Aspects of Critical Phenomena in Curved Space

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Zusammenfassung

In dieser Promotionsschrift untersuchen wir Beispiele für kritisches Verhalten von Quantensystemen, die an eine Hintergrundmetrik gekoppelt sind. Die Analyse wird mit dem doppelten Zweck durchgeführt, Informationen über Materiefreiheitsgrade und Gravitationsfreiheitsgrade zu extrahieren. Verschiedene Formulierungen der Renormalisierungsgruppe werden überprüft und angewendet, um die kritischen Eigenschaften der untersuchten Systeme zu extrahieren. Insbesondere konzentrieren wir uns auf zwei Klassen von Systemen.

Zunächst analysieren wir das Verhalten von chiraler fermionischer Materie auf negativ gekrümmten Räumen, die aufgrund der Hintergrundkrümmung typischerweise eine verstärkte Tendenz zur chiralen Symmetriebrechung aufweisen. Dieses Phänomen ist als Gravitationskatalyse bekannt. In dieser Arbeit wird argumentiert, dass die Gravitationskatalyse aufgrund der hohen Energien in der Nähe des Planck-Regimes zur Vorhersage von schwerer fermionischer Materieteilchen führen könnte, die mit den aktuellen Beobachtungen der Teilchenphysik unvereinbar wären. Wir führen daher eine skalenabhängige Analyse durch, um eine Reihe von Parametern zu identifizieren, die die Gravitationskatalyse ausschließen und zu einem Kriterium für die Falsifikation von Modellen der Quantengravitation führen können. Die skalenabhängige Analyse hat zwei Vorteile: Zum einen ermöglicht sie eine allgemeinere Behandlung des Phänomens, da wir keine Annahmen über die globale Struktur der Raumzeit treffen müssen. Zum anderen wird der Wettbewerb zwischen ihnen explizit dargestellt als die Abschirmung der Infrarotmoden aufgrund einer Regularisierungsskala und deren Verstärkung aufgrund der Hintergrundkrümmung. Wir finden, dass jede Theorie der Quantengravitation, die eine Formulierung im Sinne einer lokalen Feldtheorie der Metrik in der Nähe der Planck-Skala zulässt, unsere Einschränkung respektieren oder einen alternativen Mechanismus zur Lösung des Problems bereitstellen muss.

Die zweite Klasse von Systemen, die wir untersuchen, ist ein unendlicher Turm von Skalarmodellen mit multikritischen Eigenschaften. Hier analysieren wir ihren Renormalisierungsfluss in führender Reihenfolge unter Verwendung einer kovarianten Version der Epsilon-Entwicklung in der dimensionellen Regularisierung. Wir untersuchen dann ihre analytische Fortsetzung unterhalb der oberen kritischen Dimension, wobei wir dem Fixpunktwert der nicht-minimalen Kopplung zum Hintergrund, der sich als konformer Wert herausstellt, besondere Aufmerksamkeit schenken. Dank einer verbesserten Parametrisierung der Wirkung können wir die Gültigkeit unserer Analyse im Zusammenhang mit zweidimensionaler Quantengravitation gekoppelt an eine große Anzahl von Skalarfeldern und der statistischen Feldtheorie im gekrümmten Raum gegen bekannte Ergebnisse testen. Wir diskutieren schließlich die Möglichkeit, den konformen Wert für die nicht-minimale Schwerkraftkopplung auch in der nächsthöheren Ordnung durch spezifische Renormal-

isierungsbedingungen im Zusammenhang mit verwandten Arbeiten zu erzwingen. Wir lassen diese Hypothese vorerst ungeprüft.

Summary

In this thesis we study examples of critical behavior of quantum systems coupled to a background metric. The analysis is performed with the double purpose of extracting information about matter degrees of freedom and gravitational degrees of freedom. Different formulations of the renormalization group are reviewed and applied to extract the critical properties of the systems under examination. In particular we focus on two classes of systems.

First we analyze the behavior of chiral fermionic matter on negatively curved space which typically exhibit an enhanced tendency towards chiral symmetry breaking due to the background curvature. This phenomenon is known as gravitational catalysis. In this thesis we argue that, due to the high energies involved near the Planck regime, gravitational catalysis could result in the generation of heavy fermionic matter particles which would be inconsistent with the current observations of particle physics. We therefore perform a scale dependent analysis to identify a set of parameters which rules out gravitational catalysis and can lead to a criterion to for the falsification of models of quantum gravity. The advantage of the scale dependent analysis is twofold: On one hand it allows for a more general treatment of the phenomenon as we do not need to perform any assumption on the global structure of the spacetime, on the other hand it makes explicit the competition between the screening of the infrared modes due to a regularization scale and their enhancement due to the background curvature. We claim that any theory of quantum gravity that admits a formulation in terms of a local field theory of the metric in vicinity to the Planck scale should respect our constraint or provide an alternative mechanism to solve the problem.

The second class of systems we study is a tower of infinite scalar models with multicritical properties. Here we analyze their renormalization flow at leading order employing a covariant version of the epsilon expansion in dimensional regularization. We then investigate their analytical extension below the upper critical dimension paying particular attention to the fixed point value of the non minimal coupling to the background, which turns out to be the conformal value. Thanks to an improved parametrization of the action we can test the validity of our analysis against known results in the context of two dimensional quantum gravity couple to a large number of scalar fields and of statistical field theory in curved space. We finally speculate on the possibility of imposing the conformal value for the non-minimal coupling to gravity also at the next-to-leading order by means of specific renormalization condition in connection to related works. We leave this hypothesis untested for the moment.

Contents

1. Introduction

One of the most fascinating and challenging branches of modern physics is the search for a quantum theory of gravity [1, 2]. The unification of the two formalisms of general relativity (GR) [3, 4] and quantum field theory (QFT) [5] has proven itself to be surprisingly difficult. On one side there is a very elegant classical theory of gravity based on the geometrical properties of spacetime. Understanding GR led to a revolution in the way we describe physics through the introduction of the equivalence principle and the requirement of general covariance for the physical equations. On the other side we have a very well established framework for quantum effects in highly energetic matter which has provided some of the most precise predictions for experimental measurements so far [6]. With the introduction of QFT we had to renew the way we think about the nature of matter itself.

Neither one of these two theories, though, is fully consistent by itself. In a way, they both predict their own limitations, suggesting that an interplay of the two is needed. In fact, general relativity admits solutions where the geometrical description of spacetime becomes singular, losing every possible physical interpretation [7, 8]. Even if an attempt of solving this issue was given via the cosmic censorship hypothesis, formulated by Penrose [9], several counterexamples in terms of naked singularities that could be realized in nature where given in the literature[10]. Moreover, the current theory of gravity shows large deviations from observations of its theoretical predictions already at the level of galactic scales, a problem that leads to the conjecture of so far unobserved particles of dark matter [11, 12]. At first sight both these topics seem to require the introduction of the quantum formalism: dark matter as a new type of matter field to be incorporated in the standard model (SM) of particle physics, which currently represents our best description of fundamental matter; while the naked singularity problem as a phenomenon where the energy scales involved are larger than the Planck scale at which GR is expected to break down. For what concerns quantum field theory a consistent embedding of gravity within the formalism is still lacking. Beside the obvious theoretical generalization, such improvement could lead to a step forward in understanding problems in particle physics. In particular, the particle spectrum of the standard model, as well as the relative abundance of matter over antimatter, are results of early universe physics where fluctuations of spacetime play a fundamental role.

Within the formalism of QFT every theory aiming at being a fundamental description of nature needs to successfully pass the renormalization test [5, 13]. The failure to incorporate gravity into the standard model, as a quantum theory of the metric generated by the Einstein-Hilbert action, realizes as the non-renormalizability of general relativity at the perturbative level [14, 15]. The lack of a fundamental description for quantum gravity has set the stage for the birth of new models, each introducing new technical features in addition to the common analysis performed in quantum field theory. Examples are string theory [16–19], loop quantum gravity (LQG) [20–22], causal dynamical triangulations (CDT) [23, 24], all of which introduce some sort of non-locality at the fundamental description of spacetime; and asymptotically safe gravity (AS) [25–31], which conjectures the renormalizability of a local theory of the metric at the non-perturbative level. In the asymptotic safety scenario the renormalization flow of quantum gravity is endowed with a non-Gaussian fixed point which defines a fundamental strongly interacting field theory. Regardless of the underlying ultraviolet (UV) theory, any attempt to quantize gravity is supposed to predict an infrared (IR) scenario compatible with general relativity *and* the standard model of particle physics both at the level of the framework and of the observations (even if only effectively).

As the inclusion of gravity can drastically change the phase diagram of matter fields, the inclusion of matter degrees of freedom in a gravitational theory can strongly affect the low energy behavior of the model. A study of the interplay between gravitational and matter degrees of freedom can unfold new features for both gravity and quantum field theory. One of the best tools available to perform such analysis is the renormalization group (RG). The advantage of RG methods is that they take into account by construction the role of the different scales involved in those processes where both gravity and matter are important. The physical information stored in the renormalization group becomes manifest when the system undergoes critical phenomena [32, 33].

A study of critical phenomena and phase transitions reveals how models defined in terms of very different degrees of freedom may in fact have the same renormalization properties and similar phase diagrams. When this happens we say that two models fall into the same *universality class* [34, 35]. A very illustrative example of universality class is given by the φ^4 field theory which shares the same critical behavior as several models of magnetic systems on the lattice, e.g. the Ising model (we review this example in chapter 2). When two models fall into the same universality class they can thus be used to effectively describe different regimes of the same systems. As a consequence we have that a renormalization study of (Euclidean) field theories can provide relevant results for both high energy quantum systems and low energy statistical systems.

The purpose of this thesis is to expand the scientific knowledge about critical phenomena for field theories coupled to gravity and examine the consequences that arise for both matter degrees of freedom and gravitational degrees of freedom.

In chapter 2 we recall most of the basic aspects of critical phenomena and phase transitions. The analysis is mostly based on a thermodynamical approach which provides a straightforward physical interpretation of the formalism. A particular attention is paid to the role of symmetries and spontaneous symmetry breaking (SSB) and to the Landau theory of phase transitions [36]. The concept of universality is then introduced via a description of (thermal) fluctuations and the way they affect the behavior of a system. The framework of statistical field theory plays a big part in the analysis of thermodynamical systems at criticality and its connection to the formalism of quantum field theory is pointed out in appendix C.

In chapter 3 we introduce the renormalization group and its most relevant formulations. The Kadanoff-Wilson approach [32, 33, 37–39] is analyzed and employed to clarify the physical relevance of the renormalization group. The functional renormalization group (FRG) [40–51] is then introduced and its improved mathematical framework is studied. These two formulations are non-perturbative and provide us with (to some extent scheme dependent) pieces of information about the global properties of the theory space under examination. The connection to critical phenomena and universality is made through the fixed point analysis and the perturbative renormalization group in its ε-expansion formulation [13], which is reviewed for the case of a φ^4-theory. We then point out the connection between the FRG scheme and the $\overline{MS}$ scheme of perturbation theory following the analysis of [52]. A description of phase transitions for fermionic systems is then carried out studying the breaking of chiral symmetry in theories endowed with four fermion self interactions and the consequent mass generation. Finally, at the end of chapter 3 we introduce gravity and point out the main aspects one needs to take into account when coupling the system to gravity.

Chapter 4 is dedicated to the study of gravitational catalysis [53–72] and contains part of the original work of this thesis. The term gravitational catalysis describes the breaking of chiral symmetry and subsequent fermionic mass generation induced by a curved spacetime background. The phenomenon is known to occur generically in fermionic systems of any dimension for various negatively curved spacetimes even at the weakest fermionic attraction.

Gravitational catalysis can be understood as a consequence of dimensional reduction of the fluctuation spectrum. For instance in d-dimensional hyperbolic space, the low lying modes of the Dirac operator exhibit a reduction from d to $1+1$ dimensions [70–72]. Hence, the long-range dynamics of any self-interaction of the fermions (be it fundamental, effective or induced) involving a chiral symmetry-breaking channel behaves like the corresponding model in $1+1$ dimensions, e.g., the Gross-Neveu [73] or the Nambu-Jona-Lasinio model [74, 75], which both exhibit chiral symmetry breaking and fermionic mass generation.

In the present thesis, we argue that gravitational catalysis may play a malign role for the interplay of quantum gravitational and fermionic matter degrees of freedom in the high energy regime near the Planck scale. As suggested in [76], the observational fact of the existence of light chiral fermions in our universe puts implicit bounds on the properties of the quantum gravitational interactions: if quantum gravity near the Planck scale was such that it triggered chiral symmetry breaking, the low energy particle sector of our universe would generically be characterized by massive fermions with Planck scale masses. As gravity couples equally to all matter degrees of freedom, it thus would seem difficult to understand the existence of light chiral fermions.

Despite the fact that gravity represents an attractive interaction among particles, gravitational fluctuations in a quantum field theory setting surprisingly do not trigger chiral symmetry breaking [76, 77]. In this respect, gravity differs substantially from gauge theories or Yukawa interactions. Therefore, the existence of light fermions appears compatible even with a high energy regime of strongly coupled gravity, as long as an effective field theory description for quantum gravity is suitable. In particular, the asymptotic safety scenario of quantum gravity passes this consistency test [78–88], also in conjunction with further gauge interactions [89–92]. In asymptotic safety it is even possible to study the interplay of non-chiral fermions and gravity [93], demonstrating that chiral fermions are favored by simple asymptotic safety scenarios. Certain asymptotically safe gravity-matter scenarios even exhibit an enhanced predictive power [94–101].

Whereas most of the studies about asymptotically safe gravity have essentially been performed on flat space, with curvature dependent calculations coming up only recently [102, 103], the gravitational catalysis mechanism is active on negatively curved spacetimes. In this picture, the consistency of quantum gravity and light fermions thus is not so much a matter of gravitational fluctuations and their interplay with matter, but of the effective spacetime resulting from quantum gravity itself.

In chapter 4 a scale dependent analysis of the effective potential of four fermion theories coupled to a negatively curved background is employed to constrain the average curvature of patches of spacetime. In order to elucidate the connection to quantum gravity we apply the result to the asymptotic safety scenario, showing how the bound on the curvature can be rephrased in a constraint for the number of fermionic degrees of freedom of the underlying model of particle physics. The results, though suffering from scheme dependence, appear to be consistent with the current observations of particle physics.

Finally, in chapter 5 we perform a study of the renormalization properties of scalar field theories in curved space exhibiting a multicritical phase diagram. Even though these models are extensively studied in flat space [104–112] and some relevant cases coupled to gravity is already known in the literature [113–117], the analysis presented in this thesis extends these results to incorporate an infinite tower of models. The approach is heavily based on technical improvements of dimensional regularization which allow for a covariant formulation of the method [115–118] (see appendices F and G for further details). The fixed point analysis of these models can be analytically extended below their upper critical dimensions (which happens to be fractional for the vast majority of these theories). Therefore a prediction for the leading order quantum correction to the non-minimal coupling of scalar fields to curvature degrees of freedom is given in $d = 2$. A functional parametrization of the models permits us to reproduce known results concerning two-dimensional quantum gravity coupled to c scalar fields in the limit of large c.

The compilation of this thesis is solely due to the author. However, a large part of the work presented here has been published in a number of articles and in collaboration with other authors. Chapter 4 relies on a paper written with Holger Gies [119]. The work of chapter 5 is based on a collaboration with Omar Zanusso [120].

2. Critical phenomena

In this chapter we report the main aspects of phase transitions and critical phenomena starting from the description usually given in statistical mechanics. Even though most of the results will be understood in a classical framework, the connection emerging in the thermodynamical limit with the formalism of (euclidean) field theory allows us to generalize the study to quantum systems (see appendices A, B and C for a discussion). The two main scenarios we will have in mind in the following are fluid dynamics and magnetic systems. Even though they are very different at the microscopical level these two examples share a lot of features regarding their phase diagrams and offer complementary perspectives on critical phenomena. Most of the notions listed in the following can be found in the majority of textbooks about statistical thermodynamics and statistical field theory. The discussion presented in this chapter mainly follows the exposition of [34, 35]. Interesting discussions can be found also in [36, 121–123].

2.1. General aspects of phase transitions

The first question we should ask ourselves in order to understand the transition between two different phases is under which conditions are these able to coexist. If we define the temperature of a thermodynamical system as T, its pressure as p and its chemical potential as μ, the thermal, mechanical and chemical equilibrium are represented by following set of equations:

$$T_1 = T_2 \,, \quad p_1 = p_2 \,, \quad \mu_1 = \mu_2 \,, \tag{2.1}$$

where the indices refer to phase “1” and “2” respectively. As expressed by the Gibbs-Duhem relation (B.12), the chemical potential as a function of temperature and pressure represents one of the fundamental properties of fluids at equilibrium (see the discussion in appendix B) and is in turn equal to the Gibbs free energy per single particle. We can thus employ (B.12) in order to describe the physical system and the equilibrium would then be rephrased as:

$$\mu_1(T, p) = \mu_2(T, p) \equiv g(T, p) \,, \tag{2.2}$$

where we dropped the indices for temperature and pressure since their values are shared by the two phases along the transition[1]. In order to inspect the latent heat involved in the transition let us consider the enthalpy of the system. Following the conventions of appendix B one has that $H = U + pV$ is related to the Gibbs free energy by a Legendre transform on the temperature/entropy couple:

$$G = \mu N = H - TS\,, \tag{2.3}$$

and the condition (2.2) leads to:

$$\frac{H_2}{N} - \frac{H_1}{N} = T(s_2 - s_1)\,, \tag{2.4}$$

H_i and s_i being the enthalpy and entropy per particle in the phase i respectively. If the above quantity is not zero, then we have that $s_1 \neq s_2$, highlighting how the two phases correspond to different degrees of order of the system. Each configuration will then be favored over the other depending on the specific values of the chemical potential. An

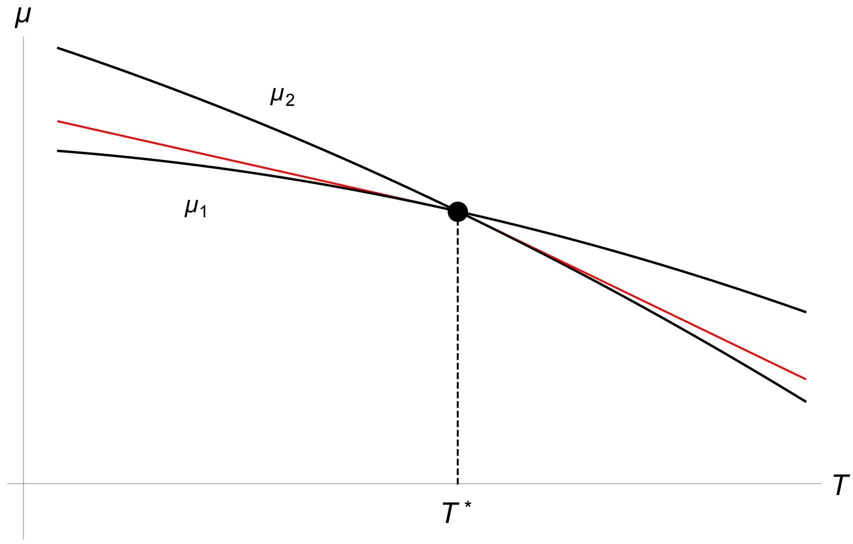


Fig. 2.1.: Typical behavior of the chemical potential in proximity of a continuous phase transition. The black solid curves represent the dependence of the chemical potential of each phase on the temperature, while the red lines point out the discontinuous change of the entropy at the transition temperature.

illustrative picture of the transition is shown in figure 2.1. The values of the chemical potential for the two phases equate at some temperature T^* and each phase results more stable on one of the two side of the temperature space (namely when the chemical

[1] At first sight it may seem from (2.2) that the coexistence of more then three phases is impossible, since the system of equations describing the equilibrium would then be overdetermined. Even though this is indeed the general case, a fine-tuning of the model could lead to such a consequence. Theories exhibiting phase diagrams of this type are known as multicritical models, and we will discuss a subclass of these systems later in this work.

potential is lower). The chemical potential of the system will, hence, present a cusp and the entropy will be discontinuous at the transition point:

$$\left.\frac{\partial \mu_1}{\partial T}\right|_p > \left.\frac{\partial \mu_2}{\partial T}\right|_p \Rightarrow s_1 < s_2 \,. \tag{2.5}$$

A similar behavior will be shown by the specific volume $v = \left.\frac{\partial \mu}{\partial p}\right|_T$ as well.

Following the coexistence line for increasing temperature, it often happens that the discontinuity disappears at some point, we call this point *critical point* and the value of the temperature there gives the *critical temperature* T_c. For values of the temperature larger than T_c is possible to shift between the two phases continuously, without encountering any discontinuity or separation between the two. This, as shown in figure 2.2 is a common feature of both classical fluids and magnetic systems. While in the case of fluids the discontinuity appears across the line separating the liquid and the vapor phases (starting from the triple point, where the three phases are at equilibrium, and ending at the critical point), in the case of magnetic systems, such as the Ising model or the Heisenberg model, the transition is between two different alignment of the spin variables (*magnetization*) depending on the orientation of an external magnetic field h:

$$m(T,h) = \left.\frac{\partial g(T,h)}{\partial h}\right|_T , \tag{2.6}$$

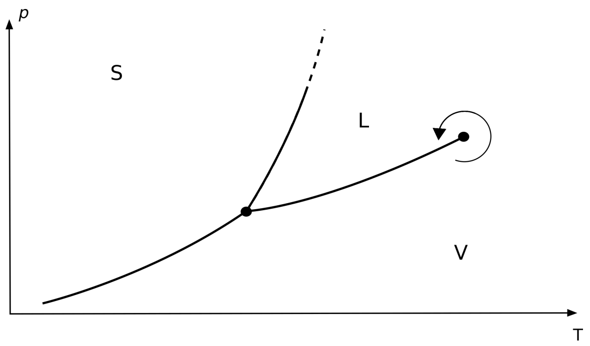
$$\Delta m(T,h) = \lim_{h\to 0}\{m(h>0) - m(h<0)\} > 0 \quad \text{for} \quad T < T_c \,. \tag{2.7}$$

In the latter case we can also tune the external magnetic field to vanish and decrease the temperature from above to below the critical point. What we will observe is a transition of the system from a disordered phase with vanishing magnetization to an ordered phase where a spontaneous magnetization grows from zero to a finite value as the temperature decreases from T_c to zero. Even though the transition is continuous (in the sense that the magnetization is continuous across the critical point), the second derivatives of the Gibbs free energy, namely the *magnetic susceptibility* and the *specific heat*

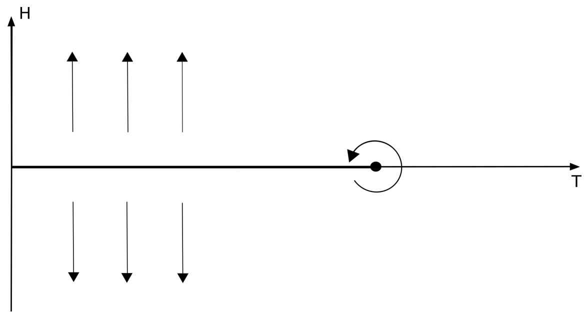
$$\chi = -\left.\frac{\partial^2 G}{\partial h^2}\right|_{T,h=0} ; \qquad C_{h=0} = -T\left.\frac{\partial^2 G}{\partial T^2}\right|_{h=0} , \tag{2.8}$$

are discontinuous and divergent.

The set of phenomenological observations we just pointed out led to the so called Ehrenfest classifications of phase transitions [34]. According to this classification a phase transition is said to be *of order* n if the free energy associated with the system and its first $n-1$ derivatives are continuous, while a discontinuity manifests for some derivative of the n-th order. Reinterpreting the phase space of fluids and magnetic systems under this perspective we have that across the equilibrium line of two phases (for example



(a)



(b)

Fig. 2.2.: Phase diagrams for classical fluids (2.2a) and magnetic systems (2.2b). Both class of systems exhibit a discontinuous phase transition. The discontinuity terminates at a critical point beyond which a continuous transition is possible.

the liquid-vapor transition between the triple point and the critical point) the system undergoes a first order phase transition, while at the critical point (e.g. for a magnetic system at zero external field) we typically have a second order phase transition. Finally, for $T > T_c$ we can switch between two phases smoothly[2].

It should be clear now that the key point in describing phase transitions is the free energy of the system and, in particular, its analytical properties. This can be understood since in general the free energy contains competing terms of energy and entropy which favor an ordered and a disordered phase respectively. As explained in appendix B the free energy is given by the logarithm of the partition function and in the thermodynamical limit[3] can, thus, be expressed in terms of different thermodynamical potentials depending on the type of ensemble that better suits the experimental apparatus. For the sake of simplicity and without loss of generality we will proceed having in mind the grand canonical ensemble, which represents a good description in the case where two phases at equilibrium can exchange energy in the form of heat, work and matter. An equivalent analysis in terms of other ensembles is nevertheless possible.

[2]It is worthwhile to mention that these considerations are not fully general and rather depend on the system. While the transition above (and far from) the critical point does not show any critical behavior, examples of infinite order phase transitions at the critical point are known, e.g. the Kosterlitz–Thouless transition [124].

[3]As expressed by the Lee-Yang theorems, a phase transition only occurs in the thermodynamical limit.

Given a grand canonical partition function

$$\mathcal{Z} = \mathcal{Z}(T, V, \mu)\,, \tag{2.9}$$

we shall call a *phase transition point* any singular point of the specific grand canonical potential [34]:

$$\Phi(T, v, z) = -\mathrm{Tlim}\left\{\frac{1}{\beta V}\log\mathcal{Z}(T, V, \mu)\right\}\,, \tag{2.10}$$

where z is the fugacity $z = e^{\beta\mu}$, $v = V/N$ the specific volume and Tlim denotes the thermodynamical limit.

2.2. Spontaneous symmetry breaking

As we pointed out in the previous section, a phase transition is often associated with a change on the degree of order of the system. The proper way to describe this phenomenon is in terms of reducing the symmetry of the system from a group $\mathbb{G}$ to a (possibly trivial) subgroup [34]. This procedure can be done either by introducing an interaction that breaks the symmetry directly at the level of the microscopical description of the model (in which case we say that the symmetry is explicitly broken) or we rely on some effect leading to a reduction of the symmetry space dynamically, namely, a spontaneous symmetry breaking (SSB).

In order to make the previous considerations formal suppose $\mathbb{G}$ to be a global symmetry group for the Hamiltonian describing a thermodynamic system. Then one says that the symmetry has been spontaneously broken iff there exist some not $\mathbb{G}$-invariant observable M such that in the (stable) thermodynamic equilibrium state

$$m \equiv \langle M\rangle \neq 0\,. \tag{2.11}$$

The quantity m will then be called *order parameter* and its value characterizes the two phases separated by SSB.

A few remarks are in order:

1. If the stabilizer of the order parameter is a non-trivial subgroup $\mathbb{H}$ of $\mathbb{G}$ then the symmetry is reduced from $\mathbb{G}$ to $\mathbb{H}$, and we can identify the order parameter space with the coset space $\mathbb{X} = \mathbb{G}/\mathbb{H}$, where $\mathbb{H}$ acts as the identity. Since the action of $\mathbb{G}$ on the order parameter space is often transitive any symmetry breaking value of m can be reached acting on any other one with an element in the appropriate coset of $\mathbb{G}$,

2. As the expectation value $\langle M\rangle$ is implicitly defined in terms of a distribution ρ

associated with the equilibrium state ω[4], it follows from the theory of invariant measures that if the state is symmetric under the action of $\mathbb{G}$ then the expectation value of M will always be zero. This implies that if we have SSB it has to happen at the level of the ground state[5]. As long as the volume of the system is finite the partition function can be written as sum of Boltzmann factors $e^{-\beta H}$ and the state will inherit all the symmetries of the Hamiltonian without developing any singularity: no phase transition can be observed. The situation might change, though, in the thermodynamical limit since the previous sum becomes a series[6]. Thus, if we make explicit the dependence of m on the number of particles and introduce symmetry breaking interaction parametrized by the field h, we will have [34, 122]:

$$\lim_{N\to\infty}\lim_{h\to 0} m_N(h) = \lim_{N\to\infty} 0 = 0\,, \tag{2.12}$$

while switching the order of the two limits:

$$\lim_{h\to 0}\lim_{N\to\infty} m_N(h) \neq 0\,, \tag{2.13}$$

3. With a glance at the quantum formalism we can understand the previous comment in terms of the cyclicity of the Fock vacuum: developing a new stable equilibrium thermodynamical state translates, upon Wick rotation, into the generation of new vacuum states with a reduced symmetry. The entire spectrum of excitations generated through Fock's construction of the Hilbert space (as well as the set of creation and annihilation operators) will hence suffer from the symmetry breaking and constitute a new phase reflecting the non-analyticity of the path integral at the phase transition. As we will explain in the next session, once that a description of the system in terms of an effective potential is available, phase transitions correspond to the development of new minima of the potential.

2.3. Landau description of phase transitions

The aspects pointed out until here are easy to understand in the context of Landau theory of phase transitions [36]. Even though Landau's approach relies on a mean field approximation it still represents a unified theory of phase transitions and manages to

[4] A state ω can be regarded as a functional on the abelian algebra of observables A and defined such that $\omega(A) = \langle A\rangle_\omega$.

[5] In particular, the symmetry breaking does not happen at the level of the Hamiltonian which, indeed, remains $\mathbb{G}$-invariant.

[6] Even though in the context of classical statistical mechanics this coincides with having an infinite space volume, the same behavior happens in general when the number of degrees of freedom is sent to infinity, e.g. in field theory.

capture many of their most relevant features. Since critical phenomena are driven by infrared modes we can safely define a continuous theory in terms of fields even when the original model is defined on a lattice. The order parameter becomes then an *order parameter field* denoted by $\phi(x)$. The idea behind the theory of Landau is that we can construct a functional $\mathcal{A}$ of the field such that $\mathcal{A}$ respects all the symmetries of the model, i.e. $\mathcal{A}$ is invariant under the action of a group $\mathbb{G}$. The partition function can be written as:

$$\mathcal{Z}(h;T) = \mathcal{N} e^{-\beta F[h]} = \mathcal{N} \exp\left\{-\beta\mathcal{A}[\phi] + \beta \int \mathrm{d}^d x\; h(x)\,\phi(x)\right\}\,, \tag{2.14}$$

where $\mathcal{N}$ is a normalization constant and F represents the free energy of the system:

$$\frac{\delta}{\delta h(x)} F[h] = \frac{\delta}{\delta h}\left[\mathcal{A}[\phi] - \int \mathrm{d}^d y\; h(y)\,\phi(y)\right] = -\phi(x)\,. \tag{2.15}$$

The field h plays the role of a source for the order parameter field and explicitly breaks the symmetry group $\mathbb{G}$ to the stabilizer of ϕ, nevertheless for vanishing h the partition function is fully symmetric. Upon expansion of the functional $\mathcal{A}$ in powers of the order parameter and in its derivatives is possible to extract a potential for the model and study the critical phenomena in the fashion described at the end of the previous section.

We are now able to state the assumptions of Landau's formulation:

1. In the continuum limit, the partition function of the model can be written in the form (2.14), where ϕ represents the order parameter of the phase transition and h couples linearly to it. The functional $\mathcal{A}$ will then be called *effective action*.

2. The $\mathbb{G}$-invariant functional $\mathcal{A}$ can be expressed as an integral of a *local* density, namely a $\mathbb{G}$-invariant function of ϕ and its derivative known as Landau function:

$$\mathcal{A}[\phi] = \int \mathrm{d}^d x\; \mathcal{L}(\phi(x), \partial_\alpha \phi(x))\,. \tag{2.16}$$

3. A satisfactory description of the critical behavior of the system can be obtained by expanding the Landau function and keeping only the lowest orders[7].

In order to better understand how the approach works and how the new degrees of freedom are connected to the fundamental one let us briefly review how Landau's approach describes magnetic systems like the Ising model at criticality [34–36].

The partition function of the Ising model reads:

$$Z[h] = \sum_{\{s_i\}} \exp\left\{\sum_{i,j} J_{ij} s_i s_j + \sum_i h_i s_i\right\}\,, \tag{2.17}$$

[7]How many orders are enough, though, depends heavily on the system and its dimensionality.

where the variables s_i represent the spin alignment of a particle sitting at the i-th site of a lattice along some given direction ($s_i = \pm 1$) and J_{ij} gauges the strength of the interactions between two different sites. The spin-spin interaction term is symmetric under the group $\mathbb{Z}_2$ and, indeed, h_i represents the symmetry breaking field, namely an external magnetic field introducing a preferred orientation. At this level we can identify the order by inspecting the first derivative of the Gibbs[8] free energy:

$$m = \left.\frac{\partial}{\partial h_i}\right|_{h_i=0} \log Z[h] = \frac{1}{Z[0]}\left.\frac{\partial}{\partial h_i}\right|_{h_i=0} Z[h] = \langle s_i \rangle \,. \tag{2.18}$$

Using the following identity valid for Gaussian integrals:

$$\int_{-\infty}^{\infty} \prod_i \mathrm{d}\phi_i \ \exp\left[-\frac{1}{4}\sum_{i,j}\phi_i J^{-1}_{ij}\phi_j + \sum_i \phi_i s_i\right] = \mathcal{N}\exp\left[\sum_{i,j} J_{ij} s_i s_j\right] , \tag{2.19}$$

$\mathcal{N}$ being a normalization constant, one can rewrite (2.17) as[9]

$$\begin{aligned} Z[h] &= \sum_{\{s_i\}} \int_{-\infty}^{\infty} \prod_i \mathrm{d}\phi_i \ \exp\left[-\frac{1}{4}\sum_{i,j}\phi_i J^{-1}_{ij}\phi_j + \sum_i (h_i+\phi_i) s_i\right] && (2.20)\\ &= \int_{-\infty}^{\infty} \prod_i \mathrm{d}\phi_i \ \exp\left[-\frac{1}{4}\sum_{i,j}(\phi_i - h_i) J^{-1}_{ij}(\phi_j - h_j)\right] \sum_{\{s_i\}} \exp\left[\sum_i \phi_i s_i\right] . && (2.21)\end{aligned}$$

The sum over the spin variables is now much simplified and can be performed explicitly leading to:

$$\sum_{\{s_i\}} \exp\left[\sum_i \phi_i s_i\right] = \prod_i (2 \cosh\,\phi_i) \propto \exp\left[\sum_i \log\left(\cosh\,\phi_i\right)\right] , \tag{2.22}$$

and through a last linear transformation $\phi_i \mapsto \frac{1}{2}J^{-1}_{ij}\phi_j$ the partition function is now turned into:

$$\begin{aligned} Z[h] =& e^{-\frac{1}{4}\sum_{i,j} h_i J^{-1}_{ij} h_j} \\ &\times \int \prod_i \mathrm{d}\phi_i \ \exp\left[-\sum_{i,j} J_{ij}\phi_i\phi_j + \sum_i \log(\cosh(2J_{ik}\phi_k)) + \sum_i h_i\phi_i\right] . \end{aligned} \tag{2.23}$$

At this point we have not performed the continuum limit yet. The field variables ϕ_i and h_i can take values over the entire real axis, but still enter in the action through the lattice sites: $\phi_i = \phi(\mathbf{r}_i)$. Assuming the external field h_i to be small we can approximate the

[8]See appendix B for a justification of using Gibbs free energy in order to discuss the Ising model.

