ON INFINITE MULTIPLICATIVE SIDON SETS

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Abstract

We prove that if A is an infinite multiplicative Sidon set, then $\liminf_{n\to\infty} \frac{|A(n)|-\pi(n)}{\frac{n^{3/4}}{(\log n)^3}} < \infty$ and construct an infinite multiplicative Sidon set satisfying $\liminf_{n\to\infty} \frac{|A(n)|-\pi(n)}{\frac{n^{3/4}}{(\log n)^3}} > 0$.

Key words and phrases: multiplicative Sidon set, asymptotic density

1. Introduction

Throughout the paper we are going to use the notions $[n] = \{1, 2, \dots, n\}$ and $A(n) = A \cap [n]$ for $n \in \mathbb{Z}^+$, $A \subseteq \mathbb{Z}^+$.

A set A of positive integers is called a *multiplicative Sidon set*, if for every s the equation $xy = s$ has at most one solution (up to ordering) with $x, y \in A$. Let $G(n)$ denote the maximal possible size of a multiplicative Sidon set contained in $[n]$. In [3] Erdős showed that $\pi(n) + c_1 n^{3/4}/(\log n)^{3/2} \le G(n) \le \pi(n) + c_2 n^{3/4}$ (with some $c_1, c_2 > 0$). 31 years later Erdős [4] himself improved this upper bound to $\pi(n) + c_2 n^{3/4}/(\log n)^{3/2}$. Hence, in the lower and upper bounds of $G(n)$ not only the main terms are the same, but the error terms only differ in a constant factor.

A generalization of multiplicative Sidon sets is multiplicative k-Sidon sets where we require that the equation $a_1 a_2 \dots a_k = b_1 b_2 \dots b_k$ does not have a solution with distinct elements taken from the given set. In [7] the maximal possible size of a (multiplicative) k-Sidon subset of $[n]$ was determined asymptotically precisely, furthermore, lower- and upper bounds were given on the error term.

A closely related problem of Erdős-Sárközy-T. Sós and Győri is the following: They examined how many elements of the set $[n]$ can be chosen in such a way that

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ppp@cs.bme.hu. This author was supported by the National Research, Development and Innovation Office NKFIH (Grant Nr. PD115978 and K124171) and the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. This work is connected to the scientific program of the "Development of quality-oriented and harmonized R+D+I strategy and functional model at BME" project, supported by the New Hungary Development Plan (Project ID: TÁMOP-4.2.1/B-09/1/KMR-2010-0002).

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none of the $2k$-element products is a perfect square. Note that if a set satisfies this property, then it is a multiplicative k-Sidon set, since if the equation $a_1a_2\dots a_k = b_1b_2\dots b_k$ has a solution of distinct elements, then the product of these $2k$ numbers is a perfect square. For more details, see [5], [6], [7].

Another related question of Erdős asks for the maximal size of a set of integers not containing $k+1$ different numbers such that $a_0 \mid a_1a_2\dots a_k$. This question is connected to the minimal possible size of a multiplicative basis of order k. For more details, see [1], [2], [8].

In this paper the maximal possible asymptotic density of a multiplicative Sidon set is investigated. According to the result of Erdős, if $A \subseteq \mathbb{Z}^+$ is a multiplicative Sidon set, then for every n we have $A(n) \le \pi(n) + c_2 n^{3/4}/(\log n)^{3/2}$ and the set of primes is, of course, a multiplicative Sidon set for which $|A(n)| = \pi(n)$ for every n.

It is not difficult to construct a multiplicative Sidon set for which

$$\limsup_{n\to\infty} \frac{|A(n)| - \pi(n)}{n^{3/4}/(\log n)^{3/2}} > 0,$$

that is, for infinitely many values of n the set $A(n)$ can be "large". In this paper our aim is to study how large $|A(n)| - \pi(n)$ can be for *all* (sufficiently large) values of n. That is, how "large" a function $f(n)$ can be, if $\liminf\limits_{n\to\infty} \frac{|A(n)|-\pi(n)}{f(n)} > 0$. We are going to show in the following theorems that the "largest" (up to a constant factor) $f(n)$ for which this holds is $f(n) = \frac{n^{3/4}}{(\log n)^3}$.

More precisely, the following theorems are going to be proven:

Theorem 1. *Let A be an infinite multiplicative Sidon set. If*

$$\limsup_{n\to\infty} \frac{|A(n)| - \pi(n)}{\frac{n^{3/4}}{(\log n)^3}} \ge 73643,$$

then we have

$$\liminf_{n\to\infty} \frac{|A(n)| - \pi(n)}{\frac{n}{(\log n)^{48}}} < 0.$$

Theorem 1 immediately implies the following corollary:

Corollary 2. *Let A be an infinite multiplicative Sidon set. Then we have*

$$\liminf_{n\to\infty} \frac{|A(n)| - \pi(n)}{\frac{n^{3/4}}{(\log n)^3}} < 73643.$$

Note that Theorem 1 is logically stronger than this corollary, under the assumption on the $\limsup$ we obtain that there must be indeed a negative deviation of order of magnitude $\frac{n}{(\log n)^{48}}$ compared to the prime counting function $\pi(n)$. Verifying only the corollary would not simplify our proof, so we decided to give a proof for Theorem 1.

Theorem 3. *There exists a multiplicative Sidon set $A \subseteq \mathbb{N}$ such that*

$$\liminf_{n\to\infty} \frac{|A(n)| - \pi(n)}{\frac{n^{3/4}}{(\log n)^3}} > \frac{1}{196608}.$$

2. Proofs

Proof of Theorem 1. Let $A \subseteq \mathbb{Z}^+$ be an infinite multiplicative Sidon set. Throughout this proof p and p_i denote prime numbers. For $a \in A$, let

$$a = p_1 p_2 \dots p_s, \text{ where } 2 \le p_1 \le p_2 \le \dots \le p_s. \tag{1}$$

The characteristic function $\chi_{A,n}$ is defined as

$$\chi_{A,n}(p) = \begin{cases} 1, & \text{if there exists an } a \in A(n) \text{ such that } p|a \\ 0, & \text{if } p \nmid a \text{ for every } a \in A(n) \end{cases}$$

Erdős [3] proved that every $a \le n$ may be written in the form $a = uv$, where $v \le u$ and $u \le n^{2/3}$ or u is a prime number.

The following subsets of $A(n)$ play a crucial role in the proof. For every $l \ge 0$ let

$$A_l^*(n) = \left\{ a : a \in A(n) \text{ and } \exists u, v \text{ such that } a = uv, v \le u, n^{1/3} \le v \le \frac{n^{1/2}}{(\log n)^l} \right\}$$

and

$$A^{**}(n) = \{a : a \in A(n) \text{ and there exist } u, v \text{ such that } a = uv, v \le u, v \le n^{1/3}, \\ u \le n^{2/3} \text{ or } u \text{ is a prime number}\}$$

The proof contains many cases and subcases, therefore we give a brief summary of the strategy for the reader's convenience. To prove the theorem it is enough to show that

$$|A_6^*(n) \cup A^{**}(n)| \le \pi(n) + 11 \cdot \frac{n^{3/4}}{(\log n)^3} \tag{2}$$

if n is large enough and

$$\liminf_{n\to\infty} \frac{|A(n) \setminus (A_6^*(n) \cup A^{**}(n))|}{\frac{n^{3/4}}{(\log n)^3}} < 73632. \tag{3}$$

To verify these bounds it suffices to prove the following five statements:

- Firstly, we are going to prove that

$$|A_l^*(n)| \le \frac{10 n^{3/4}}{(\log n)^{l/2}}, \tag{4}$$

if n is large enough (depending on l). Note that we are going to use this estimation in two cases: $l = 0$ and $l = 6$. The case $l = 6$ is necessary for (2) and the case $l = 0$ is for (7).

