APPROXIMATING SET MULTI-COVERS

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Abstract. Johnson and Lovász and Stein proved independently that any hypergraph satisfies $\tau \le (1+\ln\Delta)\tau^*$, where τ is the transversal number, τ^* is its fractional version, and Δ denotes the maximum degree. We prove $\tau_f \le 3.153\tau^* \max\{\ln\Delta, f\}$ for the f-fold transversal number τ_f. Similarly to Johnson, Lovász and Stein, we also show that this bound can be achieved non-probabilistically, using a greedy algorithm.

As a combinatorial application, we prove an estimate on how fast τ_f/f converges to τ^*. As a geometric application, we obtain an upper bound on the minimal density of an f-fold covering of the d-dimensional Euclidean space by translates of any convex body.

1. Introduction and Preliminaries

A *hypergraph* is a pair $(X, \mathcal{F})$, where X is a finite set and $\mathcal{F} \subseteq 2^X$ is a family of some subsets of X. We call the elements of X *vertices*, and the members of $\mathcal{F}$ *edges* of the hypergraph. When a vertex is contained in an edge, we may say that 'the vertex covers the edge', or that 'the edge covers the vertex'.

Let f be a positive integer. An *f-fold transversal* of $(X, \mathcal{F})$ is a multiset A of X such that each member of $\mathcal{F}$ contains at least f elements (with multiplicity). The *f-fold transversal number* τ_f of $(X, \mathcal{F})$ is the minimum cardinality (with multiplicity) of an f-fold transversal. A 1-transversal is called a transversal, and the 1-transversal number is called the transversal number, and is denoted by $\tau = \tau_1$.

A *fractional transversal* is a function $w : X \longrightarrow [0,1]$ with $\sum_{x:x\in F} w(x) \ge 1$ for all $F \in \mathcal{F}$. The *fractional transversal number* of $(X, \mathcal{F})$ is

$$\tau^* = \tau^*(\mathcal{F}) := \inf\left\{ \sum_{x:x\in X} w(x) \ : \ w \text{ is a fractional transversal} \right\}.$$

Clearly, $\tau^* \le \tau$. In the opposite direction, Johnson [Joh74], Lovász [Lov75] and Stein [Ste74] independently proved that

$$\tau \le (1+\ln\Delta)\tau^*, \tag{1}$$

where Δ denotes the *maximum degree* of $(X, \mathcal{F})$, that is, the maximum number of edges a vertex is contained in. They showed that the greedy algorithm, that is, picking vertices of X one by one, in such a way that we always pick one that is contained in the largest number of uncovered edges, yields a transversal set whose cardinality does not exceed the right hand side in (1). For more background, see Füredi's survey [Für88].

Our main result is an extension of this theorem to f-fold transversals.

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Theorem 1.1. *Let $\lambda \in (0,1)$ and let f be a positive integer. Then, with the above notation,*

$$\tau_f \leq \frac{1-\lambda^f}{1-\lambda}\tau^*(1+\ln\Delta-(f-1)\ln\lambda), \tag{2}$$

moreover, for rational λ, the greedy algorithm using appropriate weights, yields an f-fold transversal of cardinality not exceeding the right hand side of (2)*.*

Substituting $\lambda = 0.287643$ (which is a bit less than $1/e$), we obtain

Corollary 1.2. *With the above notation, we have*

$$\tau_f \leq 3.153\tau^* \max\{\ln\Delta, f\}. \tag{3}$$

This result may be interpreted in two ways. First, it gives an algorithm that approximates the integer programming (IP) problem of finding τ_f, with a better bound on the output of the algorithm than the obvious estimate $\tau_f \leq f\tau \leq f\tau^*(1+\ln\Delta)$.

A similar result was obtained by Rajagopalan and Vazirani in [RV98] (an improvement of [Dob82]), where, the (multi)-set (multi)-cover problem is considered, that is, the goal is to cover vertices by sets. This is simply the combinatorial dual (and therefore, equivalent) formulation of our problem. In [RV98], each set can be chosen at most once. They present generalizations of the greedy algorithm of [Joh74], [Lov75] and [Ste74], and prove that it finds an approximation of the (multi)-set (multi)-cover problem within an $\ln\Delta$ factor of the optimal solution of the corresponding linear programming (LP) problem. Moreover, they give parallelized versions of the algorithms.

