# ON INVARIANT SETS IN LAGRANGIAN GRAPHS

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ABSTRACT. In this exposition, we show that a Hamiltonian is always constant on a compact invariant connected subset which lies in a Lagrangian graph provided that the Hamiltonian and the graph are smooth enough. We also provide some counterexamples for the case that the Hamiltonians are not smooth enough.

## 1. INTRODUCTION

Let $M$ be a closed, connected $C^\infty$ manifold of dimension $d$, and $T^*M$ be the cotangent bundle of $M$. We always assume that Hamiltonian $T^*M \to \mathbb{R}$ is $C^r$ smooth ($r \geq 1$). We denote the associated Hamiltonian vector field and Hamiltonian flow by $X_H$ and $\phi_H^t$ respectively.

In Hamiltonian dynamics, the following result is well known:

**Let $\Gamma$ be an invariant (under the Hamiltonian flow $\phi_H^t$) $C^1$ Lagrangian graph, then H is constant on $\Gamma$.**

In fact, if $\Gamma$ is only Lipschitz, the result still holds [7], i.e.,

**Proposition 1.** *Let $\Gamma$ be an invariant (under the Hamiltonian flow $\phi_H^t$) Lipschitz Lagrangian graph, then $H$ is constant on $\Gamma$.*

We always assume the Lagrangian graphs we consider are at least $C^1$, unless other stated. After this proposition, it is naturally then to pose the following problem:

**Problem 1.1.** If $\Lambda$ is a compact, connected, invariant (under $\phi_H^t$) set, and $\Lambda \subseteq \Gamma$, then is $H$ constant on $\Lambda$?

In the case $\Lambda \neq \Gamma$, the answer to this problem is not obvious, since the structure of $\Lambda$ could be very complicated. We will study this problem concretely in this short exposition.

We denote the projection of $\Lambda$ into $M$ by $\Lambda_0$.

More precisely, we have:

**Theorem 1.** *If $h(q) := H(q, \Gamma(q)) \in C^{d',s}(M, \mathbb{R})$ with $d' \geq d$, or $d' = d-1$ and $s = 1$, then $H$ is constant on $\Lambda$.*

*Remark* 1.1. Actually the conclusion of the former theorem still holds under weaker conditions, for example $h \in C^{d-1,Zygmund}$, i.e., the $d-1$ order derivatives of $h$ is smooth in the sense of Zygmund (see [6] for details).

We say $\Gamma$ is a Lipschitz Lagrangian graph, if $\Gamma$ coincides with the differential of a $C^{1,1}$ function locally. Then, we have

*Remark* 1.2. In the case of 1 degree of freedom, one can show that if $\Lambda$ is a compact, connected, invariant set under $\phi_H^t$, and $\Lambda$ lies in a Lipschitz Lagrangian graph, then $H|_\Lambda$ is constant.

*Remark* 1.3. If $\Lambda_0$ admits some special structures, e.g., Lipschitz lamination, lower Hausdorff dimension, semi-analytic or semi-algebraic, then $H$ is still constant on $\Lambda$ under some weaker (than Theorem 1) smooth hypothesis. We refer to [3],[4], for more details.

Among these cases stated in Remark 1.3, the most interesting case is

**Theorem 2.** *If for any two points in $\Lambda$, there is a rectifiable path in $\Lambda$ which connects them, then $H$ is constant on $\Lambda$.*

In the case that $H$ is not so smooth, we have the following:

**Theorem 3.** *Assume that $d \geq 2$. For $d' < d-1$ and $s \in [0,1]$ or $d' = d-1$ and $s \in [0,1)$ , there exist examples with $H \in C^{d',s}(T^*M, \mathbb{R})$ such that $H$ is non constant on $\Lambda$.*

*Remark* 1.4. These examples show that the condition in Theorem 1 is optimal in some sense.