- Secondly, we are going to prove that for every positive integer n we have

$$|A^{**}(n)| \leq \sum_{n^{2/3} < p \leq n} \chi_{A,n}(p) + 4n^{2/3}. \tag{5}$$

- Thirdly, we are going to show that if n is large enough, then

$$A(n) \setminus (A_6^*(n) \cup A^{**}(n)) \subseteq A_1(n) \cup A_2(n), \tag{6}$$

where

$$A_1(n) := \{a : a \in A(n), a \geq n/(\log n)^{12}, a = dp_i p_{i+1} \dots p_s, d \leq (\log n)^{12}, \frac{n^{1/6}}{(\log n)^6} \leq p_i \leq p_{i+1} \leq \dots \leq p_s \leq n^{1/2}(\log n)^6, p_s \geq \frac{n^{1/2}}{(\log n)^6}\}$$

and

$$A_2(n) := \{a : a \in A(n), a \geq n/(\log n)^{12}, a = dp_{s-3}p_{s-2}p_{s-1}p_s, d \leq (\log n)^{12}, \frac{n^{1/4}}{(\log n)^9} \leq p_{s-3} \leq p_{s-2} \leq p_{s-1} \leq p_s \leq n^{1/4}(\log n)^9\}.$$

Note that according to (1) the number d denotes the product of the $i-1$ smallest prime divisors of a in the definition of $A_1(n)$, while in the case of $A_2(n)$ the number d is the product of the $s-4$ smallest prime factors of a, specially, in this case $s \geq 4$.

- Fourthly, we are going to show that the inequality

$$\limsup_{n\to\infty} \frac{|A_1(n)|}{\frac{n^{3/4}}{(\log n)^3}} > 0$$

implies

$$\liminf_{n\to\infty} \frac{|A(n)| - \pi(n)}{\frac{n}{(\log n)^{48}}} < 0. \tag{7}$$

- Finally, we are going to prove the inequality

$$\limsup_{n\to\infty} \frac{|A_2(n)|}{\frac{n^{3/4}}{(\log n)^3}} < 73632. \tag{8}$$

Note that inequality (2) follows from (4) and (5) and inequality (3) follows from (6), (7) and (8).

In order to prove (4) and (5) we are going to use Lemma 2. of [4]:

Lemma 4. *Let $G = (V, E)$ be a graph having t_1 vertices $x_1, \dots, x_{t_1}$. Assume that each edge of G is incident to one of the vertices x_i, $1 \le i \le t_2 < t_1$, and that G contains no rectangle (i.e. no circuit of four edges, the rectangle will be denoted by C_4). Then*

$$|E| \le t_1 + t_1 \left[\frac{t_2}{t_1^{1/2}}\right] + t_2^2 \left(1 + \left[\frac{t_2}{t_1^{1/2}}\right]\right)^{-1} \le t_1 + 2t_1^{1/2} t_2.$$

Proof of (4)**.** Let $L = l \log_2 \log n$. According to the definition of $A_l^*(n)$, every $a \in A_l^*(n)$ can be expressed as $a = uv$ where $v \le u$ and $n^{1/3} \le v \le \frac{n^{1/2}}{(\log n)^l}$. This representation might not be uniquely determined, let us choose for every $a \in A_l^*(n)$ the decomposition where v is minimal. As $n^{1/3} \le v \le \frac{n^{1/2}}{(\log n)^l}$, there is a unique integer $r \in \left[0, \frac{1}{6}\log_2 n\right]$ such that

$$\frac{n^{1/2}}{2^{r+L+1}} < v \le \frac{n^{1/2}}{2^{r+L}}. \tag{9}$$

Let us take an $r \in \left[0, \frac{1}{6}\log_2 n\right]$ and pick those elements $a \in A(n)$ for which the chosen decomposition $a = uv$ satisfies (9) with this choice for r.

In this case we have $u \le n^{1/2} 2^{r+L+1} \le 2n^{2/3}$. Define the graph $G_r = (V_r, E_r)$ as follows: The vertices are $1, 2, \dots, \lfloor n^{1/2} 2^{r+L+1} \rfloor$. There is an edge between u and v, if $a = uv$ is the chosen representation for some $a \in A_l^*(n)$ satisfying (9).

The graph G_r is C_4-free, otherwise for some $v_1, v_2, u_1, u_2 \in V$ we would have $(v_1, u_1), (v_1, u_2), (v_2, u_1), (v_2, u_2) \in E$. This would imply that $v_1u_1, v_1u_2, v_2u_1, v_2u_2 \in A$, but $(u_1v_1)(u_2v_2) = (u_1v_2)(u_2v_1)$ contradicts the multiplicative Sidon property.

Clearly, G_r satisfies the conditions of Lemma 4 with $t_2 = \frac{n^{1/2}}{2^{r+L}}$ and $t_1 = n^{1/2} 2^{r+L+1}$. This yields that the number of the edges in graph G_r is at most

$$|E_r| \le t_1 + 2t_1^{1/2} t_2 \le 2n^{2/3} + 2 \cdot n^{1/4} 2^{\frac{r}{2} + \frac{L}{2} + \frac{1}{2}} \cdot \frac{n^{1/2}}{2^{r+L}} = 2n^{2/3} + \sqrt{8} \cdot \frac{n^{3/4}}{2^{\frac{r}{2} + \frac{L}{2}}}.$$

The number of those $a \in A(n)$ for which $a = uv$ with $v = u$ is at most $n^{1/2}$, therefore

$$|A_l^*(n)| \le n^{1/2} + \sum_{0 \le r \le \frac{1}{6}\log_2 n} |E_r| \le$$

$$\le n^{1/2} + 2\left(\frac{1}{6}\log_2 n + 1\right) n^{2/3} + \frac{n^{3/4}}{(\log n)^{l/2}} \sum_{r=0}^{\infty} \frac{\sqrt{8}}{2^{r/2}} \le 10 \frac{n^{3/4}}{(\log n)^{l/2}},$$

if n is large enough, which proves (4).

Proof of (5)**.** As a next step, we are going to prove (5). For every $a \in A^{**}(n)$ let us choose the representation $a = uv$, where

- $v \le u$,
- $v \le n^{1/3}$,
- $u \le n^{2/3}$ or u is a prime number
- and v is minimal.

The previous Lemma 4 is applied again. Define the graph $G = (V, E)$ where the vertices are

- the integers up to $n^{2/3}$,
- those primes p from the interval $]n^{2/3}, n]$ for which there exists an $a \in A(n)$ such that $p|a$
- and an extra vertex.

There is an edge between u and v, if the following conditions hold:

- $1 \le v < n^{1/3}$,
- $u \le n^{2/3}$ or u is a prime number,
- $v < u$
- and $a = uv$ is a chosen representation for some $a \in A^{**}(n)$.

The graph G is C_4-free, otherwise for some $u_1, u_2, v_1, v_2 \in V$,

$$(u_1, v_1), (u_2, v_1), (u_1, v_2), (u_2, v_2) \in E$$

we have

$$u_1v_1, u_2v_1, u_1v_2, u_2v_2 \in A(n),$$

but

$$(u_1v_1)(u_2v_2) = (u_1v_2)(u_2v_1)$$

contradicts the multiplicative Sidon property. Thus Lemma 4 can be applied for G with

$$t_1 = \lfloor n^{2/3} \rfloor + \left(\sum_{n^{2/3} < p \le n} \chi_{A,n}(p) \right) + 1, \quad t_2 = \lfloor n^{1/3} \rfloor.$$

In this case we have $\left\lfloor \frac{t_2}{t_1^{1/2}} \right\rfloor = 0$. (Note that the extra vertex was added in order to guarantee this.) The number of those $a \in A(n)$ for which $a = uv$ with $v = u$ is at most $n^{1/2}$, therefore

$$|A^{**}(n)| \leq \sqrt{n} + |E| \leq \sqrt{n} + \left(\sum_{n^{2/3} < p \leq n} \chi_{A,n}(p) \right) + \lfloor n^{2/3} \rfloor + 1 + \lfloor n^{1/3} \rfloor^2 \leq$$

$$\leq \left(\sum_{n^{2/3} < p \leq n} \chi_{A,n}(p) \right) + 4n^{2/3},$$

which proves (5).