The main difference between [RV98] and the present paper is that there, the optimal solution of an IP problem is compared to the optimal solution of the LP-relaxation of the same IP problem, whereas here, we compare τ_f with τ^*, where the latter is the optimal solution of a weaker LP problem: the problem with $f = 1$.

We note that, using the fact that $f\tau^* \leq \tau_f$, (3) also implies that the *performance ratio* (that is, the ratio of the value obtained by the algorithm to the optimal value, in the worst case) of our algorithm is constant when $\ln\Delta \leq f$. Compare this with [BDS04, Lemma 1 in Section 3.1], where it is shown that, even for large f, the standard greedy algorithm yields a performance ratio of $\Omega(\ln m)$, where m is the number of sets in the hypergraph. Further recent results on the performance ratio of another modified greedy algorithm for variants of the set cover problem can be found in [FK06]. See also Chapter 2 of the book [Vaz01] by Vazirani.

The second interpretation of our result is the following. It is easy to see that $\frac{\tau_f}{f}$ converges to τ^* as f tends to infinity. Now, (3) quantifies the speed of this convergence in some sense. In particular, it yields that for $f = \ln\Delta$ we have $\frac{\tau_f}{f} \leq 3.153\tau^*$. We have better approximation for larger f.

Corollary 1.3. *For every $0 < \varepsilon \leq 1$, if we set $f := \left\lceil \frac{2(1+\ln\Delta)}{\varepsilon(1-\lambda)} \right\rceil$, where $0 < \lambda < 1$ is such that $-\ln\lambda/(1-\lambda) \leq 1+\varepsilon/2$, then the f-fold transversal constructed in Theorem 1.1 yields a fractional transversal which gives*

$$\tau^* \leq \frac{\tau_f}{f} \leq \tau^*(1+\varepsilon).$$

We prove Theorem 1.1 and Corollary 1.3 in Section 2, where, at the end, we discuss the running time of our algorithm.

1.1. **A geometric application.** Next, we turn to a classical geometric covering problem. Rogers [Rog57] showed that for any convex body K in $\mathbb{R}^d$, there is a covering of $\mathbb{R}^d$ with translates of K of density at most

$$d \ln d + d \ln \ln d + 5d. \tag{4}$$

For the definition of density cf. [PA95]. G. Fejes Tóth [FT76, FT79] gave the non-trivial lower bound $c_d f$ for the density of an f-fold covering of $\mathbb{R}^d$ by Euclidean unit balls (with some $c_d > 1$). For more information on multiple coverings in geometry, see the survey [FT04]. As an application of Theorem 1.1, we give a similar estimate for f-fold coverings.

Theorem 1.4. *Let $K \subseteq \mathbb{R}^d$ be a convex body and $f \geq 1$ an integer. Then there is an arrangement of translates of K with density at most*

$$(1 + o(1)) \cdot 3.153 \max \{d \ln d, f\},$$

where every point of $\mathbb{R}^d$ is covered at least f times.

The key in proving Theorem 1.4 is a general statement, Theorem 3.1, presented in Section 3. Both theorems are proved the same way as corresponding results in [Nas14], where the case $f = 1$ is considered.

Earlier versions of Theorems 1.4 and 3.1 were proved in [FNN16]. There, in place of the main result of the present paper, a probabilistic argument is used which yields quantitatively weaker bounds. The quantitative gain here comes from the fact that in the probabilistic bound on τ_f presented in [FNN16], one has the size of the edge set $\mathcal{F}$ as opposed to the maximum degree Δ, which is what we have in (3).

2. Proof of Theorem 1.1

2.1. **The algorithm.** First, we imagine that each member of $\mathcal{F}$ has f bank notes, the denominations are $\$1, \$\lambda, \ldots, \$\lambda^{f-1}$, where $\lambda < 1$ is fixed. We pick vertices one by one. At each step, we pick a vertex, and each edge that contains it pays the largest bank note that it has. So, each edge pays \$1 for the first vertex selected from it, then $\$\lambda$ for the second, etc., and finally, $\$\lambda^{f-1}$ for the f-th vertex that it contains. Later on, it does not pay for any additional selected vertex that it contains. Now, we follow the greedy algorithm: at each step, we pick the vertex that yields the largest payout at that step. We finish once each edge is covered at least f times, that is, when we collected all the money.

