the sake of completeness, we present the relevant proof here with all the necessary references. We note that instead of Proposition 2.5, one could use a result of Capoyleas [Cap96], according to which, for any ρ-spindle convex set Y, we have that $Y + \mathbf{B}[Y, \rho]$ is a set of constant width 2ρ, and then combine it with the fact that the ball of radius ρ is of the largest $k-$th intrinsic volume among sets of constant width 2ρ (cf. [Sch14, Section 7.4]).

Proof of Proposition 2.5. Lemma 3.1 of [BLNP07] implies in a straighforward way that Y slides freely inside $\mathbf{B}[o, \rho]$. Thus, Theorem 3.2.2 of [Sch14, Section 3.2] yields that Y is a summand of $\mathbf{B}[o, \rho]$ and so, using Lemma 3.1.8 of [Sch14, Section 3.1] we get right away that

$$Y + (\mathbf{B}[o, \rho] \sim Y) = \mathbf{B}[o, \rho],$$

where $\sim$ refers to the Minkowski difference with $\mathbf{B}[o, \rho] \sim Y := \cap_{y \in Y}(\mathbf{B}[o, \rho] - y)$. Thus, we are left to observe that $\cap_{y \in Y}(\mathbf{B}[o, \rho] - y) = -\mathbf{B}[Y, \rho]$. □

We will need the following fact later, the proof is an exercise for the reader.

$$\mathbf{B}[\mathbf{q}] \subseteq \mathbf{B}\left[\bigcup_{i=1}^{N} \mathbf{B}[q_i, \mu], 1 + \mu\right], \tag{9}$$

for any $\mathbf{q} \in \mathbb{E}^{d \times N}$ and $\mu > 0$.

2.3. **A non-trivial bound on g.** The key in the proof of Theorem 2.1 is the following lemma.

Lemma 2.6. *Let $d, N \in \mathbb{Z}^+, k \in [d]$ and $\lambda \in (0, \sqrt{2}]$. Then*

$$g_k(d,N,\lambda) \leq \max\left\{0, \left(1-\left(N^{1/d}-1\right)\frac{\lambda}{2}\right)^k \mathrm{V}_k\left(\mathbf{B}[o]\right)\right\}. \tag{10}$$

Proof of Lemma 2.6. Let $\mathbf{p} \in \mathbb{E}^{d\times N}$ be such that $|p_i - p_j| \geq \lambda$ for all $i, j \in [N], i \neq j$. We will assume that $\mathrm{cr}(\mathbf{p}) \leq 1$, otherwise, $\mathbf{B}[\mathbf{p}] = \emptyset$, and there is nothing to prove.

To denote the union of non-overlapping (that is, interior-disjoint) convex sets, we use the $\bigsqcup$ operator.

Using (9) with $\mu = \lambda/2$, we obtain

$$\mathrm{V}_k\left(\mathbf{B}[\mathbf{p}]\right) \leq \mathrm{V}_k\left(\mathbf{B}\left[\bigsqcup_{i=1}^{N}\mathbf{B}\left[p_i,\frac{\lambda}{2}\right], 1+\frac{\lambda}{2}\right]\right) \quad = \quad (\text{using } \mathrm{cr}(\mathbf{p}) \leq 1, \text{and } (7))$$

$$\mathrm{V}_k\left(\mathbf{B}\left[\mathrm{conv}_{1+\lambda/2}\left(\bigsqcup_{i=1}^{N}\mathbf{B}\left[p_i,\frac{\lambda}{2}\right]\right), 1+\frac{\lambda}{2}\right]\right) \leq \qquad (\text{by } (8))$$

$$\left[\left(1+\frac{\lambda}{2}\right)\mathrm{V}_k\left(\mathbf{B}[o]\right)^{1/k} - \mathrm{V}_k\left(\mathrm{conv}_{1+\lambda/2}\left(\bigsqcup_{i=1}^{N}\mathbf{B}\left[p_i,\frac{\lambda}{2}\right]\right)\right)^{1/k}\right]^k \leq$$

$$\left[\left(1+\frac{\lambda}{2}\right)\mathrm{V}_k\left(\mathbf{B}[o]\right)^{1/k} - \frac{\lambda}{2}N^{1/d}\mathrm{V}_k\left(\mathbf{B}[o]\right)^{1/k}\right]^k,$$

where, in the last step, we used the following. We have

$$\mathrm{V}_d\left(\mathrm{conv}_{1+\lambda/2}\left(\bigsqcup_{i=1}^{N}\mathbf{B}\left[p_i,\frac{\lambda}{2}\right]\right)\right) \geq \mathrm{V}_d\left((N^{1/d}\lambda/2)\mathbf{B}[o]\right).$$