[9]For the sake of keeping the exposition light we neglect the constant $\mathcal{N}$ for a while in abuse of the equality notation.

exponential in front of (2.23) by $1+o(h^2)$. By sending the discrete lattice to a continuous system we replace the sums over discrete indices with integrals over spacetime variable and we formally implement the functional measure. This procedure is implicitly justified if we are considering long-range fluctuations with wave length much bigger than the lattice spacing. Employing the following polynomial expansion:

$$\log(\cosh(x)) = \frac{1}{2}x^2 - \frac{1}{12}x^4 + o(x^6)\,, \tag{2.24}$$

we are able to check the behavior of the above model in its momentum space representation and identify the correct terms that need to be retained for the Landau approach:

$$\begin{aligned} Z[h] \sim \int \mathcal{D}\phi\ \exp\Big[&-\int \mathrm{d}\mathbf{k}\,\Big(J(\mathbf{k}) - 2|J(\mathbf{k})|^2\Big)|\phi(\mathbf{k})|^2\Big] \\ &\times \exp\Big[-\frac{4}{3}\int \{\mathrm{d}\mathbf{k}_i\}_{i=1}^3\ J(-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\phi(-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\prod_{i=1}^{3} J(\mathbf{k}_i)\phi(\mathbf{k}_i) \\ &+ \int \mathrm{d}\mathbf{k}\, h(-\mathbf{k})\phi(\mathbf{k})\Big]\,, \end{aligned} \tag{2.25}$$

where use of the reality condition for the Fourier modes has been made in order to rearrange the quadratic part. As a first approximation we can fix the coupling J to be a constant for the quartic interaction and expand its dependence on $\mathbf{k}$ (isotropically) up to the second order for what concerns the quadratic part of the field action

$$J(\mathbf{k}) \simeq J_0(1-\rho^2 k^2)\,. \tag{2.26}$$

This approximation basically amounts to retaining the minimum requirements to have dynamics and a momentum independent interaction[10]. The partition function now reads:

$$\begin{aligned} Z[h] = \int \mathcal{D}\phi\ \exp\Big[&-J_0\int \mathrm{d}\mathbf{k}\,\Big((1-2J_0) + (4J_0-1)\rho^2k^2\Big)|\phi(\mathbf{k})|^2 \\ &-\frac{4}{3}J_0^4\int \mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2\mathrm{d}\mathbf{k}_3\ \phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\phi(\mathbf{k}_3)\phi(-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) \\ &+\int \mathrm{d}\mathbf{k}\, h(-\mathbf{k})\phi(\mathbf{k})\Big]\,. \end{aligned} \tag{2.27}$$

Since phenomenologically we observe that J_0 increases for decreasing temperature, we can

[10] We will come back to this dependency in the context of renormalization for a more complete analysis.

define a critical temperature T_c for which the momentum free quadratic part vanishes:

$$\begin{aligned} 1-2J_0 &= \frac{T-T_c}{T_c}\,, \\ 4J_0-1 &= 1+o(T-T_c)\,, \\ J_0 &= \frac{1}{2}+o(T-T_c)\,. \end{aligned} \tag{2.28}$$

The last set of considerations closes the circle to map the Ising model to a field theory provided a set of definitions that make the temperature dependence explicit, namely[11]

$$\varphi(\mathbf{k}) = \frac{\rho}{\sqrt{\beta}}\,\phi(\mathbf{k})\,, \quad \tilde{h}(\mathbf{k}) = \frac{1}{\rho\sqrt{\beta}}\,h(\mathbf{k})\,, \quad m(T)^2 = \frac{1}{\rho^2}\frac{T-T_c}{T_c}\,. \tag{2.29}$$

The final expression for the functional Z turns out to be:

$$Z[\tilde{h}] = \int \mathcal{D}\varphi\ \exp\Big[-\beta\int \mathrm{d}\mathbf{x}\,\Big(\frac{1}{2}(\partial_j\varphi)^2(\mathbf{x})+\frac{m^2}{2}\varphi^2(\mathbf{x})+\frac{\lambda}{4!}\varphi^4(\mathbf{x})\Big)+\beta\int \tilde{h}(\mathbf{x})\varphi(\mathbf{x})\Big]\,, \tag{2.30}$$

where we transformed back the action to the direct space and defined the coupling λ[12] in order to collect all the constants in front of the interaction. One can thus define the classical action for our field theory to satisfy:

$$\begin{aligned} Z[h] &= \int \mathcal{D}\varphi\ e^{-\beta S[\varphi]+\beta\int h\varphi}\,, \\ S[\varphi] &= \int \mathrm{d}\mathbf{x}\ \left(\frac{1}{2}(\partial_j\varphi)^2(\mathbf{x})+\frac{m^2}{2}\varphi^2(\mathbf{x})+\frac{\lambda}{4!}\varphi^4(\mathbf{x})\right)\,. \end{aligned} \tag{2.31}$$

Once we completed the continuum limit we need to find a procedure to extract the effective action. A common approach is to rely on a mean field approximation. In this case one substitutes the path integral by its major contributions meaning that the effective action evaluated at a given state will take the form of the classical action computed at the same state, where the configuration of the field is given by its expectation value:

$$\Phi(\mathbf{x}) = \langle\phi(\mathbf{x})\rangle\,. \tag{2.32}$$

In other words, considering an expansion for the effective action the mean field approximation states that this has the following form [34, 35, 122]:

$$\mathcal{A}[\Phi] = S[\Phi] + \text{subleading correction} \tag{2.33}$$

[11] Needless to say, the unfortunate choice of symbol m here is due to the identification with the mass in the field theory formalism and needs not to be confused with the magnetization of a lattice spin system.

[12] The 4! factor was inserted for it represents a more convenient parametrization of the coupling.

and only the first term is considered. Consequently, our ansatz for $\mathcal{A}$ (sometimes called *truncation* since we discarded higher order interaction coming from the expansion of $\cosh(x)$) will be:

$$\mathcal{A}[\Phi] = \int \mathrm{d}\mathbf{x} \, \left[\frac{1}{2}(\partial_j \Phi)^2(\mathbf{x}) + \frac{m^2}{2}\Phi^2(\mathbf{x}) + \frac{\lambda}{4!}\Phi^4(\mathbf{x}) \right] . \tag{2.34}$$

One can easily check that the expectation value (2.32) transforms non trivially under the action of $\mathbb{Z}_2$ implying that for a symmetric state:

$$\Phi = \langle \phi \rangle = \langle -\phi \rangle = -\Phi \quad \Rightarrow \quad \Phi = 0 \, , \tag{2.35}$$

and, thus, it represents a legitimate order parameter for the spontaneous symmetry breaking.

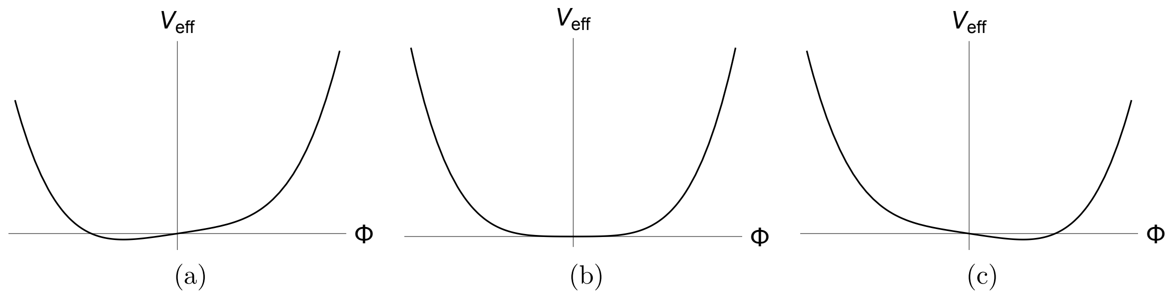


Fig. 2.3.: The figure represents the shape of the effective potential for the Ising model at $T > T_c$ for three different values of the external magnetic field, namely Fig. 2.3a: $h(\mathbf{x}) < 0$; Fig. 2.3b: $h(\mathbf{x}) = 0$; Fig. 2.3c: $h(\mathbf{x}) > 0$.

Since the truncation of the effective potential is of the same form of a classical action we can split the Landau function into a kinetic part ($\simeq (\partial_i \varphi)^2$) and an effective potential

$$V_{eff} = \frac{m^2}{2}\Phi^2(\mathbf{x}) + \frac{\lambda}{4!}\Phi^4(\mathbf{x}) \tag{2.36}$$

and identify the stable states with the minima of the potential. Figure 2.3 shows the shape of V_{eff} for a magnetic system above the critical temperature. In this scenario there is only one minimum which is shifted by the external magnetic field according to its sign. Since the order parameter represents the average of the spin variable over the system, a minimum occurring for positive Φ (Fig. 2.3c) represents a positive orientation for the majority of the sites and an induced magnetization of the system and similarly for a minimum occurring in the negative half plane (Fig. 2.3a). The transition between the two configurations is modulated by $h(\mathbf{x})$ and pass continuously across the symmetric (disordered) phase at $h(\mathbf{x}) = 0$, where the minimum is located at $\Phi(\mathbf{x}) = 0$ and the potential around it gets flatter the more we approach the critical temperature.

The interpretation in terms of the polynomial form of the effective potential is straightforward: the quartic term ensures the asymptotic stability as excitations of the order parameter increase so that, for positive λ, the Hamiltonian effective action is bounded from below; the quadratic term is proportional to the Hessian of the potential at the origin of the field space:

$$V_{eff}|_{\Phi^2} = \frac{1}{2}\frac{\partial^2 V_{eff}}{\partial \Phi^2}\bigg|_{\Phi=0} \Phi^2(\mathbf{x}) \,, \tag{2.37}$$

$$m^2 = \frac{\partial^2 V_{eff}}{\partial \Phi^2}\bigg|_{\Phi=0} \tag{2.38}$$

and since $\lim_{T\to T_c} m^2(T) = 0$ we have that the potential flattens around its minimum at the critical point. When the temperature decreases further the Hessian changes sign causing $\Phi = 0$ to become a maximum and simple calculation shows that two new minima at $\Phi(\mathbf{x}) = \pm\sqrt{-\frac{6m^2}{\lambda}}$ that in the symmetric phase were complex now turn into real. Similarly to the previous situation, the role of selecting a ground state is given to the

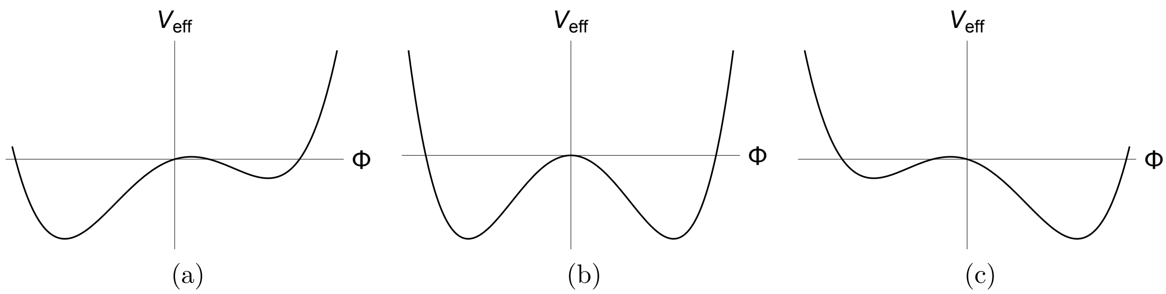


Fig. 2.4.: The figure shows the shape of the effective potential for the Ising model at $T < T_c$ for three different values of the external magnetic field, namely Fig. 2.4a: $h(\mathbf{x}) < 0$; Fig. 2.4b: $h(\mathbf{x}) = 0$; Fig. 2.4c: $h(\mathbf{x}) > 0$.

magnetic field which now rises one of the two minima (which becomes metastable) and lowers the other (Fig.s 2.4a and 2.4c). Decreasing the value of the $h(\mathbf{x})$ continuously from positive to negative amounts to switch between the minima and at $h(\mathbf{x}) = 0$ they will be equivalent and the symmetry of the potential will be restored. Tuning $h(\mathbf{x})$ to zero and decreasing the temperature across the critical point the symmetric well splits continuously into the two new minima but its derivative with respect to the magnetic field (magnetic susceptibility) results infinitely discontinuous, signaling a second order phase transition.

The same picture presented above can be generalized to effective potentials expressed by higher order polynomials [36, 121]. An example is given by the following model:

$$V_{eff} = \frac{m^2}{2}\Phi^2(\mathbf{x}) + \frac{\lambda}{4!}\Phi^4(\mathbf{x}) + \frac{\xi}{6!}\Phi^6(\mathbf{x}) \,. \tag{2.39}$$

The qualitative discussion of such a model follows the same line as above but the new interaction introduces a richer structure. In fact, a potential of the form (2.39) can be endowed with up to three local minima (see Fig. 2.5) each of which corresponds to a different phase. Depending on the values of the magnetic field and of the couplings λ and

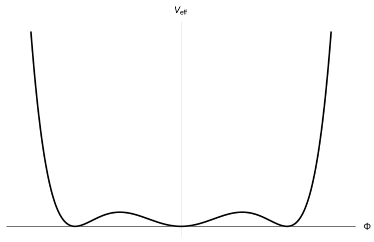


Fig. 2.5.: Schematic representation of the configuration of the effective potential at the tricritical point.

ξ the potential may have one, two or three *global* minima. The case where there is only one global minimum is trivial. The case of two minima is equivalent to the discussion about the quartic potential and represents the coexistence of two phases. The case where all the three minima are equivalent represents the equilibrium of three different phases and is called *tricritical* point.

Further generalizations with higher polynomials can of course be defined and are collectively called *multicritical* models since at criticality they describe the meeting point of many different phases, and they represent one of the natural generalizations of Ising model. An extensive discussion of the properties of multicritical models in Landau theory can be found, for example, in [121]. Chapter 5 will be dedicated to a discussion of some of these models and their coupling to gravitational degrees of freedom.

It should be now clear how the Landau approach represents a powerful method to analyze critical phenomena and provides us with direct interpretations about the mathematical description of phase transitions.

2.4. Critical exponents, scaling and universality

As we pointed out at the end of section 2.1, phase transitions and critical phenomena are associated with singularities of the free energy of the system. If we consider the free energy per unit volume g[13] it will depend on two variables, namely the reduced temperature t and the magnetic field h [34]:

$$g = g(t, h)\,, \qquad t = \frac{T - T_c}{T_c}\,, \tag{2.40}$$

[13]For the sake of simplicity we keep referring implicitly to an Ising like system, so that the free energy will be the Gibbs free energy and the external field will be a magnetic field. We stress once more that this identification is heavily system dependent, though the formalism is not.

showing a critical point at $t = h = 0$. In a neighborhood of the critical point we assume that the free energy can be decomposed into a regular and a divergent part:

$$g(t, h) = g_{\text{reg}}(t, h) + g_{\text{div}}(t, h)\,, \tag{2.41}$$

with g_{reg} analytic at $(0, 0)$. The singular component g_{div} is the one responsible for critical phenomena and non-analytic behaviors off physical quantities. These behaviors are typically described by exponential laws and characterized by a set of *critical exponents* [34–36, 121–123]. The most common and interesting exponents are given by the following scalings:

$$\begin{array}{lll} \text{magnetic susceptibility } h=0\text{:} & \chi \propto |t|^{-\gamma} & (t \lessgtr 0) \\ \text{specific heat at } h=0\text{:} & C_{h=0} \propto |t|^{-\alpha} & (t \lessgtr 0) \\ \text{magnetization at } h=0\text{:} & m_{h=0} \propto |t|^{\beta} & (t < 0) \\ \text{magnetization at } T=T_c\text{:} & m_{h=0} \propto |h|^{1/\delta} & (t = 0) \\ \text{correlation length:} & \xi \propto |t|^{-\nu} & (t \lessgtr 0) \\ \text{correlation function:} & \mathcal{G}^c(\mathbf{r}) \propto r^{-\tau} e^{-r/\xi} & (t \lessgtr 0) \\ \text{correlation function at } T=T_c\text{:} & \mathcal{G}^c(\mathbf{r}) \propto r^{-d+2-\eta} & (t = 0) \end{array} \tag{2.42}$$

where we introduced the connected correlation function (or connected Green function):

$$\mathcal{G}^c(\mathbf{r}) = \langle s(\mathbf{x}+\mathbf{r}) s(\mathbf{x}) \rangle - \langle s(\mathbf{x}+\mathbf{r}) \rangle \langle s(\mathbf{x}) \rangle\,. \tag{2.43}$$

The astonishing observation is that theories with different microscopical degrees of freedom share the same set of critical exponents and, thus, exhibit the same critical behavior. When such a phenomenon happens we say that two theories fall into the same *universality class*. An example of this relation was shown in the previous section where we identified the behaviors of the Ising model and a φ^4 scalar field theory.

In order to clarify how critical exponents depend on the microscopical details of a theory let us implement a field theoretical setting. It should be clear now that the central object to be studied are fluctuations and how do they affect correlators and their scaling properties. Let us suppose that there exist a countable set of observables $\mathcal{O}_i$ constituting a basis for perturbations of the microscopical free Hamiltonian H_o[14]

$$\begin{aligned} H[\varphi] &= H_o[\varphi] + \delta H[\varphi]\,, \\ \delta H[\varphi] &= \sum_i \lambda_i \int \mathrm{d}^d\mathbf{x}\, \mathcal{O}_i(\varphi(\mathbf{x}))\,, \end{aligned} \tag{2.44}$$

and that each of these operators transforms under a rescaling of the coordinates according

[14] The following discussion about the scaling behavior of fluctuations is largely based on [121].

to:

$$\mathcal{O}_i(\alpha\mathbf{x}) = \alpha^{-\Delta_i}\mathcal{O}_i(\mathbf{x}) \quad \Rightarrow \quad \mathcal{O}_i \sim |\mathbf{x}|^{-\Delta_i}\,. \tag{2.45}$$

We call Δ_i the scaling dimension of the observable $\mathcal{O}_i$. Then we are able to compute the asymptotic behavior of the correction to, say, the two-point correlation function[15] due to the new observable as:

$$\mathcal{G}(\mathbf{x}) = \mathcal{G}_o(\mathbf{x}) + \delta\mathcal{G}(\mathbf{x}) \sim \int \mathcal{D}\varphi\ \varphi(\mathbf{x})\varphi(\mathbf{0})(1-\delta H)e^{-H_o}\,, \tag{2.46}$$

from which it is straightforward to see that at first order in the couplings λ_i we have:

$$\delta\mathcal{G} = -\sum_i \lambda_i \int \mathrm{d}^d\mathbf{y}\ \langle\varphi(\mathbf{x})\varphi(\mathbf{0})\mathcal{O}_i(\mathbf{y})\rangle\,, \tag{2.47}$$

where in this case the brackets $\langle\cdots\rangle$ represents expectation values over the unperturbed system.

By decomposing the two fields on the basis of observables as:

$$\varphi(\mathbf{x})\varphi(\mathbf{0}) = \sum_l \mu_l|\mathbf{x}|^{\Delta_l-2\Delta_\varphi}\mathcal{O}_l(\mathbf{0})\,, \tag{2.48}$$

where the scaling dimension of the free field is $\Delta_\varphi = \frac{d-2}{2}$, and taking into account the properties of the measure one finds that (2.47) scales as:

$$\begin{aligned}\delta\mathcal{G} &= -\sum_i \lambda_i \int \mathrm{d}^d\mathbf{y}\ \sum_l \mu_l|\mathbf{x}|^{\Delta_l-2\Delta_\varphi}\langle\mathcal{O}_l(\mathbf{0})\mathcal{O}_i(\mathbf{y})\rangle \\ &\sim -\sum_{i,l}\lambda_i\mu_l|\mathbf{x}|^{d-\Delta_i-2\Delta_\varphi}\,.\end{aligned} \tag{2.49}$$

In order to make explicit the contribution of the above result we should compare it with the scaling properties of the free two-point function $\mathcal{G}_o \sim |\mathbf{x}|^{-2\Delta_\varphi}$ to find:

$$\frac{\delta\mathcal{G}}{\mathcal{G}_o} \sim -\sum_{i,l}\lambda_i\mu_l|\mathbf{x}|^{d-\Delta_i}\,. \tag{2.50}$$

If we inspect the terms entering in this result for different i (changing the index l does not change the order of the contribution) we notice right away the role of the scaling dimensions in the long-range dynamics: if $\Delta_i > d$ the correction due to $\mathcal{O}_i$ is small and the sum converges, leaving the critical exponents unchanged from the free case. In the case where $\Delta_i < d$ the contribution of the fluctuations grows and the result of

[15] We will first look at the correlation functions generated by the partition function rather than the free energy.

(2.50) represents only the first term of a series in powers of $\lambda|\mathbf{x}|^{d-\Delta_i}$ and the critical exponents are different from the free case. With a glimpse into renormalization we will name observables falling in the first category as *irrelevant* (since they do not affect the long-range dynamics), while those falling into the second category will be called *relevant*. In both the previous cases, a change in the critical behavior can be induced only in a discontinuous way by turning on a new interaction described by an observable with scaling dimension smaller than the dimensionality of the system (which typically modifies the symmetry of the Hamiltonian) and the change does not depend on the microscopical details of the theory (like the value of the coupling constant or type of fields involved in the correction). We will, thus, say that the phase transition is *universal*.

A third possibility might happen, namely the case where the dimension of the perturbation coincides with that of the system: $\Delta_i = d$. To study this case we go back to equation (2.47) and write:

$$\varphi(\mathbf{0})\mathcal{O}_i(\mathbf{y}) = a\varphi(\mathbf{0})|\mathbf{x}|^{-d} \quad + \quad \text{subleading terms}\,, \tag{2.51}$$

for some coefficient a. Upon integration, we observe a logarithmic divergence:

$$\delta\mathcal{G} = -\lambda_i\Omega_d a \log|\mathbf{x}|\mathcal{G}_o\,, \tag{2.52}$$

Ω_d being the volume of the unit sphere in d dimensions. This last scenario is being referred to as *marginal* perturbation and can signal either the violation of scale invariance or the variation of the critical exponents. In fact, by applying an infinitesimal perturbation to the scaling exponents of the free propagator it is easy to find a similar divergence:

$$\mathcal{G} \sim |\mathbf{x}|^{-2(\Delta_\varphi+\delta\Delta)} \sim \mathcal{G}_o - 2\delta\Delta\log|\mathbf{x}|\mathcal{G}_o\,. \tag{2.53}$$

By comparison with (2.52) we infer that the critical exponents depend continuously on the detail of the microscopic theory once that a marginal perturbation is included and the phase transition is therefore not universal.

If we now turn our attention back to the free energy F and its perturbations, we have at first order:

$$\int \mathcal{D}\varphi\ (1-\delta H)\, e^{-H_o} = e^{-F_o}(1-\delta F)\,. \tag{2.54}$$

The previous expression can be solved for δF:

$$\delta F = \frac{1}{Z_o}\langle\delta H\rangle\,. \tag{2.55}$$

Since every possible scaling due to the Boltzmann weight in the numerator of the last equality is compensated by the same scaling in the denominator, equation (2.55) shows that the classification of corrections to the microscopical Hamiltonian in terms of relevant, irrelevant and marginal fluctuations reflects itself directly on the level of perturbations of the free energy.

2.4.1. High temperature expansion

We conclude this section with a brief presentation of an approximate method for the computation of critical exponent, namely the high temperature expansion [121]. Suppose, for example, that we want to compute the critical exponent γ for a given model. Imaging to approach the critical point from the symmetric phase (i.e. higher temperatures) we can formally expand the susceptibility around $T = \infty$:

$$\chi = T \sum_{n=0}^{\infty} \frac{a_n}{T^n} \, . \tag{2.56}$$

At the phase transition χ will be singular implying that the critical temperature will be given by the (inverse) radius of convergence of the series:

$$T_c = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \, . \tag{2.57}$$

The character of the singularity can, hence, be determined by studying the asymptotic behavior of the coefficients a_n. Using the ansatz (2.42) then one has:

$$a_n \sim \frac{\Gamma(n+\gamma)}{\Gamma(n+1)\Gamma(\gamma)} T_c^n \, , \tag{2.58}$$

where Γ represents the Euler gamma function. For $n >> 1$ and using the properties of Γ we get to

$$\frac{a_{n+1}}{a_n} = \left(1 + \frac{\gamma}{n}\right) T_c \, . \tag{2.59}$$

The last equation shows that the asymptotics of the coefficients for the high temperature expansion provides both the critical temperature and the critical exponent. Even though we just presented only the computation of γ a similar procedure can be employed for the other exponents as well, provided that we expand the corresponding thermodynamical quantities.

The high temperature expansion is definitely not the best theoretical tool since the coefficients of the expansion need to be computed case by case and the whole universality character is hidden (as opposed to the renormalization approach), nevertheless it makes explicit the connection between the high energy physics and the critical phenomena taking place at lower scales. Moreover, having in mind the connection between classical thermodynamics and quantum field theory pointed out in appendix C, the high temperature expansion is connected to the perturbative expansion of the path integral in powers of β (or $\hbar$ in the Minkowskian formulation). The requirement that the coefficients are finite is therefore related to study the resummation of the loop expansion of the correlators of the theory and their renormalization in the perturbative approach.

3. Renormalization approach to critical phenomena

The observations pointed out up to now suggest a few characteristics that will be at the core of renormalization. The first aspect to take into account is that the long-range dynamics of fluctuations is responsible for critical phenomena and phase transitions. This matches the interpretation of the transition as a sudden change of the path integral properties from its stable form at the microscopical level, where the energies involved are very high and we are in the ultraviolet (UV) regime, to a new configuration taking over in the infrared (IR) regime, where the energies involved are low and the wavelengths of relevant fluctuations are large enough to correlate points far apart within the system giving rise to collective excitations. In this perspective it is clear how, especially for a second order phase transition, the key ingredient governing the transition is the correlation length ξ. At the critical point ξ diverges and correlated clusters of all sizes appear. In this way, zooming in and out the system (as long as we do not get to the atomic scales where the degrees of freedom are still of the microscopical nature) one would see the same picture. This is the way the system exhibits scale invariance from a certain finite scale on.

In synergy with these considerations we have the concept of universality. The fact that different microscopical theories show the same critical behavior at lower energies depending only on few parameters implies a loss of information about the fundamental degrees of freedom. The more we look at the system in its infrared physics the harder it is to infer the fundamental description. In a way, we can think about the system as an impressionist painting where the color strokes have a very specific shape (e.g. points, stars, polygons...). If we observe from very near the canvas we can see different shapes for different paintings. As we make a few step back, the shape of the stroke becomes more and more fuzzy, and we start to recognize the forms and subjects composing the painting. If two paintings represent the same scene (we could say that the colors have the same distribution) they would already appear the same, otherwise (supposing the art pieces are very large, say infinite) we can get even farther until we totally lose the notion of the intermediate figures too and the whole canvas appears as a confused abstract color layer. At this level the relative abundance of each color is the only information we need in order to predict what the painting will look like. If two canvas are made with strokes that can be either red or yellow with some probability (the same for both the paintings), when watched from very far they will both appear of the same shade of orange even though strokes and intermediate shapes can be very different. In comparison with a thermodynamical system, the role of the temperature[16] is to change the probability distribution between red and yellow. The idea of loosing information about the fundamental degrees of freedom once that we look at the large scale physics of the system takes the name of *coarse graining* and is formally implemented via the *renormalization group*.

In the following sections we will describe the main formulations of the renormalization group. We will present the topic employing the momentum space representation of the fields since this will allow for a straightforward understanding in terms of their Fourier modes. Of course this description relies on translation invariance and therefore fails to be valid once that gravity is included. Nevertheless, once that the interpretation of the whole machinery is clear we will be able to rephrase it in direct space and adapt it to the case of a curved background manifold.

For each of the approaches we are about to examine there is an extensive literature that the reader may refer to. The discussion about Wilsonian group follows the presentation of [121], other useful readings can be found in [5, 32–34, 38, 39]. For what concerns the functional renormalization group excellent reviews are [42–51]. Perturbative renormalization is of course standard textbook material. Interesting treatments can be found in [5, 13, 34, 121, 125].

3.1. Non-perturbative approach: Wilsonian formulation

The formulation of the renormalization group given by Wilson [32, 33] implements coarse graining by integrating the fast modes (with high momenta) in the partition function obtaining a theory that only depends on the slow (low momenta) part of the spectrum. We will refer to this procedure as *integrating out* the fast momenta since the theory will not explicitly depend on them anymore. Since we want to compare the strength of correlation functions for different modes a rescale of the fields and their momenta will be necessary. This will lead to a transformation of the Hamiltonian that encodes the relevance of the Fourier components and allows comparing them[17]. Since the plan of this thesis is to study the renormalization of quantum field theories we will now switch to the field theoretical language and describe the theory in terms of an action S instead of the Hamiltonian, where from the context it will be clear whether we refer to the Euclidean formulation or the Minkowskian one.

We will suppose to have a coarse grained field $\phi(x)$ for which the Fourier representation

[16] In a quantum system this would be the role of the energy of the interaction.

[17] Notice that what is important is not the value of the momentum itself but rather the dependence of the theory on the distribution of momenta.

reads:

$$\phi(x) = \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}\, \phi(\mathbf{q}) e^{i\mathbf{q}\mathbf{x}}\,, \tag{3.1}$$

where the Fourier modes $\phi(\mathbf{q})$ are null for momenta with norm lower than some value q_0. Starting from a field which is already coarse grained may have different interpretations: from a lattice point of view this would be the only notion of field available, from an effective field theory perspective this could represent an optimization of the description of the theory for some specific energy scale of phenomenon. In any case we should be interested in investigating the ultraviolet completion of the theory.

In order to study the dependence of the partition function on the parameter q_0 we write the probability distribution for the mode $\phi(q)$ as:

$$\rho = A_0 \exp\left[-S(\phi, q_0)\right]\,, \tag{3.2}$$

for some normalization constant A_0. It is possible to define a parameter k such that $0 < k < 1$ and formally integrate the distribution (3.2) for those momenta of norm between $k\, q_0$ and q_0:

$$\rho_k = A_k \exp\left[-S(\phi, k\, q_0)\right] = A_0 \int \prod_{q=k\, q_0}^{q_0} \mathrm{d}\phi(\mathbf{q})\ \exp\left[-S(\phi, q_0)\right]\,. \tag{3.3}$$

If the action is known, the transformation allows computing $H(\phi, k\, q_0)$ for any value of the parameter k. The transformation we just described is exactly a coarse graining transformation, and we will indicate it as $G(k)$:

$$S(\phi, k\, q) = G(k) S(\phi, q)\,. \tag{3.4}$$

The repeated application of coarse graining is endowed with a semigroup structure:

$$G(k_1)G(k_2) = G(k_1 k_2)\,, \tag{3.5}$$

but is lacking a well defined inverse transformation.

We act on top of the coarse graining with a scale transformation of the $\mathbf{q}$ variables and the fields:

$$\begin{aligned} \mathbf{q} &\to \mathbf{q}' = k^{-1}\mathbf{q}\,, \\ \phi(\mathbf{q}) &\to \phi'(k^{-1}\mathbf{q}) = Z(k)^{-1}\phi(\mathbf{q})\,. \end{aligned} \tag{3.6}$$

If we represent the set of equations (3.6) as $D(k)$ we have that the action transforms as follows:

$$D(k)S(\phi(\mathbf{q}), q) = S(Z(k)\phi'(k^{-1}\mathbf{q}), k^{-1}q) \equiv S'(\phi'(\mathbf{q}), k^{-1}q)\,. \tag{3.7}$$

We call the consecutive application of $G(k)$ and $D(k)$ a renormalization operation:

$$R(k) = D(k)G(k)\,, \tag{3.8}$$

for which the action is transformed according to:

$$R(k)S(\phi(\mathbf{q}, q_0) = S'(\phi'(\mathbf{q}), q_0)\,. \tag{3.9}$$

The first transformation $G(k)$ reduces the portion of the momentum space where the Fourier components of the field are active, while the second transformation $D(k)$ resizes the domain to the original dimension modifying the field accordingly and leading to a new action that can be compared with the previous one. The set of transformations $R(k)$ forms a semigroup as well but the established habit is to refer to it as the *renormalization group*.

Let us try to understand how the Wilsonian renormalization group acts on the action by studying a Gaussian initial distribution for the fluctuating field. Even though this case might seem very naive it represents the distribution for a system far from criticality where the thermodynamical regime is dominant. We thus consider an action of the form:

$$S(\phi, q_0) = \frac{1}{2}\int_{q<q_0} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}\,(m + cq^2)|\phi(\mathbf{q})|^2\,, \tag{3.10}$$

c being the speed of propagation of the fluctuations of the system. Since there is no interaction between terms with different momenta the effect of the coarse graining transformation is just to reduce the integration domain:

$$G(k)S(\phi, q_0) = S(\phi, q_0) = \frac{1}{2}\int_{q<k\,q_0} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}\,(m + cq^2)|\phi(\mathbf{q})|^2\,. \tag{3.11}$$

Applying a scale transformation we suddenly obtain a redefinition of the coupling constant of the theory:

$$R(k)S(\phi, q_0) = D(k)G(k)S(\phi, q_0) = \frac{k^d Z^2}{2}\int_{q<q_0} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}\,(m + ck^2q^2)|\phi(\mathbf{q})|^2\,. \tag{3.12}$$

Thanks to the semigroup property of the renormalization transformations we can study the limiting action simply taking the limit $k \to 0$. For $m \neq 0$ we obtain a finite limit if

$Z = k^{-\frac{d}{2}}$:

$$S^* = \frac{1}{2} \int_{q<q_0} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \, m|\phi(\mathbf{q})|^2 \,, \tag{3.13}$$

while for $m = 0$ one finds the condition $Z = k^{-\frac{d+2}{2}}$.

If we are looking at the physics in a neighborhood of the critical region fluctuations do not exhibit a thermodynamical behavior and the previous analysis becomes much more complicated. It is useful to study the effect of an infinitesimal renormalization transformation. Since all the effect of the renormalization group acts on the action through the parameter k we will use a lighter notation and write $S = S(k)$. If we assume that we already computed the flow of the action from the initial condition $S(1)$ all the way down to $S(k)$ and we want to perform a further infinitesimal renormalization step we have:

$$S(k - \delta k) = S(k(1 - \frac{\delta k}{k})) = R(1 - \frac{\delta k}{k})S(k) \,. \tag{3.14}$$

Expanding the previous equation at first order leads directly:

$$\frac{\mathrm{d}R(k)}{\mathrm{d}k}\Big|_{k=1} S(k) = k\frac{\partial S(k)}{\partial k} = \frac{\partial S(k)}{\partial \log k} \,. \tag{3.15}$$

The left hand side of the previous equation represents a functional of the action and thus depends on k only through the implicit form of the action. We can thus define the following quantities:

$$\beta[S] = \frac{\mathrm{d}R(k)}{\mathrm{d}k}\Big|_{k=1} S \,. \tag{3.16}$$

such that equation (3.15) can now be written as:

$$\frac{\partial S}{\partial t} = \beta[S] \,, t = \log k \,. \tag{3.17}$$

Even though equation (3.17) looks simple it is actually a system of non-linear differential equations for the constants defining the action S. A fixed point of the flow of the theory can now be described as a root of (3.17):

$$\beta[S^*] = 0 \,. \tag{3.18}$$

We leave for later the discussion of the properties of solutions of the flow in proximity to the fixed point.

3.2. Non-perturbative approach: Functional Renormalization Group

Another successful non-perturbative formulation of the renormalization method is given by the Functional Renormalization Group (FRG) [40–51]. As is clear by the name of this approach, FRG relies heavily on functional method and works out the renormalization group at the level of the path integral instead of the microscopical action.

While in the Wilsonian approach we enforce a loss of information about the microscopic degrees of freedom creating a theory for the slow modes, in FRG we adopt the opposite point of view. In fact, in this case, one constructs a truncation that is supposed to be a good description for the physics at some very high energy scale (defined by a ultraviolet cut-off Λ) and freezes the propagation of slow modes up to some infrared cutoff k. Letting the infrared cutoff flow to zero we have an inclusion of less energetic modes that should reflect the behavior of the theory at larger scales. The relevant modes will turn out to control the flow of the theory while the inclusion of the irrelevant ones will affect the phase diagram quantitatively but not qualitatively.

The idea behind this approach is that the fundamental physics is determined solely by the microscopical degrees of freedom while the effective theories taking over at lower energies are just the result of how modes combine themselves once that longer wavelengths are included. The infrared physics, thus, arises dynamically rather than as a loss of information.

In order to clarify the implementation of this method let us consider a path integral for a scalar field theory defined by the action $S[\phi]$:

$$Z[J] = \int \mathcal{D}\phi \; e^{-S[\phi]+\int \mathrm{d}x \; J(x)\phi(x)} \,, \tag{3.19}$$

where $J(x)$ represents a classical source for the scalar field and we assume that the functional measure entails an ultraviolet cutoff at some scale Λ. In order to decouple the slow modes from the long range dynamics we assign them a very big mass via the insertion of a infrared cutoff function of the form:

$$\Delta S_k[\phi] = \frac{1}{2}\int \mathrm{d}x\mathrm{d}y \; \phi(x) R_k(x-y)\phi(y) \,. \tag{3.20}$$

The regularized path integral will thus depend on the scale k and will be of the form:

$$Z_k[J] = e^{W_k[J]} = \int \mathcal{D}\phi \; e^{-S[\phi]-\Delta S_k[\phi]+\int \mathrm{d}x \; J(x)\phi(x)} \,. \tag{3.21}$$

The momentum space representation of the regulator function $R_k(q)$ will have to satisfy the following requirements:

1. when the infrared cutoff is removed ($k = 0$) the path integral should coincide with the original one:

$$Z_{k=0}[J] = Z[J] \quad \Leftrightarrow \quad R_{k=0}(q) = 0\,, \tag{3.22}$$

2. when $k = \Lambda$ all the slow modes should be decoupled, meaning that the regulator has to diverge for all momenta:

$$R_{k=\Lambda}(q) = \infty \quad \forall q\,. \tag{3.23}$$

   Of course, in the case where the ultraviolet cutoff can be removed, the divergence of the regulator will take place in the limit $k \to \infty$. An approximate method to freeze all the modes at the scale Λ is to choose a regulator of order Λ^2 for all momenta,

3. when $0 < k < \Lambda$ the fast modes should remain unaffected by the regulator:

$$R_k(|q| > k) \sim 0\,. \tag{3.24}$$

In order to properly take the regulator into account when considering the effective action, we will have to modify the Legendre transform we use in the appendix A and define:

$$\Gamma_k[\varphi] = \int \mathrm{d}x\; J(x)\varphi(x) - W_k[J] - \Delta S_k[\varphi]\,, \tag{3.25}$$

where φ is the expectation value of the field ϕ evaluated through the regulated path integral:

$$\varphi(x) = \frac{\delta W_k}{\delta J(x)}\bigg|_{J=0} = \frac{\langle\phi\rangle_k}{\langle 1\rangle_k}\,. \tag{3.26}$$

We call the functional Γ_k *effective average action*.