Proof of (6)**.** To prove statement (6), first let us note that if $a \leq \frac{n}{(\log n)^{12}}$, then $a \in A_6^*(n) \cup A^{**}(n)$. To see this, by the Erdős' argument, let us take the decomposition $a = uv$, where $v \leq u$ and $u \leq n^{2/3}$ or u is a prime number. The condition $v \leq u$ implies $v \leq \frac{n^{1/2}}{(\log n)^6}$. Hence,

- for $n^{1/3} \leq v \leq \frac{n^{1/2}}{(\log n)^6}$ we have $a \in A_6^*(n)$
- for $v < n^{1/3}$ we have either $u \leq n^{2/3}$ or u is a prime number, therefore $a \in A^{**}(n)$.

From now on, we are going to assume that $a > \frac{n}{(\log n)^{12}}$. Five cases are going to be distinguished depending on the size of p_{s-1} and p_s.

Case 1 $p_s \geq n^{1/2}(\log n)^6$.
The choice $v = \frac{a}{p_s}$, $u = p_s$ shows that $a \in A_6^*(n) \cup A^{**}(n)$.

Case 2 There exists a p_i such that $n^{1/3} \leq p_i \leq \frac{n^{1/2}}{(\log n)^6}$.
The choice $v = p_i$ and $u = \frac{a}{p_i}$ shows that $a \in A_6^*(n)$.

Case 3 $\frac{n^{1/2}}{(\log n)^6} < p_{s-1} \leq p_s < n^{1/2}(\log n)^6$.
In this case $a = dp_{s-1}p_s$, where $d < (\log n)^{12}$. Hence we have $a \in A_1(n)$.

Case 4 $\frac{n^{1/2}}{(\log n)^6} < p_s < n^{1/2}(\log n)^6$ and $p_{s-1} < n^{1/3}$.

- If $\prod_{p_l < \frac{n^{1/6}}{(\log n)^6}} p_l > (\log n)^{12}$, then for some j we have $p_1 p_2 \dots p_{j-1} p_s < n^{1/2}(\log n)^6$ and $p_1 p_2 \dots p_j p_s \geq n^{1/2}(\log n)^6$, but in this case $p_1 p_2 \dots p_j p_s \leq n^{2/3}$, which implies that for $u = p_1 p_2 \dots p_j p_s$ and $v = \frac{a}{u}$ we have $a \in A_6^*(n) \cup A^{**}(n)$.
- Otherwise $a = dp_i \dots p_s$, where $d = \prod_{p_l < \frac{n^{1/6}}{(\log n)^6}} p_l \leq (\log n)^{12}$ and $\frac{n^{1/6}}{(\log n)^6} \leq p_i \leq \dots \leq p_s$. Hence we have $a \in A_1(n)$.

Case 5 $p_s < n^{1/3}$.
There exists a k such that $p_{k+1}p_{k+2}\dots p_s < n^{1/3}$ but $p_kp_{k+1}\dots p_s \ge n^{1/3}$. Note that $p_kp_{k+1}\dots p_s \le n^{2/3}$, since $p_k \le p_s < n^{1/3}$.

- If $n^{1/3} \le p_kp_{k+1}\dots p_s \le \frac{n^{1/2}}{(\log n)^6}$, then $v = p_kp_{k+1}\dots p_s$ and $u = \frac{a}{v}$ shows that $a \in A_6^*(n)$.
- If $n^{1/2}(\log n)^6 \le p_kp_{k+1}\dots p_s \le n^{2/3}$, then $u = p_kp_{k+1}\dots p_s$ and $v = \frac{a}{p}$ shows that $a \in A_6^*(n) \cup A^{**}(n)$.
- Finally, let us assume that $\frac{n^{1/2}}{(\log n)^6} < p_kp_{k+1}\dots p_s < n^{1/2}(\log n)^6$. If $\prod_{p_l < \frac{n^{1/6}}{(\log n)^6}} p_l > (\log n)^{12}$, then for some j we have $p_1p_2\dots p_{j-1}p_k\dots p_s < n^{1/2}(\log n)^6$ and $p_1p_2\dots p_jp_k\dots p_s \ge n^{1/2}(\log n)^6$, but in this case $p_1p_2\dots p_jp_k\dots p_s \le n^{2/3}$, thus $u = p_1p_2\dots p_jp_k\dots p_s$ and $v = \frac{a}{u}$ shows that $a \in A_6^*(n) \cup A^{**}(n)$. Therefore, it suffices to prove the statement in the case when $a = dp_i\dots p_s$, where $d = \prod_{p_l < \frac{n^{1/6}}{(\log n)^6}} p_l = p_1p_2\dots p_{i-1} \le (\log n)^{12}$ and $\frac{n^{1/6}}{(\log n)^6} \le p_i \le \dots \le p_s < n^{1/3}$. In this case the value of $s-i+1$, that is, the number of the "large" prime factors of a can be $3,4,5$ or 6, so $a = dp_{s-2}p_{s-1}p_s$ or $a = dp_{s-3}p_{s-2}p_{s-1}p_s$ or $a = dp_{s-4}p_{s-3}p_{s-2}p_{s-1}p_s$ or $a = dp_{s-5}p_{s-4}p_{s-3}p_{s-2}p_{s-1}p_s$. Now, we are going to check these subcases separately.

Subcase 1. $a = dp_{s-2}p_{s-1}p_s$.
Let $u = p_{s-2}p_{s-1}$ and $v = dp_s$. As

$$v = dp_s < n^{1/3}(\log n)^{12} < \frac{n^{1/2}}{(\log n)^6}$$

and

$$n^{2/3} > p_{s-2}p_{s-1} = \frac{a}{dp_s} > \frac{n/(\log n)^{12}}{n^{1/3}(\log n)^{12}} = \frac{n^{2/3}}{(\log n)^{24}} > n^{1/2}(\log n)^6,$$

the decomposition $a = uv$ shows that $a \in A_6^*(n) \cup A^{**}(n)$.

Subcase 2. $a = dp_{s-3}p_{s-2}p_{s-1}p_s$.

- If $p_{s-1}p_s \ge n^{1/2}(\log n)^6$, then for $u = p_{s-1}p_s$ and $v = \frac{a}{u}$ we have

  $$p_{s-1}p_s < n^{2/3}$$

  and

  $$v = \frac{a}{u} \le \frac{n}{n^{1/2}(\log n)^6} = \frac{n^{1/2}}{(\log n)^6},$$

  so $a \in A_6^*(n) \cup A^{**}(n)$.

- If $n^{1/4}(\log n)^9 < p_s < n^{1/3}$ and $p_{s-1}p_s < n^{1/2}(\log n)^6$, then $p_{s-1} < \frac{n^{1/4}}{(\log n)^3}$, thus $v = p_{s-3}p_{s-2} < \frac{n^{1/2}}{(\log n)^6}$ and $u = dp_{s-1}p_s < (\log n)^{12}n^{1/2}(\log n)^6 \leq n^{2/3}$ shows that $a \in A_6^*(n) \cup A^{**}(n)$.

- We may assume that $p_s \leq n^{1/4}(\log n)^9$.

  - If $p_{s-3}p_{s-2} \leq \frac{n^{1/2}}{(\log n)^6}$, then $u = dp_{s-1}p_s \leq (\log n)^{12}(n^{1/4}(\log n)^9)^2 \leq n^{2/3}$ and $v = p_{s-3}p_{s-2}$ shows that $a \in A_6^*(n) \cup A^{**}(n)$.
  - If $p_{s-3} < \frac{n^{1/4}}{(\log n)^9}$ and $p_{s-3}p_{s-2} > \frac{n^{1/2}}{(\log n)^6}$, then $p_{s-2} \geq n^{1/4}(\log n)^3$, therefore $n^{1/2}(\log n)^6 \leq p_{s-1}p_s \leq n^{2/3}$, thus $u = p_{s-1}p_s$ and $v = dp_{s-3}p_{s-2}$ shows that $a \in A_6^*(n) \cup A^{**}(n)$.
  - Therefore, we may assume that $a = dp_{s-3}p_{s-2}p_{s-1}p_s$ where

$$d = \prod_{p_l < \frac{n^{1/6}}{(\log n)^6}} p_l = p_1p_2\dots p_{s-4} \leq (\log n)^{12}$$

    and

$$\frac{n^{1/4}}{(\log n)^9} \leq p_{s-3} \leq p_{s-2} \leq p_{s-1} \leq p_s \leq n^{1/4}(\log n)^9,$$

    that is, $a \in A_2(n)$.