Thus, by a general form of the isoperimetric inequality (cf. [Sch14, Section 7.4.]) stating that among all convex bodies of given (positive) volume precisely the balls have the smallest $k-$th intrinsic volume for $k = 1, \ldots, d-1$, we have

$$\mathrm{V}_k\left(\mathrm{conv}_{1+\lambda/2}\left(\bigsqcup_{i=1}^{N}\mathbf{B}\left[p_i,\frac{\lambda}{2}\right]\right)\right) \geq \mathrm{V}_k\left((N^{1/d}\lambda/2)\mathbf{B}[o]\right).$$

Finally, (10) follows. □

2.4. **Proof of Theorem 2.1.** (a) follows from Lemma 2.3. To prove (b), we assume that $\lambda \leq \sqrt{2}$.

By (5), we have

$$\left(\frac{f_k(d,N,\lambda)}{\mathrm{V}_k\left(\mathbf{B}[o]\right)}\right)^{1/k} \geq 1 - \sqrt{\frac{2d}{d+1}}\frac{\lambda}{2}. \tag{11}$$

On the other hand, (10) yields that either $g_k(d,N,\lambda)=0$, or

$$\left(\frac{g_k(d,N,\lambda)}{\mathrm{V}_k\left(\mathbf{B}[o]\right)}\right)^{1/k} \leq 1-\left(N^{1/d}-1\right)\frac{\lambda}{2}. \tag{12}$$

Comparing (11) and (12) completes the proof of (b), and thus, the proof of Theorem 2.1.

3. Proof of Theorem 1.2

Theorem 1.2 clearly follows from the next theorem. For the statement we shall need the following notation and formula. Take a regular d-dimensional simplex of edge length 2 in $\mathbb{E}^d$ and then draw a d-dimensional unit ball around each vertex of the simplex. Let σ_d denote the ratio of the volume of the portion of the simplex covered by balls to the volume of the simplex. It is well known that $\sigma_d=\left(\frac{1+o(1)}{e}\right)d2^{-\frac{d}{2}}$, cf. [Rog64].

Theorem 3.1. *Let $d,N\in\mathbb{Z}^+$, and let $\mathbf{q}\in\mathbb{E}^{d\times N}$ be a uniform contraction of $\mathbf{p}\in\mathbb{E}^{d\times N}$ with some separating value $\lambda\in(0,2)$.*
(a) If $\lambda\in[\sqrt{2},2)$ and $N\geq\left(1+\frac{\lambda}{2}\right)^d\frac{d+2}{2}$, then (2) holds.
(b) If $\lambda\in[0,\sqrt{2})$ and
$N\geq\left(1+\frac{2}{\lambda}\right)^d\sigma_d=\left(\frac{1}{\sqrt{2}}+\frac{\sqrt{2}}{\lambda}\right)^d\left(\frac{1+o(1)}{e}\right)d$, then (2) holds.
(c) If $\lambda\in[0,1/d^3)$ and $N\geq(2d^2+1)^d$, then (2) holds.

In this section, we prove Theorem 3.1.

The diameter of $\bigcup_{i=1}^N\mathbf{B}[q_i]$ is at most $2+\lambda$. Thus, the isodiametric inequality (cf. [Sch14, Section 7.2.]) implies that

$$\mathrm{V}_d\left(\bigcup_{i=1}^N\mathbf{B}[q_i]\right)\leq\left(1+\frac{\lambda}{2}\right)^d\kappa_d. \tag{13}$$

On the other hand, $\{\mathbf{B}[p_i,\lambda/2]\ :\ i=1,\dots,N\}$ is a packing of balls.