It will be useful to workout the relations between the regulated functionals. The first variation of Γ_k will have a correction due to the regulator function:

$$\frac{\delta\Gamma_k}{\delta\varphi(x)} = J(x) - \int \mathrm{d}y\; R_k(x-y)\varphi(y) \tag{3.27}$$

$$= J(x) - \frac{\delta\Delta S_k[\phi]}{\delta\phi(x)}\bigg|_{\phi=\varphi}\,. \tag{3.28}$$

This result can be employed to investigate the relation between the second variations of

W_k and Γ_k:

$$\begin{aligned}\delta(x-y) =& \frac{\delta\varphi(x)}{\delta\varphi(y)} = \int \mathrm{d}z \, \frac{\delta^2 W_k}{\delta J(x) J(z)} \frac{\delta J(z)}{\delta\varphi(y)} \\ =& \int \mathrm{d}z \, \frac{\delta^2 W_k}{\delta J(x) J(z)} \left[\frac{\delta^2 \Gamma_k}{\delta\varphi(z)\varphi(y)} + R_k(z-y) \right] ,\end{aligned} \tag{3.29}$$

showing that

$$W_k^{(2)} = \left[\Gamma_k^{(2)} + R_k \right]^{-1} . \tag{3.30}$$

Let us turn to study the exponential of the effective average action:

$$\begin{aligned}e^{-\Gamma_k[\varphi]} =& e^{W_k[J] - \int J\varphi + \Delta S_k[\varphi]} \\ =& \int \mathcal{D}\phi \, \exp\left\{ -S[\phi] - \Delta S_k[\phi] + \int \mathrm{d}x \, J(x)[\phi(x) - \varphi(x)] + \Delta S_k[\varphi] \right\} .\end{aligned} \tag{3.31}$$

We can define the fluctuation field $\chi(x) = \phi(x) - \varphi(x)$ for which we have $\langle \chi(x) \rangle_k = 0$ by construction. Changing variable from from ϕ to χ does not affect the functional measure since it is just a translation in field space. Expanding the regulator term and keeping in mind that $\Delta S_k[\phi]$ is quadratic in the fields we find

$$\begin{aligned}&\Delta S_k[\varphi + \chi] - \Delta S_k[\varphi] \\ &= \Delta S_k[\varphi] + \int \mathrm{d}x \, \frac{\delta \Delta S_k[\phi]}{\delta\phi(x)}\Big|_{\phi=\varphi} \chi(x) + \int \mathrm{d}x\mathrm{d}y \, \chi(x) \frac{\delta^2 \Delta S_k[\phi]}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=\varphi} \chi(y) - \Delta S_k[\varphi] \\ &= \int \mathrm{d}x \, \frac{\delta \Delta S_k[\phi]}{\delta\phi(x)}\Big|_{\phi=\varphi} \chi(x) + \Delta S_k[\chi] .\end{aligned} \tag{3.32}$$

Inserting the last expansion into (3.31) we can rearrange the path integral as

$$\begin{aligned}e^{-\Gamma_k[\varphi]} =& \int \mathcal{D}\chi \, \exp\left\{ -S[\varphi + \chi] - \Delta S_k[\chi] + \int \mathrm{d}x \, \left[J(x) - \frac{\delta \Delta S_k[\phi]}{\delta\phi(x)}\Big|_{\phi=\varphi} \right] \chi(x) \right\} \\ =& \int \mathcal{D}\chi \, \exp\left\{ -S[\varphi + \chi] - \Delta S_k[\chi] + \int \mathrm{d}x \, \frac{\delta\Gamma_k[\varphi]}{\delta\varphi(x)} \chi(x) \right\} .\end{aligned} \tag{3.33}$$

We are now able to extract a flow equation for the effective average action simply taking the derivative with respect to the parameter k on both sides of the last equation:

$$k\partial_k \Gamma_k = e^{\Gamma_k[\varphi]} \int \mathcal{D}\chi \, k\partial_k \left(\Delta S_k[\chi] - \int \mathrm{d}x \, \frac{\delta\Gamma_k[\varphi]}{\delta\varphi(x)} \chi(x) \right) e^{-S[\varphi+\chi] - \Delta S_k[\chi] + \int \frac{\delta\Gamma_k}{\delta\varphi}\chi} . \tag{3.34}$$

It should be obvious that averages computed using the distribution (3.33) are proportional to the regulated expectation values since they differ only by a constant, namely $\exp\{\int J\varphi\}$, that can be absorbed in the path integral normalization and that is eliminated upon inclusion of the normalization $\exp\{\Gamma_k\}$. Hence, equation (3.34) can be rewritten as:

$$k\partial_k\Gamma_k = \frac{\langle k\partial_k \Delta S_k[\chi]\rangle_k - \langle k\partial_k \int \mathrm{d}x \, \frac{\delta\Gamma_k[\varphi]}{\delta\varphi(x)}\chi(x)\rangle_k}{\langle 1\rangle_k} \, . \tag{3.35}$$

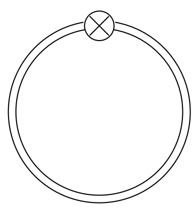
The second term on the right hand side is proportional to $\langle\chi(x)\rangle_k$ and can thus be set to zero. The first term yelds:

$$\begin{aligned}
\frac{\langle k\partial_k \Delta S_k[\chi]\rangle_k}{\langle 1\rangle_k} =& \frac{1}{2}\int \mathrm{d}x\mathrm{d}y \; k\partial_k R_k(x-y)\frac{\langle\chi(x)\chi(y)\rangle_k}{\langle 1\rangle_k} \\
=& \frac{1}{2}\int \mathrm{d}x\mathrm{d}y \; k\partial_k R_k(x-y)\left[\frac{\langle\phi(x)\phi(y)\rangle_k}{\langle 1\rangle_k} - \frac{\langle\phi\rangle_k^2}{\langle 1\rangle_k^2}\right] \\
=& \frac{1}{2}\int \mathrm{d}x\mathrm{d}y \; k\partial_k R_k(x-y)\frac{\delta^2 W_k}{\delta J(x)\delta J(y)} \\
=& \frac{1}{2}\int \mathrm{d}x\mathrm{d}y \; k\partial_k R_k(x-y)\left[\Gamma^{(2)} + R_k\right]^{-1}(x,y) \, .
\end{aligned} \tag{3.36}$$

Collecting all the results we can write the flow equation for the effective average action as:

$$k\partial_k\Gamma_k = \frac{1}{2}\mathrm{STr}\left[k\partial_k R_k\left(\Gamma_k^{(2)} + R_k\right)^{-1}\right] , \tag{3.37}$$

where the super trace STr means that the trace over fermion sector is computed with en extra minus due to the fermionic loop. Equation (3.37) is known as Wetterich equation [40] and is an equation for the exact renormalization group. The term within the round brackets represents the kinetic term of the full regulated theory and hence its inverse coincides with the full propagator. Since we trace the propagator against an insertion of the regulator function the Wetterich equation is a one-loop equation. Nevertheless, the one-loop involved in (3.37) does not consists in an approximation since no expansion of the path integral was done in order to obtain the result. It is therefore possible to give a diagrammatic representation of equation (3.37) as follows:

$k\partial_k\Gamma_k = \frac{1}{2}$  (3.38)

where with a double line we indicate the full propagator of the theory (as opposed to the one appearing in the perturbative expansion of the path integral) and with crossed circle we refer to the insertion of the derivative of the regulator.

Equation (3.37) does not rely on any truncation of the effective average action and, thus, represents a differential equation for a functional containing all possible operators that are compatible with the symmetries of the theory. The number of these operators is usually infinite and in order to do real computations we need to force a projection of Γ_k onto a truncation. Is therefor important to identify the marginal and relevant operators of the theory and to systematically add on top of them irrelevant deformations.

Moreover, the approach suffers from a scheme dependence inherited by the fact that the regulator function is included by hand. Nevertheless this last ambiguity is removed once the cutoff k is removed. As we will clarify in the next session most of the interesting physics is encoded in the fixed point structure of the renormalization flow. Here the cutoff can be safely removed without encountering any divergence and therefore any result depending on the existence of a fixed point and on the flow in a neighborhood of it would maintain its universal feature.

3.3. Perturbation theory and fixed point analysis

So far we have been mostly dealing with infrared physics, nevertheless the renormalization picture offers us a broader perspective on where the ultraviolet completion an the infrared phase structure of a model are just two faces of the same coin. The connection between the high energy regime and critical phenomena was already clarified at the end of section 2.4 where we saw that the critical exponents, as well as the critical temperature, can be found with an expansion of thermodynamical observables around $T = \infty$. From a quantum field theory perspective this method corresponds to compute correlators from a perturbative expansion of the path integral. Nevertheless, once that the computation is pushed beyond tree level, correlation functions tend to show divergences in the UV spectrum. This raises the question of whether a UV completion of the theory exists or not. Formally we tackle the problem by studying the renormalizability of the theory in terms of our ability to remove the divergences of the correlators.

From the point of view of the renormalization group the UV completion of a model is, again, formulated in terms of the fixed point structure of the renormalization flow. In figure 3.1 we have an example of a renormalization flow for a theory parametrized by two operators. The fixed point I has one relevant direction (depicted in red) and one irrelevant direction (in blue) which represents two different operators of the theory. We can think about these two operators as a basis for our theory space[18]. It is straightforward to notice how a tuning of the initial value of the red operator (for example between the three possible initial conditions S_1, S_2, and S_3) leads to two different infrared behaviors

[18]At least in a neighborhood of the point I

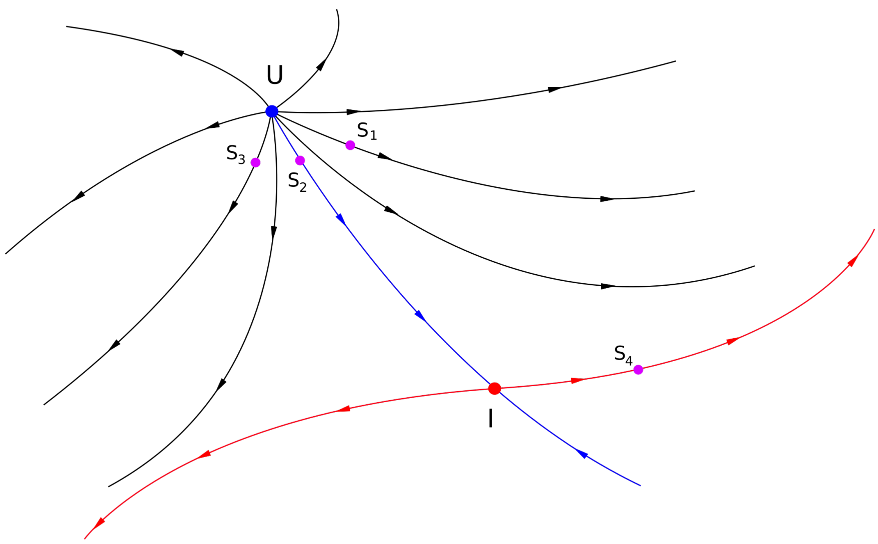


Fig. 3.1.: Example of the renormalization flow of a theory, arrows point towards the infrared. The fixed point U represents a ultraviolet sink for the flow while the fixed point I can be a infrared or ultraviolet depending on the initial conditions of the flow. The blue line represents an irrelevant coupling and identifies the critical surface of the phase diagram, while the red line is a relevant coupling and, thus, controls the infrared behavior of the model.

and distinguishes between two possible phases. The predictivity of a theory lies in the presence of fixed points with a finite number of relevant directions since tuning them determines the low energy physics. The blue trajectory separates the two possible infrared outcomes and hits the fixed point I. This feature characterizes it as the critical surface of the flow and scale invariance for the infrared regime of the model S_2 is manifest at the fixed point I. It is important to realize that RG transformations are not physical: physics always happens at the UV and IR scales at the same time and the renormalization group only parametrizes the regime we are looking at. A physical transformation on the system (for the Ising model this could represent a tuning of the temperature and the external magnetic field) would cause a change of the initial conditions and is typically orthogonal to the critical surface.

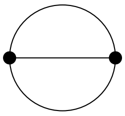
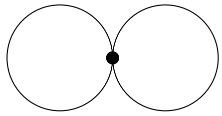
The fixed point U is a UV-sink for the flow in the upper half plane of the theory space. This implies that in all the models with initial conditions on a trajectory connected to U the ultraviolet cutoff can be safely removed. In the present case, the point U lies on the critical surface and therefore represents a possible UV completion for both the phases. In a different case we would expect a further phase transition for the divergent part of the flow.

Finally, it is important to notice how the fixed point I might as well be regarded as an ultraviolet completion for a theory with initial condition id S_4. In this very special case the theory would exhibit only one possible phase in its infrared regime and, in order to be provided, requires a tuning of all the parameters of the model (relevant and irrelevant).

It is now clear how a perturbative expansion of the path integral can generate divergent terms that need to be renormalized and how the renormalizability of the theory depends on the presence of an appropriate fixed point. Let us consider a perturbative expansion of the effective action. The functional $\Gamma[\phi]$ can be expanded in powers $\hbar$ (which we briefly reinsert in order to keep track of the order of expansion) corresponding to the number of loops in vacuum diagrams [5, 125]:

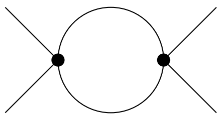
$$\Gamma[\varphi] = S[\varphi] + \sum_{L=1}^{\infty} \hbar^L \Gamma_L[\varphi]\,, \tag{3.39}$$

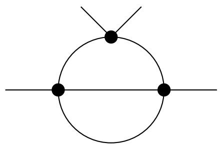
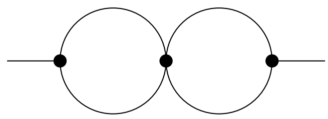
$$\Gamma_1[\phi] = \frac{1}{2}\mathrm{Tr}\log S^{(2)}[\phi]\,, \tag{3.40}$$

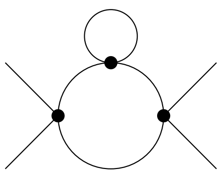
$\Gamma_2[\varphi] = -\frac{1}{12}$  $+\frac{1}{8}$  (3.41)

As pointed out in appendix C the functional derivatives of Γ generate 1PI Feynman graphs and, in this respect, $\frac{\delta^m}{\delta\varphi^m}\Gamma_n[\varphi]$ generates n-loop contributions to the strongly connected m-point functions[19]. For example, in the case of the φ^4-theory with the truncation (2.34)[20], the fourth variation of the effective action $\Gamma^{(4)}$ includes all the quantum fluctuations corrections to the quartic interaction. Diagrammatically we have:

$$\Gamma_0^{(4)}[\varphi] = S^{(4)}[\varphi] = \bar{\lambda}. \tag{3.42}$$

$\Gamma_1^{(4)}[\varphi] = \frac{3}{2}$  (3.43)

$\Gamma_2[\varphi]^{(4)} = 3$  $+\frac{3}{4}$ 

$+\frac{3}{2}$  (3.44)

where factor of 3 takes into account the three possible channels for a 4-point function. As we will shortly see, the loop contributions lead to divergences. This is due to the fact that the theory is propagating incorrect degrees of freedom (bare fields and couplings) which we will have to renormalize. The program of perturbative renormalization mainly goes through three steps [5, 125]:

1. Compute regularized amplitudes:

[19] Of course, since this procedure acts through the insertion of vertices of the theory, the generated graphs will have to be compatible with the truncation of the theory.

[20] Of course upon the identification $\Phi = \varphi$.

in this step we identify and isolate the divergent part of diagrams appearing in the loop expansion of $\Gamma^{(n)}$. This can be achieved either via the insertion of a cutoff or with dimensional regularization (we will focus on the last method).

2. Give a prescription that relates bare quantities to renormalized ones:
since we are dealing with singular quantities we will have to face an evident ambiguity concerning the finite parts of our computations which encode the definition of physical quantities. The freedom we have in defining the mapping between physical and bare quantities will be referred to as scheme dependence of the renormalization group.

3. Rewrite the amplitudes in terms of the new couplings:
this last step will provide amplitudes that are finite by construction and complete the renormalization process.

In order to clarify these steps let us have a closer look to the one loop renormalization of the self interaction of a φ^4-theory. Relying once again on the momentum space representation and calling the diagram in equation (3.43) $\mathcal{A}(q)$ where q is the total incoming momentum, we have:

$$\mathcal{A}(q) = \bar{\lambda}^2 \int \frac{\mathrm{d}^d p}{(2\pi)^{(d)}} \frac{1}{p^2+m^2} \frac{1}{(p+q)^2+m^2}\,, \tag{3.45}$$

where p is the loop momentum. The diagram is convergent for $d<4$, hence, we regularize it by analytically extend the dimensionality of the system slightly below the upper critical dimension. We therefore insert a new parameter ε and shift the computation to $d=4-\varepsilon$ [5, 13, 121, 125]. To keep track of the dimension of the couplings we also add an arbitrary mass parameter μ and define the dimensionless coupling λ as:

$$\lambda = \bar{\lambda}\mu^{-\varepsilon}\,. \tag{3.46}$$

Employing a generalization of Feynman parametrization for the integral in (3.45) [5, 125]:

$$\begin{aligned}\int \frac{\mathrm{d}^d p}{(2\pi)^{(d)}} \frac{1}{(p^2+m^2)^a} \frac{1}{[(p+q)^2+m^2]^b} &= \\ = \frac{\Gamma(a+b-\frac{d}{2})}{(4\pi)^{d/2}\Gamma(a)\Gamma(b)} \int_0^1 \mathrm{d}z\; z^{b-1}(1-z)^{a-1}[q^2 z(1-z)+m^2]^{\frac{d}{2}-a-b}\,,\end{aligned} \tag{3.47}$$

one can rewrite the $\mathcal{A}(q)$ as:

$$\begin{aligned}\mathcal{A}(q) =&\bar{\lambda}^2 \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}z \; \frac{1}{[q^2 z(1-z)+m^2]^{2-d/2}} \\ =&\lambda \mu^{\varepsilon} \frac{\Gamma\left(\frac{\varepsilon}{2}\right)}{(4\pi)^2} \int_0^1 \mathrm{d}z \; \left[\frac{4\pi\mu^2}{q^2 z(1-z)+m^2}\right]^{\frac{\varepsilon}{2}} .\end{aligned} \tag{3.48}$$

The divergence for $d = 4$ is evident from the pole of the Euler gamma function as ε is sent to zero, while the integral is finite for any value of d provided that the mass is not vanishing. Expanding the result in powers of ε we find:

$$\mathcal{A}(q) = \frac{\lambda\mu^{\varepsilon}}{(4\pi)^2}\left[\frac{2}{\varepsilon} - \gamma + \int_0^1 \mathrm{d}z \; \log\left(\frac{4\pi\mu^2}{q^2 z(1-z)+m^2}\right) + o(\varepsilon)\right] , \tag{3.49}$$

where γ is the Euler-Mascheroni constant. At this stage we completed the first step in perturbative renormalization. The divergence of the one loop correction to the coupling $\bar{\lambda}$ was identified and isolated in the form of a pole in the ε expansion in dimensional regularization.

In the last result the mass parameter μ only affects the finite part of the diagram (in the limit $\varepsilon \to 0$) showing the freedom in the renormalization procedure. We can in general define the form factor:

$$f(q^2, m, \mu) = \int_0^1 \mathrm{d}z \; \log\left(\frac{4\pi\mu^2}{q^2 z(1-z)+m^2}\right) . \tag{3.50}$$

Taking into account the three possible channels available for a $2 \to 2$ scattering (all of which are described at one loop by the same diagram we just evaluated) and the coefficient in (3.43) we find:

$$\begin{aligned}\Gamma^{(4)}(q_i) =&S^{(4)} + \Gamma^{(4)}_1 + \text{ higher loops} \\ =&\mu^{\varepsilon}\lambda + \frac{3\mu^{\varepsilon}\lambda^2}{(4\pi)^2\varepsilon} + \frac{\mu^{\varepsilon}\lambda^2}{2(4\pi)^2}\left[f(s,m,\mu)+f(t,m,\mu)+f(u,m,\mu)-3\gamma\right] + \text{h.l.} ,\end{aligned} \tag{3.51}$$

where the index i in the argument of $\Gamma^{(4)}$ runs over the external legs of the diagram: $i = 1, \dots, 4$.

Since we have an ambiguity in the finite part of the amplitude we need to give a prescription to relate bare and renormalized functions. As stated before this is tantamount to defining the physical coupling. In the case we are looking at, we could choose to define

$\bar{\lambda}_{\text{phys}}$ (at one loop) via a scheme that is independent on the incoming momentum:

$$\bar{\lambda}_{\text{phys}} = \Gamma^{(4)}(q_i = 0) = \lambda\mu^{\varepsilon}\left\{1 + \frac{3\bar{\lambda}}{2(4\pi)^2}\left[\frac{2}{\varepsilon} - \gamma + f(0, m, \mu)\right]\right\}. \tag{3.52}$$

Having a brief look at the action for the φ^4-theory we expect that at least three quantities need renormalization: the coupling λ, the mass m and the field φ. With a procedure similar to the one described until here we can study the renormalization of the kinetic term from the two point function $\Gamma^{(2)}$. The momentum independent renormalization conditions for the theory can be written as:

$$\Gamma^{(2)}(q = 0) = \bar{m}_{\text{phys}}\,, \qquad \left.\frac{\partial\Gamma^{(2)}(q)}{\partial q^2}\right|_{q=0} = 1\,, \qquad \Gamma^{(4)}(q_i = 0) = \bar{\lambda}_{\text{phys}}\,, \tag{3.53}$$

and are known as Coleman-Weinberg renormalization conditions [126].

Even though this choice looks very natural from a mathematical perspective it does not represent a very physical scheme because of the pathological choice of zero momentum. One might thus be led to choose a different prescription which relies on a more physical configuration. Typically this is performed according to the initial setup of an experiment which can provide an experimental value for the coupling at some energy in some configuration.

Finally we need to rewrite the amplitudes in terms of the renormalized couplings. This can be done by formally inverting equation (3.52) and inserting $\lambda = \lambda(\bar{\lambda}_{\text{phys}})$ in (3.51). The resulting 4-point function will be one-loop finite by construction. To the order $\bar{\lambda}^2_{\text{phys}}$ we have:

$$\Gamma^{(4)}(q_i) = \bar{\lambda}_{\text{phys}} + \mu^{-\varepsilon}\frac{\bar{\lambda}^2_{\text{phys}}}{2(4\pi)^2}\left[f(s, m, \mu) + f(t, m, \mu) + f(u, m, \mu) - 3f(0, m, \mu)\right]\,, \tag{3.54}$$

for which is straightforward to verify that $\Gamma^{(4)}(0) = \bar{\lambda}_{\text{phys}}$.

The philosophy behind the procedure described so far is that the original couplings are actually divergent and renormalization gets rid of unwanted singularities by subtracting the singular part of the couplings and redefining the correlators in terms of the renormalized parameters of the theory. The couplings of the theory exhibit now a dependence on the arbitrary mass parameter μ which can be identified with the energy scale of the observed phenomena. Hence, the derivative with respect to this parameter encodes the evolution of the theory with respect to the scale at which we observe it. Having fixed the definition of $\bar{\lambda}_{\text{phys}}$ to a given scale, this will serve as an initial condition for the renormalizatioin flow. Upon inversion of (3.52) we find (to lowest order in λ):

$$\beta_{\lambda} = \frac{3\lambda^2}{16\pi^2}\,. \tag{3.55}$$

It is natural to wonder whether it is possible to redefine the parameters directly at the level of the action. We are thus wondering whether there exist an renormalized action $S_R[\varphi_R]$ and a renormalized functional $\Gamma_R[\varphi_R]$, such that the n-point function generated by Γ_R are finite by construction. We therefore introduce a *counterterm* action $S_{c.t.}$ and postulate that it satisfies [5, 125]:

$$S[\varphi] = S_R[\varphi_R] + S_{c.t.}[\varphi_R]\,. \tag{3.56}$$

In order for this approach to result in a properly defined action formulation of our quantum field theory in terms of renormalized objects we need to require that $S_R[\varphi_R]$ and $S_{c.t.}[\varphi_R]$ have the same functional form. This parametrization implicitly defines the counterterms to coincide with the divergent part of the n-point function up to scheme dependence. In the case of the scalar theory we are looking at we write:

$$S_{c.t.}[\varphi_R] = \int \mathrm{d}x\ \left[\frac{1}{2}(Z_\varphi - 1)(\partial\varphi_R)^2 + \frac{1}{2}(Z_m - 1)m_R^2\varphi_R^2 + \frac{\bar{\lambda}_R}{4!}(Z_\lambda - 1)\varphi_R^4\right]\,. \tag{3.57}$$

We thus have the following mapping between renormalized and bare quantities:

$$\varphi = \sqrt{Z_\varphi}\varphi_R\,, \qquad m = \sqrt{\frac{Z_m}{Z_\varphi}}m_R\,, \qquad \bar{\lambda} = \frac{Z_\lambda}{Z_\varphi^2}\bar{\lambda}_R\,. \tag{3.58}$$

If we expand the renormalization constants Z_i in a loop series similar to what we did in (3.39)

$$Z_i = 1 + \sum_{L=1}^{\infty} \hbar^L \delta Z_{i,L}\,, \tag{3.59}$$

we obtain an expansion for the counterterm action which we graphically represent as:

$$S_{c.t.}[\varphi_R] = \ \hbar\ \boxed{1}\ + \hbar^2\ \boxed{2} + \dots\ . \tag{3.60}$$

Each term of the above series is responsible for removing a divergence in the expansion of the effective action (3.39). The procedure of renormalization at higher order loop expansion follows the same idea as the one loop case described so far but some subtleties have to be taken into account. Beside the obvious technical difficulties concerning the computation of the loop integrals, an important role is played by subdivergences. Defining the superficial degree of divergence ω as[21]

$$\omega = dL - 2I\,, \tag{3.61}$$

[21] The definition given here only holds for the case of purely bosonic fields, however it can easily generalized to include spinors.

where L is the number of loops and I the number of internal legs, it turns out that subdivergences are due to subgraphs with $\omega \leq 0$. While the cancellation of the leading poles of n-loop diagrams is carried over by the n-loop counterterms, the cancellation of subdivergences is implemented via m-loop counterterms inserted in $(n-m)$-loop diagrams (of course $m < n$). For example one can compute again the diagram represented in (3.43) but inserting the one loop coupling (3.52) in one of the two vertices obtaining a next to leading contribution to the counterterms. This cancellation is, though, not granted at all and needs to be checked for each model.

At this point one last thing needs to be checked. The method of subtraction of divergences via counterterms relies on the possibility to keep the same truncation all along the full loop expansion, meaning that all the n-point correlators can be renormalized via a finite set of counterterms. If this is possible the theory is said to be renormalizable and predictive since only a finite amount of parameters need to be measured in order to fix the scheme dependence of the theory. Requiring such a possibility is fully equivalent to require that a predictive fixed point has a finite number of relevant directions.

It is important to understand the connection between the perturbative formulation of renormalization group and the FRG framework presented in section 3.2. In [52] the same analysis we just applied to the effective action was performed on the effective average action Γ_k of the functional renormalization group. Plugging the loop expansion of Γ_k inside the Wetterich equation one obtains an infinite tower of differential equations that can be solved recursively:

$$
\begin{aligned}
\partial_t S_B[\varphi] &= 0\,, \\
\partial_t \Gamma_{1,k}[\varphi] &= \frac{1}{2}\mathrm{STr}\left[\partial_t R_k \left(S_B^{(2)} + R_k\right)^{-1}\right]\,, \\
\partial_t \Gamma_{2,k}[\varphi] &= \frac{1}{2}\mathrm{STr}\left[\Gamma_{1,k}^{(2)} \partial_t \left(S_B^{(2)} + R_k\right)^{-1}\right]\,, \\
&\vdots
\end{aligned}
\tag{3.62}
$$

The integration of these equations, though, requires a commutation of the trace with the derivative with respect to t. In doing so one introduces ultraviolet divergences (previously regularized by the presence of $\partial_t R_k$) which require regularization. This can be achieved via dimensional regularization. The introduction of the scale μ then allows studying the interplay between the two approaches. It turns out that the leading divergence of the ε expansion of each diagram does not depend on the parameter k of the FRG. This means that the universal part of the β-functions obtained via the Wetterich equation coincide with the $\overline{MS}$ scheme. Nevertheless the full β-functions (and the full renormalization flow) will have non-universal properties. This is due to the mass dependence pf the FRG scheme. The ability of mapping the results obtained via the functional renormalization group to those of the $\overline{MS}$ scheme represents the technical foundation of what is known as *functional perturbative RG* [105, 106].

3.4. Fermionic systems and chiral symmetry breaking

So far we have only dealt with bosonic (scalar) degrees of freedom, nevertheless fermions are not a stranger to critical phenomena. In order to introduce phase transitions in fermionic systems let us consider theories of self interacting spinors. We can think of the interaction of these models as the point-like limit of 4-point functions generated by a more fundamental theory of fermions interacting via the exchange of a bosonic mediator where we integrated away the bosonic degrees of freedom (see figure 3.2). The result consists of

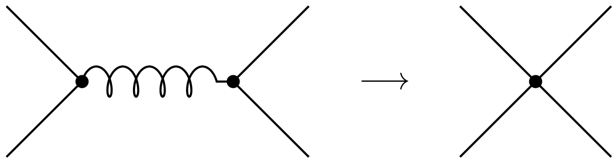


Fig. 3.2.: Graphic representation of the point-like limit in four point functions in QCD.

self interactions of quartic order in the fermionic fields. If we denote the fields by $\psi(x)$ and $\bar{\psi}(x) = \psi^\dagger\gamma_4$, where the Dirac matrices satisfy the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu,\nu}\mathbb{1}$ for $\mu, \nu = 1 \dots 4$, we can describe a four fermion interaction following in the form:

$$S_{\text{int}}[\bar{\psi}, \psi] = \lambda \int \mathrm{d}^4x \, \left(\bar{\psi}(x)\mathcal{O}_i\psi(x)\right)^2 , \tag{3.63}$$

The interaction channels $\mathcal{O}_i$ are listed and explained in appendix D.

The simplest model we can construct is known as the Gross-Neveu model [73]:

$$S_{\text{GN}}[\bar{\psi}, \psi] = \int \mathrm{d}^4x \, \left\{\bar{\psi} i\partial\!\!\!/\psi + \frac{\lambda}{2}(\bar{\psi}\psi)^2\right\} . \tag{3.64}$$

This theory has a discrete symmetry sometimes referred to as discrete chiral symmetry which is explicitly broken by the inclusion of a bare mass term: $\psi \to \gamma_5\psi$, $\bar{\psi} \to -\bar{\psi}\gamma_5$. This action can easily be studied using mean-field techniques. In fact (3.64) is equivalent to:

$$S[\sigma, \bar{\psi}, \psi] = \int \mathrm{d}^4x \, \left\{\frac{1}{2}g^2\sigma^2 + \bar{\psi}\left(i\partial\!\!\!/ + ih\sigma\right)\psi\right\} , \tag{3.65}$$

where g and h are coupling constants. The equivalence becomes evident through the elimination of the auxiliary field $\sigma(x)$ by means of the equations of motion and the identification:

$$\lambda = \frac{h^2}{g^2} . \tag{3.66}$$

Equation (3.66) shows that only the ratio of the two couplings is physical. The expectation value of $\sigma(x)$ is directly proportional to the chiral condensate $\langle\bar{\psi}\psi\rangle$ and can therefore be used as order parameter for the phase transition. Since the fermionic degrees of freedom appear only quadratically in the action they can be integrated out exactly from the path integral. We thus extract the following effective action:

$$S_{\text{eff}}[\sigma] = \int \mathrm{d}^4x \, \left\{ \frac{1}{2} g^2 \sigma^2 - \mathrm{Tr} \log \left(i\partial\!\!\!/ + ih\sigma \right) \right\} . \tag{3.67}$$

The method we followed to extract the effective action is know as *bosonization* of a fermionic theory and is based on the Hubbard-Stratonovich transformation [127]. Since the dependence on the condensate is now explicit, this method is particularly suited to study the broken phase. We can look for the minimum of the effective potential of (3.67). Assuming that the stable ground state coincides with a constant field configuration $\sigma(x) = \sigma_0$ one has:

$$\begin{aligned} \sigma_0 &= \frac{1}{g^2} \mathrm{Tr} \left[ih \left(i\partial\!\!\!/ + ih\sigma_0 \right)^{-1} \right] \\ &= \frac{1}{g^2} \mathrm{Tr} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \, ih \left(ip\!\!\!/ + ih\sigma_0 \right)^{-1} . \end{aligned} \tag{3.68}$$

Using the fact that the Dirac matrices are traceless and explicitly inverting the operator under the sign of integration we finally get to the result:

$$\sigma_0 = d_\gamma \left(\frac{h}{g} \right)^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\sigma_0}{p^2 + h^2 \sigma_0^2} , \tag{3.69}$$

d_γ being the dimension of the Dirac space. The previous equation still admits a trivial solution but a deeper inspection reveals that it is unstable. A second solution for $\sigma_0 \neq 0$ is the actual minimum of the effective potential. Hence, the ground state is dynamically shifted towards non-zero values of the chiral condensate breaking the symmetry and generating an effective mass. Therefore, equation (3.69) takes the name of *gap equation*. The expression for σ_0 is logarithmically divergent in the UV (in $d = 4$) and requires renormalization.

Another interesting model is given by the Nambu-Jona-Lasinio (NJL) in four dimensions. This model was first introduced in [74, 75] in order to describe the low energy physics of fundamental particles in analogy with the theory of superconductivity.

The simplest version of the NJL model we can take into account is described by the following action[22]

$$S_{\rm NJL}[\bar{\psi},\psi] = \int \mathrm{d}^4x \left\{ \bar{\psi} i\partial\!\!\!/ \psi + \frac{\lambda}{2}\left[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2 \right] \right\} . \tag{3.70}$$

Using the projectors on the right and left components of the Dirac fields:

$$P_{R,L} = \frac{\mathbb{1} \pm \gamma_5}{2} , \tag{3.71}$$

for which $\psi_{R,L} = P_{R,L}\psi$ and $\bar{\psi}_{R,L} = \bar{\psi}P_{L,R}$, it is easy to see that the action can be rearranged as:

$$S_{\rm NJL}[\bar{\psi},\psi] = \int \mathrm{d}^4x \, \left\{ \bar{\psi}_L i\partial\!\!\!/ \psi_L + \bar{\psi}_R i\partial\!\!\!/ \psi_R + 2\lambda(\bar{\psi}_L\psi_R)(\bar{\psi}_R\psi_L) \right\} , \tag{3.72}$$

which exhibits a manifest invariance under independent transformations of the left and right components by a complex phase: $\psi_L(x) \to e^{i\theta_L}\psi_L(x)$, $\psi_R(x) \to e^{i\theta_R}\psi_R(x)$. Hence the symmetry group of the model is $U(1)\times U(1)$ which includes the symmetry of the GN model as a special case and is called (continuous) chiral symmetry. It is clear that a mass term would explicitly break also this symmetry since left and right components would mix. Here we focus on dynamical a breaking of the symmetry leading to an effective mass generated as a chiral condensate. We refer to such a mechanism as chiral symmetry breaking (χSB).