Subcase 3. $a = dp_{s-4}p_{s-3}p_{s-2}p_{s-1}p_s$.
The inequality

$$n \geq p_{s-4}p_{s-3}p_{s-2}p_{s-1}p_s = \frac{(p_{s-4}p_{s-3}p_s)(p_{s-2}p_{s-1}p_s)}{p_s} > \frac{(p_{s-4}p_{s-3}p_s)^2}{n^{1/3}},$$

yields $p_{s-4}p_{s-3}p_s \leq n^{2/3}$. We claim that $dp_{s-2}p_{s-1} \leq \frac{n^{1/2}}{(\log n)^6}$.
For the sake of contradiction, let us assume that $dp_{s-2}p_{s-1} > \frac{n^{1/2}}{(\log n)^6}$. This would imply

$$\frac{n^{1/2}}{(\log n)^6} < dp_{s-2}p_{s-1} \leq (\log n)^{12}p_{s-1}^2,$$

whence $\frac{n^{1/4}}{(\log n)^9} \leq p_{s-1} \leq p_s$. Now,

$$\frac{n^{1/2}}{(\log n)^6} < dp_{s-2}p_{s-1} = \frac{a}{p_{s-4}p_{s-3}p_s} \leq \frac{n}{\left(\frac{n^{1/6}}{(\log n)^6}\right)^2 \frac{n^{1/4}}{(\log n)^9}} = n^{5/12}(\log n)^{21}$$

is a contradiction. Hence, $dp_{s-2}p_{s-1} \leq \frac{n^{1/2}}{(\log n)^6}$.
The choice $u = p_{s-4}p_{s-3}p_s$ and $v = dp_{s-2}p_{s-1}$ shows that $a \in A_6^*(n) \cup A^{**}(n)$.

Subcase 4. $a = dp_{s-5}p_{s-4}p_{s-3}p_{s-2}p_{s-1}p_s$.
First of all,

$$n^{1/2}(\log n)^6 \le \frac{n^{2/3}}{(\log n)^{24}} \le p_{s-5}p_{s-4}p_{s-3}p_{s-2},$$

thus

$$dp_{s-1}p_s \le \frac{n}{p_{s-5}p_{s-4}p_{s-3}p_{s-2}} \le \frac{n}{n^{1/2}(\log n)^6} = \frac{n^{1/2}}{(\log n)^6}.$$

Also,

$$n \ge p_{s-5}p_{s-4}p_{s-3}p_{s-2}p_{s-1}p_s \ge (p_{s-5}p_{s-4}p_{s-3}p_{s-2})^{3/2},$$

which yields the bound $p_{s-5}p_{s-4}p_{s-3}p_{s-2} \le n^{2/3}$.
Hence $u = p_{s-5}p_{s-4}p_{s-3}p_{s-2}$ and $v = dp_{s-1}p_s$ shows that $a \in A_6^*(n) \cup A^{**}(n)$. This completes the proof of statement (6).

Proof of (7)**.** Now, we continue with proving statement (7). We claim that it is enough to prove that for every $c > 0$ there exists an $N_0 = N_0(c)$ such that for every $n \ge N_0$ and

$$|A_1(n)| = |\{a : a \in A(n), a = dp_i \dots p_s, d \le (\log n)^{12}, \\ \frac{n^{1/6}}{(\log n)^6} \le p_i \le \dots \le p_s < n^{1/2}(\log n)^6\}| > c \cdot \frac{n^{3/4}}{(\log n)^3} \tag{10}$$

there exists an $m \in \left[\frac{n^{1/2}}{(\log n)^6}, n^{1/2}(\log n)^6\right]$ such that

$$\frac{|A(m)| - \pi(m)}{\frac{m}{(\log m)^{48}}} \le -\frac{c^2}{10 \cdot 2^{51} + 1}. \tag{11}$$

First we are going to check that this statement implies statement (7), then we are going to prove it.

If the condition of (7) holds, then there is a $c > 0$ and infinite sequence $n_1 < n_2 < \dots$ such that

$$|A_1(n_j)| > c\frac{n_j^{3/4}}{(\log n_j)^3}.$$

According to our claim for every large enough j there is an $m_j \in \left[\frac{n_j^{1/2}}{(\log n_j)^8}, n_j^{1/2}(\log n_j)^8\right]$ such that $\frac{|A(m_j)| - \pi(m_j)}{\frac{m_j}{(\log m_j)^{48}}} \le -\frac{c^2}{10 \cdot 2^{51}+1}$. Therefore, $\liminf_{n\to\infty} \frac{|A(n)| - \pi(n)}{\frac{n^{3/4}}{(\log n)^{48}}} \le -\frac{c^2}{10 \cdot 2^{51} + 1}$.
Hence, it suffices to prove our claim.

If (10) holds, then there exists an integer $d \in \left[1, (\log n)^{12}\right]$ such that

$$|\{a : a \in A(n), a = dp_i \dots p_s, \frac{n^{1/6}}{(\log n)^6} \le p_i \le \dots \le p_s < n^{1/2}(\log n)^6\}| > \\ > c\frac{n^{3/4}}{(\log n)^{15}} \quad (12)$$

Let us fix such an integer d. Let us define a bipartite graph $G = (V, E)$ as follows. Let $V = V_1 \cup V_2$, where V_1 contains the prime number p if there exists an $a \in A(n)$ such that $a = dp_i \dots p_s$ and $\frac{n^{1/6}}{(\log n)^6} \le p_i \le \dots \le p_s = p < n^{1/2}(\log n)^6$ and V_2 contains the integers $p_i \dots p_{s-1}$. There is an edge between $v_1 \in V_1$ and $v_2 \in V_2$ if and only if $dv_1v_2 \in A(n)$.

Let $V_2 = \{v_1^{(2)}, v_2^{(2)}, \dots\}$. Let us denote the degree of $v_j^{(2)}$ by $\deg(v_j^{(2)})$. We may assume that $\deg(v_1^{(2)}) \ge \deg(v_2^{(2)}) \ge \dots$. Let P be the set of prime numbers. Let $P_j \subset P$ such that $p \in P_j$ if and only if the vertex $v_j^{(2)}$ is connected to p in the graph G. Clearly, we have $|P_j| = \deg(v_j^{(2)})$.

We claim that G is C_4-free. If there is a C_4, then there are $p_s, p'_{s'} \in V_1$ and $p_i \dots p_{s-1}, p'_i \dots p'_{s'-1} \in V_2$ such that

$$dp_i \dots p_{s-1}p_s, dp_i \dots p_{s-1}p'_{s'}, dp'_i \dots p'_{s'-1}p_s, dp'_i \dots p'_{s'-1}p'_{s'} \in A,$$

but

$$((dp_i \dots p_{s-1})p_s)((dp'_i \dots p'_{s'-1})p'_{s'}) = ((dp_i \dots p_{s-1})p'_{s'})((dp'_i \dots p'_{s'-1})p_s)$$

would contradict the multiplicative Sidon property. Therefore, G is C_4-free, so

$$|P_j \cap P_k| \le 1, \quad \text{for } j \ne k. \quad (13)$$

If $p_s, p'_s \in P_j$, then $p_s \notin A(n)$ or $p'_s \notin A(n)$ because otherwise

$$(d(p_i \dots p_{s-1})p_s)p'_s = (d(p_i \dots p_{s-1})p'_s)p_s$$

would contradict the multiplicative Sidon property, because $dp_i \dots p_{s-1} = \frac{a}{p_s} \ge \frac{\frac{n}{(\log n)^{12}}}{n^{1/2}(\log n)^6} = \frac{n^{1/2}}{(\log n)^{18}} > 1$ if n is large enough. Hence,