2.2. **Notation.** Given a positive integer f, we define the *truncated exponential function* denoted by ${}^k\lambda$ as follows: for any $\lambda > 0$, and any $0 \leq k < f$, let ${}^k\lambda = \lambda^k$, and let ${}^k\lambda = 0$ for any $k \geq f$. Note that the value of f is implicitly present in any formula involving the truncated exponential function.

For each $F \in \mathcal{F}$, let $k(F)$ denote the number of chosen vertices (with multiplicity) contained in F. We call the function $k : \mathcal{F} \to \mathbb{Z}^{\geq 0}$ the *current state*, where $\mathbb{Z}^{\geq 0}$ is the set of non-negative integers. At the start, k is identically zero.

Given a function $k : \mathcal{F} \to \mathbb{Z}^{\geq 0}$, we define the *value of a vertex $x \in X$* with respect to k as

$$v_k(x) := \sum_{F : x \in F \in \mathcal{F}} {}^{k(F)}\lambda.$$

The *total remaining value* of k is defined as

$$v(k) := \sum_{F : F \in \mathcal{F}} \sum_{i=k(F)}^{f} {}^{i}\lambda,$$

which is the total pay out that will be earned in the subsequent steps.

2.3. **Fractional matchings.** A *fractional matching* of the hypergraph $(X,\mathcal{F})$ is a function $w:\mathcal{F}\longrightarrow[0,1]$ with $\sum_{F:x\in F\in\mathcal{F}} w(F)\le 1$ for all $x\in X$. The *fractional matching number* of $(X,\mathcal{F})$ is

$$\nu^*=\nu^*(\mathcal{F}):=\sup\left\{\sum_{F:F\in\mathcal{F}} w(F)\ :\ w\text{ is a fractional matching}\right\}.$$

By the duality of linear programming, $\nu^*=\tau^*$.

We will need the following simple observation.

Lemma 2.1. *Let $z>0$, and $\ell:\mathcal{F}\to\mathbb{Z}^{\ge 0}$ be such that $v_\ell(x)\le z$ for any $x\in X$. Then we have*

$$\frac{v(\ell)}{z}\le(1+\lambda+\ldots+\lambda^{f-1})\nu^*(\mathcal{F})=\frac{1-\lambda^f}{1-\lambda}\nu^*(\mathcal{F}).$$

Proof of Lemma 2.1. Let

$$w(F):=\frac{\sum\limits_{i=\ell(F)}^{f}{}^{i}\lambda}{z(1+\lambda+\ldots+\lambda^{f-1})},\ \text{for any } F\in\mathcal{F}.$$

First, we show that w is a fractional matching. Indeed, fix an $x\in X$.

$$\sum_{F:x\in F\in\mathcal{F}} w(F)=\frac{1}{z}\sum_{F:x\in F\in\mathcal{F}}\frac{\sum\limits_{i=\ell(F)}^{f}{}^{i}\lambda}{1+\lambda+\ldots+\lambda^{f-1}}\le$$
$$\le\frac{1}{z}\sum_{F:x\in F\in\mathcal{F}}{}^{\ell(F)}\lambda=\frac{v_\ell(x)}{z}\le 1.$$

Second, the total weight is

$$\sum_{F:F\in\mathcal{F}} w(F)=\frac{1}{z(1+\lambda+\ldots+\lambda^{f-1})}\sum_{F:F\in\mathcal{F}}\sum_{i=\ell(F)}^{f}{}^{i}\lambda=\frac{v(\ell)}{z(1+\lambda+\ldots+\lambda^{f-1})},$$

finishing the proof of the Lemma. □

2.4. **Finally, we count the steps of the algorithm.** We may assume that $\lambda=p/q\in(0,1)$ with $p,q\in\mathbb{Z}^+$. If λ is irrational, then the statement of Theorem 1.1 follows by continuity. Clearly, q^{f-1} is a common denominator for the pay outs at each step.

At the start, $k_0(F):=k(F)=0$ for all $F\in\mathcal{F}$. We group the steps according to the \$-amount (that is, $v_k(x)$) that we get at each.

In the first t_1 steps, each vertex x that we pick has value $v_k(x)=\Delta=:z_1$, where, we recall, Δ is the maximum degree in the hypergraph. Let $k_1:\mathcal{F}\to\mathbb{Z}^{\ge 0}$ denote the current state after the first t_1 steps.