It is possible to implement an FRG approach to study the renormalization flow of λ. Defining a sharp regulator of the form [128, 129]

$$R_\psi(p) = p\!\!\!/ \left(\sqrt{\frac{k^2}{p^2}} - 1 \right) \theta(k^2 - p^2) , \tag{3.73}$$

k being the infrared cutoff, and employing the usual ansatz for the UV truncation of the effective action:

$$\Gamma_k[\bar{\Psi},\Psi] = S_{\rm NJL}[\bar{\Psi},\Psi] + \int \frac{\mathrm{d}^4p}{(2\pi)^4} \, \bar{\Psi} R_\psi(p) \Psi , \tag{3.74}$$

one finds that the regularized propagator is a 2 by 2 matrix over the field space:

$$\Gamma_k^{(1,1)} = \begin{pmatrix} \frac{\vec{\delta}}{\delta\Psi^T} \\ \frac{\vec{\delta}}{\delta\bar{\Psi}} \end{pmatrix} \Gamma_k[\bar{\Psi},\Psi] \left(\frac{\overleftarrow{\delta}}{\delta\Psi} , \quad \frac{\overleftarrow{\delta}}{\delta\bar{\Psi}^T} \right) . \tag{3.75}$$

The β-function for the coupling λ can then be extracted projecting the flow equation onto

[22] For the moment we consider only one fermionic flavor and ignore the relation between different channels provided by the Fierz identities (see appendix D), since the most significant aspects of chiral symmetry breaking are still captured in this approximation.

the interaction channel of the NJL model. For the cutoff function (3.73) the β-function results in[23] (see [50] for a review on the topic):

$$\beta_\lambda \equiv \partial_t \lambda = 2\lambda - \frac{1}{2\pi^2}\lambda^2 \,. \tag{3.76}$$

The quadratic structure of the β-function could be guessed from the structure of the one loop diagram contributing to the vertex function which can only have two bare vertex insertions. Beside the Gaussian theory, hence, there is a non-trivial fixed point of the

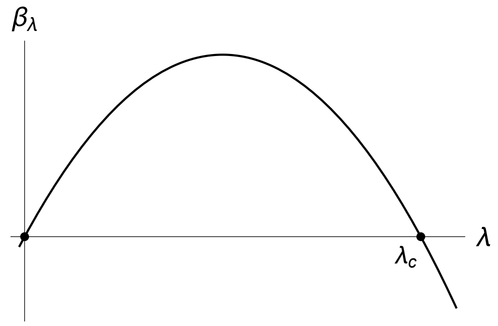


Fig. 3.3.: Profile of the β-function for the fermionic self interaction. As is evident from the plot, there is a non-Gaussian fixed point playing the role of a critical initial value for the coupling flow.

renormalization flow (see figure 3.3). This fixed point separates two possible infrared limits and, thus, two phases. If the initial condition for λ is smaller than $4\pi^2$ the model flows towards the free theory, otherwise the flow diverges. Therefor we can use the value of the non-Gaussian fixed point to define a critical coupling:

$$\lambda_c = 4\pi^2 \,. \tag{3.77}$$

Integrating the β-function for an initial condition $\lambda = \lambda^{UV}$ at the cutoff scale Λ, the solution looks like:

$$\begin{aligned} \lambda(k) &= \lambda^{UV} \left[\left(\frac{\Lambda}{k}\right)^{\Theta} \left(1 - \frac{\lambda^{UV}}{\lambda_c}\right) + \frac{\lambda^{UV}}{\lambda_c} \right]^{-1} , \\ \Theta &\equiv -\frac{\partial}{\partial \lambda}\beta_\lambda \Big|_{\lambda=\lambda_c} = 2 \,. \end{aligned} \tag{3.78}$$

Inspecting (3.78) for $\lambda^{UV} > \lambda_c$ we notice that the coupling eventually diverges at a finite

[23] For the sake of simplicity we drop the contribution of the wavefunction renormalization.

value of the renormalization scale k[24]:

$$k = \Lambda \sqrt{\frac{\lambda^{UV}/\lambda_c - 1}{\lambda^{UV}/\lambda_c}}\,, \tag{3.79}$$

which identifies the scale of the transition between a symmetric and a broken regime. In the following parts of this thesis we will review and study the contribution to this transition due to the presence of gravity taking into account the inclusion of many fermionic flavors.

3.5. Introducing gravity

In the last section of this chapter we point out the main aspects concerning the generalization of what we explained so far to the case where gravity is included. Even without considering gravitational fluctuations there are several details purely due to the presence of a non-flat background manifold that need to be taken into account.

The first complication we encounter is that we lack translational invariance and, hence, a momentum space representation of quantum field theory is not available. We therefore need a formalism that allows us to perform computations directly in coordinate space. Most of the computations we perform in quantum field theory and, in particular, in the framework of renormalization rely on the possibility of explicitly using the propagator. More generally we often need to compute traces of differential operators of some kind. Once that we switch on gravity, though, translational invariance of the background is lost and the Fourier transform with it. We therefore need a technique to represent such differential operators, which does not rely on momentum space. Let us therefore consider an elliptic differential operator $\mathcal{O}$[25] If we assume a basis for the orthogonal eigenfunctions of $\mathcal{O}$, $\phi_m(x)$, such that

$$\begin{aligned} &\mathcal{O}\phi_m = \lambda_m \phi_m(x)\,, \\ &\int \mathrm{d}^d x\ \phi_m(x) \mathbb{M} \phi_n(x) = \delta_{mn}\,, \end{aligned} \tag{3.80}$$

where $\mathbb{M}$ is an appropriate metric on the space of functions, then we can employ the spectral theorem to decompose a function of the operator $\mathcal{O}$ over its eigenspaces:

$$f(\mathcal{O}) = \int \mathrm{d}\mu(\lambda_m) f(\lambda_m) \mathbb{P}_m\,, \tag{3.81}$$

$\mu(\lambda_m)$ representing the spectral measure associated with $\mathcal{O}$ and $\mathbb{P}_m$ the projector oin the

[24] Here we assume that the initial condition is given at a very large scale Λ, larger than the transition scale.

[25] The requirement of $\mathcal{O}$ being elliptic only implies that the Cauchy problem is well defined.

appropriate subspace. If we further assume that f has a well defined inverse Laplace transform $\mathcal{L}^{-1}(f)$, such that

$$f(t) = \int_0^\infty \mathrm{d}s\, \mathcal{L}^{-1}(f)(s)e^{-st}\,, \tag{3.82}$$

then equation (3.81) becomes

$$f(\mathcal{O}) = \int_0^\infty \mathrm{d}s \int \mathrm{d}\mu(\lambda_m)\mathcal{L}^{-1}(f)(s)e^{-s\lambda_m}\mathbb{P}_m \tag{3.83}$$

$$= \int_0^\infty \mathrm{d}s\, \mathcal{L}^{-1}(f)(s)e^{-s\mathcal{O}}\,. \tag{3.84}$$

Tracing on both sides provides us with a formula for functional traces:

$$\mathrm{Tr}f(\mathcal{O}) = \int_0^\infty \mathrm{d}s\, \mathcal{L}^{-1}(f)(s)\mathrm{Tr}e^{-s\mathcal{O}}\,. \tag{3.85}$$

The operator under the sign of integration is the formal solution of a diffusion equation where the variable s represents the diffusion time:

$$(\mathcal{O} + \frac{d}{ds})e^{-s\mathcal{O}} = 0\,. \tag{3.86}$$

Therefor we define the heat kernel $\mathbb{K}_\mathcal{O}$ of the operator $\mathcal{O}$ as:

$$\mathbb{K}_\mathcal{O}(s;x,x') = \langle x|e^{-s\mathcal{O}}|x'\rangle\,, \tag{3.87}$$

with initial condition

$$\lim_{s\to 0} e^{-s\mathcal{O}} = \delta(x,x')\,. \tag{3.88}$$

In particular, if $\mathcal{O}$ is the kinetic operator of an action, the heat kernel allows for a direct space representation of the propagator:

$$G(x,x') \equiv \mathcal{O}^{-1}(x,x') = \int_0^\infty \mathrm{d}s\, \mathbb{K}_\mathcal{O}(s;x,x')\,. \tag{3.89}$$

The price that we have to pay, though, is that we need to know the spectrum of the operator $\mathcal{O}$ in order to perform any computation. It is nevertheless possible to give an asymptotic expansion of the heat kernel valid for small propertime values (see appendix F).

The second main input we shall take into account is the presence of new operators due to the coupling of dynamical fields to the background curvature. In this work we choose to perform dimensional analysis keeping the metric dimensionless[26]. In this way the basic curvature operators have the same dimensions regardless of the index distribution:

$$[R(x)] = [R_{\mu\nu}(x)] = [R_{\mu\nu\rho\sigma}(x)] = 2\,. \tag{3.90}$$

We notice that the non-minimal coupling of a canonically normalized scalar field to the Ricci scalar curvature $\xi\varphi^2(x)R(x)$ turns out to be a marginal operator in any dimension: $[\xi] = 0$. As a consequence we will expect the coupling ξ to affect the renormalization flow of scalar field theories on curved space as well as their critical behavior. This investigation constitutes an important part of the last chapter of this thesis.

As a second example of how the coupling to the background curvature can lead to non-trivial physics we consider fermions on curved space. The common way of dealing with such a theory is to define a set of maps between the local general coordinate frame and a locally inertial (Lorentz) frame [130–133]. These maps are called tetrads $e^\mu_a(x)$ and they transform as covariant vectors with respect to general coordinate transformations through the index μ, while the index a transforms according to vector representations of Lorentz group. They satisfy the following relations:

$$g_{\mu\nu}(x)e^\mu_a(x)e^\nu_b(x) = \eta_{ab}\,, \tag{3.91}$$

where η_{ab} is the flat metric. In this formulation gravity is seen as a gauge theory with gauge group given by the inertial frame group (d dimensional Lorentz group for a Lorentz signature or $SO(d)$ in Euclidean formulation). The introduction of tetrads allows us to define a set of Dirac matrices for spinors on curved space starting from the more familiar formulation on flat background:

$$\gamma^\mu(x) = e^\mu_a(x)\gamma^a\,, \tag{3.92}$$

$$\{\gamma^\mu(x), \gamma^\nu(x)\} = 2\mathbb{1}e^\mu_a(x)e^\nu_b(x)\eta_{ab} = 2\mathbb{1}g^{\mu\nu}(x)\,. \tag{3.93}$$

An equivalent perspective would be to start from the definition of a Clifford algebra for a generic metric $g_{\mu\nu}$ and formulate the theory leaving the charts from the spacetime manifold to a flat space be implicitly defined via the metric. This approach is known in the literature as spin-base invariance [134–138]. Of course in both these cases the bundle structure on the spacetime is enriched by fermionic degrees of freedom. Therefore the generalized covariant derivative receives a contribution from both the Levi-Civita connection Γ, acting on the tangent bundle, and the spin connection ω_μ, acting on the spin bundle, such that for a generic tensor T^a_ν carrying a representation of the tangent

[26]Since the physical quantity is the dimensionality of the square interval $dl^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$ it is possible (and equivalent) to assign mass dimensions to the metric field and keep dimensionless coordinates.

bundle through the index ν and a spin representation through the index a we have:

$$\nabla_\mu T^a_\nu = \partial_\mu T^a_\nu - \Gamma^\lambda_{\mu\nu} T^a_\lambda + \omega_\mu{}^a{}_b T^b{}_\nu \,. \tag{3.94}$$

The covariant Dirac operator is then constructed via contraction of the covariant derivative and Dirac matrices:

$$\not{\nabla} = \gamma^\mu(x)\nabla_\mu \,. \tag{3.95}$$

While in the tetrad formalism there is a natural representation of the spin connection in terms of the Levi-Civita connection:

$$\omega_\mu{}^a{}_b = -e^\nu_b \partial_\mu e^a_\nu + \Gamma^\lambda_{\mu\nu} e^a_\lambda e^\nu_b \,, \tag{3.96}$$

in the spin-base invariance framework there are more possible representation and metric compatibility has to be required.

A lot of information about spinors in curved space can be deduced from the spectrum of the Dirac operator and its heat kernel. In appendix E we summarize the case of hyperbolic spaces (particularly relevant for this work) and its application to the Heat Kernel. From equations (E.11) and (E.12) is already evident that the heat kernel of the Dirac operator contains several contributions coupling spinors to gravity. In [70, 139] it was shown that the infrared regime of the spectrum faces an effective dimensional reduction from $d+1$ to $1+1$ dimensions. This result affects in particular modes with self interactions described by the covariant version of (3.63) since they are relevant in the infrared.

The problem of phase transitions for self interacting fermions on a two-dimensional manifold was studied in several works. In [53, 54] a mean field analysis of the Gross-Neveu model with N_f spinor fields showed a curvature induced chiral phase transition. The effective potential in fact is found to be:

$$V(\sigma) = \frac{\sigma^2}{2}\left[1 - \frac{\lambda}{2\pi}\left(2 + \log\frac{2\lambda\mu^2}{N_f R}\right)\right] + \frac{N_f R}{8\pi}\log\Gamma\left(1 + \frac{2\lambda\sigma^2}{N_f R}\right) \,, \tag{3.97}$$

$\sigma(x)$ being the auxiliary fields satisfying (on-shell) $\sigma = \sqrt{\frac{\lambda}{N_f}}\bar{\psi}\psi$, λ is the coupling for the quartic self interaction and R the scalar background curvature. Already for small values of R we see that the effective potential develops a minimum for $\sigma = \sigma_0$ with

$$\begin{aligned} \sigma_0^2 &= \bar{\sigma}_0\left(1 - \frac{RN_f}{4\lambda\bar{\sigma}_0^2}\right) \,, \\ \bar{\sigma}_0 &= \mu^2 \exp\left\{2 - \frac{2\pi}{\lambda}\right\} \end{aligned} \tag{3.98}$$

for an arbitrary renormalization scale μ, while $\bar{\sigma}_0$ represents the condensate in flat space. The computation was performed assuming a positive constant curvature (spherical background). We notice that large values of R help the restoration of chiral symmetry. A first attempt to generalize the result to hyperbolic spaces already suggests that the negative curvature favorites the broken phase.

A similar result was found for a larger class of models in two dimensions in [56]. Here, a study of the temperature and curvature dependence of chiral symmetry breaking was performed for a theory of massless fermions interacting with gauge fields, scalar fields and pseudoscalar fields. For a manifold with constant positive curvature the gravitational contribution to the chiral condensate can be computed and studied via an expansion:

$$\langle \psi^\dagger P_R \psi \rangle = \bar{\alpha} \exp\left\{ -\frac{\pi}{12e^2} R \right\} \quad \text{as} \quad \frac{R}{m_\gamma} \to 0 \, , \tag{3.99}$$

and

$$\langle \psi^\dagger P_R \psi \rangle = \bar{\alpha} \left(\frac{R}{2m_\gamma} \right)^{\frac{\pi}{2\pi g^2}} \exp\left\{ -\frac{\pi}{4e^2} R - \frac{\pi m_\gamma^2}{4e^2} \gamma \right\} \quad \text{as} \quad \frac{R}{m_\gamma} \to \infty \, , \tag{3.100}$$

with $\bar{\alpha}$ the flat space value of the condensate $\langle \psi^\dagger P_R \psi \rangle$, m_γ the dynamically generated photon mass, g the pseudoscalar interaction and γ the Euler-Mascheroni constant. Again we see that the large positive curvature is responsible for an exponential damping of the chiral condensate.

Heuristically we can understand the difference between positive and negative curvature noticing that the size of a closed background represents a natural cutoff for the infrared modes. These are exactly the degrees of freedom dragging the system towards the phase transition and in this configuration their contribution is partially suppressed. Conversely, hyperbolic geometry enhances infrared effects leading to the symmetry breaking with weaker interactions.

Having in mind the dimensional reduction described above and considering the evidences of the phase transition in $1+1$ dimensions the natural expectation is that models of fermionic self interactions will face chiral symmetry breaking regardless of the dimensionality. In the next chapter we will discuss some evidence of this behavior coming from a functional renormalization group analysis and discuss how this leads to the formulation of a possible constraint for quantum gravity.

4. Gravitational catalysis of chiral symmetry breaking

In this chapter we study the role of gravity in dynamical symmetry breaking for chiral models of fermionic matter on hyperbolic spaces. A key ingredient of the study is the implementation of heat kernel techniques. The spinor heat kernel is introduced in appendix E following the work of [140, 141]. The dependence of the phenomenon on the dimensionality of the spacetime is investigated in general for the odd dimensional case. The even dimensional scenario requires the implementation of numerical methods, therefore we performed the analysis only for the most interesting cases. Finally the result is employed to reduce the set of physical solutions for the ultraviolet fixed point of the cosmological constant in the context of asymptotically safe quantum gravity.

4.1. Gravitational Catalysis

The great interest of the community in fermionic self interactions is mostly due to the great renormalization properties they exhibit. Interactions of the form 3.63 share asymptotic freedom with QCD, a feature that can be investigated in the large N_f limit, and still they provide a dynamical mechanism of mass generation. The phase diagrams of these models were extensively studied for both pure fermionic models and theories coupled to other fields. An example of coupling that plays an important role in chiral symmetry breaking is given by the introduction of a magnetic field. The enhancement of mass generation via an external magnetic field takes the name of *magnetic catalysis* [142–149]. This phenomenon is a result of an effective dimensional reduction of the dynamics of the system fro d to $d-2$ dimension. We can see that a lot of features of magnetic catalysis are qualitatively matching the discussion made about gravity at the end of section 3.5[27]. Therefore the curvature contribution to chiral symmetry breaking takes the name of *gravitational catalysis*[53–72, 139, 150].

As we pointed out at the end of the last chapter the natural expectation is to detect gravitational catalysis in four fermion theories in any dimension. Evidences for this phenomenon have been provided in several works. In [55, 151] the effective potential

[27] One of the fundamental aspects of catalyzed chiral symmetry breaking is that the external force should be attractive in order to favor bound states.

of the four dimensional NJL model in curved spacetime was studied using Riemann normal coordinates and Schwinger's propert time in the $1/N_f$ expansion. In [57] the same analysis was carried over for various models of four fermion theories in $d = 2$ and $d = 3$. Similarly to the scenario taking place in two dimensions we have that positive curvature contributes to symmetry restoration while negative values of R lead to an enhancement of chiral symmetry breaking. In particular in the closed topology case is possible to identify a critical value R_{cr} such that the symmetry is restored for $R > R_{cr}$. In particular the symmetry is always restored for large positive curvature and alway broken for large negative curvature.

Surprisingly, similar effects appear also when the curvature is only appearing in the space like sections of the manifold. This is the case of the Lobachevsky plane where it was shown ([71]) that the dynamical mass generated in the Gross-Neveu model receives contributions from the spatial curvature in the weak coupling regime.

These results turned out to be in accordance with the picture provided by a renormalization approach to the phase transition. In [150] a FRG study of the Gross-Neveu model was performed for both the hyperbolic space in $d = 3$ and the Lobachevsky plane. The shape of β-function for the self interaction coupling appears to be parametrized by the RG scale k which always appears in the ratio $\frac{R}{k^2}$. This causes the collapse of the β-function in the deep infrared regime and the non-Gaussian fixed point falls into the Gaussian one. Consequently there is no real notion of subcritical initial condition for the coupling since, at some scale $k = k_o$, the critical value λ_{cr} will cross the initial condition and render it supercritical (see figure 4.1 for a sketch of the phenomenon).

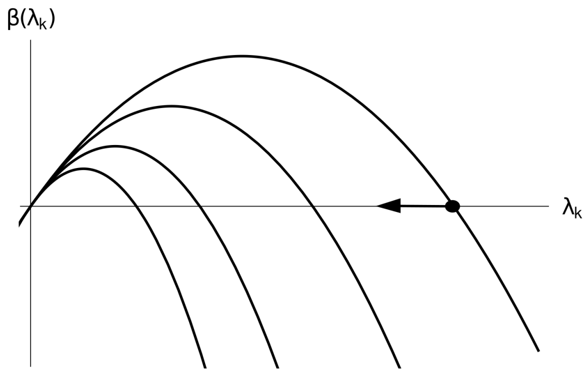


Fig. 4.1.: Sketch of the dependence of the β-function for the Gross-Neveu model in hyperbolic three dimensional space on the scale k. The deeper we go in the infrared regime the more the critical value of the coupling is pushed toward the Gaussian fixed point.

From a phenomenological point of view an important question arises. Supposing that the average curvature of the spacetime is indeed negative and gravity contributes to the fermion mass generation, the hierarchy of fundamental interactions would force the transition to happen at high energy scales[28] where gravity would be strong interacting.

[28] From this perspective, asymptotic freedom of QCD suppresses the gauge modes and the color structure

At this point the condensate is expected to receive contributions fro the cutoff of the theory (this is typical also in magnetic catalysis). Since the natural cutoff for gravity is the Planck scale we should observe masses not far from the Planck mass. Nevertheless the observation we have so far for the masses of fundamental fermionic particles differ a lot of orders of magnitude from this expectation. Is gravitational catalysis really compatible with the observations of light fermions? This question was originally asked in [76, 152] and represents the main subject of the investigation of this chapter. The prediction of a negatively curved manifold is common to several quantum gravity scenario. The key approach will be to identify a set of parameters of the theory such that gravitational catalysis does not occur. This would imply a possible constraint for quantum gravity via the infrared predictions of specific theories.

4.2. Mean field analysis

Let us start from a fermionic matter sector with a global chiral symmetry group $U(N_\mathrm{f})_\mathrm{R} \times U(N_\mathrm{f})_\mathrm{L}$. As before N_f represents the number of fermion species. This is reminiscent to the fermionic sector of the standard model subject to the strong interaction with N_f counting the number of flavors times the number of colors. Even without any further gauge interactions, gravitational fluctuations, say in the (trans-)Planckian regime, will induce effective fermionic self interactions. With gravity preserving chiral symmetry, a Fierz complete local fermionic self interaction to fourth order in the fields is parametrized by the action [76, 153]

$$S[\bar\psi,\psi] = \int_x \left\{ \bar\psi \nabla\!\!\!/\, \psi + \frac{\bar\lambda_-}{2}\left[\left(\bar\psi^a\gamma_\mu\psi^a\right)^2 + \left(\bar\psi^a\gamma_\mu\gamma_5\psi^a\right)^2\right] \right. \\ \left. + \frac{\bar\lambda_+}{2}\left[\left(\bar\psi^a\gamma_\mu\psi^a\right)^2 - \left(\bar\psi^a\gamma_\mu\gamma_5\psi^a\right)^2\right]\right\}, \tag{4.1}$$

where the Latin indices represent different flavor species and $\nabla\!\!\!/$ is the covariant Dirac operator. Denoting the vector interaction channel term with $(V) = (\bar\psi\gamma_\mu\psi)^2$ and the axial one with $(A) = -(\bar\psi\gamma_\mu\gamma_5\psi)^2$, we expect the transition to be triggered by the $(V)+(A)$ term which is equivalent to

$$(V) + (A) = -2[(S') - (P')] \tag{4.2}$$

simply counts the number of fermions in the model with no relevance difference from the flavor symmetry.

by means of a Fierz transformation (see appendix D). Here, (S') and (P') denote the scalar and pseudo scalar channels in the space of flavor nonsinglet terms,

$$\begin{aligned}(S') &= (\bar{\psi}^a\psi^b)^2 = (\bar{\psi}^a\psi^b)(\bar{\psi}^b\psi^a),\\ (P') &= (\bar{\psi}^a\gamma_5\psi^b)^2 = (\bar{\psi}^a\gamma_5\psi^b)(\bar{\psi}^b\gamma_5\psi^a)\,.\end{aligned} \tag{4.3}$$

In fact, the structure $(S')-(P')$ is familiar from the NJL model and further generic models of chiral symmetry breaking. In such models, the onset of chiral symmetry breaking is signaled by this channel becoming RG relevant. Hence, we concentrate in the following on the NJL channel and ignore the $(V)-(A)$ channel. The latter is expected to stay RG irrelevant across a possible phase transition, justifying to approximate $\lambda_- \simeq 0$ for the purpose of detecting the onset of symmetry breaking.

Using the projectors on the left and right chiral components (3.71), in a total similar fashion as we did in section 3.4, the NJL channel can also be written as the following interacting Lagrangian

$$\mathcal{L}_{\rm int}(\bar{\psi},\psi) = -2\bar{\lambda}(\bar{\psi}^a_{\rm L}\psi^b_{\rm R})(\bar{\psi}^b_{\rm R}\psi^a_{\rm L})\,, \quad \bar{\lambda} = 2\bar{\lambda}_+. \tag{4.4}$$

By means of a Hubbard-Stratonovich transformation, the interaction term can also be expressed in terms of a Yukawa interaction with an auxiliary scalar field,

$$\mathcal{L}_{\rm int}(\phi,\bar{\psi},\psi) = \bar{\psi}^a[P_{\rm L}(\phi^\dagger)_{ab} + P_{\rm R}\phi_{ab}]\psi^b + \frac{1}{2\bar{\lambda}}\mathrm{tr}(\phi^\dagger\phi)\,. \tag{4.5}$$

The equivalence of (4.5) with (4.4) becomes obvious with the help of the equations of motion for the chiral matrix fields ϕ and $\phi^\dagger$,

$$\begin{aligned}\phi_{ab} &= -2\bar{\lambda}\bar{\psi}^b_{\rm R}\psi^a_{\rm L}\,,\\ (\phi^\dagger)_{ab} &= -2\bar{\lambda}\bar{\psi}^b_{\rm L}\psi^a_{\rm R}\,.\end{aligned} \tag{4.6}$$

Note that the inclusion of flavor degrees of freedom reflects in the auxiliary field being a tensor. From (4.5), it is obvious that the Dirac particles can acquire a mass if chiral symmetry gets broken by a nonzero expectation value of the field ϕ_{ab}. The precise breaking pattern is fixed by the nonzero components of $\langle\phi_{ab}\rangle$ which in turn is determined by the minima of the effective potential for ϕ. In the following, we assume a diagonal breaking pattern, $\phi_{ab} = \phi_0\delta_{ab}$ with constant order parameter ϕ_0, which for $|\phi_0| > 0$ breaks the chiral group down to a residual vector symmetry familiar from QCD-like theories. In the form of (4.5) read together with the fermion kinetic term, we can integrate out the fermionic degrees of freedom and obtain the standard mean field expression for the effective potential of the order parameter

$$U(\phi_0) = \frac{N_\mathrm{f}}{2\bar{\lambda}}\phi_0^2 - N_\mathrm{f} \log\det(\nabla\!\!\!\!/ + \phi_0) = \frac{N_\mathrm{f}}{2\bar{\lambda}}\phi_0^2 - \frac{N_\mathrm{f}}{2}\mathrm{Tr}\log(-\nabla\!\!\!\!/^2 + \phi_0^2)\,, \tag{4.7}$$

where we have made use of γ_5-hermiticity of the Dirac operator in the last step. Since we are considering a homogeneous order parameter, the trace (as well as log det) is understood to be already normalized by a spacetime volume factor, such that we are considering local quantities throughout the computations. Using the Schwinger propertime representation as explained in section 3.5, we write

$$U(\phi) = \frac{N_\mathrm{f}}{2\bar{\lambda}}\phi_0^2 + \frac{N_\mathrm{f}}{2}\int_0^\infty \frac{ds}{s}\, e^{-\phi_0^2 s}\mathrm{Tr}\, e^{\nabla\!\!\!\!/^2 s}\,. \tag{4.8}$$

where we encounter the trace of the heat kernel of the squared Dirac operator on the manifold under consideration,

$$\mathrm{Tr}\, e^{\nabla\!\!\!\!/^2 s} = \mathrm{Tr}\, \mathbb{K}(x,x';s) =: K_s\,, \tag{4.9}$$

$$\frac{\partial}{\partial s}\mathbb{K} = \nabla\!\!\!\!/^2\mathbb{K}, \quad \lim_{s\to 0^+}\mathbb{K}(x,x';s) = \frac{\delta(x-x')}{\sqrt{g}}. \tag{4.10}$$

In our analysis, the information about the nature of spacetime enters through the trace (4.9). As this trace parametrizes the contributions of fermionic fluctuations on all scales, the explicit evaluation of (4.8) would contain information about both the local and global structure of spacetime.

Though the propertime integration has been introduced as an auxiliary representation, the integrand can be interpreted as the result of a diffusion process of a fictitious particle on the spacetime within propagation time s [154, 155]. The trace enforces that the diffusion path is closed. For a finite propertime s, the fictitious particle traces out a closed path in spacetime which is localized around a point x under consideration. This path can be considered as the spacetime path of a virtual fermionic fluctuation; this perspective can also be made explicit by introducing a Feynman path integral representation of the heat kernel (worldline formalism) [156–160]. For instance, the mean average distance of the diffusing particle from its center of mass in flat space is $d = \sqrt{s/6}$ [161], indicating that $\sqrt{s}$ can be considered as a typical length scale of the fluctuations at a fixed value of s.

Aiming at a statement about spacetime in the (trans-)Planckian regime, we do not want to make an assumption about its global properties, but intend to consider only local patches of spacetime. This is possible by means of an RG type analysis of (4.8). For this, we introduce a propertime regulator function f_k inside the propertime integral

[162, 163],

$$f_k = e^{-(k^2 s)^p} . \tag{4.11}$$

Here, the power $p > 0$ is a parameter specifying the details of the regularization and k corresponds to an IR momentum space regularization scale. For instance, for $p \to \infty$, all long range contributions for length scales $\sqrt{s} > 1/k$ are cut off sharply. For finite p, the length scale $1/k$ becomes a smooth long range cutoff. The case $p = 1$ is special as it corresponds precisely to a Callan-Symanzik regularization scheme. In the limit $k \to 0$, the insertion factor becomes $f_{k\to 0} = 1$ and the regularization is removed. This procedure is in total analogy to the FRG approach described in section 3.2. The correspondence can be made formal simply applying the propertime representation to the regularized propagator $\left(\Gamma_k^{(2)} + R_k\right)^{-1}$. Starting from the bare potential U_Λ at a high momentum scale $k = \Lambda$, the potential at any IR scale k_{IR} can be constructed from

$$U_{k_{\mathrm{IR}}} = U_\Lambda - \int_{k_{\mathrm{IR}}}^{\Lambda} dk\, \partial_k U_k\,, \quad U_\Lambda = \frac{N_{\mathrm{f}}}{2\bar{\lambda}_\Lambda}\phi_0^2\,, \tag{4.12}$$

where the RG flow of the potential can be computed from:

$$\partial_k U_k = \frac{N_{\mathrm{f}}}{2}\int_0^\infty \frac{ds}{s}\, e^{-\phi_0^2 s}\partial_k f_k K_s\,. \tag{4.13}$$

The subscript Λ to the bare coupling $\bar{\lambda}$ in equation (4.12) stresses the fact that the process of defining a model goes through the definition of the initial condition for the interaction coupling at the high scale Λ.

Since $\partial_k f_k \sim s^p$ for small s, also the short range fluctuations are suppressed in (4.13), such that the effect of the fermionic fluctuations can be studied in a Kadanoff-Wilson spirit length scale by length scale. The evaluation of one RG step $\sim \partial_k U_k$, typically receives contributions from length scales $\sqrt{s} \sim 1/k$. This implies that we do not have to know the global structure of the spacetime, but our assumptions about the spacetime properties need to hold only over these covariant length scales. More specifically, we assume below that the spacetime can locally be approximated as maximally symmetric.

Though the analysis of the chiral interactions leading to (4.1) has been performed in $d = 4$ dimensional spacetime, the study of the flow of the order parameter potential of (4.13) can be performed in any dimension. What might differ is the relation to the symmetry breaking channel which, in other dimensions, can be more involved or not even unique, see [164] for an analysis in $d = 3$. In higher dimensions, the perturbative non-renormalizability of Yukawa theories suggests that more relevant operators appear near the Gaussian fixed point. The corresponding regularization of UV divergences may require higher values of p for a stronger suppression of UV modes. Independently of these technical complications, our analysis can in principle be performed for any value of d.

4.3. $d = 3$ dimensional space

Let us begin with an analysis of the RG flow of the potential for the case of $d = 3$ spacetime dimensions. This case is highly instructive from the viewpoint of the method: it can be treated analytically in all detail, and does not involve further relevant operators. Since gravitational catalysis can occur for negative curvature, we consider spacetimes that can locally be approximated by a hyperbolic space for Euclidean signature, corresponding to AdS spacetime for a Lorentzian signature. The analysis could similarly be performed for spacetimes with negative curvature in the purely spatial part with quantitatively rather similar results [71, 150]. In $d = 3$, the trace of the heat kernel reads [140, 141]

$$K_s = \frac{1}{8\pi^{\frac{3}{2}} s^{\frac{3}{2}}} \left(1 + \frac{1}{2}\kappa^2 s\right), \tag{4.14}$$

where

$$\kappa^2 = -\frac{R}{d(d-1)} = -\frac{R}{6} \geq 0, \tag{4.15}$$

denotes the local curvature parameter related to the Ricci scalar R.

Including the propertime regularization leads to an effective, scale-dependent potential of the following form:

$$U_k = \frac{N_\mathrm{f}}{2\bar{\lambda}}\phi_0^2 + \frac{N_\mathrm{f}}{2(4\pi)^{\frac{3}{2}}} \int_0^\infty \frac{ds}{s^{\frac{5}{2}}}\, e^{-\phi_0^2 s} f_k \left(1 + \frac{1}{2}\kappa^2 s\right). \tag{4.16}$$

In $d = 3$, the Callan-Symanzik regulator is known to be sufficient to control the RG flow of our model. Thus, let us first choose the exponent $p = 1$ for simplicity; the result for general p will be given below. The regularized flow of the potential with respect to the scale k then reads

$$\begin{aligned} \partial_k U_k(\phi) =& -\frac{2kN_\mathrm{f}}{2(4\pi)^{\frac{3}{2}}} \left[\int_0^\infty \frac{ds}{s^{\frac{3}{2}}}\, e^{-k^2 s}\left(e^{-\phi_0^2 s} - 1\right) + \frac{\kappa^2}{2}\int_0^\infty \frac{ds}{s^{\frac{1}{2}}}\, e^{-k^2 s}\left(e^{-\phi_0^2 s} - 1\right)\right] \\ =& \frac{N_\mathrm{f}}{4\pi}\left[k^2\left(\sqrt{1 + \frac{\phi_0^2}{k^2}} - 1\right) - \frac{\kappa^2}{4}\left(\frac{k}{\sqrt{\phi_0^2 + k^2}} - 1\right)\right]. \end{aligned} \tag{4.17}$$

Upon insertion into (4.12), the effective potential at the scale k_IR can be computed, yielding

$$\begin{aligned} U_{k_\mathrm{IR}} =& -\frac{N_\mathrm{f}}{2}\phi_0^2\left(\frac{1}{\bar{\lambda}_\mathrm{cr}} - \frac{1}{\bar{\lambda}_\Lambda} - \frac{k_\mathrm{IR}}{4\pi}\right) + \frac{N_\mathrm{f}}{12\pi}\left((\phi_0^2 + k_\mathrm{IR}^2)^{\frac{3}{2}} - \frac{3}{2}k_\mathrm{IR}\phi_0^2 - k_\mathrm{IR}^3\right) \\ & - \frac{N_\mathrm{f}}{16\pi}\kappa^2\left(\sqrt{\phi_0^2 + k_\mathrm{IR}^2} - k_\mathrm{IR}\right), \end{aligned} \tag{4.18}$$

where we have introduced the (scheme dependent) critical coupling $\bar{\lambda}_{\text{cr}} = 4\pi/\Lambda$, and dropped terms of order $\mathcal{O}(1/\Lambda)$.

The physics described by this effective potential can be read off term by term. The first bracket controls the mass-like term in the potential[29]. For subcritical coupling $\bar{\lambda}_\Lambda < \bar{\lambda}_{\text{cr}}$, the mass-like term remains positive for any k_{IR} implying that the system in flat space remains in the symmetric phase with a minimum $\phi_0 = 0$ and does not develop fermion masses. For supercritical couplings $\bar{\lambda}_\Lambda > \bar{\lambda}_{\text{cr}}$, the mass-like term becomes negative below a certain critical IR scale k_{IR}, indicating that the potential develops a nontrivial minimum $\phi_0^2 > 0$. The system hence exhibits chiral symmetry breaking and fermion mass generation already in flat space with a mechanism rather similar to the one illustrated in Landau theory in section 2.3. The second term does not contribute to the mass-like part of the potential $\sim \phi_0^2$. This is easily verified upon Taylor expansion. For large ϕ_0 it grows like $\sim +\phi_0^3$, ensuring stability of the potential. The last bracket represents the gravitational contribution, being manifestly negative. In the limit $k_{\text{IR}} \to 0$, it is linear in the field ϕ_0 and thus dominates for small field amplitudes. In this way, it induces a nonzero ϕ_0 and inevitably drives the system to chiral symmetry breaking and fermion mass generation.

However, gravitational catalysis receives its relevant contributions from the deep IR, i.e., the long-wavelength modes. In order to dominate the mass spectrum, the curvature has to be such that the hyperbolic space is an adequate description also on large length scales. Within our RG description, we make the less severe assumption that the hyperbolic space is an adequate description only up to lengths scales of order $1/k_{\text{IR}}$. Whether or not the potential develops a nonzero minimum then is decided by the competition between the first and the third term of (4.18).