## 2. Proof of Theorem 1

The following lemma is an easy consequence of the flow-invariance of $\Lambda$:

**Lemma 2.1.** *If $\Gamma$ is $C^1$ smooth, then $\Lambda_0$ is contained in the critical set of $h$.*

*Proof.* We will prove $dh(q_0) = 0$ for any point $q_0 \in \Lambda_0$. For this, we only need to show that $dh(q_0) \cdot v = 0$ for any $v \in T_{q_0}M$. Now we also regard $\Gamma$ as a map from $M$ to $T^*M$, then $dh(q_0) \cdot v = dH(q_0, \Gamma(q_0)) \cdot \Gamma_* v$, here $\Gamma_* v \in T_{(q_0, \Gamma(q_0))}\Gamma$. Since $\Lambda$ is invariant under the flow $\phi_H^t$, we have $X_H(q_0, \Gamma(q_0)) \in T_{(q_0, \Gamma(q_0))}\Gamma$. Note that $T_{(q_0, \Gamma(q_0))}\Gamma$ is a Lagrangian subspace, we have

$$dh(q_0) \cdot v = dH(q_0, \Gamma(q_0)) \cdot \Gamma_* v = -\omega(X_H, \Gamma_* v) = 0.$$

$\square$

Clearly, we may generalize Lemma 2.1 to

**Lemma 2.2.** *If $\Gamma$ is Lipschitz, then every differentiable points contained in $\Lambda_0$ is critical for $h$.*

Now we begin to prove Theorem 1.

Suppose $H$ is not constant on $\Lambda$. This means that $h$ is not constant on its critical point set $\Lambda_0$. Note that $\Lambda_0$ is connected, so the Lebesgue measure of the set of critical values of $h$ is positive. This contradicts to Bates' improved Morse-Sard's theorem [1].

## 3. Proof of Theorem 2

Of course, it is a direct consequence of Norton's improved Morse-Sard's theorem [4]. However, we present a slightly different proof here.

For any two points $(q_1, p_1), (q_2, p_2)$ on $\Lambda$, denote by $\beta$ the rectifiable path connects them. Note that $\beta \in \Lambda$, and $\Lambda$ is invariant, so $dH \cdot \dot{\beta}(t) = 0$, at each differential point, (here, we choose $t$ as the parameter of arc length). Thus

$$H((q_2, p_2)) - H((q_1, p_1)) = \int dH \cdot \dot{\beta}(t) = 0.$$

## 4. Proof of Theorem 3

In [9], Whitney constructed a function $f(q) \in C^{d-1}$ on $d$ ($\geq 2$) dimension manifold $M$ such that there exists a connected set $\Lambda_0$ with $df(q) = 0$ for every $q \in \Lambda_0$, but $f$ is not constant on $\Lambda_0$. In [5], Norton showed more in this direction the existence of a large class of Whitney-type examples for $f \in C^{d-1,s}$ with $0 \leq s < 1$.

By using these Whitney-Norton type examples, we can construct examples Theorem 3 required.

In fact, for any $s \in [0,1)$, there exists a $C^{d-1,s}$ function $f(q)$ and a connected subset $\Lambda_0 \subset M$ such that $df(q) = 0$, $\forall q \in \Lambda_0$, but $f(q)$ is not constant on $\Lambda_0$. Moreover, we may assume that $\Lambda_0$ is contained in a coordinate neighborhood $U$, by changing $f$ outside if necessary. Shrinking $U$ if necessary, we may introduce an auxiliary $C^\infty$ Riemannian metric $g$ such that $g$ is Euclidiean on $U$.