$$|P_j \setminus A(n^{1/2}(\log n)^6)| \ge |P_j| - 1. \quad (14)$$

Using inequalities (13) and (14) we get that

$$\begin{aligned}
&|(P_1 \cup P_2 \cup \cdots \cup P_t) \setminus A(n^{1/2}(\log n)^8)| = \\
&= |(P_1 \cup (P_2 \setminus P_1) \cup (P_3 \setminus (P_1 \cup P_2)) \cup \cdots \cup (P_k \setminus (\cup_{j=1}^{k-1} P_j)) \cup \cdots \cup (P_t \setminus (\cup_{j=1}^{t-1} P_j))) \setminus A| = \\
&= \sum_{k=1}^{t} |(P_k \setminus (\cup_{j=1}^{k-1} P_j)) \setminus A| \geq \\
&\geq \sum_{k=1}^{t} (|(P_k \setminus (\cup_{j=1}^{k-1} P_j))| - 1) \geq \sum_{k=1}^{t} (|P_k| - (k-1) - 1) = \sum_{k=1}^{t} (|P_k| - k) \quad (15)
\end{aligned}$$

According to (12) and the definition of the graph G we

$$c \cdot \frac{n^{3/4}}{(\log n)^{15}} \leq |E| = \sum_j |\deg(v_j^{(2)})| = \sum_j |P_j|. \quad (16)$$

We are going to prove that

$$\deg\left(v^{(2)}_{\left[\frac{c}{2} \cdot \frac{n^{1/4}}{(\log n)^{21}}\right]}\right) \geq \left[\frac{c}{2} \cdot \frac{n^{1/4}}{(\log n)^{21}}\right].$$

For the sake of contradiction let us suppose the opposite. Let us split the sum on the right-hand side of (16) into two parts:

$$c \cdot \frac{n^{3/4}}{(\log n)^{15}} \leq \sum_j \deg(v_j^{(2)}) = \sum_{j \leq \left[\frac{c}{2} \cdot \frac{n^{1/4}}{(\log n)^{21}}\right]} \deg(v_j^{(2)}) + \sum_{j > \left[\frac{c}{2} \cdot \frac{n^{1/4}}{(\log n)^{21}}\right]} \deg(v_j^{(2)}). \quad (17)$$

It is well known that $\pi(n^{1/2}(\log n)^6) < \frac{n^{1/2}(\log n)^6}{2}$, if n is large enough, therefore $\deg(v_j^{(2)}) \leq |V_1| < \frac{n^{1/2}(\log n)^6}{2}$. Hence

$$\sum_{j \leq \left[\frac{c}{2} \cdot \frac{n^{1/4}}{(\log n)^{21}}\right]} \deg(v_j^{(2)}) \leq \frac{c}{2} \cdot \frac{n^{1/4}}{(\log n)^{21}} \cdot \frac{n^{1/2}(\log n)^6}{2} = \frac{c}{4} \cdot \frac{n^{3/4}}{(\log n)^{15}}. \quad (18)$$

Also, $|V_2| \leq n^{1/2}(\log n)^6$, since $p_s \geq \frac{n^{1/2}}{(\log n)^6}$ implies that $p_i \ldots p_{s-1} \leq n^{1/2}(\log n)^6$. Therefore,

$$\sum_{j > \left[\frac{c}{2} \cdot \frac{n^{1/4}}{(\log n)^{21}}\right]} \deg(v_j^{(2)}) \leq \frac{c}{2} \cdot \frac{n^{1/4}}{(\log n)^{21}} \cdot n^{1/2}(\log n)^6 = \frac{c}{2} \cdot \frac{n^{3/4}}{(\log n)^{15}}. \quad (19)$$

Hence, (17), (18) and (19) would imply

$$c \cdot \frac{n^{3/4}}{(\log n)^{15}} < \frac{3c}{4} \cdot \frac{n^{3/4}}{(\log n)^{15}},$$

which is a contradiction.

Thus,

$$\left|(P_1 \cup \dots \cup P_{\left[\frac{c}{2}\cdot\frac{n^{1/4}}{(\log n)^{21}}\right]} \setminus A(n^{1/2}(\log n)^8))\right| \geq \sum_{i \leq \left[\frac{c}{2}\cdot\frac{n^{1/4}}{(\log n)^{21}}\right]} (\deg(v_i^{(2)}) - i) \geq$$

$$\left[\frac{c}{2}\cdot\frac{n^{1/4}}{(\log n)^{21}}\right]^2 - \left(\frac{\left[\frac{c}{2}\cdot\frac{n^{1/4}}{(\log n)^{21}}\right]+1}{2}\right) > \frac{c^2 n^{1/2}}{10(\log n)^{42}},$$

if n is large enough.

As

$$\sum_{-6\log_2\log n - 1 \leq k \leq 6\log_2\log n} \frac{\frac{n^{1/2}}{2^k}}{(\log\frac{n^{1/2}}{2^k})^{48}} < 2^{51}\cdot\frac{n^{1/2}}{(\log n)^{42}},$$

there exists an integer $k \in [-6\log_2\log n - 1, 6\log_2\log n]$ such that

$$\left|\left(P\left(\frac{n^{1/2}}{2^k}\right) \setminus P\left(\frac{n^{1/2}}{2^{k+1}}\right)\right) \setminus A\left(\frac{n^{1/2}}{2^k}\right)\right| \geq \frac{c^2\frac{n^{1/2}}{2^k}}{10\cdot 2^{51}\cdot(\log\frac{n^{1/2}}{2^k})^{48}},$$

if n is large enough. Let us fix such a k. If $p \in \left(P\left(\frac{n^{1/2}}{2^k}\right) \setminus P\left(\frac{n^{1/2}}{2^{k+1}}\right)\right) \setminus A\left(\frac{n^{1/2}}{2^k}\right)$, then $\chi_{A,\frac{n^{1/2}}{2^k}}(p) = 0$, since $p \notin A$ and $2p > \frac{n^{1/2}}{2^k}$.

By Erdős' argument every $a \in A\left(\frac{n^{1/2}}{2^k}\right)$ can be written in the form $a = uv$, where $v \leq u$ and $u \leq \left(\frac{n^{1/2}}{2^k}\right)^{2/3}$ or u is a prime number, thus $A\left(\frac{n^{1/2}}{2^k}\right) \subseteq A_0^*\left(\frac{n^{1/2}}{2^k}\right) \cup A^{**}\left(\frac{n^{1/2}}{2^k}\right)$. Therefore, by using (4) and (5) we obtain that

$$|A\left(\frac{n^{1/2}}{2^k}\right)| \leq |A_0^*\left(\frac{n^{1/2}}{2^k}\right)| + |A^{**}\left(\frac{n^{1/2}}{2^k}\right)| \leq$$

$$\leq \sum_{\left(\frac{n^{1/2}}{2^k}\right)^{2/3} < p \leq \left(\frac{n^{1/2}}{2^k}\right)} \chi_{A,n}(p) + 11\left(\frac{n^{1/2}}{2^k}\right)^{3/4},$$

if n is large enough.

Using this upper bound we get

$$\left|A\left(\frac{n^{1/2}}{2^k}\right)\right| \leq \pi\left(\frac{n^{1/2}}{2^k}\right) - \frac{c^2\frac{n^{1/2}}{2^k}}{10\cdot 2^{51}\cdot(\log\frac{n^{1/2}}{2^k})^{48}} + 11\left(\frac{n^{1/2}}{2^k}\right)^{3/4} \leq$$

$$\leq \pi\left(\frac{n^{1/2}}{2^k}\right) - \frac{c^2\frac{n^{1/2}}{2^k}}{(10\cdot 2^{51}+1)\cdot(\log\frac{n^{1/2}}{2^k})^{48}},$$

if n is large enough. The choice $m = \frac{n^{1/2}}{2^k}$ satisfies (11), thus statement (7) holds.