Since we are interested in curvature induced symmetry breaking, we assume that the fermionic interactions are subcritical, $\bar{\lambda}_\Lambda \leq \bar{\lambda}_{\text{cr}}$, such that the mass-like term in the first line is bounded from below by

$$-\frac{N_{\text{f}}}{2}\phi_0^2\left(\frac{1}{\bar{\lambda}_{\text{cr}}} - \frac{1}{\bar{\lambda}_\Lambda} - \frac{k_{\text{IR}}}{4\pi}\right) \geq \frac{N_{\text{f}} k_{\text{IR}}}{8\pi}\phi_0^2. \tag{4.19}$$

The only other term contributing to the Hessian of the potential around $\phi_0 = 0$ arises from the curvature dependent part of (4.18):

$$-\frac{N_{\text{f}}}{16\pi}\kappa^2\left(\sqrt{\phi_0^2 + k_{\text{IR}}^2} - k_{\text{IR}}\right) = -\frac{N_{\text{f}} k_{\text{IR}}}{32\pi}\frac{\kappa^2}{k_{\text{IR}}^2}\phi_0^2 + \mathcal{O}(\phi_0^4) \tag{4.20}$$

Comparing the last two equations tells us that gravitational catalysis does not induce chiral symmetry breaking and fermion mass generation as long as the hyperbolic curvature

[29] Not to be confused with an indication of mass generation, a quadratic term in the order parameter field is proportional to the expectation value of four spinors.

parameter satisfies

$$\frac{\kappa^2}{k_{\rm IR}^2} \leq 4. \tag{4.21}$$

In terms of the negative scalar (spacetime) curvature, this implies

$$|R| = 6\kappa^2 \leq 24 k_{\rm IR}^2\,, \quad (\text{for } R<0)\,. \tag{4.22}$$

If the last inequality is respected the occurrence of a nontrivial minimum for the effective potential is inhibited and fermion mass generation is less likely to happen as a consequence of gravity[30]. Equation (4.22) represents our first example of a curvature bound from gravitational catalysis: in line with our assumptions we conclude that a fermionic particle physics system will not be plagued by curvature induced chiral symmetry breaking, as long as the local curvature of spacetime patches averaged over the scale of $1/k_{\rm IR}$ satisfies the bound (4.22).

Some comments are in order:

1. From the derivation, it is obvious that a study of the mass-like term $\sim \phi_0^2$ is sufficient to obtain a curvature bound. Of course, the global structure of an effective potential could be such that a nontrivial minimum exists even for a positive mass-like term. This could be the case of a first order phase transition. In that case, the true curvature bound would even be stronger than the one derived from the mass-like term. (In the present $d=3$ dimensional system, this does not happen at mean field level.).

2. The curvature bound is independent of the self-couplings due to our estimate performed in (4.19). The equal sign holds for bare couplings exactly tuned to criticality, i.e., the maximum value of the self interaction that does not lead to chiral symmetry breaking in the IR. Therefore, the bound limits the regime where the system is safe from the formation of the chiral condensate through gravitational catalysis. Whether or not fermion mass generation sets in once the bound is violated depends on further details of the system such as the fermion couplings.

3. The bound is naively scheme dependent in the sense that the prefactor (24 in the present case) depends on the way the fluctuation averaging procedure is performed. In the calculation so far, we used a Callan-Symanzik regulator that suppresses long wavelength modes beyond the scale $1/k_{\rm IR}$ exponentially. In fact, the calculation can straightforwardly be performed for the general regulator (4.11). For general p, we obtain

$$\frac{\kappa^2}{k_{\rm IR}^2} \leq \frac{2\Gamma(1-\frac{1}{2p})}{\Gamma(1+\frac{1}{2p})}. \tag{4.23}$$

[30] Of course it can still happen due to other interactions. This is not problematic, though, since the dynamical masses would differ many orders from the Planck mass.

For $p = 1$, we obtain (4.21) and (4.22) again, whereas in the sharp cutoff limit ($p \to \infty$) we find

$$|R| \leq 12k_{\mathrm{IR}}^2\,, \quad (\text{for } p \to \infty, R < 0). \tag{4.24}$$

Comparing this to (4.22), the curvature bound naively seems to be stronger for $p \to \infty$. However, this simply reflects the fact that the length scale of the fluctuations $1/k_{\mathrm{IR}}$ is effectively shorter for the sharp cutoff than for the smooth exponential regulator, where the fluctuations extending even further out are only suppressed but not cut off. Hence, it is plausible to say that $k_{\mathrm{IR}}|_{p\to\infty}$ is effectively larger than $k_{\mathrm{IR}}|_{p=1}$[31]. This goes hand in hand with the inversely behaving prefactor. We consider this as an indication that the curvature bound itself might have a scheme independent meaning: the scheme dependence of the prefactors in the bound should be viewed as a parametrization of the fluctuation averaging process that has to be matched with the procedure determining the averaged curvature.

4.4. $d = 4$ dimensional space

Let us now turn to the physically more relevant case of $d = 4$ dimensional spacetime. The analysis is conceptually complicated by the appearance of two more relevant operators coming along with physical couplings. It is technically more involved because of the structure of the heat kernel. Nevertheless, it is possible to capture the essential behavior analytically by making use asymptotic heat kernel expansions and a simple interpolation. The full result is, of course, analyzed below by straightforward numerical integration. We start with the representation of the heat kernel trace as a one parameter integral [140, 141]

$$K_s = \frac{2}{(4\pi s)^2}\int_0^\infty du\, e^{-u^2} u(u^2+\kappa^2 s)\coth(\frac{\pi u}{\kappa\sqrt{s}})\,. \tag{4.25}$$

Using the asymptotic expansions of the coth function, cf. Eqs. (E.13) and (E.14), the weak and strong curvature expansions of the heat kernel read

$$K_s = \frac{2}{(4\pi s)^2}(1+\kappa^2 s+\dots), \quad \kappa^2 s \ll 1, \tag{4.26}$$

$$K_s = \frac{1}{(4\pi s)^2}\Big(\frac{\kappa^3 s^{\frac{3}{2}}}{\pi} + \frac{3+\pi^2}{6\sqrt{\pi}}\kappa s^{\frac{1}{2}}+\dots\Big), \quad \kappa^2 s \gg 1\,. \tag{4.27}$$

[31] A first check of this can be done by regard at f_k as a distribution and inspecting the average $\langle\frac{1}{\sqrt{s}}\rangle$ were it peaks. For our ansatz we find $\langle\frac{1}{\sqrt{s}}\rangle = 2\frac{\Gamma(1+\frac{1}{2p})}{k_{\mathrm{IR}}}$ which, for fixed nuerical value of k_{IR} is monotonically increasing for integer p values. The global minimum is located between $p = 1$ and $p = 2$.

For a simple qualitative, still asymptotically exact estimate, we use an interpolating approximation of the heat kernel that allows for a fully analytical treatment,

$$K_s \simeq \frac{2}{(4\pi s)^2}(1+\kappa^2 s+\frac{\kappa^3 s^{\frac{3}{2}}}{\pi}). \tag{4.28}$$

Upon insertion of the heat kernel into Eqs. (4.12) and (4.13), a first difference to the $d=3$ case is the occurrence of a logarithmic UV divergence of the type $\sim \phi_0^4 \ln\Lambda$. This is expected as ϕ^4 is a marginal operator in $d=4$, the coupling of which corresponds to a new and independent physical parameter. The proper definition of the particle system requires to also define an initial condition for the flow of this operator, i.e., to put a counterterm at the high scale Λ. This is then fixed by demanding for a specific physical renormalized value for the ϕ^4 coupling in a long range experiment.

For our purposes, these details are, in fact, not relevant, as the ϕ^4 coupling cannot inhibit chiral symmetry breaking. Once, the mass-like term $\sim \phi_0^2$ triggers the onset of a chiral condensate, the ϕ^4 coupling will take influence on the final value of the condensate ϕ_0; this is, however, irrelevant for the curvature bound. For consistency, we only assume that the renormalized ϕ^4 coupling is such that the potential is stable towards large fields.

As we have seen in the $d=3$ case, we can obtain a curvature bound by solely studying the ϕ_0^2 term of the potential. Using the approximate form of the heat kernel (4.28), we obtain the analytic estimate to this order:

$$\begin{aligned} U_{k_{\mathrm{IR}}}\Big|_{\phi_0^2} = &-\frac{N_{\mathrm{f}}\phi_0^2}{2}\Big[\frac{1}{\bar\lambda_{\mathrm{cr}}}-\frac{1}{\bar\lambda_{\Lambda}}-\Gamma\Big(1-\frac{1}{p}\Big)\frac{k_{\mathrm{IR}}^2}{(4\pi)^2}+2\frac{\kappa^3}{k_{\mathrm{IR}}}\frac{\Gamma\Big(1+\frac{1}{2p}\Big)}{\sqrt{\pi}}\Big] \\ &-\frac{N_{\mathrm{f}}\phi_0^2}{(4\pi)^2}\kappa^2\log\Big(\frac{\Lambda}{k_{\mathrm{IR}}}\Big), \end{aligned} \tag{4.29}$$

again dropping terms of order $\mathcal{O}(1/\Lambda)$. As before, the diverging contribution coming from the flat part of the heat kernel is indicative of the critical value of the coupling constant,

$$\bar\lambda_{\mathrm{cr}} = \frac{(4\pi)^2}{\Lambda^2\Gamma\Big(1-\frac{1}{p}\Big)}, \tag{4.30}$$

As a new feature in $d=4$, we observe a new logarithmically divergent term $\sim \ln\Lambda$ in (4.29). This term corresponds to a new, power counting marginal operator of the form $\phi^2 R$, which again comes along with a new physical parameter to be fixed by renormalization. Hence, we introduce an initial condition for this operator at the high scale with a bare coupling ξ_Λ:

$$U_\Lambda|_{\phi^2 R} = N_{\mathrm{f}}\xi_\Lambda\phi^2 R. \tag{4.31}$$

Upon inclusion of (4.31), the effective potential at the scale k_{IR} receives an overall contribution of the form

$$U_{k_{\rm IR}}|_{\phi^2 R} = -\left(\xi_\Lambda N_{\rm f} + \frac{3N_{\rm f}}{4\pi^2}\log\left(\frac{\Lambda}{k_{\rm IR}}\right)\right)\phi_0^2|R|$$
$$\equiv - N_{\rm f}\xi_{k_{\rm IR}}\phi_0^2|R|\,, \tag{4.32}$$

where we have made use of the relation $\kappa^2 = \frac{|R|}{12}$, $R < 0$, in $d = 4$. Here we have introduced the long range parameter $\xi_{k_{\rm IR}}$ that, in principle, has to be fixed by a physical measurement. For our analysis, we will consider it as a free parameter. As a consequence, the curvature bound depends parametrically on this physical coupling. Assuming again, that the fermion self interactions are subcritical $\bar\lambda_\Lambda \le \bar\lambda_{\rm cr}$, we obtain again a bound on the curvature parameter for which no chiral symmetry breaking occurs:

$$\frac{\kappa^3}{k_{\rm IR}^3} + \frac{4}{3}\frac{\pi^{\frac{5}{2}}}{\Gamma\Big(1+\frac{1}{2p}\Big)}\xi_{k_{\rm IR}}\frac{\kappa^2}{k_{\rm IR}^2} \le \frac{\sqrt{\pi}}{2}\frac{\Gamma\Big(1-\frac{1}{p}\Big)}{\Gamma\Big(1+\frac{1}{2p}\Big)}\,. \tag{4.33}$$

The divergence of the right hand side for $p \to 1$, where the bound seems to disappear, is an artifact of the Callan-Symanzik regulator which is insufficient to control all UV divergences in $d = 4$. In order to stay away from this artifact, we consider regulators in the range $p \in [2, \infty]$.

For a comparison with the $d = 3$ case, let us first set $\xi_{k_{\rm IR}} = 0$ and consider limiting regulator values,

$$\frac{\kappa^3}{k_{\rm IR}^3}\Big|_{p=2} \le \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{4})}, \qquad \frac{\kappa^3}{k_{\rm IR}^3}\Big|_{p\to\infty} \le \frac{\sqrt{\pi}}{2}. \tag{4.34}$$

From (4.33), it is obvious that the bound gets stronger (weaker) for positive (negative) coupling $\xi_{k_{\rm IR}}$. Most importantly, there is a nontrivial bound for any finite value of $\xi_{k_{\rm IR}}$.

While (4.34) has been derived analytically based on the interpolating approximation (4.28) for the heat kernel, a full calculation can be performed numerically. For this, we first have to isolate the divergent pieces by hand and treat them analytically as before. In fact, all divergent parts are related to the small curvature expansion of the heat kernel, i.e., to the expansion coefficients displayed in (4.26). Treating them separately as before leaves us with a triple integral over the heat kernel parameter u in (4.25), the propertime s and the RG scale k. A transition to dimensionless integration variables $t = \kappa^2 s$ and $\sigma = k/\kappa$ yields an integral representation depending only on the dimensionless parameter ratio $\kappa/k_{\rm IR}$. The mass-like term of the effective potential then acquires the form

$$U_{k_{\rm IR}}\Big|_{\phi_0^2} = -\frac{N_{\rm f}\phi_0^2}{2}\Big[\frac{1}{\bar\lambda_{\rm cr}} - \frac{1}{\bar\lambda_\Lambda} + \kappa^2\mathcal{A}\Big(\frac{\kappa}{k_{\rm IR}};p\Big)\Big]$$
$$- 12N_{\rm f}\xi_{k_{\rm IR}}\phi_0^2\kappa^2\,, \tag{4.35}$$

with the function $\mathcal{A}$ to be evaluated by numerical integration. Assuming subcritical fermion interactions $\bar{\lambda}_\Lambda \leq \bar{\lambda}_{\rm cr}$, the curvature bound can be expressed as

$$\frac{1}{2}\mathcal{A}\Big(\frac{\kappa}{k_{\rm IR}};p\Big) + 12\xi_{k_{\rm IR}} \geq 0\,, \tag{4.36}$$

in order to avoid fermion mass generation from gravitational catalysis. The function

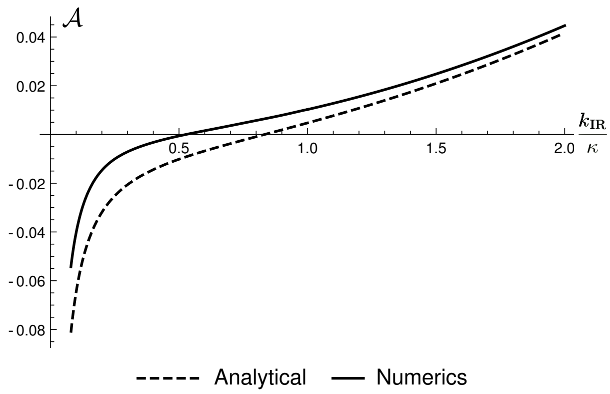


Fig. 4.2.: Scaling of the mass-like term of the effective potential: the function $\mathcal{A}$, cf. (4.35), (solid line) as a function of the inverse curvature parameter $k_{\rm IR}/\kappa$ is compared to the analytical approximation obtained, cf. (4.29), (dashed line) for the case $p = 2$. For the case of $\xi_{k_{\rm IR}} = 0$, positive values of $\mathcal{A}$ are compatible with the absence of chiral symmetry breaking and the existence of chiral fermions at low energies. The zero crossing corresponds to the curvature bound for gravitational catalysis.

$\mathcal{A}$ is plotted in Fig. 4.2 as a function of $k_{\rm IR}/\kappa$ for $p = 2$ (solid line). For comparison, the dashed line represents the result from the analytical interpolation matching the full behavior qualitatively for all curvatures. The strong and weak curvature asymptotics matches very well: we have checked that the leading powerlaws for both results are the same with coefficients agreeing within an error below the 1% level. In the intermediate curvature region, the deviations between the numerical result and the analytical estimate are larger.

For $\xi_{k_{\rm IR}} = 0$, the zero of the curve marks the curvature bound, since positive values of $\mathcal{A}$ are compatible with the absence of chiral symmetry breaking. From the numerical analysis we obtain the curvature bound,

$$\frac{\kappa}{k_{\rm IR}}\Big|_{p=2} \leq 1.8998, \qquad \frac{\kappa}{k_{\rm IR}}\Big|_{p\to\infty} \leq 1.5757. \tag{4.37}$$

for the two limiting regulators, showing that the full solutions deviate from the approximated ones by about 40%.

A finite $\xi_{k_{\rm IR}}$ parameter corresponds to a linear vertical shift of the graph in Fig. 4.2 and a corresponding shift of the zero crossing marking the curvature bound. Figure 4.3 shows the curvature of the effective potential at the origin (normalized by $N_{\rm f}\kappa^2/2$) as a function of the curvature parameter $\kappa/k_{\rm IR}$ for various values of $\xi_{k_{\rm IR}}$. Positive values are

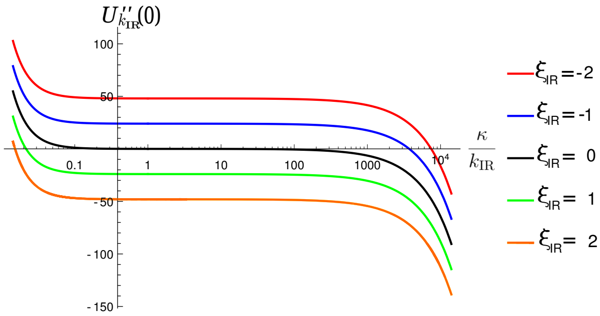


Fig. 4.3.: Scaling behavior of the curvature of the effective potential at the origin (normalized by $N_{\rm f}\kappa^2/2$) as a function of the curvature parameter $\kappa/k_{\rm IR}$ for the case $p = 2$ and different values of the marginal coupling $\xi_{k_{\rm IR}}$. The zero crossing marks the curvature bound which is strengthened for increasing values of $\xi_{k_{\rm IR}}$.

compatible with the existence of light chiral fermions.

4.5. Higher dimensions

4.5.1. $d = 6$

It is instructive to also study curvature bounds in higher dimensions. Perturbative non-renormalizability implies that further relevant operators and thus physical couplings have to be accounted for; still, for any finite dimension, also the number of additional couplings is finite at mean field level. As before, we have to pay attention only to those operators that couple to the mass-like term in the effective potential. Other operators do not directly take influence on the curvature bound for chiral symmetry.

In $d = 6$ dimensional spacetime, one further divergence of this type is encountered requiring to consider one more physical parameter. As before, the divergences are in correspondence with the small propertime expansion of the heat kernel for which we need to retain only the 0-th order of the hyperbolic cotangent expansion inside the heat kernel,

$$\begin{aligned} K_s^{\rm div} =& \frac{1}{(4\pi s)^3}\int_0^\infty du\, e^{-u^2} u(u^2+\kappa^2 s)(u^2+4\kappa^2 s) \\ =& \frac{1}{(4\pi s)^3}(1+\frac{5}{2}\kappa^2 s + 2\kappa^4 s^2)\,. \end{aligned} \tag{4.38}$$

The divergencies associated with the curvature dependent terms are controlled by initial conditions for the two operators

$$U_\Lambda|_{\phi^2 R,\phi^2 R^2} = N_{\rm f}\xi_\Lambda \phi_0^2 R + N_{\rm f}\chi_\Lambda \phi_0^2 R^2\,. \tag{4.39}$$

Adding these two operators to the terms arising from (4.38), yields the following contributions to the mass-like term in the effective potential:

$$\begin{aligned} U^{\rm div}_{k_{\rm IR}}\Big|_{\phi_0^2} &= -\frac{N_{\rm f}\phi_0^2}{2}\bigg\{\frac{\Lambda^4-k_{\rm IR}^4}{2(4\pi)^3}\Gamma\Big(1-\frac{2}{p}\Big)+\kappa^2\Big[60\xi_\Lambda+5\frac{\Lambda^2-k_{\rm IR}^2}{2(4\pi)^3}\Gamma\Big(1-\frac{1}{p}\Big)\Big] \\ &\quad +\kappa^4\Big[-1800\chi_\Lambda+\frac{4}{(4\pi)^3}\log\Big(\frac{\Lambda}{k_{\rm IR}}\Big)\Big]\bigg\} \\ &\equiv -\frac{N_{\rm f}\phi_0^2}{2}\bigg\{\frac{1}{\bar\lambda_{\rm cr}}-\frac{k_{\rm IR}^4}{(4\pi)^3}\Gamma\Big(1-\frac{2}{p}\Big)-2\xi_{k_{\rm IR}}R-2\chi_{k_{\rm IR}}R^2\bigg\}\,. \end{aligned} \tag{4.40}$$

Here, we have used that $\kappa^2=\frac{|R|}{d(d-1)}=\frac{|R|}{30}$ in $d=6$, and identified the critical coupling

$$\bar\lambda_{\rm cr}=\frac{2(4\pi)^3}{\Lambda^4\Gamma\Big(1-\frac{2}{p}\Big)}\,. \tag{4.41}$$

The parameter ξ_Λ has positive mass dimensions ($[\xi_\Lambda]=2$) and thus the operator ϕ_0^2R is now a power counting relevant operator, while $\phi_0^2R^2$ is marginal and the corresponding coupling χ_Λ has vanishing mass dimensions. The curvature dependent terms in the last line of (4.40) are finite and need to be fixed by a measurement. As before, the divergence hidden in the critical coupling will be balanced by the initial condition for the bare coupling $\bar\lambda_\Lambda$.

This concludes the analytical treatment of the divergent parts. The remaining regular part of the effective potential can then be integrated straightforwardly by numerical means as in the four dimensional case. In order to stay away from regulator artifacts, we choose the regulator parameter in the range $p\in[4,\infty]$. With the usual assumption of subcriticality, the dependence of the resulting mass-like term of the effective potential (normalized by $N_{\rm f}\kappa^2/2$) as a function of the curvature parameter $\kappa/k_{\rm IR}$ for the case $p=4$ and all further couplings $\xi_{k_{\rm IR}},\chi_{k_{\rm IR}}$ set to zero is depicted in Fig. 4.4.

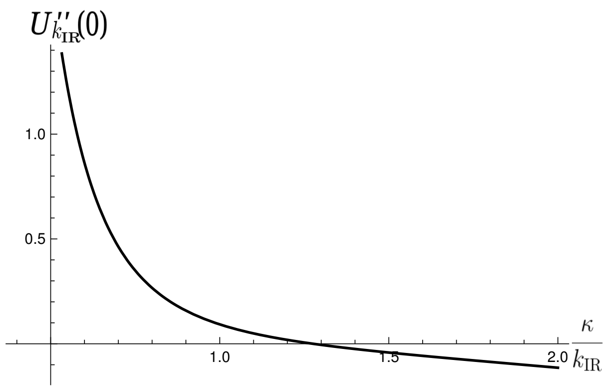


Fig. 4.4.: $d=6$ scaling behavior of the curvature of the effective potential at the origin (normalized by $N_{\rm f}\kappa^2/2$) as a function of the curvature parameter $\kappa/k_{\rm IR}$ for the case $p=4$ and $\xi_{k_{\rm IR}}=0=\chi_{k_{\rm IR}}$. Positive values are compatible with the existence of light chiral fermions. The zero crossing marks the curvature bound.

For a fair comparison of the curvature bounds for different spacetime dimensions, two conditions need to be met:

1. The physical parameters have to be chosen such that the relevant operator content is comparable.

2. The same p parameter needs to be employed for the regularization procedure.

For the first condition, we simply set all independent scalar curvature couplings to zero, $\xi_{k_{\mathrm{IR}}} = 0 = \chi_{k_{\mathrm{IR}}}$. For the second condition, we first check $p = 4$ for numerical simplicity. This results in

$$\begin{aligned} \frac{\kappa}{k_{\mathrm{IR}}} &\leq 1.7039\,, \quad d = 4\,, \quad \xi_{k_{\mathrm{IR}}} = 0\,, \\ \frac{\kappa}{k_{\mathrm{IR}}} &\leq 1.2763\,, \quad d = 6\,, \quad \xi_{k_{\mathrm{IR}}} = 0\,, \quad \chi_{k_{\mathrm{IR}}} = 0\,. \end{aligned} \tag{4.42}$$

The same analysis can be performed in the $p \to \infty$ limit. This scenario can be implemented noticing that the cutoff function reduces to a Heaviside θ function centered in $s = \frac{1}{k^2}$ and its derivative is therefore a Dirac δ distribution. In six dimensions, we obtain

$$\left.\frac{\kappa}{k_{\mathrm{IR}}}\right|_{p\to\infty} \leq 1.0561\,. \tag{4.43}$$

We observe that, for both values of p, the bound for the dimensionless curvature parameter decreases with increasing the spacetime dimensions (compare with (4.37)). We verify this circumstantial evidence in the next section for all odd dimensions. A general discussion follows below.

4.5.2. Odd dimensions: $d = 2n + 1$

The odd dimensional case is more easily analytically accessible thanks to the absence of the hyperbolic cotangent in the heat kernel (cf. (E.11) and (E.12)). In line with the preceding studies, we associate the curvature bound with a possible sign change of the mass-like term in the effective potential. Thus, it suffices to focus on the ϕ_0^2 order of the effective potential. Inserting (E.11) into (4.12) and expanding in powers of the field, we obtain

$$\begin{aligned} U_{k_{\mathrm{IR}}}\Big|_{\phi_0^2} =& U_\Lambda\Big|_{\phi_0^2} - \frac{N_{\mathrm{f}}\phi_0^2}{2}\left[\frac{4p\kappa^{d-2}}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\int_{\frac{k_{\mathrm{IR}}}{\kappa}}^{\frac{\Lambda}{\kappa}} d\sigma \int_0^\infty dt\; t^{p-\frac{d}{2}}\sigma^{2p-1}e^{-(\sigma^2 t)^p}\times\right. \\ &\left.\times\int_0^\infty du\; e^{-u^2}\prod_{j=\frac{1}{2}}^{\frac{d}{2}-1}(u^2+j^2 t)\right], \end{aligned} \tag{4.44}$$

where we have defined the dimensionless integration variable as $\sigma = \frac{k}{\kappa}$ and $t = \kappa^2 s$. The effective potential can be decomposed into the following building blocks,

$$U_{k_{\mathrm{IR}}} = U_\Lambda\Big|_{\kappa=0} + U_{k_{\mathrm{IR}}}\Big|_{\kappa=0} + U^{\mathrm{reg}}_{k_{\mathrm{IR}},\kappa} + U_\Lambda\big|_{\phi^2 R^n} + U_{k_{\mathrm{IR}}}\big|_{\phi^2 R^n}, \tag{4.45}$$

which we discuss separately in the following, concentrating on their quadratic part. The first two terms correspond to the contribution from the flat space physics. By renormalizing the fermionic self interaction, these terms exhibit the balance between the bare coupling $\bar\lambda_\Lambda$ and the leading cutoff divergence. The latter arises from the monomial containing the highest power of u in the product in (4.44) $\sim u^{d-1}$, and can be summarized in the definition of the critical coupling

$$\bar\lambda_{\mathrm{cr}} = \frac{(4\pi)^{\frac{d}{2}}(d-2)}{2\Lambda^{d-2}\Gamma\Big(1-\frac{d}{2p}+\frac{1}{p}\Big)}. \tag{4.46}$$

For the flat space part, we thus obtain

$$U_\Lambda\Big|_{\phi_0^2,\kappa=0} + U_{k_{\mathrm{IR}}}\Big|_{\phi_0^2,\kappa=0} = \frac{N_{\mathrm{f}}\phi_0^2}{2}\left[\frac{1}{\bar\lambda_\Lambda} - \frac{1}{\bar\lambda_{\mathrm{cr}}} + \frac{2\Gamma\Big(1-\frac{d}{2p}+\frac{1}{p}\Big)}{(4\pi)^{\frac{d}{2}}(d-2)}k_{\mathrm{IR}}^{d-2}\right]. \tag{4.47}$$

The only a priori UV-regular term in (4.44) comes from the u-independent monomial arising from the product inside the last integral. It contains the relevant curvature dependence for gravitational catalysis:

$$\begin{aligned} U^{\mathrm{reg}}_{k_{\mathrm{IR}},\kappa}\Big|_{\phi_0^2} &= -\frac{N_{\mathrm{f}}\phi_0^2}{2}\frac{4p\kappa^{d-2}}{(4\pi)^{\frac{d}{2}}\Gamma\Big(\frac{d}{2}\Big)}\int_{\frac{k_{\mathrm{IR}}}{\kappa}}^{\frac{\Lambda}{\kappa}} d\sigma \int_0^\infty dt\, t^{p-\frac{d}{2}}\sigma^{2p-1}e^{-(\sigma^2 t)^p}\int_0^\infty du\, e^{-u^2}\frac{\Gamma^2\Big(\frac{d}{2}\Big)}{\pi}t^{\frac{d-1}{2}} \\ &= -\frac{N_{\mathrm{f}}\phi_0^2}{2}\frac{2\Gamma\Big(\frac{d}{2}\Big)\Gamma\Big(1+\frac{1}{2p}\Big)}{(4\pi)^{\frac{d}{2}}\sqrt{\pi}}\frac{\kappa^{d-1}}{k_{\mathrm{IR}}}, \end{aligned} \tag{4.48}$$

where we have taken the limit $\Lambda \to \infty$ in the last line.

All other monomials in the product of (4.44) carry UV divergencies, thus indicating the necessity to provide initial conditions for further operators. In total, we need $\frac{d-3}{2}$ operators with scalar curvature couplings and correspondingly many physical parameters to be fixed by a measurement. The required operators are of the form $N_{\mathrm{f}}\xi_{\Lambda,m}\phi^2 R^n$. Here, we choose conventions such that the index m corresponds to a specific monomial in the above expression and $\xi_{\Lambda,m}$ parametrizing the initial condition for the bare coupling to be fixed. In order to analyze these contributions, we represent the polynomial part of the heat kernel as

$$\prod_{j=\frac{1}{2}}^{\frac{d}{2}-1}(u^2+j^2t)=\sum_{m=0}^{\frac{d-1}{2}}C_m u^{2m}t^{\frac{d-1}{2}-m}\,, \tag{4.49}$$

where C_m denotes the numerical coefficients arising from the product. The resulting curvature dependence for each m results in a power R^n with $n=\frac{d-1}{2}-m$. The $m=0$ term corresponds to the regular monomial computed in (4.48), while the $m=\frac{d-1}{2}$ term equals the curvature independent part of the heat kernel, already dealt with in (4.47). The remaining terms with $1\leq m\leq(d-3)/2$ make up for the last two terms in our decomposition (4.45) of the effective potential. They correspond to new operators where the matter sector is coupled to the background scalar curvature. The computation of the effective potential yields

$$\begin{aligned}U_\Lambda\big|_{\phi_0^2R^n}+U_{k_{\rm IR}}\big|_{\phi_0^2R^n}=&-N_{\rm f}\phi_0^2\sum_{m=1}^{\frac{d-3}{2}}\kappa^{d-1-2m}\Big\{(-1)^{\frac{d-1}{2}-m-1}\xi_{\Lambda,m}[d(d-1)]^{\frac{d-1}{2}-m}\\&+\frac{C_m\Gamma\Big(m+\frac{1}{2}\Big)\Gamma\Big(1-\frac{m}{p}+\frac{1}{2p}\Big)}{(4\pi)^{\frac{d}{2}}\Gamma\Big(\frac{d}{2}\Big)(2m-1)}(\Lambda^{2m-1}-k_{\rm IR}^{2m-1})\Big\}\\ \equiv&-N_{\rm f}\phi_0^2\sum_{m=1}^{\frac{d-3}{2}}(-1)^{\frac{d-1}{2}-m-1}\kappa^{d-1-2m}\xi_{k_{\rm IR},m},\end{aligned} \tag{4.50}$$

As before, the Λ-dependent terms combine with the bare couplings such that the long range interactions $\xi_{k_{\rm IR},m}$ are formed; for a physical system, the latter are finite and have to be fixed by a measurement. It is clear that possible curvature bounds will depend on these couplings. For the reason of comparing theories with different dimensionality, we set all these couplings to zero $\xi_{k_{\rm IR},m}=0$ at the scale $k_{\rm IR}$. Let us study two cases explicitly.

$d=5$

Inserting Eqs. (4.47) and (4.48) into (4.45) and using that (4.50) gives a vanishing contribution for $\xi_{k_{\rm IR},m}=0$, it is straightforward to obtain the following result for the mass-like term of the scale dependent effective potential in $d=5$ dimensional spacetime:

$$U_{k_{\rm IR}}^{\rm d=5}\Big|_{\phi_0^2}=\frac{N_{\rm f}\phi_0^2}{2}\Big[\frac{1}{\bar\lambda_\Lambda}-\frac{1}{\bar\lambda_{\rm cr}}+\frac{\Gamma\big(1-\frac{3}{2p}\big)}{48\pi^{\frac{5}{2}}}k_{\rm IR}^3-\frac{3\Gamma\big(1+\frac{1}{2p}\big)}{64\pi^{\frac{5}{2}}}\frac{\kappa^4}{k_{\rm IR}}\Big]\,. \tag{4.51}$$

Assuming again a subcritical coupling as initial condition of the flow, we can identify the bound for the ratio between the curvature parameter and the averaging scale, below which symmetry breaking is not catalyzed gravitationally:

$$\left(\frac{\kappa}{k_{\mathrm{IR}}}\right)^4 \leq \frac{4}{9}\frac{\Gamma\left(1-\frac{3}{2p}\right)}{\Gamma\left(1+\frac{1}{2p}\right)}\,. \tag{4.52}$$

In order to stay away from artifacts arising from insufficient regulators, we choose p in the interval $p \in [2, \infty]$. For the two extremal cases, we have:

$$\frac{\kappa}{k_{\mathrm{IR}}} \leq \frac{2}{\sqrt{3}} \simeq 1.154 \quad \text{for} \quad p = 2\,, \tag{4.53}$$

$$\frac{\kappa}{k_{\mathrm{IR}}} \leq \sqrt{\frac{2}{3}} \simeq 0.816 \quad \text{for} \quad p = \infty\,. \tag{4.54}$$

$d = 7$

Similarly, the mass-like term of the effective potential in $d = 7$ dimensional spacetime reads

$$U^{\mathrm{d}=7}_{k_{\mathrm{IR}}}\Big|_{\phi_0^2} = \frac{N_{\mathrm{f}}\phi_0^2}{2}\left[\frac{1}{\bar\lambda_\Lambda} - \frac{1}{\bar\lambda_{\mathrm{cr}}} + \frac{\Gamma\left(1-\frac{5}{2p}\right)}{320\pi^{\frac{7}{2}}}k_{\mathrm{IR}}^5 - \frac{15\Gamma\left(1+\frac{1}{2p}\right)}{512\pi^{\frac{7}{2}}}\frac{\kappa^6}{k_{\mathrm{IR}}}\right]. \tag{4.55}$$

This time, a range of admissible regulators includes $p \in [3, \infty]$. Assuming a subcritical coupling, we can again read off the curvature bounds which for the extremal regulators are given by

$$\frac{\kappa}{k_{\mathrm{IR}}} \leq 0.928\,, \quad \text{for} \quad p = 3\,, \tag{4.56}$$

$$\frac{\kappa}{k_{\mathrm{IR}}} \leq 0.689\,, \quad \text{for} \quad p = \infty\,. \tag{4.57}$$

Dimensional dependence

As is obvious from all these examples, the curvature bound arises from a competition between the screening of the long range modes parametrized by the last term in (4.47) and the dominant curvature term given by (4.48). For general d, we need to use the regulator with $p \to \infty$ to ensure that we stay away from regularization artifacts in any d. In order to perform a meaningful comparison, we set all possible nonzero scalar curvature interactions terms $\sim \xi_{k_{\mathrm{IR}},m}$ to zero. For this, the curvature bound can be expressed as follows:

$$\frac{\kappa}{k_{\mathrm{IR}}} \leq \frac{1}{\sigma_0} \equiv \left(\frac{\sqrt{\pi}}{\Gamma\left(\frac{d}{2}\right)(d-2)}\right)^{\frac{1}{d-1}}\,, \tag{4.58}$$

exhibiting a monotonically decreasing behavior as is visible in Fig. 4.5. Asymptotically,

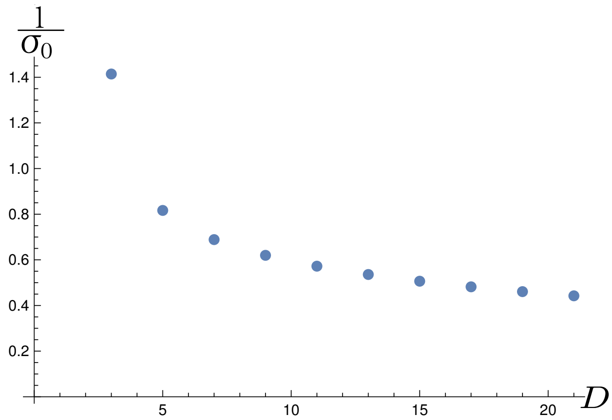


Fig. 4.5.: Curvature bound (4.58) as a function of the spacetime dimensions in the odd dimensional case for the regulator with $p \to \infty$.

the bound decays as $\sim 1/\sqrt{d}$.