Then, in the second group of steps, we make t_2 steps, at each picking a vertex $x\in V$ of value $v_k(x)=\Delta-q^{1-f}=:z_2$, where k changes at each step. Let $k_2:\mathcal{F}\to\mathbb{Z}^{\ge 0}$ denote the current state after the first t_1+t_2 steps.

In the j-th group of steps, we make t_j steps, at each picking a vertex $x\in V$ of value $v_k(x)=\Delta-(j-1)q^{1-f}=:z_j$. Let $k_j:\mathcal{F}\to\mathbb{Z}^{\ge 0}$ denote the current state after the first $t_1+\ldots+t_j$ steps.

Obviously, $t_j\ge 0$, moreover some t_j may be zero. For instance (the reader may check as an exercise), if $f>1$, then $t_2=0$. For the last group, we have $j=q^{f-1}\Delta-p^{f-1}+1=:N$.

Notice that $v_{k_j}(x) \le z_{j+1}$ for any $x \in V$. Therefore, by Lemma 2.1, we have

$$\frac{v(k_j)}{z_{j+1}} \le \frac{1-\lambda^f}{1-\lambda}\nu^*(\mathcal{F}). \tag{5}$$

Clearly,

$$v(k_j) = \sum_{i=j+1}^{N} t_i z_i, \quad \text{for } 0 \le j \le N-1. \tag{6}$$

In total, we choose $t_1+t_2+\cdots+t_N$ vertices (that is the cardinality of A with multiplicity), and they form an f-fold transversal of $(X, \mathcal{F})$. Thus, by (6) and (5), we obtain

$$\tau_f \le t_1+t_2+\cdots+t_N =$$
$$= \left(\frac{v(k_0)}{z_1} + \sum_{j=1}^{N-1} v(k_j)\left(\frac{1}{z_{j+1}} - \frac{1}{z_j}\right)\right) = \frac{v(k_0)}{z_1} + \sum_{j=1}^{N-1} \frac{v(k_j)q^{1-f}}{z_{j+1}z_j} \le$$
$$\le \frac{1-\lambda^f}{1-\lambda}\nu^*(\mathcal{F})\left(1+\sum_{j=1}^{N-1}\frac{q^{1-f}}{z_j}\right) = \frac{1-\lambda^f}{1-\lambda}\tau^*(\mathcal{F})\left(1+\sum_{k=p^{f-1}+1}^{q^{f-1}\Delta}\frac{1}{k}\right) \le$$
$$\le \frac{1-\lambda^f}{1-\lambda}\tau^*(\mathcal{F})(1+\ln\Delta-(f-1)\ln\lambda),$$

which completes the proof of Theorem 1.1.

2.5. **Proof of Corollary 1.3.** An f-fold transversal $A \subset X$ (A is a multiset) easily yields a fractional transversal: one sets the weight $w(x) = \frac{|\{x:x\in A\}|}{f}$ (cardinality counted with multiplicity) for any vertex $x \in X$. The total weight that we get from our construction in Theorem 1.1 is then

$$\tau^*(\mathcal{F}) \le \sum_{x:x\in V} w(x) \le \tau^*(\mathcal{F})\frac{1+\ln\Delta-(f-1)\ln\lambda}{f(1-\lambda)} \le \tau^*(\mathcal{F})(1+\varepsilon).$$

2.6. **Running time.** Let n denote the number of vertices and m be the number of edges of the hypergraph. The adjacency matrix and f are the inputs of the algorithm. As preprocessing, for each vertex, we create a list of edges that contain it (at most Δ), which takes nm operations. We keep track of the current state in an array k of length m.

At each step, the following operations are performed. Computing the value of a vertex takes the addition of at most Δ numbers. Thus, finding the vertex of maximal value is $n\Delta$ operations. Picking that vertex means decreasing at most Δ entries of the array k by one. We make at most $\frac{1-\lambda^f}{1-\lambda}\tau^*(\mathcal{F})(1+\ln\Delta-(f-1)\ln\lambda)$ steps.