Now we define the Hamiltonian:

$$H(q,p) = f(q) + \frac{1}{2}|p|^2,$$

where

$$q = (q_1, q_2, \cdots, q_d), p = (p_1, p_2, \cdots, p_d)$$

are local coordinates of $T^*M$, and $|\cdot|$ is induced by the Riemannian metric $g$. The Hamiltonian equation is:

$$\dot{q} = \frac{\partial H(q,p)}{\partial p} = p, \quad \dot{p} = -\frac{\partial H(q,p)}{\partial q} = h(q).$$

Let $\Lambda = (\Lambda_0, 0)$, then $\Lambda$ is contained in the zero section of $T^*M$. It is easy to check that $\Lambda$ is invariant under the flow $\phi_H^t$. But $H|_\Lambda = h|_{\Lambda_0}$ is not constant by the definition of $f$.

*Remark* 4.1. If, we take Hamiltonian to be

$$H(q,p) = f(q) + \frac{1}{2}|p - \Gamma|^2,$$

here $\Gamma$ is any Lagrangian graph, then the required invariant critical set $\Lambda \subset \Gamma$.

*Remark* 4.2. In this example, the invariant set $\Lambda$ consists only of fixed points. In fact, we can also construct examples such that $\Lambda$ support non-Dirac measures:

For instance, consider the standard 4-torus. Let $f(q_1, q_2, q_3)$ be a function of Whitney-Norton type on 3-sub-torus, (denote the associated connected critical set by $\Lambda_1$), as discussed above. Now let the Hamiltonian be

$$H(q_1, q_2, q_3, q_4, p_1, p_2, p_3, p_4) = f(q_1, q_1, q_3) + \frac{1}{2}(p_1^2 + p_2^2 + p_3^2 + (p_4 + 1)^2),$$

then $\Lambda_0 = \Lambda_1 \times \mathbb{T}$ is the required projected invariant set, and

$$\Lambda = \{(q_1, q_2, q_3, q_4, 0, 0, 0, 0) : (q_1, q_2, q_3) \in \Lambda_1\}.$$

Clearly, $\Lambda$ is contained in the zero section, and the Hamiltonian flow is not stationary on $\Lambda$.

## 5. Problems

In the example in Theorem 3, the section is $C^\infty$, but the Hamiltonian $H$ is finite smooth. It is more interesting if one can construct counterexamples with infinitely smooth Hamiltonian and finite smooth Lagrangian graph. For this purpose, we pose the following problems:

**Problem 5.1.** Can one construct an explicit example of $H$ of $C^\infty$, which admits a compact, connected invariant set $\Lambda$ in a Lagrangian graph $\Gamma$ of finite smooth, such that $H$ is not constant on $\Lambda$?

We call a graph $\Gamma$ is $C^{0,s}$ Lagrangian, if $\Gamma$ coincides with a differential of a $C^{1,s}$ function locally. As a negative side of Proposition 1, we also pose

**Problem 5.2.** Can one construct an explicit example of $H$, which admits an invariant $C^{0,s}$ (here $0 \le s < 1$) Lagrangian graph $\Gamma$, such that $H$ is not constant on $\Gamma$?

*Remark* 5.1. For Tonelli Hamiltonians, solutions of the associated Hamilton-Jacobi equation have the following nice property: a $C^1$ solution must be $C^{1,1}$, [2]. So, if one can construct a $C^{0,s}$ ($0 \le s < 1$)), non-Lipschitz invariant (under the flow of $\phi_H^t$, H is Tonelli Hamiltonian) Lagrangian graph $\Gamma$, then $H$ is not constant automatically.

## 6. Appendix

In this appendix, we give a proof of Proposition 1, which is slightly different from [7].

Let $h$ be the function as in Theorem 1, then $h$ is a Lipschitz function on $M$, and $dh = 0$ at any differentiable point. For any two points $q_0, q_1$, we can choose an absolutely continuous curve $\gamma : [0,1] \to M$ with $\gamma(0) = q_0, \gamma(1) = q_1$ and $h$ is differentiable on $\gamma$ almost everywhere. Hence, $h(q_0) = h(q_1)$. Thus, $h$ constant on $M$, and $H$ is constant on $\Gamma$, consequently.a

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