Of course, as soon as the couplings $\sim \xi_{k_{\mathrm{IR}},m}$ are switched on, the bound can be shifted in both directions depending on the precise parameter values. Nevertheless it cannot be washed away completely.

This result inspires to develop the following scenario: Let us assume that some fundamental theory of spacetime and matter can have a high energy phase of arbitrary dimension and allows for a regime where a metric description applies. If the theory in addition exhibits fluctuating values of curvature $\kappa \sim \mathcal{O}(1)$ when averaged over local patches, our results suggest that it is unlikely to find higher dimensional regions that admit massless or light fermions in the long range physics. Upon the onset of gravitational catalysis, higher dimensional regions would then generically go along with a massive fermionic particle content and without explicit chiral symmetry.

4.6. Constraining quantum gravity

As an illustration for the application of our curvature bound, we use a specific quantum gravity scenario in $d = 4$ dimensional spacetime: asymptotically safe gravity [25–31, 165]. In this scenario, Einstein's gravity arises as the low energy limit of a quantum field theory of the metric, the high energy behavior of which is controlled by a non-Gaussian fixed point in the space of relevant couplings. Typically the renormalization flow of local metric theories is computed employing FRG techniques. The gauge degrees of freedom of the spin 2 field are introducing several complications with respect to the analysis presented in section 3.2, the most important of which is the fact that the regulator breaks gauge invariance. The normal procedure is then to rely on background field gauge [166, 167] with a fiducial but arbitrary background metric and compute the renormalization flow for the fluctuating field on the background. This breaks the gauge group down to the group of gauge symmetry for the background metric. After the cutoff is removed the result can be reinterpreted in terms of the full quantum fields and the theory should recover the full gauge invariance.

Among the universal properties of asymptotically safe (pure) gravity [168–173] we have that the critical exponents around the ultraviolet fixed point are complex and trajectories result in spirals falling into the fixed point[32]. This behavior, though, changes upon inclusions of fermionic matter. In this case the position of the interacting fixed point is pushed more and more towards smaller values of the cosmological constant until, eventually, this becomes negative (hence describing a hyperbolic scenario) and the critical exponents become real.

For simplicity, we confine ourselves to the theory space spanned by the Einstein-Hilbert action. A more comprehensive analysis suggests the existence of one further relevant operator overlapping with an R^2-term in the action [174–187]. For a first glance at the consequences of the curvature bound, we also ignore the influence of the scalar curvature operator $\sim \phi^2 R$, which is, in principle, calculable within asymptotic safety from the fermionic operator content, i.e., schematically $\sim (\bar{\psi}\psi)^2 R$.

In the simple Einstein-Hilbert truncation, the background metric itself is a solution to the equations of motion derived from the scale dependent effective action [155],

$$R_{\mu\nu}(\langle g \rangle_k) = \bar{\Lambda}_k \langle g_{\mu\nu} \rangle_k, \tag{4.59}$$

where $\bar{\Lambda}_k$ denotes the scale dependent cosmological constant, and k is the coarse graining or resolution scale, at which the spacetime is considered. Here, we have assumed the absence of any explicit matter sources since we expect the symmetry to be unbroken in the UV. The asymptotic safety scenario provides us with a prediction for the RG trajectories for the cosmological constant $\bar{\Lambda}_k$, as well as for the UV fixed point value $\lim_{k\to\infty} \bar{\Lambda}_k/k^2 = \lambda_*$ being a finite number. In the fixed point regime, the solution to (4.59) is given by

$$\frac{R}{k^2} = 4\lambda_*. \tag{4.60}$$

This shows that the sign of the curvature in the fixed point regime is dictated by the sign of the fixed point value of the cosmological constant. Equation (4.60) exemplifies the self similarity property of physical observables in the fixed point regime: the curvature is proportional to the scale k at which the curvature is measured. Since the fixed point value λ_* may come out negative for an increasing number of fermionic degrees of freedom the spacetime structure appears locally as negatively curved for large N_{f}. The asymptotic safety scenario including matter degrees of freedom hence predicts that a local patch of

[32] The critical exponents themselves should be universal. Nevertheless the necessary approximations introduce deviations from the universal values.

spacetime in the (trans-Planckian) fixed point regime satisfies

$$\frac{\kappa^2}{k^2} = \frac{|\lambda_*|}{3}, \quad \text{for } \lambda_* < 0. \tag{4.61}$$

Now, the precise value of the fixed point λ_* is scheme dependent, see, e.g., [168–173] for comparative studies. With regard to (4.60) this is natural, since the result of a curvature measurement is expected to depend on the coarse graining procedure that is used to average over metric fluctuations. This is precisely the type of scheme dependence, we expect to cancel the scheme dependence of our curvature bounds in order to arrive at a scheme independent answer to the question as to whether or not there is gravitational catalysis in a given theory.

For the remainder of the section, we simply identify the gravitational RG coarse graining scale k with the scale k_{IR} used for our curvature bounds and use results obtained in the asymptotic safety literature. In fact, the typical fixed point scenario can already be discovered within a simple one-loop calculation [31, 169], yielding the fixed point values for the cosmological constant and the dimensionless Newton constant

$$\lambda_* = \frac{3}{4}\frac{2+d_\lambda}{46-d_g}, \quad g_* = \frac{12\pi}{46-d_g}. \tag{4.62}$$

Here, we used the results obtained from a so called type IIa cutoff [31]. The two parameters d_g and d_λ are determined by the number of (free) matter degrees of freedom,

$$d_g = N_{\text{S}} - 4N_{\text{V}} + 2N_{\text{f}}, \quad d_\lambda = N_{\text{S}} + 2N_{\text{V}} - 4N_{\text{f}}, \tag{4.63}$$

where N_{S} denotes the number of real scalar fields, N_{V} the number of gauge vector bosons and, as before, N_{f} the number of Dirac fermion flavors.

For gravitational catalysis to be potentially active at all, we need a negative fixed point value $\lambda_* < 0$, implying

$$N_{\text{f}} > \frac{1}{2} + \frac{N_{\text{S}}}{4} + \frac{N_{\text{V}}}{2}. \tag{4.64}$$

This criterion is satisfied for the standard model with $N_{\text{S}} = 4$, $N_{\text{V}} = 12$ and $N_{\text{f}} = 45/2$, as well as typical generalizations with right handed neutrino components, axion or simple scalar dark matter models. It is also generically satisfied for supersymmetric models; for instance, for the MSSM with two Higgs doublets, we have $N_{\text{S}} = 53$, $N_{\text{V}} = 12$ and $N_{\text{f}} = 65/2$. This exemplifies that the curvature bound should be monitored in asymptotically safe gravity-matter systems. However, the criterion (4.64) is typically not satisfied for GUT-like non-supersymmetric theories where the contribution from larger number of gauge bosons and Higgs fields for the necessary symmetry breaking exceeds that of the fermion flavors.

For a given number of scalars and vectors, increasing the number of flavors drives the fixed point λ_* towards more negative values. Using (4.61) with $k = k_{\mathrm{IR}}$, the averaged curvature can eventually violate the curvature bound. Hence, the curvature bound translates into an upper bound $N_{\mathrm{f}} \leq N_{\mathrm{f,gc}}$ on the number of fermion flavors in order not to be inflicted by chiral symmetry breaking from gravitational catalysis. For instance, for a purely fermionic matter content, $N_{\mathrm{S}} = 0 = N_{\mathrm{V}}$, we find $N_{\mathrm{f,gc}} \simeq 17.58$ for $p \to \infty$, and $N_{\mathrm{f,gc}} \simeq 18.31$ for $p = 2$, cf. (4.37). The scheme dependence of our curvature bound thus has only a mild influence on the critical fermion number.

Similarly, fixing the bosonic matter content to that of the standard model, $N_{\mathrm{S}} = 4$, $N_{\mathrm{V}} = 12$, the corresponding critical fermion number is $N_{\mathrm{f,gc}} \simeq 35.97$ for $p \to \infty$. This would still allow for a fourth generation of standard model flavors, but exclude a fifth generation.

Interestingly, the MSSM with $N_{\mathrm{S}} = 53$ and $N_{\mathrm{V}} = 12$ would imply a critical flavor number of $N_{\mathrm{f,gc}} \simeq 20.3$ far below the fermionic content of the model $N_{\mathrm{f}} = 65/2$, thus indicating a possible tension between asymptotically safe gravity and a particle physics matter content of that of the MSSM because of gravitational catalysis.

This analysis based on a simple one-loop calculation on the gravity side may be somewhat over simplistic. In fact, a number of more sophisticated analyses have been performed for asymptotically safe gravity in conjunction with matter systems. A first study on the consistency of asymptotic safety with matter [82] was based on the background field approximation with some improvements for the anomalous dimensions. Using their fixed point results, we find $N_{\mathrm{f,gc}} \simeq 8.21$ for a purely fermionic model ($N_{\mathrm{S}} = 0 = N_{\mathrm{V}}$), and $N_{\mathrm{f,gc}} \simeq 26.5$ for the standard model with $N_{\mathrm{S}} = 4$, $N_{\mathrm{V}} = 12$ (and anomalous dimensions set to zero). The latter result includes the standard model fermion content without and with right handed neutrino partners, but does not offer room for a fourth generation. For the MSSM and other models there is not even a gravitational fixed point according to [82]. Even if we artificially reduce the number of fermion flavors, we do not find a suitable fixed point above $N_{\mathrm{f}} \simeq 17$. Here, λ_* has become negative but the curvature bound is still satisfied.

The fixed point scenario found in [83, 92] is different. The calculation distinguishes between the background field and the dynamical fluctuation field. The flow of the dynamical couplings which is driven by the dynamical correlators [188] is found to have a gravitational UV fixed point for any matter content that has been accessible in this study. This scenario hence does not rule out any particle physics content from the side of UV compatibility with quantum gravity. Still, the predictions for the background field couplings are qualitatively similar to those of [82]. The fixed point results of [83] upon insertion into (4.61) and a comparison with the curvature bound suggest $N_{\mathrm{f,gc}} \simeq 48.7$ for $p \to \infty$ for a purely fermionic model with $N_{\mathrm{S}} = 0 = N_{\mathrm{V}}$; for $p \to 2$, the results of [83] lead to $N_{\mathrm{f,gc}} \simeq 50.9$.

An analysis of gravity-matter systems was performed in [87] using an ADM decompo-

	$N_{\mathrm{f,gc}}$		
	PF	SM+N_{f}	MSSM+N_{f}
one-loop approx. (type IIa) [31, 169]	17.58	35.97	20.3
background field approximation [82]	8.21	26.5	no FP
RG flow on foliated spacetimes [87]	9.27	27.67	10.01
dynamical FRG [83]	48.7		

Tab. 4.1.: Summary of the critical number of fermion species $N_{\mathrm{f,gc}}$ below which particle theories are safe from chiral symmetry breaking through gravitational catalysis, using the $p \to \infty$ regulator. Results are shown for theories with a purely fermionic matter content (PF), the standard model and the MSSM artificially varying the number of fermions (SM+N_{f}, MSSM+N_{f}). For an estimate of the UV properties of quantum spacetime, we use various literature results obtained within the asymptotic safety scenario of quantum gravity, see main text for details.

sition of the gravitational degrees of freedom, yielding an RG flow on foliated spacetimes. For both, gravitational as well as matter degrees of freedom, a type I regulator was used. As argued by the authors, the use of different regulators can be viewed as yielding a different map of the number of degrees of freedom N_{S}, N_{V} and N_{f} onto the parameters d_g and d_λ; e.g., for the type I regulator, one gets [31, 87] $d_g = N_{\mathrm{S}} - N_{\mathrm{V}} - N_{\mathrm{f}}$. It has been argued that the type II regulator should be used for fermions in order to regulate the fluctuation spectrum of the Dirac operator in a proper fashion [31, 189]. The basic difference between these two approaches can be traced back to the well known identity:

$$\nabla\!\!\!/^2 = -\Box + \frac{R}{4}\,. \tag{4.65}$$

The type I regulator regards the right hand side of the previous equation as a function of $-\Box$ of the form $g(x) = x + \frac{R}{4}$. The FRG cutoff function would then be a function of just the box operator $R_k = R_k(-\Box)$ and the Wetterich equation would read

$$k\partial_k \Gamma_k = \frac{1}{2}\mathrm{STr}\left[k\partial_k R_k(-\Box)\left(-\Box + R_k(-\Box) + \frac{R}{4}\right)^{-1}\right]. \tag{4.66}$$

Applying the Schwinger representation we would deal with the heat kernel of $-\Box$ and the Legendre transform of

$$\frac{k\partial_k R_k(x)}{x + R_k(x) + \frac{R}{4}}\,. \tag{4.67}$$

The type II regulator instead would see the right hand side of (4.65) as a single operator with its own spectrum and heat kernel and the Wetterich equation would see a replacement of $R_k(-\Box)$ with $R_k(-\Box+\frac{R}{4})$. The conjecture that favorites one regularization over the other comes from a comparison with results on maximally symmetric spaces, where the spectrum of the squared Dirac operator is known exactly. Hence, we use the flows of [87] but with a definition of the parameters d_g and d_λ as in (4.63). To leading order, this corresponds to a type I regularization of the gravity fluctuations but a type II regulator for the matter degrees of freedom.

In this case, the possible onset of gravitational catalysis for a purely fermionic model with $N_S = 0 = N_V$ occurs at a critical flavor number $N_{\mathrm{f,gc}} = 9.27$ for $p \to \infty$ ($N_{\mathrm{f,gc}} = 9.84$ for $p = 2$). For a standard model like theory ($N_S = 4$, $N_V = 12$), we have $N_{\mathrm{f,gc}} = 27.67$ $p \to \infty$ ($N_{\mathrm{f,gc}} = 28.71$ for $p = 2$). Finally, the minimally supersymmetric extension of the standard model would lead to $N_{\mathrm{f,gc}} = 10.01$ for $p \to \infty$ ($N_{\mathrm{f,gc}} = 10.27$ for $p = 2$), if we artificially allow N_{f} to vary independently in this model. Therefore the MSSM in this approximation is an example for a model where gravitational catalysis could lead to large fermion mass generation in the trans-Planckian regime; in fact, if N_{f} is set to the physical value $N_{\mathrm{f}} = 65/2$, the MSSM matter content in this setting does not lead to a fixed point suitable for asymptotically safe quantum gravity, see also [173].

We summarize the critical values for the fermion numbers $N_{\mathrm{f,gc}}$ for $p \to \infty$ for the possible onset of gravitational catalysis derived within the various approximations for an asymptotically safe quantum gravity scenario in Tab. 4.1. Whereas the standard model (e.g., also including right handed neutrinos) satisfies the bound from gravitational catalysis in each of thee approximations, a standard model with a fourth fermion generation could already be affected by gravitational catalysis. Supersymmetric versions of the standard model show already some tension with the bound within asymptotically safe gravity.

Using the results of [87] as described above, we display the various regions in the space of matter theories parametrized by d_g and d_λ, cf. (4.63), in Fig. 4.6. In the upper orange shaded region, the criterion analogous to (4.64) is not satisfied (in the calculation of [87], it corresponds to $d_\lambda > -16/3$); here, we expect a spacetime in the fixed point regime which is positively curved and thus not affected by gravitational catalysis. The curvature bound translates into a line in the d_g, d_λ plane, with the (white) region above that line satisfying the bound. We observe that the lines for different regulators $p \in [2, \infty]$ are rather similar and deviate significantly only for extreme particle numbers. The purely fermionic model (PF) and the standard model are represented by dots in the plane. The lines attached to the dots correspond to increasing the fermion number in these models. The purely fermionic model starts at $N_{\mathrm{f}} = 0$, while the standard model starts at its physical value $N_{\mathrm{f}} = 45/2$. The MSSM with $N_{\mathrm{f}} = 65/2$ would lie deep inside the black region to the right where no fixed point suitable for asymptotically safe gravity exists [87, 173]. Note that the evolution of the standard model and the purely fermionic theory are described by parallel lines since increasing the number of fermions have the same effect on the number of modes of the theories. For a supersymmetric model, new fermionic modes go hand in hand with new bosonic degrees of freedom and the slope of the line describing its evolution in the theory space would be different.

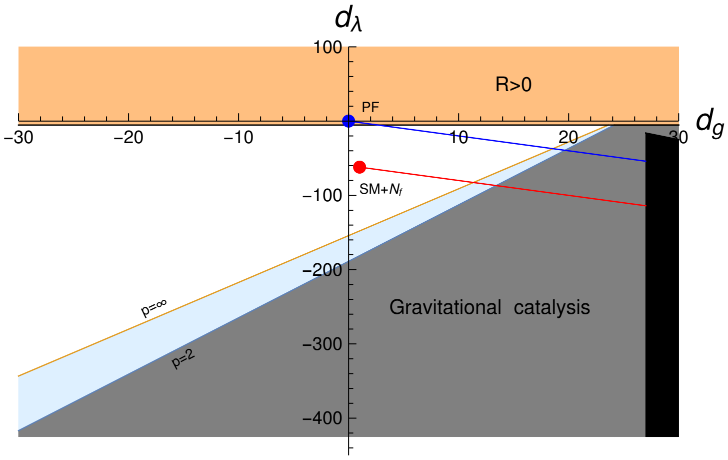


Fig. 4.6.: Relevance of gravitational catalysis in different regions in the space of matter theories parametrized by d_g and d_λ, cf. (4.63). In the bright orange region we have $\lambda_* > 0$, indicating that gravitational catalysis does not occur. Our curvature bound results in a line with the (white) region above that line satisfying the bound; we observe a mild regulator dependence for different regulator parameters $p \in [2, \infty]$. The dots indicate a purely fermionic model (PF) and the standard model (SM). The lines originating by these theories (resp. blue and red) represent their evolution for increasing N_f. The gray region indicates the region where gravitational catalysis can occur. The black region does not have a non-Gaussian fixed point suitable for asymptotic safety [87].

Let us close this section with two remarks: the first remark concerns the regularization scheme dependence which occurs at various places in this calculation. In case of a fully consistent calculation this scheme dependence would cancel in the final result for $N_{\mathrm{f,gc}}$. However, since different parts in the present estimates are performed with different regulators, we observe various sources of scheme dependence. Whereas the scheme dependence arising from our mean field RG calculation parametrized by p is rather mild, a change of the regulator from type II to type I in the asymptotic safety scenario can change the dependence on the fermion flavor content significantly as studied in the literature [31, 189]. Since our fermionic mean field RG calculation corresponds to a type II regularization, we find it reassuring that a consistent use of type II regulators for the fermions leads to qualitatively and partly quantitatively similar results in the various approximations.

Second, the asymptotic safety scenario suggests that at least one further relevant operator of R^2 type should be included in the fixed point regime. As this could take a significant quantitative influence on the effective equation of motion in the fixed point regime, cf. (4.59), the relevance of the curvature bound for the asymptotic safety scenario may also change qualitatively. With these reservations in mind, the present discussion should be viewed as an example how the curvature bound from gravitational catalysis could potentially be used to constrain combined scenarios of quantum gravity and quantum matter.

5. Scalar fields and covariant perturbation theory

This chapter is dedicated to the investigation of multicritical behavior in scalar field theories near criticality. The analysis is performed coupling the theories to a generic background manifold. The renormalization study carried over to determine the critical structure of the phase diagram is based on dimensional regularization in its covariant formulation. Thanks to a functional formalism it is possible to treat at the same time an infinite tower of models and classify the corresponding universality classes. A particular attention is payed to non-minimal coupling introduced in section 3.5.

5.1. Multicritical scalar field theories

The multicritical scalar models with φ^{2n} interaction are the simplest and most straightforward generalization of the φ^4 model. Much like the φ^4 field theory captures the critical properties of an universality class of models that includes the ferromagnetic Ising Hamiltonian, the φ^{2n} field theory can be thought as describing a generalization in which the Ising's spin domains of plus or minus sign are potentially replaced by n distinct vacuum states which become degenerate at the critical temperature.

The RG flow of the φ^{2n} models has been explored at length in the literature: perturbatively [104–106], nonperturbatively [107–109], and with non-canonical kinetic terms [110, 111]. For the most part the renormalization of the multicritical models generalizes the one of the φ^4 model, but the leading contributions to the critical exponents are determined by multiloop computations in which the number of loops increases with n [105].

Throughout this thesis we always regarded the dimensionality of the system as given a priori and tried to understand which operators need to be included in the truncation in order to have a comprehensive description of critical phenomena. Following a somehow opposite point of view we may wonder, given a certain truncation, in which dimension does the theory exhibit non-trivial critical behavior and in which dimension does it become Gaussian. According to the same principles described in section 2.4 we can define for each multicritical model an upper critical dimension d_n such that the φ^{2n} interactions describe theories that are Gaussian for $d > d_n$ and logarithmic at $d = d_n$, but have non-trivial critical exponents for $d < d_n$. Moreover, it is well known that a consistent perturbative expansion in the coupling can be constructed at the upper critical dimension

$$d_n = \frac{2n}{n-1}\,. \tag{5.1}$$

It is easy to check that, as expected, the case $n = 2$ corresponds to the φ^4 interaction which has upper critical dimension $d = 4$ [13]. A common practice is to compute such perturbative expansion and critical exponents in the ε-expansion described in 3.3, in which one introduces the constant $\varepsilon = d_n - d$ and uses it to parametrize the displacement of the critical point from the Gaussian theory at $d = d_n$ [13].

In section 3.3 we showed how to use the ε-expansion to extract the critical behavior near the upper critical dimension of the φ^4 model and to define the theory in the limit $\varepsilon \to 0$. Nevertheless, the parametrization allows for an analytic continuation of the theory far from the critical dimension.

For example, among the φ^{2n} models the only ones with integer upper critical dimension correspond to the case $n = 2$ discussed above and the case $n = 3$, with a φ^6 interaction known to describe the universal features of the tricritical Ising model with upper critical dimension $d = 3$ [190]. All other models have purely fractional upper critical dimensions [191] which asymptotically tend to $d = 2$. As a consequence $d = 2$ is the first physical dimension in which *all* the models φ^{2n} are nontrivial; the continuation to two dimensions is particularly relevant because they are known to interpolate with the unitary minimal models arising as representations of the infinite dimensional Virasoro algebra [192, 193].

The only multicritical model that has nontrivial exponents in $d = 3$ is the φ^4 one unless one includes the multicritical non-unitary models φ^{2n+1} [194].Specifically, φ^3 and φ^5 have upper critical dimensions $d = 6$ [195] and $d = \frac{10}{3}$ respectively, but they require the tuning of an imaginary-valued magnetic field at criticality [196–198]. It is important to mention that the $d = 2$ realizations of these models are all “far away” in a perturbative sense from their Gaussian points even though $d_n \to 2$ for $n \to \infty$ [105]. Nevertheless, the simple existence of the sequence of multicritical theories provides a very interesting and valuable link between purely field theoretical realizations and CFT representations [106, 199].

One natural and potentially interesting generalization of the above discussion is the study of the renormalization of the φ^{2n} models in *curved space*. As explained in section 3.5, if a scalar field theory is coupled with a background geometry, simple dimensional analysis reveals that there is a new non-minimal marginal interaction with the curvature: $\frac{1}{2}\xi\varphi^2 R$. One expects that in curved space the perturbative construction should thus accommodate for some mixing between the φ^{2n} and $\varphi^2 R$ operators regardless of n. In other words, the non-minimal interaction $\varphi^2 R$ holds a special status in that it is always canonically marginal.

A guess on the value that the coupling ξ can take at a curved space generalization of the critical point could be made as follows: Consider a non-minimally coupled “free” scalar field with quadratic action

$$S_0[\varphi] = \frac{1}{2} \int \mathrm{d}^d x \Big\{ g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \xi \varphi^2 R \Big\} \,. \tag{5.2}$$

Ideally, the above action captures the Gaussian limit of the φ^{2n} models which is realized exactly at the upper critical dimension. The non-minimal action is invariant under a conformal Weyl rescaling $g'_{\mu\nu} = \Omega^2(x) g_{\mu\nu}$ and $\varphi'(x) = \Omega^{1-\frac{d}{2}}(x)\varphi(x)$ iff the coupling ξ takes the *conformal value*

$$\xi_c = \frac{d-2}{4(d-1)} \,. \tag{5.3}$$

Since conformal invariance implies scale invariance, the non-minimal action (5.2) is thus scale invariant when ξ takes the conformal value, but it is also expected to be a description of the critical (scale-invariant) φ^{2n} model when the interaction becomes Gaussian at the upper critical dimension.

Putting everything together we make the hypothesis which we intend to investigate: the critical point of the coupling ξ at the upper critical dimension, which emerges as fixed point of the renormalization group, is the conformal value (5.3)[33].

More generally, one would be tempted to extend the above statement to any dimension *below* the upper critical dimension, having expressed the desire of analytically continuing these models to $d = 2$. In this case, we would want to know the conditions under which the conformal value (5.3) is always the critical value for ξ even when the φ^{2n} interaction is non-Gaussian below d_c. For this purpose it is instructive to recall the investigation by Brown and Collins [114], in which it is shown that at the leading order our hypothesis is true in the special case of the φ^4 model, but beyond the leading order one has to exploit the freedom of subtracting additional finite parts proportional to the leading counter terms [115]. An analog renormalization condition has also been adopted for the φ^3 model [116] in $d = 6$, and it plays an important role in preserving conformal invariance in [113, 117]. Notice that, strictly speaking, the said two examples have not been concerned with the analytic continuation below the upper critical dimensions $d = 4$ and $d = 6$, while our interest is to bring the multicritical models down to $d = 2$ which does require continuation. Assuming that we have the same freedom in changing the renormalization condition, one might be tempted to conjecture the fact that the critical point of the coupling ξ determined through the ε-expansion below the upper critical dimension can always be tuned to match the conformal value beyond the leading order. An explicit check of this hypothesis goes beyond the purpose of this thesis.

[33] The conformal invariance of (5.2) is actually expected to be anomalous [200], but for our purposes it is sufficient that scale invariance survives the quantization process. In even dimensions, the anomaly is signaled by special non-local contributions appearing in the effective action [201, 202].

5.2. Renormalization

We are interested in a simple self interacting canonically normalized scalar field φ which is non-minimally coupled to a background metric $g_{\mu\nu}$ in d dimensions. The straightforward bare action is

$$S[\varphi] = \int \mathrm{d}^d x \sqrt{g} \Big\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\phi) + F(\phi) R \Big\} . \tag{5.4}$$

Using the bare action we can formally construct the path integral. For later convenience we shall do it in the background field approach [166, 167], thus by integrating the fluctuations χ over an arbitrary background ϕ as follows

$$Z = \int \mathcal{D}\chi \; \mathrm{e}^{-S[\phi+\chi]} . \tag{5.5}$$

The background field method will allow us to renormalize the theory by means of solely vacuum diagrams[34].

In flat space it is possible to construct a meaningful perturbative expansion for potentials $V(\phi)$ which are polynomials of order $2n$ below the upper critical dimensions (5.1). If we parametrize $V(\phi) = \frac{\lambda}{(2n)!}\phi^{2n} + \dots$ and $F(\phi) = \frac{\xi}{2}\phi^2 + \dots$ the ϕ^{2n} and $\phi^2 R$ operators are expected to mix because of statistical or quantum mechanical fluctuations. Thus our truncation includes all the relevant and naively marginal operators of the model.

Since the order of the non-minimal interaction is only two, we can incorporate it easily in a quadratic part of the bare action

$$S_0[\chi] = \frac{1}{2} \int \mathrm{d}^d x \sqrt{g} \chi \left(-\nabla^\mu \partial_\mu + F''(\phi) R \right) \chi . \tag{5.6}$$

Note that in principle the full quadratic part of the action (5.4) should receive contributions also from the expansion of the potential around the background. Nevertheless, in perturbation theory we can retain this insertion and treat it as a vertex. According to the dimensionality, there are two possible leading contributions if the action of the path integral is expanded perturbatively around $S_0[\chi]$ for the ϕ^{2n} models in powers of $V(\phi)$: the linear and the quadratic contributions,

$$\begin{aligned} Z = \int \mathcal{D}\chi \; e^{-S_0[\chi]} \Big[1 + \int \mathrm{d}x \; V\left(\phi+\chi\right)(x) \\ + \frac{1}{2} \int \mathrm{d}x \mathrm{d}x' \; V\left(\phi+\chi\right)(x) V\left(\phi+\chi\right)(x') \Big] . \end{aligned} \tag{5.7}$$

At the linear order and Taylor-expanding the potential itself we have a generalized

[34] Actually the diagrams have in principle external legs but they appear only in the form of the background field which has no dynamics and thus plays the role of simple constants for the perturbative expansion.

tadpole-like contribution

$$-\int \mathrm{d}^d x \sqrt{g(x)} \sum_{0\leq r\leq n} \frac{1}{(2r)!}\, G(x,x)^r\, V^{(2r)}(\phi(x))\,, \tag{5.8}$$

in which the number of closed lines is constrained to be even because of trivial topological reasons. In dimensional regularization the linear term contributes to the renormalization of the potential only if $r = 1$ and $d = 2$ as we show later in subsection 5.2.3. At the quadratic order we have instead

$$\frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d x' \sqrt{g(x)g(x')} \sum_{0\leq r\leq 2n} \frac{1}{r!} V^{(r)}(\phi(x))\, G(x,x')^r\, V^{(r)}(\phi(x'))\,. \tag{5.9}$$

In (5.8) and (5.9) we introduced $G(x,x')$ which is the Green function associated to the operator of the quadratic part of the action

$$\begin{aligned} \mathcal{O} &= -g^{\mu\nu}\nabla_\mu \partial_\nu + F''(\phi) R\,, \\ \mathcal{O}_x G(x,x') &= \delta^{(d)}(x,x')\,. \end{aligned} \tag{5.10}$$

As explained in section 3.5 it is possible to give a covariant representation of the Green function for an operator of Laplace-type using the heat kernel. Even though for an exact treatment of the propagator we should know the full spectrum of the heat kernel, including non-local contributions arising in the long range processes, for the purposes of perturbation theory an asymptotic expansion for small propertime values is enough. The proper construction of such expansion is given in appendix F. For our present needs, the representation simply shows that the Green function can be expanded

$$G(x,x') \;=\; G_0(x,x') + a_1(x,x')\; G_1(x,x') + \dots\,, \tag{5.11}$$

in which we purposely neglected all further contributions which do not affect the relevant operators. The leading $G_0(x,x')$ term can be understood as a covariant generalization of the standard Green function of flat space (see Appendix F for more details), while $a_1(x,x')$ is the first correction due to curvatures and multiplies the subleading correction to the propagator $G_1(x,x')$.

In the following subsection we consider first the renormalization of the general ϕ^{2n} universality class for $n \geq 3$, while the case $n = 2$ is deferred for later. The reason for this is that the case $n = 2$ is special when it comes to the renormalization of the function $F(\phi)$. In particular, the results for the general ϕ^{2n} case often cannot be continued to $n = 2$ because the subleading correction to the propagator is powerlaw for each $d = d_n$ with $n \geq 3$, but it is logarithmic in $d = d_{n=2} = 4$. If the analytic continuation is performed anyway, there is thus an additional "unbalanced" singularity which is seen as an additional $1/(n-2)$ pole in the beta functions.

5.2.1. ϕ^{2n} universality class

The leading quadratic contribution to the path integral (5.9) is not a one-loop contribution for all $n \geq 3$ models, but rather it involves $(r-1)$-loops, which is a marked distinction from the more familiar analyses of ϕ^4 and Yang-Mills theory below the upper critical dimension $d = 4$. To highlight this fact let us consider the first element of this family, which is ϕ^6 for $n = 3$ and which has been already renormalized in curved space in [203]: the leading contributions to the renormalization of the couplings come from two loop diagrams and in general contributions come from every other loop order [204].

In general, not all loop contributions to (5.9) lead to $1/\varepsilon$ poles for all values of n. Using the methods described in Appendix G and dimensional analysis, it is possible to infer that $1/\varepsilon$ poles arise for the cases $r = n$ and $r = 2n-1$, corresponding to $(n-1)$- and $(2n-2)$-loop diagram respectively likewise flat space [105, 106]. In the case $r = n$, the contribution arises solely from r lines of the leading $G_0(x, x')$ term of the Green function. In the second case the diagram can be either composed by $2n-1$ lines of $G_0(x, x')$, or by $2n-2$ lines of $G_0(x, x')$ and one of $G_1(x, x')$. In practice, this makes for three multiloop diagrams that must be evaluated by the methods described in Appendix G. The diagrams are depicted in Fig. 5.1. Referring to 5.1a as $\mathcal{A}$, 5.1b as $\mathcal{B}$ and 5.1c as $\mathcal{C}$ we have that in $d = d_n - \varepsilon$ the three diagrams evaluate to

$$
\begin{aligned}
\mathcal{A} =& \frac{1}{2n!} \int V^{(n)}(\phi)\; G_0^n \; V^{(n)}(\phi') \sim c_n^{n-1} \mu^{(1-n)\varepsilon} \frac{1}{4n!\;\varepsilon} V^{(n)}(\phi)^2 \\
\mathcal{B} =& \frac{1}{2(2n-1)!} \int V^{(2n-1)}(\phi)\; G_0^{2n-1}\; V^{(2n-1)}(\phi') \\
& \sim -c_n^{2n-2} \mu^{2(1-n)\varepsilon} \frac{(n-1)}{16(2n)!\;\varepsilon} \int \Big\{ V^{(2n)}(\phi)^2 (\partial\phi)^2 - \frac{2n-3}{6} V^{(2n-1)}(\phi)^2 R \Big\} \\
\mathcal{C} =& \frac{1}{2(2n-2)!} \int V^{(2n-1)}(\phi)\; G_0^{2n-2} G_1 a_1 \; V^{(2n-1)}(\phi') \\
& \sim c_n^{2n-2} \mu^{2(1-n)\varepsilon} \frac{n(n-1)(2n-1)}{16(n-2)\;(2n)!\varepsilon} \int \Big\{ F''(\phi) - \frac{1}{6} \Big\} V^{(2n-1)}(\phi)^2 R
\end{aligned}
\tag{5.12}
$$

in which we suppress several coordinate indices on the left hand side for brevity. We integrated by parts one derivative to cast the kinetic-like term of the second diagram in a suitable form, and defined the constant

$$
c_n = \frac{1}{4\pi} \frac{1}{\pi^{\frac{1}{n-1}}} \Gamma\left(\frac{1}{n-1}\right) . \tag{5.13}
$$

The results of (5.12) are essentially the counterterms which must be inserted to remove the divergences of all the relevant operators of the ϕ^{2n} model in curved space for $n \geq 2$. The pole at $n = 2$ of the last counterterm is a clear indication of why we left the ϕ^4 models out of this general discussion.

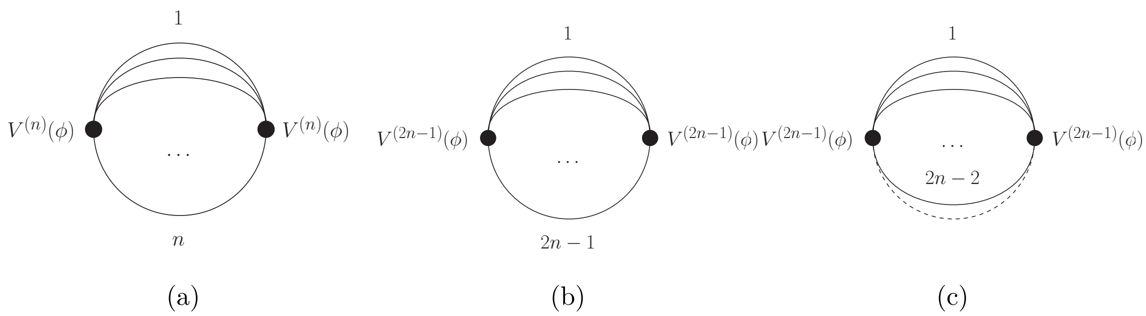


Fig. 5.1.: Diagrammatic representation of (5.12) in order of appearance. The first and second diagrams are made of n and $2n-1$ lines of the leading contribution of the Green function $G_0(x, x')$. Trivially their symmetry factors are $n!$ and $(2n-1)!$ respectively. The third diagram is again made of $2n-1$ lines, but one corresponds to the subleading $G_1(x, x')$ line which is depicted as dashed. Its symmetry factor is $(2n-2)!$ because there are $(2n-1)$ ways to choose the last line.