With the $\lambda = 0.287643$ substitution, in total, the number of operations is at most

$$nm + O(\tau^* \max\{\ln\Delta, f\}\cdot\Delta n) \le O(\max\{\ln\Delta, f\}\cdot\Delta nm).$$

3. Multiple covering of space – Proof of Theorem 1.4

We denote by $K \sim T := \{x \in \mathbb{R}^d \ : \ T + x \subseteq K\}$ the *Minkowski difference* of two sets K and T in $\mathbb{R}^d$. For $K, L \subset \mathbb{R}^d$, and $f \ge 1$ integer, we denote the *f-fold translative covering number* of L by K, that is, the minimum number of translates of K such that each point of L is contained in at least f, by $N_f(L,K)$. We denote the *fractional covering number* of L by K by $N^*(L,K) := \tau^*(\mathcal{F})$, where $\mathcal{F} := \{x - K \ : \ x \in L\}$ is a hypergraph with base set $\mathbb{R}^d$, see details in [Nas14], or [AAS15].

Theorem 3.1. *Let K, L and T be bounded Borel measurable sets in $\mathbb{R}^d$ and let $\Lambda \subset \mathbb{R}^d$ be a finite set with $L \subseteq \Lambda + T$. Then*

(7)
$$N_f(L,K) \leq \left\lceil 3.153 N^*(L-T, K \sim T) \max\left\{ \ln\left(\max_{x \in L-K} |(x+(K\sim T)) \cap \Lambda| \right), f \right\} \right\rceil .$$

If $\Lambda \subset L$, then we have

$$N_f(L,K) \leq \left\lceil 3.153 N^*(L, K \sim T) \max\left\{ \ln\left(\max_{x \in L-K} |(x+(K\sim T)) \cap \Lambda| \right), f \right\} \right\rceil . \tag{8}$$

Theorem 3.1 is the f-fold analogue of [Nas14, Theorem 1.2], where the case $f = 1$ is considered. For completeness, we give an outline the proof.

Proof of Theorem 3.1. To prove (7), consider the hypergraph with base set $\mathbb{R}^d$ and hyperedges of the form $u - (K \sim T)$, where $u \in \Lambda$. An f-fold transversal of this hypergraph clearly yields an f-fold covering of L by translates of K. A substitution into (3) yields the desired bound. We omit the proof of (8), which is very similar. □

Using this result, one may prove Theorem 1.4 following [Nas14, proof of Theorem 2.1], which is the proof of Rogers' density bound (4). We give an outline of this proof.

Proof of Theorem 1.4. Let C denote the cube $C = [-a, a]^d$, where $a > 0$ is large. Our goal is to cover C by translates of K economically. We only consider the case when $K = -K$, as treating the general case would add only minor technicalities.

Let $\delta > 0$ be fixed (to be chosen later) and let $\Lambda \subset \mathbb{R}^d$ be a finite set such that $\Lambda + \frac{\delta}{2}K$ is a saturated (ie. maximal) packing of $\frac{\delta}{2}K$ in $C - \frac{\delta}{2}K$. By the maximality of the packing, we have that $\Lambda + \delta K \supseteq C$. By considering volume, for any $x \in \mathbb{R}^d$ we have

$$|\Lambda \cap (x + (1-\delta)K)| \leq \frac{\mathrm{vol}\left((1-\delta)K + \frac{\delta}{2}K\right)}{\mathrm{vol}\left(\frac{\delta}{2}K\right)} \leq \left(\frac{2}{\delta}\right)^d . \tag{9}$$

Let $\varepsilon > 0$ be fixed. Clearly, if a is sufficiently large, then

$$N^*(C - \delta K, (1-\delta)K) \leq (1+\varepsilon)\frac{\mathrm{vol}\,(C)}{(1-\delta)^d \,\mathrm{vol}\,(K)}. \tag{10}$$

By (7), (9) and (10) we have

$$N_f(C,K) \leq \left\lceil 3.153 \frac{1+\varepsilon}{(1-\delta)^d} \frac{\mathrm{vol}\,(C)}{\mathrm{vol}\,(K)} \max\left\{ d \ln\left(\frac{2}{\delta}\right), f \right\} \right\rceil .$$

Thus, we obtain an f-fold covering of C. We repeat this covering periodically for all translates of C in a tiling of $\mathbb{R}^d$ by translates of C, which yields an f-fold covering of $\mathbb{R}^d$. The density of this covering is at most

$$N_f(C,K)\,\mathrm{vol}\,(K) \,/\, \mathrm{vol}\,(C) \leq \left\lceil 3.153 \frac{1+\varepsilon}{(1-\delta)^d} \max\left\{ d \ln\left(\frac{2}{\delta}\right), f \right\} \right\rceil .$$

We choose $\delta = \frac{2}{d \ln d}$, and a standard computation yields the desired result. □

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