Proof of (8)**.** Finally, we prove (8). We split into parts the set $A_2(n)$ as follows. Let $a = dp_{s-3}p_{s-2}p_{s-1}p_s \in A_2(n)$ be arbitrary. There exist uniquely determined integers r and w such that

$$\frac{n}{2^{r+1}} < dp_{s-3}p_{s-2}p_{s-1}p_s \leq \frac{n}{2^r},$$

$$2^w \leq d < 2^{w+1}.$$

Since $d \leq (\log n)^{12}$ and $a \geq n/(\log n)^{12}$ we have

$$0 \leq r \leq 12 \log_2 \log n,$$

$$0 \leq w \leq 12 \log_2 \log n.$$

Furthermore,

$$\frac{n}{2^{r+w+2}} < p_{s-3}p_{s-2}p_{s-1}p_s \leq \frac{n}{2^{r+w}}, \tag{20}$$

which implies that $p_{s-3}p_{s-2} \leq \frac{n^{1/2}}{2^{\frac{r}{2}+\frac{w}{2}}}$. There exists a uniquely determined integer q for which

$$\frac{n^{1/2}}{2^{\frac{r}{2}+\frac{w}{2}+q+1}} < p_{s-3}p_{s-2} \leq \frac{n^{1/2}}{2^{\frac{r}{2}+\frac{w}{2}+q}}. \tag{21}$$

The lower bound $\frac{n^{1/2}}{(\log n)^{18}} \leq p_{s-3}p_{s-2}$ implies

$$0 \leq q \leq 18 \log_2 \log n.$$

Let us collect those elements $a = dp_{s-3}p_{s-2}p_{s-1}p_s$ of $A_2(n)$ to $A_2^{(r,w,q)}(n)$, for which

- $\frac{n}{2^{r+1}} < dp_{s-3}p_{s-2}p_{s-1}p_s \leq \frac{n}{2^r}$,
- $2^w \leq d < 2^{w+1}$, and
- $\frac{n^{1/2}}{2^{\frac{r}{2}+\frac{w}{2}+q+1}} < p_{s-3}p_{s-2} \leq \frac{n^{1/2}}{2^{\frac{r}{2}+\frac{w}{2}+q}}$.

Now, $A_2(n)$ can be partitioned to the union of the $A_2^{(r,w,q)}(n)$ sets:

$$A_2(n) = \bigcup_{r=0}^{\lfloor 12 \log_2 \log n \rfloor} \bigcup_{w=0}^{\lfloor 12 \log_2 \log n \rfloor} \bigcup_{q=0}^{\lfloor 18 \log_2 \log n \rfloor} A_2^{(r,w,q)}(n).$$

We are going to give an upper bound for $|A_2^{(r,w,q)}(n)|$. Let us define the edge-coloured bipartite graph $G_{r,w,q} = (V_{r,w,q}, E_{r,w,q})$ as follows. Let $V_{r,w,q} = V_1 \cup V_2$, where

- V_1 contains the integers $p_{s-1}p_s$ if and only if there is an $a = dp_{s-3}p_{s-2}p_{s-1}p_s \in A_2^{(r,w,q)}$,
- V_2 contains the integers $p_{s-3}p_{s-2}$ if and only if there is an $a = dp_{s-3}p_{s-2}p_{s-1}p_s \in A_2^{(r,w,q)}$.

The vertices $p_{s-1}p_s \in V_1$ and $p_{s-3}p_{s-2} \in V_2$ are connected to each other if and only if there is a $d \in \left[2^w, 2^{w+1}\right)$ such that $dp_{s-3}p_{s-2}p_{s-1}p_s \in A_2^{(r,w,q)}(n)$. In this case let the color of this edge be d. (Note that there can be more edges between two vertices.) For $v_1 \in V_1$ and $2^w \le d < 2^{w+1}$ let us denote by $\deg_d(v_1)$ the number of edges of color d starting from v_1.

Let us suppose that $p_sp_{s-1}, p'_{s'}p'_{s'-1} \in V_1$ and $p_{s-3}p_{s-2}, p'_{s'-3}p'_{s'-2} \in V_2$. Then there is no C_4 on these points such that

- edges $(p_{s-1}p_s, p_{s-3}p_{s-2})$ and $(p_{s-1}p_s, p'_{s'-3}p'_{s'-2})$ are of color d,
- edges $(p'_{s'-1}p'_{s'}, p_{s-3}p_{s-2})$ and $(p'_{s'-1}p'_{s'}, p'_{s'-3}p'_{s'-2})$ are of color d',

since otherwise

$$(dp_{s-3}p_{s-2}p_{s-1}p_s)(d'p'_{s'-3}p'_{s'-2}p'_{s'-1}p'_{s'}) = (dp'_{s'-3}p'_{s'-2}p_{s-1}p_s)(d'p_{s-3}p_{s-2}p'_{s'-1}p'_{s'})$$

would contradict the multiplicative Sidon property. Hence,

$$\sum_{v_1 \in V_1, 2^w \le d < 2^{w+1}} \binom{\deg_d(v_1)}{2} \le \binom{|V_2|}{2}. \tag{22}$$

The set of pairs (v_1, d) satisfying $v_1 \in V_1$ and $2^w \le d < 2^{w+1}$ is split into two classes:

- the first class contains pairs (v_1, d) if $\deg_d(v_1) \le \left\lfloor \frac{|V_2|}{|V_1|^{1/2}2^{w/2}} \right\rfloor + 1$,
- the second class contains pairs (v_1, d) if $\deg_d(v_1) \ge \left\lfloor \frac{|V_2|}{|V_1|^{1/2}2^{w/2}} \right\rfloor + 2$.

Clearly,

$$|A_2^{(r,w,q)}(n)| = \sum_{v_1 \in V_1} \sum_{d=2^w}^{2^{w+1}-1} \deg_d(v_1) = \sum_{(v_1,d) \in class_1} \deg_d(v_1) + \sum_{(v_1,d) \in class_2} deg_d(v_1).$$

The number of pairs (v_1, d) in $class_1$ is at most $|V_1|2^w$, therefore

$$\sum_{(v_1,d) \in class_1} \deg_d(v_1) \le \left(\left\lfloor \frac{|V_2|}{|V_1|^{1/2}2^{w/2}} \right\rfloor + 1 \right) |V_1|2^w \le 2^w|V_1| + 2^{w/2}|V_1|^{1/2}|V_2|.$$

By inequality (22) we have

$$\sum_{(v_1,d)\in class_2} \deg_d(v_1) \le \frac{2}{\left\lfloor \frac{|V_2|}{|V_1|^{1/2}2^{w/2}} \right\rfloor + 1} \sum_{(v_1,d)\in class_2} \binom{\deg_d(v_1)}{2} \le$$

$$\le \frac{2}{\frac{|V_2|}{2^{w/2}|V_1|^{1/2}}} \binom{|V_2|}{2} < 2^{w/2}|V_1|^{1/2}|V_2|. \quad (23)$$

Hence we obtain that

$$|A_2^{(r,w,q)}(n)| < 2^w|V_1| + 2\cdot 2^{w/2}|V_1|^{1/2}|V_2|. \quad (24)$$

Our aim is to give upper bounds for $|V_1|$ and $|V_2|$, respectively.