Since we are just considering a leading renormalization, the computation of the renormalization group flow is straightforward because it can be obtained by simply acting on the counterterms with the logarithmic derivative with respect to the reference scale $\mu\frac{\partial}{\partial\mu}$. Naturally, we display the RG in the guise of functional equations. We also include a field dependent wavefunction $Z(\phi)$ as renormalization of the kinetic term. The wavefunction is generated by the flow and, while it includes irrelevant contributions for the most part, the use of a boundary condition for $Z(0)$ allows for the determination of the anomalous dimension of the renormalized field. At the upper critical dimension we find

$$\begin{aligned} \beta_V =& \frac{c_n^{n-1}(n-1)}{4\ n!} V^{(n)}(\phi)^2\,, \\ \beta_Z =& -\frac{c_n^{2n-2}(n-1)^2}{4\ (2n)!} V^{(2n)}(\phi)^2\,, \\ \beta_F =& -\frac{c_n^{2n-2}(n-1)^2}{8(n-2)\ (2n)!}\Big\{(n-1) - n(2n-1)F''(\phi)\Big\} V^{(2n-1)}(\phi)^2\,. \end{aligned} \tag{5.14}$$

In a rather standard fashion we switch to the dimensionless renormalized canonically-normalized field

$$\varphi = Z_0^{\frac{1}{2}} \mu^{-\frac{d-2}{2}} \phi\,, \tag{5.15}$$

which includes a rescaling by the wavefunction renormalization constant $Z_0 = Z(0)$ which is generated by β_Z. The field φ is the natural argument for the dimensionless renormalized functions $v(\varphi) = \mu^{-d} V(\phi)$, $z(\varphi) = Z_0^{-1} Z(\phi)$ and $f(\varphi) = \mu^{2-d} F(\phi)$. Their renormalization group flow is

$$\begin{aligned}\beta_v &= -dv + \frac{d-2+\eta}{2}\varphi v' + \frac{c_n^{n-1}(n-1)}{4\,n!}(v^{(n)})^2\,,\\ \beta_z &= \eta z + \frac{d-2+\eta}{2}\varphi z' - \frac{c_n^{2n-2}(n-1)^2}{4\,(2n)!}(v^{(2n)})^2\,,\\ \beta_f &= (2-d)f + \frac{d-2+\eta}{2}\varphi f' - \frac{c_n^{2n-2}(n-1)^2}{8(n-2)\,(2n)!}\Big\{(n-1)-n(2n-1)f''\Big\}(v^{(2n-1)})^2\,.\end{aligned} \tag{5.16}$$

By construction we have $z(0) = 1$, so the limit $\varphi \to 0$ can be used to determine the anomalous dimension $\eta \equiv -\partial \log Z_0/\partial \log \mu$ directly from $\beta_z|_{\varphi=0} = 0$.

5.2.2. ϕ^4 universality class

The four dimensional case is special for three main reasons. Firstly, diagrams and counterterms leading to the renormalization are not directly obtained as the analytic continuations to $n = 2$ of the results of Sect. 5.2.1. Secondly, the subleading correction to the Green function in four dimensions is logarithmic. This means that in the ε-expansion an additional divergence must be subtracted from the propagator as we show in (F.14). The difference in the behavior of the subleading part of the propagator is the reason why a $\frac{1}{n-2}$ pole appears in the third diagram of (5.12). Thirdly, a simple dimensional analysis reveals that operators quadratic in the curvatures have the same canonical dimension of the operators ϕ^4 and $\phi^2 R$, and hence must be renormalized together for consistency.

Here we try to follow the notation of [115] for the most part with some minor modification. Let us first generalize the action (5.4) to accommodate the higher curvatures

$$\begin{aligned}S[\phi] = \int \mathrm{d}^d x\sqrt{g}\Big\{&\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi) + F(\phi)R\\ &- a\,\mathcal{F} - b\,\mathcal{G} - c\,R^2 - e\,\nabla^2 R\Big\}\,,\end{aligned} \tag{5.17}$$

with the following invariants

$$\begin{aligned}\mathcal{F} &= \frac{2}{(d-2)(d-1)}R^2 - \frac{4}{d-2}R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\theta}R^{\mu\nu\rho\theta}\,,\\ \mathcal{G} &= R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\theta}R^{\mu\nu\rho\theta}\,.\end{aligned} \tag{5.18}$$

These invariants are chosen so that in four dimensions $\mathcal{G}$ integrates to a topological invariant and $\mathcal{F}$, which is the square of the Weyl tensor, transforms covariantly under scale transformations.

It is convenient to define one general function and its modification as follows

$$\begin{aligned} U(\phi,R) &= V(\phi)+F(\phi)R-a\mathcal{F}-b\mathcal{G}-cR^2-e\nabla^2 R\,,\\ \hat{U}(\phi,R) &= U(\phi,R)-\frac{1}{12}R\phi^2\,. \end{aligned} \tag{5.19}$$

At one loop, which is the leading order, the counterterm to $U(\phi,R)$ can be obtained by a simple application of the heat kernel. One finds that the leading contribution to the renormalization of $U(\phi,R)$ comes from the $a_2(x,x)$ coefficient given in (F.10)

$$-\frac{\mu^{-\varepsilon}}{(4\pi)^2\,\varepsilon}\int\left\{\frac{1}{2}\partial_\phi^2\hat{U}(\phi,R)^2+\frac{1}{120}\mathcal{F}-\frac{1}{360}\mathcal{G}\right\}, \tag{5.20}$$

while the wavefunction renormalization is a two loop effect completely analogous to the limit $n=2$ of Sect. 5.2.1. The computation of the leading beta function is straightforward

$$\begin{aligned} \beta_U &= \frac{1}{(4\pi)^2}\left\{\frac{1}{2}\partial_\phi^2\hat{U}(\phi,R)^2+\frac{1}{120}\mathcal{F}-\frac{1}{360}\mathcal{G}\right\},\\ \beta_Z &= -\frac{1}{6(4\pi)^4}V^{(4)}(\phi)^2\,. \end{aligned} \tag{5.21}$$

Returning to the original functions of (5.17) we find the functional beta functions

$$\begin{aligned} \beta_V &= \frac{1}{2(4\pi)^2}V''(\phi)^2\,, \qquad \beta_Z = -\frac{1}{6(4\pi)^4}V^{(4)}(\phi)^2\,,\\ \beta_F &= -\frac{1}{(4\pi)^2}\left\{\frac{1}{6}-F''(\phi)\right\}V''(\phi)\,. \end{aligned} \tag{5.22}$$

as well as the beta functions for the higher derivative couplings

$$\begin{aligned} \beta_a &= -\frac{1}{120(4\pi)^2}\,, & \beta_c &= \frac{1}{2(4\pi)^2}\left\{\frac{1}{6}-F''(\phi)\right\}^2,\\ \beta_b &= \frac{1}{360(4\pi)^2}\,, & \beta_e &= -\frac{1}{6(4\pi)^2}\left\{\frac{1}{5}-F''(\phi)\right\}. \end{aligned} \tag{5.23}$$

Since $F(\phi)$ is at most quadratic we have that $F''(\phi)=F''(0)$ and the couplings c and e can be treated as numbers, even though the right hand side suggests otherwise.

In order to make this section on ϕ^4 more self consistent, we briefly discuss some critical property of the above system. This discussion anticipates some points that are made later in the development of section 5.3. One can see that at the leading order the critical value for the non-minimal coupling $\xi = F''(0)$ is $\xi=\frac{1}{6}$ as one would naively expect from continuing the general conformal value (5.3) to $d=4$, thus proving the hypothesis formulated in the introduction for the special case $n=2$.

In general, it is not guaranteed that the critical value of ξ remains a fixed point beyond the leading order unless a further renormalization condition is exploited [114]. For the purpose of the analytic continuation of the theory, it would be interesting under which circumstances at two loops and for $d = 4 - \varepsilon$ the coupling takes the value

$$\xi = \frac{d-2}{4(d-1)} = \frac{1}{6} - \frac{1}{36}\varepsilon + \dots \,. \tag{5.24}$$

One can prove, using naively the dimensionally regularized scheme at next-to-leading order (NLO) and a straightforward subtraction, that the above value is *not* a fixed point to order ε. However, the freedom highlighted in [115] of redefining the potential $U(\phi, R)$ by a copy of the one loop counterterms can be exploited to ensure that (5.24) is the fixed point at NLO. The redefinition is a change of the renormalization conditions which thus defines and links the metric and the field.

5.2.3. ϕ^∞ universality class: the Sine-Gordon model

The upper critical dimension $d = 2$ emerges as the limit $d_n \to 2$ of $n \to \infty$. The renormalization of the path integral for the two-dimensional case is very simple, even though it represents a special case likewise the ϕ^4 one. It is convenient to borrow the notation from the previous section and use the full potential $U(\phi, R)$. The computation of the leading counterterms and beta functions necessitates only the use of the standard heat kernel expansion of an operator of Laplace-type, and specifically of the coefficient $a_1(x, x)$ given in (F.10). We find the leading counterterm at one loop

$$\frac{\mu^{-\varepsilon}}{4\pi\ \varepsilon} \int \partial_\phi^2 \hat{U}(\phi, R)\,, \tag{5.25}$$

and deduce the very simple RG beta functional

$$\beta_U = -\frac{1}{4\pi} \partial_\phi^2 \hat{U}(\phi, R)\,. \tag{5.26}$$

Notice that there is no anomalous dimension renormalization coming from our leading order computation.

In two dimensions the scale invariant solutions of this beta function become periodic. It has been argued that the critical solution of this RG flow in flat space is periodic and corresponds to the Sine-Gordon universality class [106]. Here we are observing a generalization to curved spacetime for zero anomalous dimension as in [205]. Let us first introduce the dimensionless potential $u(\varphi, R) = \mu^{-2} U(\varphi, \mu^2 R)$. Using the boundary conditions $u(\varphi, R) = u(-\varphi, R)$ and $\partial_\phi^2 U(\phi, R)|_{\phi=0} = m^2$, at the fixed point in $d = 2$ we find

$$u(\varphi, R) = -\frac{m^2}{8\pi} \cos\left(\sqrt{8\pi}\varphi\right) + \frac{R}{48\pi}\,. \tag{5.27}$$

Notice that we have imposed the boundary conditions as a function of the scalar curvature, therefore an implicit dependence on R might in principle be hidden in the mass $m^2 = m^2(R)$. In this way we have ensured that the result agrees both with the assumption that this solution generalizes the Sine-Gordon universality to curved space, and with the expectation that the non-minimal coupling ξ should be zero at the critical point.

5.2.4. $2d$ gravity at large-c

As a brief intermezzo we believe that it is interesting to show the relevance of the results of Sect. 5.2.3 in reproducing some well-known result of two-dimensional quantum gravity coupled to conformal matter. Let us recall that in exactly two dimensions the path integral of gravity can be determined by integrating the conformal anomaly [206], which leads to a renormalization procedure linked to a non-local action known as the Polyakov action [201]. This action is especially relevant because the spacetime integral of the Einstein term is a topological invariant in two dimension, and hence it cannot govern the dynamics of the model.

However for general d (and specifically for $d = 2 - \varepsilon$) the Einstein term is not a topological invariant, and therefore it has been argued by Kawai and Ninomiya that it should be possible to reproduce the results based on the Polyakov action by just renormalizing the Einstein action in $d = 2 - \varepsilon$ and then taking the limit $\varepsilon \to 0$ [207]. The validity of this argument was shown through the course of several papers, which ultimately lead to the two loop renormalization of the Einstein action in $d = 2 - \varepsilon$. For more details we refer to [208] and references therein; notice however that in the literature of $2d$ gravity it is often chosen $d = 2 + \varepsilon$, therefore the replacement $\varepsilon \to -\varepsilon$ is necessary when comparing results.

The renormalization of dimensionally regulated two-dimensional gravity is slightly unconventional because it has to deal with the conformal factor of the metric, otherwise one finds discontinuities when analytically continuing to $\varepsilon \to 0$ [209]. In order to describe it, let us first introduce the Einstein action interacting with c distinct conformally coupled fields ϕ_i in d dimensions

$$S[g,\phi] = \int \mathrm{d}^d x \sqrt{g} \Big\{ -\frac{1}{G} R + \frac{1}{2} \sum_i \Big(\partial_\mu \phi^i \partial^\mu \phi^i + \xi_c \phi^i \phi^i R \Big) \Big\} . \tag{5.28}$$

We require that the coupling ξ_c is determined by the conformal value (5.3) and assume that this condition can be preserved through renormalization (see the discussion of sections 5.1 and 5.3 for more details on this point). The number c is often referred to as "central charge" and it counts the effective number of matter degrees of freedom.

In two dimensions all possible metrics are related by a Weyl transformation, and therefore only their conformal mode is allowed to fluctuate. Close to two dimensions, instead, it is customary to parametrize the metric $g_{\mu\nu} \to (\varepsilon/8)^{2/\varepsilon}\psi^{4/\varepsilon} g_{\mu\nu}$ into a conformal mode ψ and a metric $g_{\mu\nu}$ which is not allowed to fluctuate in its trace part (by abuse of notation we denote the transformed metric with $g_{\mu\nu}$).[35] Using this normalization the mode ψ of the metric enjoys a Weyl invariant action, which is in form analogous to any of those of the fields ϕ_i, if not for an overall negative sign which makes ψ an unstable "scalar" degree of freedom. The idea of [211] is to transform (5.28) into

$$S[g,\psi,\phi] = \int \mathrm{d}^d x \sqrt{g} \Big\{ -\frac{1}{G} L(\psi,\phi_i) R - \frac{1}{2}\partial_\mu\psi\partial^\mu\psi + \frac{1}{2}\sum_i \partial_\mu\phi^i\partial^\mu\phi^i \Big\} . \tag{5.29}$$

and renormalize it such that the function $L(\psi,\phi_i)$ respects the conformal coupling. The new function is normalized by $L(0,0) = 1$, which is a necessary condition to read off the value of the Newton constant G.

Assuming that the instability of ψ can be cured by opportunely Wick rotating the theory, it is possible to neglect the effects of the dilaton field ψ as compared to those of the multiplet ϕ_i; moreover one can argue that for large values of c the fluctuations induced by the fields ϕ_i dominate over those of $g_{\mu\nu}$ too. In other words, for large c it should be necessary to integrate only the loops of ϕ_i, but this is exactly the multifield generalization of what we have done in section 5.2.3 upon the identification

$$-\frac{\mu^{-\varepsilon}}{G} L(\phi_i) R = U(\phi_i, R) \tag{5.30}$$

for the dimensionless versions of L and G. Notice that in the large-c limit we are dropping any parametric dependence on the mode ψ to highlight the connection with the previous section.

Now we use (5.30) inside (5.26) to determine the renormalization group flow of the renormalized G and $L(\phi_i)$. In order to separate the two beta functions we have to impose that $L(0) = 1$ along the flow. Additionally, we impose that all fields ϕ_i are coupled in the same way so that it will be sufficient to denote each one of them by ϕ. We find

$$\begin{aligned} \beta_G &= -\varepsilon G + \frac{c}{24\pi} G^2 + \frac{c}{4\pi} G\, L''(0) \\ \beta_L &= -\frac{c}{24\pi} G \Big\{ 1 - L(\phi) \Big\} + \frac{c}{4\pi} \Big\{ L(\phi)\ L''(0) - L''(\phi) \Big\} . \end{aligned} \tag{5.31}$$

The interaction with the fluctuating modes of $g_{\mu\nu}$ can change the anomalous dimension of the fields ϕ_i as $\eta \sim G$, but this contribution is also generally subleading in the limit of large central charge. We follow the strategy of [211] and parametrize $L = 1 + a\psi + b\psi^2 - \xi_c\phi^2$.

[35] We find that the best recent review of this formulation appeared in [210], in which it has been named unimodular Dirac gravity, or alternatively unimodular dilaton gravity. The gauge group $Diff^*$ of the formulation comes from the breaking of a semidirect product of diffeomorphisms and Weyl transformations which is itself isomorphic to the diffeomorphisms group $Diff \ltimes Weyl \to Diff^* \simeq Diff$, but acts on ψ and $g_{\mu\nu}$ in a nonstandard way.

It is straightforward to see that the beta functions of a and b have Gaussian solutions, thus setting all couplings except for G at the respective fixed points we obtain

$$\beta_G = -\varepsilon G + AG^2 \tag{5.32}$$

with $A = -\frac{c}{24\pi}$. This result agrees with the large-c limit of the exact leading result in which the constant A takes the value $A = \frac{25-c}{24\pi}$ [207].

Notice that the general Euclidean result hinges on our ability of solving the problem of the instability of the conformal mode, which in [211] is "Wick" rotated $\psi \to \mathrm{i}\psi$. While several solutions have been proposed there is no definite answer, nor general consensus on how to approach the problem. In fact, proposals to solve the problem without a Wick rotation of the dilaton mode have received renewed attention recently [212]. This problem can be framed in the more general discussion of finding the universality class of quantum gravity and exploring the corresponding conformal theory [213]. Here we would like to mention another less explored yet interesting possibility that was outlined in [214]: the path integral of $2d$ gravity could be "defined" starting with the path integral of a fluid $2d$ membrane embedded in d bulk dimensions (which is essentially a non-critical Nambu-Goto string) and analytically continuing to $d \to 0$. In the membrane path integral the correct counting of the degrees of freedom involves the propagation of modes of the extrinsic curvature, which play a role analogous to the gauge fixing ghosts.

5.3. Criticality

We now resume the analysis of the RG system (5.16) of Sect. 5.2.1 representing the general case of the multicritical model ϕ^{2n} for $n \geq 3$. The ϕ^4 model is an outlier, so we anticipated a brief discussion of the critical properties of its non-minimal coupling to the curvature already in Sect. 5.2.2. We find it convenient to rescale the potential

$$v(\varphi) \to \frac{4\, c_n^{1-n}}{n-1} v(\varphi) = \frac{(4\pi)^n}{n-1} \Gamma\left(\frac{1}{n-1}\right)^{1-n} v(\varphi)\,, \tag{5.33}$$

while leaving all other functions intact. The system (5.16) simplifies to

$$\begin{aligned} \beta_v =& -dv + \frac{d-2+\eta}{2}\varphi v' + \frac{1}{n!}(v^{(n)})^2\,,\\ \beta_f =&(2-d)f + \frac{d-2+\eta}{2}\varphi f' - \frac{2n(2n-1)}{(n-2)\ (2n)!}\left\{\frac{n-1}{n(2n-1)} - f''\right\}(v^{(2n-1)})^2\,. \end{aligned} \tag{5.34}$$

Using the boundary condition $z(0) = 1$ in the rescaled flow β_z, we also determine the anomalous dimension of the scalar field $\eta = 4v^{(2n)}(0)^2/(2n)!$.

The critical couplings appear as the leading couplings of the potentials $v(\varphi)$ and $f(\varphi)$.

By construction, in the minimal subtraction scheme all other couplings are dimensionful, and therefore are zero at the critical point. We therefore parametrize the potentials in terms of the two almost marginal interactions

$$v(\varphi) = \frac{\lambda}{(2n)!}\varphi^{2n}\,, \qquad f(\varphi) = \frac{\xi}{2}\varphi^2\,. \tag{5.35}$$

Using the above parametrization in (5.34), we find the following beta functions and anomalous dimension

$$\begin{aligned} \beta_\lambda &= -(n-1)\varepsilon\lambda + \eta n\lambda + \frac{(2n)!}{(n!)^2}\lambda^2\,, \qquad \eta = \frac{4}{(2n)!}\lambda^2\,, \\ \beta_\xi &= \eta\xi - \frac{4(n-1)}{(n-2)(2n)!}\lambda^2 + \frac{4n(2n-1)}{(n-2)(2n)!}\xi\lambda^2\,. \end{aligned} \tag{5.36}$$

It is clear that η contributes to the cubic order in λ of β_λ, which has no effect on the determination of the order ε of the fixed point. However η has an important effect in β_ξ because its contribution scales with the same power of λ as the other terms. Substituting η we find

$$\begin{aligned} \beta_\lambda &= -(n-1)\varepsilon\lambda + \frac{(2n)!}{(n!)^2}\lambda^2\,, \\ \beta_\xi &= \frac{8(n^2-1)}{(n-2)\;(2n)!}\left(\xi - \frac{1}{2(n+1)}\right)\lambda^2\,. \end{aligned} \tag{5.37}$$

The system has two different fixed points. On the one hand we have the Gaussian fixed point at $\lambda = 0$ which sets both beta functions to zero. In this case the natural fixed point for ξ is the subleading root of β_ξ. On the other hand we have the non Gaussian fixed point

$$\lambda^* = \frac{(n-1)\;(n!)^2}{(2n)!}\varepsilon\,, \qquad \xi^* = \frac{1}{2(n+1)}\,. \tag{5.38}$$

For both fixed points the coupling ξ takes the critical value that is expected *at the upper critical dimension*

$$\xi^* = \xi_n \equiv \frac{d_n - 2}{4(d_n-1)} = \frac{1}{2(n+1)}\,, \tag{5.39}$$

which evidently proves the hypothesis given in the introduction. Interestingly, the only outlier of our analysis is the case for $n = 2$, for which we have to use the set of beta functions coming from (5.22) as discussed in Sect. 5.2.2. However, it is straightforward to find that in this case $\xi = \frac{1}{6}$ which happens to coincide with the continuation of (5.39) to $n = 2$. It is an easy check to see that the limit $n \to \frac{3}{2}$ in (5.39) gives $\xi^* = \frac{1}{5}$ as shown in [215].

The next step would be to test if the next-to-leading order correction to the non-Gaussian fixed point of ξ matches the ε-expansion of conformal value for the coupling ξ evaluated in $d = d_n - \varepsilon$ instead of $d = d_n$. This would imply

$$\xi \overset{?}{=} \frac{d-2}{4(d-1)} = \frac{1}{2(n+1)} - \frac{(n-1)^2}{4(n+1)^2}\,\varepsilon + \dots\,, \tag{5.40}$$

which comes from the expansion of (5.3) to orders of ε using $d = d_n - \varepsilon$. Following the discussion of [114, 115], which we reproduced briefly in Sect. 5.2.2, we argue that ensuring (5.40) probably requires a special choice in the renormalization conditions leading the RG flow. In practice, the next-to-leading contributions to the RG flow can be changed by the inclusion of terms which match the counterterms (5.12) and which can be used to change the renormalization conditions leading to the fixed point value for ξ. Let us include here also a short remark on the steps that have led to (5.39). While the hypothesis formulated in the introduction stated that we expected ξ to take the conformal value at criticality, the validity of the guess is not at all obvious from the initial form of the counterterms (5.12). In particular, there is a very delicate balance among the terms appearing in the renormalization (5.16) and the anomalous dimension which produces the form of β_ξ in (5.37) and which makes evident that the conformal value (5.39) is actually the critical point.

We conclude this section by discussing the implications that the system of beta functions (5.37) has on the infrared physics. For obvious reasons, we are mostly interested in studying the renormalization group flow in a physical dimension. The first natural dimension (smaller than d_n) in which almost all models for $n \geq 3$ are nontrivial is $d = 2$, we therefore continue ε to the value $\varepsilon = \frac{2}{n-1}$ to continue the ϕ^{2n} models to the physical dimension $d = 2$. Correspondingly, the fixed point value of the coupling λ becomes $\lambda^* = \frac{2}{A_n}$ in which we define $A_n = (2n)!/(n!)^2$ which is simply the coefficient of the λ^2 term in β_λ. The flow can be integrated as follows

$$\begin{aligned} \lambda(\mu) &= \frac{\lambda_0}{\frac{\lambda_0}{\lambda^*} + \left(\frac{\mu}{\mu_0}\right)^2 \left(1 - \frac{\lambda_0}{\lambda^*}\right)}\,, \\ \xi(\mu) &= \xi_n + (\xi_0 - \xi_n)\; e^{-B_n \int_\mu^{\mu_0} d\rho\, \frac{\lambda^2(\rho)}{\rho}}\,, \end{aligned} \tag{5.41}$$

in which we introduce $B_n = 8(n^2-1)/((n-2)\;(2n)!)$ that is the coefficient of β_ξ. The flow satisfies the ultraviolet boundary conditions $\lambda_0 = \lambda(\mu_0)$ and $\xi_0 = \xi(\mu_0)$, which can be checked by setting $\mu = \mu_0$ in (5.41).

More interestingly, we can use (5.41) to explore the infrared limit $\mu = 0$. One can see trivially that the second term in the denominator of $\lambda(\mu)$ drops for $\mu = 0$ and therefore we have $\lambda(0) = \lambda^*$. Slightly less trivial is to show that for $\mu \to 0$ the integral appearing in the exponential of $\xi(\mu)$ diverges logarithmically implying that the second term drops;

we thus have $\xi(0) = \xi_n$. These results are in line with the expectation that the nontrivial fixed point (5.38)–(5.39) is of infrared nature in that it controls the large scale behavior of the model near criticality.

6. Conclusions

In this thesis we discussed several aspects of quantum systems undergoing critical phenomena in presence of a gravitational interaction. The analysis uncovers phenomenology of both geometrical degrees of freedom and matter degrees of freedom.

The gravitational catalysis of chiral symmetry breaking and fermion mass generation was investigated on patches of hyperbolic spaces, corresponding to negatively curved patches of AdS spacetimes in a Lorentzian setting.

The general phenomenon of gravitationally catalyzed symmetry breaking has long been known to be driven by long range modes and their sensitivity to negatively curved spacetimes. The novelty of this study consists in inspecting the competition between the screening of these modes by a gauge invariant IR averaging scale k_{IR} and the effect of the presence of an averaged curvature on this scale. This competition leads to a bound on the local curvature parameter $\kappa \sim \sqrt{|R|}$ in units of the averaging scale k_{IR}. Gravitational catalysis does not set in as long as the bound is satisfied.

Built on RG type arguments, our analysis applies to local patches of spacetime and hence does not require the whole spacetime to be hyperbolic, negatively curved or uniform. Rather, the resulting bound applies to each patch of space or spacetime with an averaged negative curvature. Fermion modes in spacetime patches violating the bound can be subject to gravitational catalysis. Of course, the precise location of the onset of gravitational catalysis in parameter space depends also on further induced or fundamental interactions of the fermions. In case of chiral symmetry breaking through gravitational catalysis, the fermions generically acquire masses of the order of at least k_{IR} or larger depending on the relevance of further effective interactions.

An application of these findings to a possible high energy regime of quantum gravity results in the following scenario: let us assume the existence of a, say Planck scale, regime where a metric/field theory description is already appropriate, but large curvature fluctuations are allowed to occur. Our bound disfavors the occurrence of patches of spacetime with large negative averaged curvature. In such patches, the generation of fermion masses of the order of k_{IR} could be triggered. Since k_{IR} itself can be of order Planck scale in such a regime, the fermion masses would generically be at the Planck scale upon onset of gravitational catalysis. Even worse, gravitational catalysis would naturally remove light fermions from the spectrum of particle physics models on such spacetimes. Therefore, we argue that our bounds apply to any quantum gravity scenario satisfying these assumptions that aims to be compatible with particle physics observations: if a quantum gravity scenario satisfies the bound, it is safe from gravitational catalysis in the matter sector; if not, the details of the fermion interactions matter. In the latter case, gravitational catalysis may still be avoided, if the interactions remain sufficiently weak.

As the curvature bounds refer to an IR cutoff scale k_{IR}, they are naturally scheme dependent. In fact, this scheme dependence in the first place parametrizes the details of how the fermionic long range modes are screened by the regularization scale. We observe that a finite curvature bound exists for any physically admissible regularization. Moreover, the shifts of the bound due to a change of the propertime regularization agrees with the behavior expected from the underlying propertime diffusion process. We therefore claim that the curvature bound has a scheme independent meaning. A fully scheme independent definition might eventually need to take the prescription for defining the averaged curvature of a local spacetime patch into account, which depends on the details of the underlying quantum gravity scenario.

Having performed a mean field type RG analysis, our bounds may receive corrections from further fluctuations that may be relevant at the scale k_{IR} including further independent degrees of freedom or chiral order parameter fluctuations. Such corrections can go into both directions: further interactions such as gauge or Yukawa forces typically enhance the approach to chiral symmetry breaking, whereas order parameter fluctuations can have the opposite effect. Also, thermal fluctuations can inhibit the occurrence of a chiral condensate at sufficiently high temperature. Effects that trigger symmetry breaking can effectively be summarized in terms of finite bare fermionic self interactions $\bar{\lambda}_\Lambda$ in our approach, whereas thermal fluctuations can be understood as moving the critical coupling to larger values [50, 216].

An analysis of the dependence of the bound on the dimensionality of the system has been performed. In general odd dimensions, we have derived a simple closed form expression. Since different dimensions can exhibit a different number of relevant scalar curvature operators and thus a different number of physical parameters, a meaningful comparison of theories in different dimensions is not straightforward. Assuming that all further physical parameters are essentially zero at the scale k_{IR}, we observe that the resulting curvature bound decreases with $\sim 1/\sqrt{d}$ for higher dimensions. This result shows how quantum gravity scenarios with extended higher dimensional patches of spacetime where the hypothesis for our bound are matched are less likely to admit the existence of light fermionic matter.

Unfortunately, results from quantum gravity scenarios that could be checked against our curvature bounds are rather sparse. Many approaches focus on the gravitational sector leaving matter, and fermions in particular, aside. One of the most developed approaches in this respect is asymptotically safe gravity. Concentrating on a simple picture for the UV regime of gravity using the Einstein-Hilbert action as the scaling action, our curvature bound translates into a bound on the particle content of the matter sector. In particular, the number of fermion flavors becomes constrained in order to avoid gravitational catalysis. Our simple estimates based on various literature studies of asymptotically safe gravity with matter indicate that the standard model is compatible with asymptotically safe gravity and not affected by gravitational catalysis in the trans-Planckian regime. This statement is nontrivial insofar that the matter content together with the effective Einstein equation suggest negatively curved local patches of spacetime in the fixed point regime. Still, the curvature is sufficiently weak to satisfy our bound. By contrast, our estimates suggest that the standard model with an additional fourth flavor generation would not satisfy the curvature bound within asymptotic safety. In order to obtain more reliable estimates, the curvature dependence of correlation functions and its interdependence with the matter sector in the trans-Planckian fixed point regime would be welcome.

Furthermore, the leading order renormalization of the multicritical scalar models with ϕ^{2n} interaction in curved space has been studied. The analysis shows that for almost all values of n one has to consider counterterms for the self interaction as well as for the non-minimal interaction of the form $\phi^2 R$, while some additional counterterms based on curvature invariants are needed in the special case $n = 2$. The counterterms have been obtained from a computation of the $\frac{1}{\varepsilon}$ poles of dimensionally regularized covariant Feynman diagrams of $(n-1)$-loops for the self interaction, and $(2n-1)$-loops for the self energy and the non-minimal interaction.

The result generalizes the renormalization of the ϕ^4 model in curved space, which we have considered as a special case, but it also shows that the general case functions rather differently. Specifically, the structure of the counterterms for the non-minimal coupling displays a discontinuity for $n = 2$, which corresponds to ϕ^4. We have deduced a set of functional beta functions which describes the scale dependence of a self interaction potential and a generalized non-minimal interaction with the scalar curvature.

We have used the perturbative renormalization group flow to determine standard perturbative beta functions for the two canonically marginal couplings: λ of the self interaction ϕ^{2n} and ξ of the non-minimal interaction $\phi^2 R$. The system of RG flow equations clearly shows that at the leading order the scale invariant fixed point of the non-minimal coupling ξ coincides with its dimension dependent conformal value ξ_c evaluated at the upper critical dimension. This result is in agreement with an educated guess enunciated in the introduction. Importantly, the leading critical value for the coupling ξ is an ultraviolet attractive feature of the renormalization group flow.

We have also discussed the possibility that at the next-to-leading order the ε-expansion of the fixed point value of ξ matches the expansion of the conformal value ξ_c below the upper critical dimension. Based on similar and already available results for the ϕ^4 [114] and ϕ^3 models [113, 116, 117], we argue that one has to either follow a modified version of the prescription of Brown and Collins [114], or alternatively to subtract normally while exploiting the freedom of redefining the renormalization group flow at the next-to-leading order using the counterterms of the leading order [105]. In other words, one might want to *find* the appropriate renormalization condition which ensures the validity of the conjecture for the non-minimal coupling. We believe that this condition plays an important role, especially in those cases where it is necessary to describe the model in a conformal or Weyl invariant way.

A clearer understanding of the status of this conjecture and the necessary renormalization condition requires further studies and the expansion of the analysis of this thesis to the next-to-leading order contributions to the renormalization flow.

Appendix A.

Legendre transform

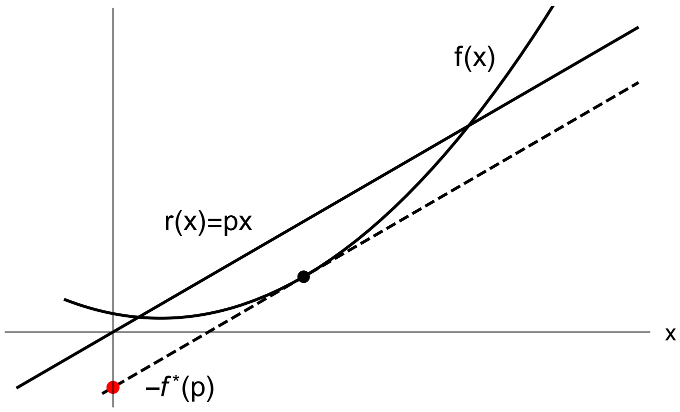


Fig. A.1.: Graphical interpretation of conjugate variables.

Given a convex function $f(x)$ we can define its Legendre transform $f^*(p)$ as [5, 34, 123]:

$$f^*(p) = \sup_x \{px - f(x)\} \ , \tag{A.1}$$

where the convexity ensures that for any given p the supremum exists for some finite x.

If $f(x)$ is differentiable it is possible to introduce an auxiliary function of both x and p:

$$g(p, x) = px - f(x) \tag{A.2}$$

for which the supremum coinsides with its maximum. Checking the extrema of g with respect to x:

$$\frac{\mathrm{d}g(p,x)}{\mathrm{d}x} = p - \frac{\mathrm{d}f(x)}{\mathrm{d}x} \equiv 0 \tag{A.3}$$

entails directly that

$$p = \frac{\mathrm{d}f(x)}{\mathrm{d}x} \, . \tag{A.4}$$

The supremum condition, thus, turns out to be equivalent to p being the derivative of f. On the other hand, the differential form of g reads:

$$\mathrm{d}g = x\mathrm{d}p + p\mathrm{d}x - \mathrm{d}f = x\mathrm{d}p \, . \tag{A.5}$$

As a consequence we have

$$x(p) = \frac{\mathrm{d}g(p,x)}{\mathrm{d}p}\,. \tag{A.6}$$

In a similar fashion we can inspect the second derivative of the Legendre transform:

$$\frac{\mathrm{d}^2 g}{\mathrm{d}p^2} = \frac{\mathrm{d}^2 x}{\mathrm{d}p^2} + 2\frac{\mathrm{d}x}{\mathrm{d}p} - \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\left(\frac{\mathrm{d}x}{\mathrm{d}p}\right)^2 - \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}^2 x}{\mathrm{d}p^2}\,, \tag{A.7}$$

imposing (A.4) and (A.6) means to identify g with f^* and we obtain:

$$\frac{\mathrm{d}^2 f^*}{\mathrm{d}p^2} = 2\frac{\mathrm{d}^2 f^*}{\mathrm{d}p^2} - \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\left(\frac{\mathrm{d}^2 f^*}{\mathrm{d}p^2}\right)^2\,, \tag{A.8}$$

i.e.

$$\frac{\mathrm{d}^2 f^*}{\mathrm{d}p^2}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 1\,. \tag{A.9}$$

This last result shows that the Legendre transform does not affect the convexity of the original function.

The generalization of this treatment to the case of functions of more variables is straightforward since the Legendre transform can be performed independently on each variable.

Appendix B.

Thermodynamics and statistical mechanics

In this appendix we recall the role of thermodynamical potentials and their relation to the statistical ensembles. The presentation follows the exposition in [123]. As stated in the main text, the system that we implicitly think about is that of a perfect fluid. Since we will work with equilibrium thermodynamics it is important to keep in mind that the differential equations we will be writing are not describing dynamical processes, but rather the change of thermodynamical quantities between two configurations infinitesimally apart. Suppose for example that we have an instrument able to directly measure the internal energy of a system and we intend to do so for different values of the volume. Ideally, after taking a measure we change the volume of a very small amount and let the system settle down to its new equilibrium configuration before taking the next measurement. The differential equations we will be dealing in this appendix represent the change of internal energy (or of other thermodynamical quantities) under similar processes. The procedure we just described is called *quasi static* process. Finally is useful to notice that, exactly because of the interpretation we just pointed out, the variables we use to compute thermodynamical functions represent the set of quantities that we have under access to during an experiment. Thus, different thermodynamical potentials better fit to different experimental setup, even though they entail the same information on the equilibrium configuration of the system.