3.1. **To prove part (a)** in Theorem 3.1, we note that Theorem 2 of [BL15] implies in a straightforward way that

$$\frac{N\left(\frac{\lambda}{2}\right)^d\kappa_d}{\mathrm{V}_d\left(\bigcup_{i=1}^N\mathbf{B}[p_i]\right)}\leq\frac{d+2}{2}\left(\frac{\lambda}{2}\right)^d.$$

holds for all $\lambda\in\left[\sqrt{2},2\right)$. Thus, we have

$$\frac{2N\kappa_d}{d+2}\leq\mathrm{V}_d\left(\bigcup_{i=1}^N\mathbf{B}[p_i]\right). \tag{14}$$

As $N\geq\left(1+\frac{\lambda}{2}\right)^d\frac{d+2}{2}$, the inequalities (13) and (14) finish the proof of part (a).

3.2. **For the proof of part (b),** we use a theorem of Rogers, discussed in the introduction of [BL15], according to which

$$\frac{N\left(\frac{\lambda}{2}\right)^d \kappa_d}{\mathrm{V}_d\left(\bigcup_{i=1}^N \mathbf{B}[p_i]\right)} \le \sigma_d.$$

holds for all $\lambda \in \left[0, \sqrt{2}\right)$. Thus, we have

$$\frac{N\left(\frac{\lambda}{2}\right)^d \kappa_d}{\sigma_d} \le \mathrm{V}_d\left(\bigcup_{i=1}^N \mathbf{B}[p_i]\right). \tag{15}$$

As $N \ge \left(1+\frac{2}{\lambda}\right)^d \sigma_d = \left(\frac{1}{\sqrt{2}}+\frac{\sqrt{2}}{\lambda}\right)^d \left(\frac{1+o(1)}{e}\right) d$, the inequalities (13) and (15) finish the proof of part (b).

3.3. **We turn to the proof of part (c).** Note that $\frac{N^{1/d}-1}{2} \ge d^2$. Thus,

$$\mathrm{V}_d\left(\bigcup_{i=1}^N \mathbf{B}[p_i, \lambda/2]\right) = N\left(\frac{\lambda}{2}\right)^d \kappa_d \ge \mathrm{V}_d\left(\mathbf{B}[o, (d^2+1/2)\lambda]\right).$$

Thus, by the isodiametric inequality, there are two points p_j and p_k, with $1 \le j < k \le N$, such that $|p_j - p_k| \ge 2d^2\lambda$. Set $h := |p_j - p_k|/2 \ge d^2\lambda$. Now, $\mathbf{B}[p_j] \cap \mathbf{B}[p_k]$ is symmetric about the perpendicular bisector hyperplane H of p_jp_k, and $D := \mathbf{B}[p_j] \cap H = \mathbf{B}[p_k] \cap H$ is a $(d-1)$-dimensional ball of radius $\sqrt{1-h^2}$. Let H^+ denote the half-space bounded by H containing p_k. Consider the sector (i.e., solid cap) $S := \mathbf{B}[p_j] \cap H^+$, and the cone $T := \mathrm{conv}(\{p_j\} \cup D)$. We have two cases.

Case 1, when $h \le \frac{1}{\sqrt{d}}$. Then clearly,

$$\mathrm{V}_d\left(\bigcup_{i=1}^N \mathbf{B}[p_i]\right) \ge \mathrm{V}_d\left(\mathbf{B}[p_j] \cup \mathbf{B}[p_k]\right) =$$

$$2\kappa_d - 2(\mathrm{V}_d(T) + \mathrm{V}_d(S)) + 2\mathrm{V}_d(T) \ge \kappa_d + 2\mathrm{V}_d(T) = \kappa_d + 2\frac{h}{d}(1-h^2)^{(d-1)/2}\kappa_{d-1}.$$

The latter expression as a function of h is increasing on the interval $[d^2\lambda, 1/\sqrt{d}]$. Thus, it is at least

$$\kappa_d + 2\frac{d^2\lambda}{d}(1-d^4\lambda^2)^{(d-1)/2}\kappa_{d-1} \ge \kappa_d\left[1 + 2d\lambda e^{-d^5\lambda^2}\right].$$

By (13), if

$$1 + 2d\lambda e^{-d^5\lambda^2} \ge \left(1+\frac{\lambda}{2}\right)^d \tag{16}$$

holds, then (2) follows. Using $\lambda \le d^{-3}$, we obtain (16), and thus, Case 1 follows.