Let us start with the upper bound for $|V_2|$: If $p_{s-3}p_{s-2} \in V_2$, then there is a uniquely determined nonnegative integer t such that

$$\frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}+\frac{q}{2}+t+1}} < p_{s-3} \le \frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}+\frac{q}{2}+t}}. \quad (25)$$

According to the definition of V_2 we have

$$\frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}+\frac{q}{2}-t+1}} < p_{s-2} \le \frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}+\frac{q}{2}-t-1}}.$$

We are going to give an upper bound for t. As

$$p_{s-1}p_s = \frac{p_{s-3}p_{s-2}p_{s-1}p_s}{p_{s-3}p_{s-2}} \le \frac{\frac{n}{2^{r+w}}}{\frac{n^{1/2}}{2^{\frac{r}{2}+\frac{w}{2}+q+1}}} = \frac{n^{1/2}}{2^{\frac{r}{2}+\frac{w}{2}-q-1}},$$

we get that

$$\frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}+\frac{q}{2}-t+1}} < p_{s-2} \le p_{s-1} \le \frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}-\frac{q}{2}-\frac{1}{2}}},$$

thus $t \le q + 1.5$, that is,

$$t \le q + 1 \le 18 \log_2 \log n + 1. \quad (26)$$

Now, with the help of the prime number theorem with error term $\pi(x) = (1 + O(\frac{1}{\log x}))\frac{x}{\log x}$ we obtain the following upper bound for those $p_{s-3}p_{s-2} \in V_2$ that satisfy (25):

$$\left(\frac{1}{2} + O\left(\frac{1}{\log n}\right)\right) \frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}+\frac{q}{2}+t} \log \frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}+\frac{q}{2}+t}}} \times$$

$$\left(\frac{3}{4} + O\left(\frac{1}{\log n}\right)\right) \frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}+\frac{q}{2}-t-1} \log \frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}+\frac{q}{2}-t-1}}}. \quad (27)$$

Here
$$\log \frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}+\frac{q}{2}+t}} = \frac{1}{4}\log n + O(\log\log n)$$
and
$$\log \frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}+\frac{q}{2}-t-1}} = \frac{1}{4}\log n + O(\log\log n).$$
Therefore, the gained upper bound is
$$\left(12 + O\left(\frac{\log\log n}{\log n}\right)\right)\frac{n^{1/2}}{2^{\frac{r}{2}+\frac{w}{2}+q}(\log n)^2}.$$
All in all, by using (26) we get the upper bound
$$|V_2| \le \left(12 + O\left(\frac{\log\log n}{\log n}\right)\right)\frac{n^{1/2}(q+2)}{2^{\frac{r}{2}+\frac{w}{2}+q}(\log n)^2}.$$
As a next step, we give an upper bound for $|V_1|$. According to (20) and (21) we have
$$\frac{n^{1/2}}{2^{\frac{r}{2}+\frac{w}{2}-q+2}} \le p_{s-1}p_s \le \frac{n^{1/2}}{2^{\frac{r}{2}+\frac{w}{2}-q-1}}, \tag{28}$$
therefore $p_{s-1} \le \frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}-\frac{q}{2}-\frac{1}{2}}}$. There is a uniquely determined integer t for which
$$\frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}-\frac{q}{2}-\frac{1}{2}+t+1}} < p_{s-1} \le \frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}-\frac{q}{2}-\frac{1}{2}+t}}. \tag{29}$$
Now, (28) and (29) implies that
$$\frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}-\frac{q}{2}+\frac{5}{2}-t}} < p_s < \frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}-\frac{q}{2}-\frac{3}{2}-t}}$$
We are going to give an upper bound for t. By (21) and (29) we get that
$$\frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}+\frac{q}{2}+\frac{1}{2}}} \le p_{s-2} \le p_{s-1} \le \frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}-\frac{q}{2}-\frac{1}{2}+t}},$$
which implies
$$t \le q+1 \le 18\log_2\log n + 2. \tag{30}$$
Now, with the help of the prime number theorem with error term $\pi(x) = (1 + O(\frac{1}{\log x}))\frac{x}{\log x}$ we obtain the following upper bound for those $p_{s-1}p_s \in V_1$ that satisfy (29):
$$\left(\frac{1}{2} + O\left(\frac{1}{\log n}\right)\right)\frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}-\frac{q}{2}-\frac{1}{2}+t}\log\frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}-\frac{q}{2}-\frac{1}{2}+t}}} \times$$
$$\left(\frac{15}{16} + O\left(\frac{1}{\log n}\right)\right)\frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}-\frac{q}{2}-\frac{3}{2}-t}\log\frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}-\frac{q}{2}-\frac{3}{2}-t}}}. \tag{31}$$

Since

$$\log \frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}-\frac{q}{2}-\frac{1}{2}+t}} = \frac{1}{4}\log n + O(\log\log n)$$

and

$$\log \frac{n^{1/4}}{2^{\frac{r}{4}+\frac{w}{4}-\frac{q}{2}-\frac{3}{2}-t}} = \frac{1}{4}\log n + O(\log\log n),$$

we obtain the upper bound

$$\left(30 + O\left(\frac{\log\log n}{\log n}\right)\right)\frac{n^{1/2}}{2^{\frac{r}{2}+\frac{w}{2}-q}(\log n)^2}.$$

By (30) we get

$$|V_1| \le \left(30 + O\left(\frac{\log\log n}{\log n}\right)\right)\frac{n^{1/2}(q+2)}{2^{\frac{r}{2}+\frac{w}{2}-q}(\log n)^2}.$$

Plugging in these bounds for $|V_1|$ and $|V_2|$ in (24) yields the following upper bound for $|A_2^{(r,w,q)}(n)|$:

$$|A_2^{(r,w,q)}(n)| \le 2^w|V_1| + 2\cdot 2^{w/2}|V_1|^{1/2}|V_2| \le \left(30 + O\left(\frac{\log\log n}{\log n}\right)\right)\frac{n^{1/2}(q+2)}{2^{\frac{r}{2}+\frac{w}{2}-q}(\log n)^2} +$$

$$2\left(30^{1/2} + O\left(\frac{\log\log n}{\log n}\right)\right)\frac{n^{1/4}(q+2)^{1/2}}{2^{\frac{r}{4}+\frac{w}{4}-\frac{q}{2}}\log n}\left(12 + O\left(\frac{\log\log n}{\log n}\right)\right)\frac{n^{1/2}(q+2)}{2^{\frac{r}{2}+\frac{w}{2}+q}(\log n)^2}2^{w/2} =$$

$$= \left(24\cdot 30^{1/2} + O\left(\frac{\log\log n}{\log n}\right)\right)\frac{n^{3/4}(q+2)^{3/2}}{2^{\frac{3r}{4}}2^{\frac{w}{4}}2^{\frac{q}{2}}(\log n)^3}.$$

Therefore,

$$|A_2(n)| \le \sum_{r=0}^{12\log_2\log n+1}\ \sum_{w=0}^{12\log_2\log n+1}\ \sum_{q=0}^{18\log_2\log n+1} |A_2^{(r,w,q)}(n)| \le$$

$$\le \sum_{r=0}^{\infty}\sum_{w=0}^{\infty}\sum_{q=0}^{\infty}\left(24\cdot 30^{1/2} + O\left(\frac{\log\log n}{\log n}\right)\right)\frac{n^{3/4}(q+2)^{3/2}}{2^{\frac{3r}{4}}2^{\frac{w}{4}}2^{\frac{q}{2}}2(\log n)^3} =$$

$$= \left(24\cdot 30^{1/2} + O\left(\frac{\log\log n}{\log n}\right)\right)\left(\sum_{r=0}^{\infty}\frac{1}{2^{\frac{3r}{4}}}\right)\left(\sum_{w=0}^{\infty}\frac{1}{2^{\frac{w}{4}}}\right)\left(\sum_{q=0}^{\infty}\frac{(q+2)^{3/2}}{2^{\frac{q}{2}}}\right)\frac{n^{3/4}}{(\log n)^3} =$$

$$= \left(73631.3\cdots + O\left(\frac{\log\log n}{\log n}\right)\right)\frac{n^{3/4}}{(\log n)^3},$$

which completes the proof. ■

Now, we continue with the proof of Theorem 3. The following lemma will play an important role in the proof:

Lemma 5. *Let S be a set of size $s \geq 56$. Then there exists a family $\mathcal{H}$ of 4-element subsets of S satisfying the following conditions:*

(i) If $H_1, H_2 \in \mathcal{H}$ and $H_1 \neq H_2$, then $|H_1 \cap H_2| \leq 2$.

(ii) If K, L, M, N are pairwise disjoint 2-element subsets of S, then at least one of the sets $K \cup L, L \cup M, M \cup N, N \cup K$ does not lie in $\mathcal{H}$.

(iii) $|\mathcal{H}| \geq s^3/24576$.

Proof. Let p be a prime in the interval $(s/8, s/4]$. Note that $p \geq 11$, since $s \geq 56$. It can be supposed that $S \supseteq \mathbb{F}_p \times [4]$. That is, it can be assumed that S contains 4 disjoint copies of $\mathbb{F}_p$, namely, A, B, C, D. We are going to define a family $\mathcal{H}$ of 4-element subsets such that each element of $\mathcal{H}$ consists of one element from A, one from B, one from C and one from D. For $a, b, c, d \in \mathbb{F}_p$ let (a, b, c, d) denote the 4-element set $\{(a, 1), (b, 2), (c, 3), (d, 4)\} \in S$. We claim that for some $\alpha \in \mathbb{F}_p$, the size of the set

$$\mathcal{H}_\alpha = \{(a, b, c, d) \in \mathbb{F}_p^4 :\ a + b + c \neq 0, a + b + d \neq 0, a + c + d \neq 0, b + c + d \neq 0, \\ ab + ac + ad + bc + bd + cd = \alpha\}$$

is at least $p^3 - 4p^2 \geq p^3/2$. The size of the set $\{(a, b, c, d) \in \mathbb{F}_p^4 :\ a + b + c = 0\}$ is p^3, and the same holds when another triple from $\{a, b, c, d\}$ adds up to 0, therefore,

$$|\{(a, b, c, d) \in \mathbb{F}_p^4 :\ a+b+c \neq 0, a+b+d \neq 0, a+c+d \neq 0, b+c+d \neq 0\}| \geq p^4 - 4p^3.$$

There are p possibilities for $\alpha = ab + ac + ad + bc + bd + cd$, which proves that for a well-chosen α we have $|\mathcal{H}_\alpha| \geq p^3/2$. Let us fix such an α and delete some elements of $\mathcal{H}_\alpha$, obtaining $\mathcal{H}$, in such a way that the multiset $\{a, b, c, d\}$ is different for each element (a, b, c, d) of $\mathcal{H}$. It can be done in such a way that $|\mathcal{H}| \geq |\mathcal{H}_\alpha|/4!$ holds. We claim that $\mathcal{H}$ satisfies the required properties.

Firstly, for checking (i) it is enough to show that the intersection of two elements of $\mathcal{H}$ can not contain exactly 3 elements. Let us assume that $(a, b, c, d_1), (a, b, c, d_2) \in \mathcal{H}$. Then $d_1 = \frac{\alpha-(ab+bc+ca)}{a+b+c} = d_2$, so two elements of $\mathcal{H}$ can't differ just in the fourth "coordinate". By symmetry, this holds for the first three "coordinates", too.

Secondly, for checking (ii) let us assume that

$$(k_1, k_2, l_1, l_2), (m_1, m_2, l_1, l_2), (m_1, m_2, n_1, n_2), (k_1, k_2, n_1, n_2) \in \mathcal{H}.$$

According to the definition of $\mathcal{H}$ the following equations hold:

$$k_1k_2 + l_1l_2 + (k_1 + k_2)(l_1 + l_2) = \alpha \tag{32}$$

$$l_1l_2 + m_1m_2 + (l_1 + l_2)(m_1 + m_2) = \alpha \tag{33}$$

$$m_1m_2 + n_1n_2 + (m_1+m_2)(n_1+n_2) = \alpha \tag{34}$$

$$n_1n_2 + k_1k_2 + (n_1+n_2)(k_1+k_2) = \alpha \tag{35}$$

Now (32) − (33) + (34) − (35) gives $(k_1+k_2-m_1-m_2)(l_1+l_2-n_1-n_2) = 0$. Without the loss of generality it can be assumed that $k_1+k_2 = m_1+m_2$. Then (32) − (33) implies that $k_1k_2 = m_1m_2$. Thus $\{k_1,k_2\} = \{m_1,m_2\}$. Therefore, $\{k_1,k_2,l_1,l_2\} = \{m_1,m_2,l_1,l_2\}$, so $(k_1,k_2,l_1,l_2) = (m_1,m_2,l_1,l_2)$, hence $K = M$.

Finally, $|\mathcal{H}| \geq |\mathcal{H}_\alpha|/24 \geq p^3/48 \geq s^3/24576$.

□

The following well-known estimations of [9] are going to be used in the proof of Theorem 3 to estimate the number of primes up to x:

Lemma 6. *For every $x \geq 17$ we have $\frac{x}{\log x} < \pi(x)$. For every $x > 1$ we have $\pi(x) \leq 1.26\frac{x}{\log x}$.*

Proof of Theorem 3.

Let P_k consist of the primes from the interval $(2^{k-1}, 2^k)$. If $k \geq 11$, then by Lemma 6

$$|P_k| = \pi(2^k) - \pi(2^{k-1}) \geq \frac{2^k}{\log 2^k} - \frac{1.26 \cdot 2^{k-1}}{\log 2^{k-1}} \geq \frac{2^k}{4\log 2^k}.$$

Let us apply Lemma 5 for $S = P_k$ and let $\mathcal{H}_k$ be the obtained collection of 4-subsets of P_k. Let $A_k = \{\prod_{p\in H} p : H \in \mathcal{H}_k\}$. Finally, let $A = \{primes\} \cup \bigcup_{k\geq 11} A_k$.

Now we show that A is a multiplicative Sidon set. Assume that $ab = cd$ for $a,b,c,d \in A$. As each element of A is either a prime or the product of 4 primes, the number of prime factors of ab (counted by multiplicity) is $\Omega(ab) = \Omega(cd) \in \{2,5,8\}$. If $\Omega(ab) = \Omega(cd) = 2$, then $\{a,b\} = \{c,d\}$, and we are done. Now let us assume that $\Omega(ab) = \Omega(cd) = 5$. Without the loss of generality it can be assumed that $\Omega(a) = \Omega(c) = 4$. Then $\Omega(\gcd(a,c)) \geq 3$, therefore $a, c \in A_k$ for some k, moreover according to property (i) (of Lemma 5) we get $a = c$. Then $b = d$ also holds, and we are done. Finally, let us assume that $\Omega(ab) = \Omega(cd) = 8$. If ab is not squarefree, that is, divisible by p^2 for some prime p, then p has to divide a,b,c,d, since all elements of A are squarefree. However, it would imply that $\frac{a}{p}|\frac{c}{p}\cdot\frac{d}{p}$, therefore $\Omega(\gcd(a,c))$ or $\Omega(\gcd(a,d))$ would be at least 3. Then, again by property (i) we obtain that $a = c$ (or $a = d$), thus $\{a,b\} = \{c,d\}$. So we can suppose that $ab = cd$ is squarefree. Property (i) and $a|cd$ imply that $\Omega(\gcd(a,c)) = \Omega(\gcd(a,d)) = 2$, so for some primes

$$a = p_1p_2p_3p_4, b = p_5p_6p_7p_8, c = p_1p_2p_5p_6, d = p_3p_4p_7p_8,$$

however this contradicts property (ii) of Lemma 5. Hence, A is a multiplicative Sidon set.

Now we show that for $n \geq 2^{44}$, we have $|A(n)| \geq \pi(n) + \frac{n^{3/4}}{196608(\log n)^3}$.

If $n \geq 2^{44}$, then $k = \left\lfloor \frac{\log_2 n}{4} \right\rfloor \geq 11$. Therefore, $|P_k| \geq \frac{2^k}{4 \log 2^k} > 56$, so Lemma 5 can be applied for the set P_k. Moreover,

$$|P_k| \geq \frac{2^k}{4 \log 2^k} \geq \frac{2^{\frac{\log_2 n}{4} - 1}}{4 \log 2^{\frac{\log_2 n}{4} - 1}} \geq \frac{n^{1/4}}{2 \log n}.$$

Therefore, $|A(n)| \geq \pi(n) + |A_k| \geq |A_k| + \frac{|P_k|^3}{24576} \geq \pi(n) + \frac{n^{3/4}}{196608(\log n)^3}$. ■

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