Case 2, when $h > \frac{1}{\sqrt{d}}$. Then

$$\mathrm{V}_d\left(\bigcup_{i=1}^N \mathbf{B}[p_i]\right) \ge \mathrm{V}_d\left(\mathbf{B}[p_j] \cup \mathbf{B}[p_k]\right) \ge 2\kappa_d - 2(\mathrm{V}_d(T) + \mathrm{V}_d(S)).$$

Using a well known estimate on the volume of a spherical cap (see e.g. [BW03]), we obtain that the latter expression is at least

$$2\kappa_d\left[1-\frac{(1-h^2)^{(d-1)/2}}{\sqrt{2\pi(d-1)}h}\right]\geq 2\kappa_d\left[1-\frac{(1-1/d)^{(d-1)/2}}{\sqrt{\pi}}\right]\geq 1.1\kappa_d.$$

As in Case 1, we compare this with (13), and obtain Case 2. This completes the proof of Theorem 3.1.

4. Proof of Theorem 1.3

We prove only (3), as (4) can be obtained in the same way.

Let us start with the point configuration $\mathbf{p}=(p_1,\ldots,p_N)$ in $\mathbb{E}^d$ having coordinates

$$p_1=(p_1^{(1)},\ldots,p_1^{(d)}),p_2=(p_2^{(1)},\ldots,p_2^{(d)}),\ldots,p_N=(p_N^{(1)},\ldots,p_N^{(d)}).$$

It is enough to consider the case when $\mathbf{q}=(q_1,\ldots,q_N)$ is such that for each $1\leq i\leq N$ and each $2\leq j\leq d$, we have

$$q_i^{(j)}=p_i^{(j)}.$$

In other words, we may assume that all the coordinates of q_i, except for the first coordinate, are equal to the corresponding coordinate of p_i. Indeed, if we prove (3) in this case, then, by repeating it for the other $d-1$ coordiantes, one completes the proof.

Let ℓ be an arbitrary line parallel to the first coordinate axis. Consider the sets

$$\ell_p:=\ell\cap\left(\bigcup_{i=1}^{N}(p_i+K_i)\right)\text{ and }\ell_q:=\ell\cap\left(\bigcup_{i=1}^{N}(q_i+K_i)\right).$$

Both sets are the union of N (not necessarily disjoint) intervals on ℓ, where the corresponding intervals are of the same length. Moreover, since each K_i is unconditional, the sequence of centers of these intervals in ℓ_q is a contraction of the sequence of centers of these intervals in ℓ_p. Now, (3) is easy to show in dimension 1 (see also [KW91]), and thus, for the total length (1-dimensional measure) of ℓ_p and ℓ_q, we have

$$\text{length}\,(\ell_p)\geq\text{length}\,(\ell_q)\,. \tag{17}$$

Let $H:=\{x=(0,x^{(2)},x^{(3)},\ldots,x^{(d)})\in\mathbb{E}^d\}$ denote the coordinate hyperplane orthogonal to the first axis, and for $x\in H$, let $\ell(x)$ denote the line parallel to the first coordiante axis that intersects H at x.

$$\mathrm{V}_d\left(\bigcup_{i=1}^{N}(p_i+K_i)\right)=\int_H\text{length}\left(\ell(x)\cap\left(\bigcup_{i=1}^{N}(p_i+K_i)\right)\right)\mathrm{d}x\overset{\text{by (17)}}{\geq}$$

$$\int_H\text{length}\left(\ell(x)\cap\left(\bigcup_{i=1}^{N}(q_i+K_i)\right)\right)\mathrm{d}x=\mathrm{V}_d\left(\bigcup_{i=1}^{N}(q_i+K_i)\right),$$

completing the proof of Theorem 1.3.

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