The fundamental equation for a fluid in the energy representation reads

$$U = U(S, V, N)\,, \tag{B.1}$$

where U is the internal energy and it depends on the entropy of the system S, the volume V and the particle number N. The differential version of (B.1) can be written as:

$$\mathrm{d}U = \left(\frac{\partial U}{\partial S}\right)_{V,N} \mathrm{d}S + \left(\frac{\partial U}{\partial V}\right)_{S,N} \mathrm{d}V + \left(\frac{\partial U}{\partial N}\right)_{V,S} \mathrm{d}N\,. \tag{B.2}$$

By comparison with the conservation of energy law:

$$\Delta U = \Delta Q + \Delta W_{mech} + \Delta W_{chem} = T\Delta S - p\Delta V + \mu\Delta N\,, \tag{B.3}$$

it is straightforward to see that we can identify the temperature T, the pressure p and the chemical potential μ with the partial derivatives of the internal energy:

$$T = \left(\frac{\partial U}{\partial S}\right)_{V,N}\,, \quad p = -\left(\frac{\partial U}{\partial V}\right)_{S,N}\,, \quad \mu = \left(\frac{\partial U}{\partial N}\right)_{V,S}\,. \tag{B.4}$$

According to this identification we have that, in the energy representation, T, p and μ play the role of parameters (as opposed to variables) and they turn out to be again functions of entropy, volume and number of particles:

$$T = T(S, V, N)\,, \tag{B.5}$$

$$p = p(S, V, N)\,, \tag{B.6}$$

$$\mu = \mu(S, V, N)\,. \tag{B.7}$$

In contrast with (B.1), the previous set of equations are not fundamental equations of thermodynamics and they depends on the detail of the system under examination, hence they are *equations of state*.

Since both the internal energy and its variables are extensive quantities the parameters turn out to be intensive. We can generalize this property of the energy introducing the notion of *homogeneous functions*. A function of m variables $f(x_1, \ldots, x_m)$ is a homogeneous function of degree n in its variables if:

$$f(\lambda x_1, \ldots, \lambda x_m) = \lambda^n f(x_1, \ldots, x_m) \tag{B.8}$$

for some number λ. The internal energy is thus an homogeneous function of degree 1 in its variables:

$$U(\lambda S, \lambda V, \lambda N) = \lambda U(S, V, N)\,. \tag{B.9}$$

Differentiating with respect to λ and setting $\lambda = 1$ one finds:

$$U = TS - pV + \mu N\,. \tag{B.10}$$

Inspecting the differential version of the last expression and imposing (B.2) and (B.4) one finds:

$$S\mathrm{d}T - V\mathrm{d}p + N\mathrm{d}\mu = 0\,. \tag{B.11}$$

Thanks to the extensiveness of entropy and volume we can rephrase the last result in terms of densities and obtain:

$$\mathrm{d}\mu = v\mathrm{d}p - s\mathrm{d}T\,, \tag{B.12}$$

where$v = \frac{V}{N}$ is the specific volume and $s = \frac{S}{N}$ the specific entropy. Equation (B.12) is called *Gibbs-Duhem* relation and is a fundamental property of thermodynamics because no equation of state was used in its derivation. Since this property is not spoiled by integrating the differentials, we have that the chemical potential as a function of pressure and temperature $\mu = \mu(p, T)$ is fundamental as well while the specific volume and entropy play the role of parameters.

Since the setup discussed so far relies on heaving experimental access to the arguments of the internal energy we might want to map the whole framework to a language depending on variables we can actually measure in an easy way. In order to do that we employ the Legendre transform described in appendix A. Given that the derivatives of the internal energy are given by (B.4) we can simply define the transform without the supremum condition. The most interesting transforms provide the following *thermodynamic* *potentials*[36]

1. Helmholtz free energy:

$$F(T, V, N) \equiv U - TS\,,$$

2. enthalpy:

$$H(S, p, N) \equiv U + pV\,,$$

3. Gibbs free energy

$$G(T, p, N) = U - TS + pV\,, \tag{B.13}$$

4. grand canonical potential (or Landau potential)

$$\Phi(T, V, \mu) = U - TS - \mu N\,. \tag{B.14}$$

Even if we did not list the full set of possible Legendre transforms of a function of three variables as the internal energy, it is worth to notice that, since the internal energy is an homogeneous function of first degree in its variables, the simultaneous Legendre transform in all of them would be zero:

$$U - TS + pV - \mu N = 0\,. \tag{B.15}$$

As a remark we point out that everything stated until this point has an equivalent formulation in the entropy representation. In this case we solve the entropy with respect to the internal energy:

$$\begin{aligned} S &= S(U, V, N)\,, \\ \mathrm{d}S &= \frac{1}{T}\mathrm{d}U + \frac{p}{T}\mathrm{d}V - \frac{\mu}{T}\mathrm{d}N\,. \end{aligned} \tag{B.16}$$

There exist, thus, a set of thermodynamical potential derived from Legendre transforms of the entropy.

Let us now try to understand the connection between statistical mechanics and thermodynamics. To this end we consider the example of a canonical ensemble. This type of system (we will call it $\mathcal{S}$) can be understood as embedded into a microcanonical one from which the degrees of freedom of the reservoir (say $\mathcal{R}$) are integrated out under the constraint that the only energy can be exchanged between the two. Since the integration takes place over the reservoir variables, the probability distribution to find the system $\mathcal{S}$ in a microstate j of energy E_j will be proportional to the volume $\Omega_{\mathcal{R}} = \Omega_{\mathcal{R}}(E_{calR})$ that

[36] Even if we define the thermodynamic potential as a Legendre transform changed by an overall sign all the listed potential reach a minimum at the equilibrium. To the careful reader this will not look like contradicting the statement that the Legendre transform preserves convexity. In fact, the statement is about the values of energy at equilibrium as a function of fixed entropy, volume and number pf particles at equilibrium. If we were to consider out of equilibrium thermodynamics, other internal variables should be included. We shall not discuss this treatment here.

the reservoir microstates occupy in the phase space:

$$P_j = c\Omega_{\mathcal{R}}(E_0 - E_j)\,, \tag{B.17}$$

where c is a normalization constant and E_0 represents the energy of the microcanonical ensemble. If we take the logarithm of P_j and expanding for small values of the energy E_j we obtain:

$$\begin{aligned}\log P_j = & \log c + \log \Omega_{\mathcal{R}}(E_0) + \frac{\partial \Omega_{\mathcal{R}}(E)}{\partial E}\Big|_{E=E_0}(-E_j) \\ & + \frac{1}{2}\frac{\partial^2 \Omega_{\mathcal{R}}}{\partial E^2}\Big|_{E=E_0} E_j^2 + o(E^3)\,,\end{aligned} \tag{B.18}$$

where the expansion id of course justified by the assumption that the reservoir is assumed to be much bigger than the system itself. Using the definition of the statistical entropy $S(E) = k_B \log \Omega(E)$ where k_B is the Boltzmann constant we have that:

$$\begin{aligned}\frac{\partial \log \Omega_{\mathcal{R}}(E)}{\partial E} &= \frac{1}{k_B T} = \beta\,, \\ \frac{\partial^2 \log \Omega_{\mathcal{R}}(E)}{\partial E^2} &= \frac{1}{k_B}\frac{\partial}{\partial E}\left(\frac{1}{T}\right) \longrightarrow 0\,,\end{aligned} \tag{B.19}$$

where use of (B.16) has been made and the limit in the second line of the last equation represents the thermodynamical limit. Hence, the expansion (B.18) becomes:

$$\log P_j = C - \beta E_j\,, \tag{B.20}$$

which leads to:

$$P_j = \frac{e^{-\beta E_j}}{\sum_k e^{-\beta E_k}}\,. \tag{B.21}$$

The partition function Z for the canonical ensemble is defined by the normalization factor of the probability distribution. If we take into account the possible energy degeneracies of the microstates we are allowed to rewrite the sum in the partition function as running over the energy values:

$$Z = \sum_k e^{-\beta E_k} = \sum_E \Omega(E) e^{-\beta E}\,. \tag{B.22}$$

Since in the thermodynamical limit (and away from criticality) the energy fluctuations become irrelevant, we can approximate the partition function by its maximal contribution knowing that the approximation will become an exact equality once that the number of particles is sent to infinity:

$$Z = \sum_E \exp\left\{\log \Omega(E) - \beta E\right\} \sim \exp\left\{\beta \sup_E \left(TS(E) - E\right)\right\}\,, \tag{B.23}$$

where we recognize the definition of the Helmholtz free energy F:

$$Z \sim e^{-\beta F}\,, \tag{B.24}$$

or, in the thermodynamical limit:

$$f(T,v) = -\frac{1}{\beta} \lim_{V,N\to\infty} \frac{1}{N} \log Z(T,V,N)\,. \tag{B.25}$$

A similar treatment can be performed for other types of system, where different constraints are imposed on the microstates probability distribution. The most remarkable results are for sure the grand canonical ensemble, the partition function of which is surprisingly given by the grand canonical potential, and the pressure ensemble described by the Gibbs free energy. This last ensemble is the one describing the Ising model. In this case we should identify the total spin of the system as the total volume (it is indeed a function of the number of sites of the system) and the external magnetic field as (minus) the pressure. The external magnetic field is of course an intensive quantity and affects the total spin which represents the response of the system to the external perturbation and, for a non zero induced magnetization, is an extensive quantity. Since in an experimental setup we think about the Ising model as embedded into a fixed external magnetic field (or generalized pressure under this identification) then the correct thermodynamical potential to describe the system is given by the Gibbs free energy.

Appendix C.

Wick rotation

This appendix is dedicated to fill the gap between the formalism of statistical field theory and quantum field theory and to set the conventions implicitly used in the text. Most of the textbooks about quantum field theory contain similar discussions. A good exposition can be found in [5]. Let us consider an action for a massive scalar field defined on a flat manifold of Lorentzian:

$$S[\varphi] = \int \mathrm{d}^d x \,\left[\frac{1}{2}\varphi\left(-\Box - m^2\right)\varphi - V(\varphi)\right], \tag{C.1}$$

where $\Box = \partial_\mu \partial^\mu$. The action is defined in such a way that the eigenvalues for the kinetic term are bounded from below for timelike momenta. For $V(\varphi) = 0$ we can easily compute the propagator in momentum space and check that the poles reproduce the relativistic dispersion relation (in natural units):

$$p_0^2 = \mathbf{p}^2 + m^2 \,.$$

If we assume that the inclusion of $V(\varphi)$ does not spoil the well definiteness of the eigenvalues problem of the equations of motion, we can expect the poles of the propagator to reproduce the relativistic dispersion relation in presence of a potential. Let us, thus, call ω the real part of the pole for p_0, i.e., the momentum space propagator develops a singularity for $p_0 = \pm\omega$. It is well known that the causal prescription to regulate this divergence consists into shifting the poles of an infinitesimal imaginary part such that they will lie in the second and fourth quadrants of the complex p_0 plane. We can thus consider the integration path shown in figure C.1a. The arch contribution vanishes as their radius is sent to infinity. Since the poles are left out of the contour we see that the integration along the real axis is equivalent to the one along the imaginary one. This is the core of the Wick rotation and it boils down to the substitution:

$$p_0 \longrightarrow i p_0 \,, \tag{C.2}$$

which renders a convergent integral for the Fourier transform of the Feynman propagator.

If we wish to perform a similar trick in direct space we first need to identify the location of the poles. If we parametrize the causal prescription in terms of an infinitesimal rotation of the complex plane:

$$\begin{aligned} p_0 &\longrightarrow e^{-i\theta} p_0 \,, \\ \sin(\theta) &= \frac{\epsilon}{\omega} \,, \end{aligned} \tag{C.3}$$

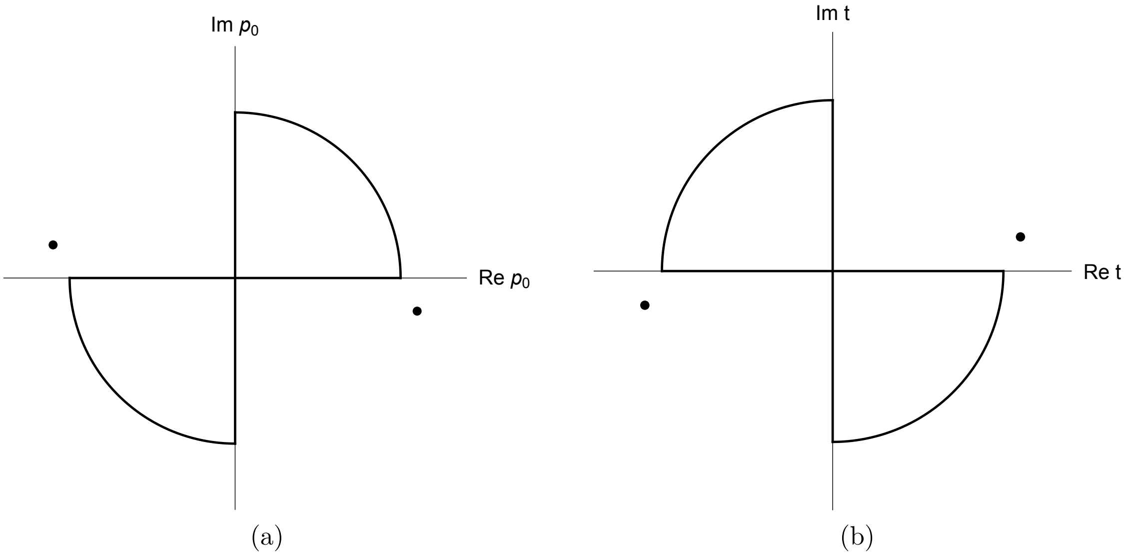


Fig. C.1.: The figure shows the typical disposition of poles in the p_0 complex plane (figure C.1a) an the t complex plane (figure C.1b)

we can identify the corresponding rotation in the direct space as:

$$t \propto \frac{\partial}{\partial p_0} \longrightarrow e^{i\theta} \frac{\partial}{\partial p_0} \,. \tag{C.4}$$

The resulting integration contour to be consider is shown in figure C.1b and corresponds to the transformation:

$$t \longrightarrow -it_{\mathcal{E}} \,. \tag{C.5}$$

The transformation of the action (C.1) can be inferred inspecting the integration measure and the transformation of the d'Alembert operator:

$$i\mathrm{d}t \longrightarrow \mathrm{d}t_{\mathcal{E}} \,, \tag{C.6}$$

$$-\Box = -\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \longrightarrow \frac{\partial}{\partial t_{\mathcal{E}}} + \frac{\partial}{\partial \mathbf{x}} \equiv \Box_{\mathcal{E}} \,, \tag{C.7}$$

$$iS[\varphi] \longrightarrow -S_{\mathcal{E}}[\varphi] = -\int \mathrm{d}^d x_{\mathcal{E}} \, \left[\frac{1}{2} \varphi \left(-\Box_{\mathcal{E}} + m^2 \right) \varphi + V(\varphi) \right] \,. \tag{C.8}$$

Hence, a Lagrangian formulation of a dynamical system is mapped into an Hamiltonian formulation of a static problem. In order to complete this parallelism we should reinsert the constants $\hbar$ and $\beta = \frac{1}{k_B T}$ necessary to keep track of the correct dimensions of fields and coupling

$$Z[J] = \int \mathcal{D}\varphi \; e^{\frac{i}{\hbar} S[\varphi] + \frac{i}{\hbar} \int J\varphi} \to \int \mathcal{D}\varphi \; e^{-\beta S_{\mathcal{E}}[\varphi] + \int J\varphi} \,. \tag{C.9}$$

In doing so we notice how they play a similar role as $\hbar$ is source for the quantum fluctuations as much as the temperature is a source for the thermal ones. The quantum path integral $Z[J]$, functional of the classical source field $J(x)$ is thus rearranged in a statistical partition function.

Strong of the functional formalism we can now define the $W[J]$ and $\Gamma[\Phi]$. The generating functional of connected Green functions $W[J] = i\hbar \log Z[J]$ is mapped to the Euclidean side under the same rule of the action to $W_{\mathcal{E}}[J] = \frac{1}{\beta}\log Z[J]$. It is now clear how $W[J]$ plays the role of the free energy of the system. This equivalence persists to the level of correlation functions. In fact the 2-point connected Green function

$$G^c(x,x') = (-i\hbar)^2 \frac{\delta^2}{\delta J(x)\delta J(x')} \log Z[J]\Big|_{J=0}, \tag{C.10}$$

becomes

$$G^c_{\mathcal{E}}(x,x') = \frac{1}{\beta^2}\frac{\delta^2}{\delta J(x)\delta J(x')} \log Z[J]\Big|_{J=0} = \frac{1}{\beta}\frac{\delta^2}{\delta J(x)\delta J(x')} W_{\mathcal{E}}[J]\Big|_{J=0} \equiv \frac{1}{\beta}\chi(x,x'), \tag{C.11}$$

$\chi(x,x')$ being the generalized susceptibility.

As we discussed in B, when the degrees of freedom are infinite the free energy takes the form of a thermodynamical potential, which specific one depending on the boundary conditions of the system and the specific experimental setup. The mathematical tool to change the fundamental variable encoding the information about the system is the Legendre transform. In the present case the conjugate variable of the classical source for the functional $W_{\mathcal{E}}[J]$[37] turns out to be the normalized expectation value of the field: $\Phi \equiv \frac{\delta W}{\delta J} = \frac{\langle\varphi(x)\rangle}{\langle 1\rangle}$. Hence:

$$\Gamma_{\mathcal{E}}[\Phi] = \int \mathrm{d}^d x\; J(x)\Phi(x) - W_{\mathcal{E}}[J], \tag{C.12}$$

for which we have the well known functional relation:

$$\int \mathrm{d}z\; \frac{\delta^2 W_{\mathcal{E}}[J]}{\delta J(z)\delta J(x)} \frac{\delta^2 \Gamma_{\mathcal{E}}[\Phi]}{\delta\Phi(y)\delta\Phi(z)} = \delta(x-y). \tag{C.13}$$

Mapping the Legendre transform (C.12) to the quantum field theory side one finds the definition of the quantum effective action to be:

$$\Gamma[\Phi] = -\int \mathrm{d}^d x\; J(x)\Phi(x) - W[J], \tag{C.14}$$

$$\int \mathrm{d}z\; \frac{\delta^2 W[J]}{\delta J(z)\delta J(x)} \frac{\delta^2 \Gamma[\Phi]}{\delta\Phi(y)\delta\Phi(z)} = \delta(x-y). \tag{C.15}$$

and we recognize in Γ the generating functional for one particle irreducible Feynman graphs.

Even though the sign in the Legendre transform might seem confusing we first need to notice that with a Lorentzian signature we have that

$$\frac{\delta W[J]}{\delta J(x)} = -\frac{\langle\varphi(x)\rangle}{\langle 1\rangle} = -\Phi(x). \tag{C.16}$$

The parallelism we just illustrated clarifies the role of Landau approach to phase tran-

[37] The Legendre transform is naturally defined in an Euclidean setting. Therefore we define it for the statistical field theory first and use Wick rotation to map it back to the Lorentzian case.

sitions and the connection to the renormalization program for the effective action.

Appendix D.

Fierz transformations

Fierz transformations are a set of identities that allow to express spinor self interactions constructed through a specific channel as a linear combinations of all the channels upon reordering of the fermionic fields. The following results can be found, e.g., in [5]. Given matrix space $\mathcal{L}$ with a basis Γ^A we have that an element of the space $M \in \mathcal{L}$ acting on a $d_{\mathcal{S}}$ dimensional vector space can be decomposed as:

$$M = \frac{1}{d_{\mathcal{S}}} \sum_A \mathrm{tr}(M\Gamma^A)\Gamma^A \,. \tag{D.1}$$

With a little algebra it is easy to extract the completeness relation and orthogonality condition for the basis:

$$\frac{1}{d_{\mathcal{S}}} \mathrm{tr}(\Gamma^A \Gamma^B) = \delta^{AB} \,, \tag{D.2}$$

$$\frac{1}{d_{\mathcal{S}}} \sum_A \Gamma^A_{ml} \Gamma^A_{ik} = \delta_{mk}\delta_{il} \,. \tag{D.3}$$

In the case of spinors $\psi \in \mathcal{S}$ over a four dimensional manifold we have that the space $\mathcal{L} = \bar{\mathcal{S}} \otimes \mathcal{S}$ is generated by the elements of the Clifford algebra:

$$\begin{aligned} &\Gamma^A \in \left\{ \mathbb{1}, \gamma_\mu, \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right], i\gamma_\mu\gamma_5, \gamma_5 \right\} \,, \\ &\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 \,. \end{aligned} \tag{D.4}$$

In this case $d_{\mathcal{S}}$ just becomes the dimension of the spin representation. Now let χ, ψ, ζ, and ξ be spinors and let us multiply equation (D.3) by $\bar{\chi}_m (\Gamma^B \psi)_k \, \bar{\zeta}_i \, (\Gamma^B \xi)_l$. This leads to a relation between quartic monomial weighted by a single element in (D.4) to a linear combination of quartic monomials with shuffled fields:

$$\left(\bar{\chi}\Gamma^B\psi\right)\left(\bar{\zeta}\Gamma^B\xi\right) = -\frac{1}{d_{\mathcal{S}}} \sum_A \left(\bar{\chi}\Gamma^A\Gamma^B\xi\right)\left(\bar{\zeta}\Gamma^A\Gamma^B\psi\right) \,. \tag{D.5}$$

If we wish to construct interactions which are invariant with respect to the manifold symmetry group we need to entail covariance with respect to the Greek indices in (D.4). Hence, it is useful to define the following interaction channels:

1. scalar channel:

$$(S) \equiv \left(\bar{\chi}\mathcal{O}_S\psi\right)\left(\bar{\zeta}\mathcal{O}_S\xi\right) = \left(\bar{\chi}\psi\right)\left(\bar{\zeta}\xi\right) \,, \tag{D.6}$$

2. vectorial channel:

$$(V) \equiv (\bar{\chi}\mathcal{O}_V\psi)\left(\bar{\zeta}\mathcal{O}_V\xi\right) = (\bar{\chi}\gamma^\mu\psi)\left(\bar{\zeta}\gamma_\mu\xi\right) \ , \tag{D.7}$$

3. tensorial channel:

$$(T) \equiv (\bar{\chi}\mathcal{O}_T\psi)\left(\bar{\zeta}\mathcal{O}_T\xi\right) = (\bar{\chi}\sigma^{\mu\nu}\psi)\left(\bar{\zeta}\sigma_{\mu\nu}\xi\right) \ , \tag{D.8}$$

4. axial channel:

$$(A) \equiv (\bar{\chi}\mathcal{O}_A\psi)\left(\bar{\zeta}\mathcal{O}_A\xi\right) = -\left(\bar{\chi}\gamma^\mu\gamma_5\psi\right)\left(\bar{\zeta}\gamma_\mu\gamma_5\xi\right) \ , \tag{D.9}$$

5. pseudoscalar channel:

$$(P) \equiv (\bar{\chi}\mathcal{O}_P\psi)\left(\bar{\zeta}\mathcal{O}_P\xi\right) = (\bar{\chi}\gamma_5\psi)\left(\bar{\zeta}\gamma_5\xi\right) \ . \tag{D.10}$$

Working out the algebra of equation (D.5) we can summarize Fierz transformations in terms of the above channels as a matrix law:

$$\begin{pmatrix} S \\ V \\ T \\ A \\ P \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -1 & \frac{1}{2} & 0 & -\frac{1}{2} & 1 \\ -\frac{3}{2} & 0 & \frac{1}{2} & 0 & -\frac{3}{2} \\ -1 & -\frac{1}{2} & 0 & \frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} S' \\ V' \\ T' \\ A' \\ P' \end{pmatrix} \ , \tag{D.11}$$

where the prime on the right hand side indicates the shuffle of spinor fields. In particular we notice that if we construct an interaction out of rotational invariant terms of the form:

$$(V) + (A) = \left(\bar{\psi}\gamma^\mu\psi\right)(\bar{\chi}\gamma_\mu\chi) - \left(\bar{\psi}\gamma^\mu\gamma_5\psi\right)(\bar{\chi}\gamma_\mu\gamma_5\chi) \ , \tag{D.12}$$

and apply Fierz transformations we end up with the following expression:

$$-2[(S') - (P')] = -2\left[\left(\bar{\psi}\chi\right)(\bar{\chi}\psi) - \left(\bar{\psi}\gamma_5\chi\right)(\bar{\chi}\gamma_5\psi)\right] \ . \tag{D.13}$$

Appendix E.

Spinor heat kernel on hyperbolic spaces

For completeness, we summarize results for the heat kernel on hyperbolic spaces in this appendix, as they are needed for the present thesis. The derivation and the conventions mostly follow the pioneering work presented in [140, 141]. Hence, we start by normalizing the inverse radius κ of the manifold to 1. We will reinstate this curvature parameter later on. The heat kernel for the squared Dirac operator on a d-dimensional hyperbolic space can be written as:

$$\mathbb{K}(x, x', s) = \mathbb{U}(x, x')\hat{f}_N(d_G, s)\,, \tag{E.1}$$

$$\hat{f}_N(y, s) = \frac{2^{d-3}\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}+1}} \int_0^\infty \varphi_\lambda(y) e^{-s\lambda^2}\mu(\lambda) d\lambda\,, \tag{E.2}$$

where $\mathbb{U}$ represents a parallel transport, and $\hat{f}_N$ is a scalar function satisfying the following equation:

$$\begin{aligned} 0 &= \Big(-\frac{\partial}{\partial s} + \Box_d - \frac{R}{4} - \frac{d-1}{4}\tanh^2(y)\Big)\hat{f}_N(y) & \text{(E.3)}\\ &\equiv \Big(-\frac{\partial}{\partial s} + L_d\Big)\hat{f}_N(y)\,, & \text{(E.4)}\end{aligned}$$

with $\Box_d$ being the radial Laplacian. The eigenfunctions φ_λ of the L_d operator with eigenvalues $-\lambda^2$ can be written as

$$\begin{aligned} &L_d\varphi_\lambda = -\lambda^2\varphi_\lambda & \text{(E.5)}\\ &\varphi_\lambda(y) = \cosh\frac{y}{2}\, {}_2F_1\Big(\frac{d}{2} + i\lambda, \frac{d}{2} - i\lambda; \frac{d}{2}; -\sinh^2\frac{y}{2}\Big). \end{aligned}$$

Here, ${}_2F_1$ denotes the hypergeometric function, while the spectral measure $\mu(\lambda)$ reads:

$$\begin{aligned} \mu(\lambda) =& \frac{\pi}{2^{2d-4}\Gamma^2\left(\frac{d}{2}\right)} & \text{(E.6)}\\ &\times \begin{cases} \prod_{j=\frac{1}{2}}^{\frac{d}{2}-1}(\lambda^2 + j^2)\,, & d \quad \text{odd}\\ \lambda\coth(\pi\lambda)\prod_{j=1}^{\frac{d}{2}-1}(\lambda^2 + j^2)\,, & d \text{ even}\,. \end{cases} \end{aligned}$$

In the main text, cf. Sect. 4.2, we only need the equal point limit of the heat kernel, with $x' \to x$ and the geodesic distance $d_G \to 0$ goes to zero, i.e., $y \to 0$ in (E.2). From equation (E.5) is clear that the coincident points limit leads to

$$\lim_{y\to 0} \varphi_\lambda(y) = 1\,, \tag{E.7}$$

while the $\mathbb{U}$ reduces to the identity. Thus, we end up with

$$K_s = \frac{2^{d-3}\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}+1}} \int_0^\infty d\lambda\; e^{-s\lambda^2} \mu(\lambda) d\lambda\,. \tag{E.8}$$

In order to reinstate the curvature parameter, we make contact with the flat space limit of the heat kernel, starting with the odd dimensional case. Plugging the definition of $\mu(\lambda)$ into equation (E.8), we get upon substitution

$$\begin{aligned} K_s^{\text{odd}} &= \frac{2}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_0^\infty d\lambda\; e^{-s\lambda^2} \prod_{j=\frac{1}{2}}^{\frac{d}{2}-1} (\lambda^2 + j^2) \\ &= \frac{2}{(4\pi s)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_0^\infty du\; e^{-u^2} \prod_{j=\frac{1}{2}}^{\frac{d}{2}-1} (u^2 + j^2 s)\,, \end{aligned} \tag{E.9}$$

and similarly for an even dimensional background:

$$\begin{aligned} K_s^{\text{even}} =& \frac{2}{(4\pi s)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_0^\infty du\; e^{-u^2} u \coth(\pi \frac{u}{\sqrt{s}}) \\ & \times \prod_{j=1}^{\frac{d}{2}-1} (u^2 + j^2 s)\,. \end{aligned} \tag{E.10}$$

Recalling that in flat spacetime the heat kernel in the coincident points limit reads $K_s = (4\pi s)^{-1}$ with s carrying mass dimension $[s] = -2$, we obtain the correct limit by rescaling the propertime inside the integrals by a sufficient power of the curvature parameter with $[\kappa] = 1$; note that the integration variables has to remain dimensionless, $[u] = 0$. We finally obtain,

$$K_s^{\text{odd}} = \frac{2}{(4\pi s)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_0^\infty du\; e^{-u^2} \prod_{j=\frac{1}{2}}^{\frac{d}{2}-1} (u^2 + j^2\kappa^2 s)\,, \tag{E.11}$$

$$\begin{aligned} K_s^{\text{even}} =& \frac{2}{(4\pi s)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_0^\infty du\; e^{-u^2} u \coth(\pi \frac{u}{\kappa\sqrt{s}}) \\ & \times \prod_{j=1}^{\frac{d}{2}-1} (u^2 + j^2\kappa^2 s)\,. \end{aligned} \tag{E.12}$$

For an analytical approximation in even dimensions, the expansion of the integrand in the two limits $s \approx 0$ and $s \approx \infty$ are useful. For small s, we rewrite the hyperbolic cotangent as:

$$\coth(\pi\frac{u}{\kappa\sqrt{s}}) =1 + 2\sum_{n=1}^{\infty} e^{-2\frac{\pi n u}{\kappa\sqrt{s}}} \,. \tag{E.13}$$

The large s regime corresponds to the small u approximation of the hyperbolic cotangent, thus, it suffices to consider the first few terms in the Laurent expansion of $\coth(\pi\frac{u}{\kappa\sqrt{s}})$ in order to capture the behavior of K_s for s around infinity,

$$\coth(\pi\frac{u}{\kappa\sqrt{s}}) = \frac{\kappa\sqrt{s}}{\pi u} + \frac{\pi u}{3\kappa\sqrt{s}} + \mathcal{O}(u^3) \,. \tag{E.14}$$

These two approximations are combined in section 4.4 to identify an analytic approximation for the heat-kernel trace in four dimensions.

Appendix F.

Covariant representation of the Green function

This and the next appendix follow roughly the presentation of [115] but summarize and adapt it to the specific purposes of this thesis. We restrict our attention to simple scalar fields, but the inclusion of internal indices and a gauge connection is straightforward. Let us consider an operator of Laplace-type

$$\mathcal{O} = -g^{\mu\nu}\nabla_\mu\partial_\nu + E \,, \tag{F.1}$$

in which we included a local endomorphism $E = E(x)$ acting multiplicatively on the scalar field's bundle. Notice that the spacetime metric $g_{\mu\nu}$ appears both through the inverse $g^{\mu\nu}$ and inside the Christoffel's symbols Γ of Levi-Civita connection $\nabla = \partial + \Gamma$.

In the background field approach to the Euclidean path integral the curved space propagator of the scalar field corresponds to the Green function of the operator $\mathcal{O}$ for an opportune choice of the endomorphism E. The Green function is defined as

$$\mathcal{O}_x G(x,x') = \delta^{(d)}(x,x') \,, \tag{F.2}$$

in which we introduced the biscalar δ-function that generalizes the usual flat space Dirac delta. The propagator that is used in the main text can be obtained by specifying the endomorphism as $E = F''(\phi)R$.

It is convenient to represent the Green function using the heat kernel method. The heat kernel function is defined as the solution of the following differential equation

$$\begin{aligned} &\partial_s \mathbb{K}(s;x,x') + \mathcal{O}_x \mathbb{K}(s;x,x') = 0 \,, \\ &\mathbb{K}(0;x,x') = \delta^{(d)}(x,x') \,. \end{aligned} \tag{F.3}$$

If we solve the diffusion equation implicitly

$$\mathbb{K}(s;x,x') = \langle x' | \, \mathrm{e}^{-s\mathcal{O}} \, | x \rangle \,, \tag{F.4}$$

then the relation of the heat kernel function with the Green function is straightforward

$$G(x,x') = \int_0^\infty \mathrm{d}s \, \mathbb{K}(s;x,x') \,. \tag{F.5}$$

For all intents and purposes the above relation should be taken as our operative definition of $G(x,x')$.

The heat kernel representation is useful because the solution admits an asymptotic expansion for small values of the parameter s, known as the Seeley-de Witt expansion, which captures the ultraviolet properties of the Green function. The expansion is generally parametrized as

$$\mathbb{K}(s;x,x') = \frac{\Delta(x,x')^{1/2}}{(4\pi s)^{d/2}}\, \mathrm{e}^{-\frac{\sigma(x,x')}{2s}} \sum_{k\geq 0} a_k(x,x')\, s^k \,. \tag{F.6}$$

We introduced several bitensors in the expansion. The most fundamental is $\sigma(x,x')$, sometimes known as Synge's or Synge-de Witt's world function, which is half of the square of the geodesic distance between the points x and x' [217]. The bitensor $\Delta(x,x')$ is known as the van Vleck determinant and is related to the world function and the determinant of the metric as

$$\Delta(x,x') = (g(x)g(x'))^{-1/2} \det\left(-\partial_\mu \partial_{\nu'}\sigma\right) \,.$$

Together, the bitensors $\sigma(x,x')$ and $\Delta(x,x')$ ensure that the leading term of the Seeley-de Witt parametrization covariantly generalizes the solution of the heat equation in flat space with $\mathcal{O} \sim -\partial^2$. Finally, the bitensors $a_k(x,x')$ are the coefficients of the asymptotic expansion and contain the geometrical information of the operator $\mathcal{O}$, which includes curvatures, connections and interactions.

It is well-known that ultraviolet properties are (and must be) local in renormalizable theories. For the case of the heat kernel and the Green function locality corresponds to $x \sim x'$ and it is captured by the so-called coincidence limit in which $x \to x'$. Given any bitensor $B(x,x')$, its coincidence limit is defined

$$[B] = \lim_{x'\to x} B(x,x') \,. \tag{F.7}$$

Notice that covariant derivatives do not generally commute with the coincidence limit $\nabla[B] \neq [\nabla B]$, but rather satisfy a modified relation [217, 218].

The coincidence limits of the bitensors $\sigma(x,x')$ and $\Delta(x,x')$ and their derivatives can be obtained by repeated differentiation of the *crucial* relations

$$\sigma_\mu \sigma^\mu = 2\sigma \,, \qquad \Delta^{1/2}\sigma_\mu{}^\mu + 2\sigma^\mu \nabla_\mu \Delta^{1/2} = d\Delta^{1/2} \,, \tag{F.8}$$

for which we suppressed bitensor coordinates and we used the notation in which subscripts of $\sigma(x,x')$ correspond to covariant derivatives. Similarly, coincidence limits of the coefficients $a_k(x,x')$ can be obtained by differentiating and inductively using

$$k a_k + \sigma^\mu \nabla_\mu a_k + \Delta^{-1/2}\mathcal{O}(\Delta^{1/2} a_{k-1}) = 0 \tag{F.9}$$

with the boundary condition $\sigma^\mu \nabla_\mu a_0 = 0$. In the relevant example of a simple scalar field the first coefficient is trivial $a_0(x,x') = 1$, because the Seeley-de Witt expansion solves the diffusion equation in flat space. We give here the first two nontrivial coincidence limits for the expansion of the operator (F.1) which are used in the computations of the main text

$$
\begin{aligned}
[a_1] &= \frac{R}{6} - E\,, \\
[a_2] &= \frac{1}{72}R^2 - \frac{1}{6}RE + \frac{1}{2}E^2 - \frac{1}{6}\nabla^2\left(E - \frac{1}{6}R\right) \\
&\quad + \frac{1}{180}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu}\right)\,.
\end{aligned} \tag{F.10}
$$

Using (F.6) in (F.5), we obtain an analog expansion of the Green function

$$
G(x,x') = \sum_{k\geq 0} G_k(x,x')a_k(x,x')\,. \tag{F.11}
$$

The leading $G_0(x,x')$ and the subleading $G_k(x,x')$ for $k \geq 1$ are bilocal contributions to the Green function and are determined by a simple integration over the heat kernel parameter s

$$
G_k(x,x') = \frac{2^{d-2-2k}}{(4\pi)^{d/2}}\frac{\Delta^{1/2}}{(2\sigma)^{d/2-1-k}}\Gamma\left(\frac{d}{2} - 1 - k\right)\,. \tag{F.12}
$$

Depending on the theory and its dimensionality, the exponent $d/2 - 1 - k$ inevitably becomes negative for a certain value of k highlighting the fact that there is only a finite number of Green's function contributions which are singular in the limit $x \sim x'$.

It is very important to point out that in (F.12) we have implicitly assumed that the Green contributions do not scale logarithmically with $\sigma(x,x')$ at the critical dimension. While they do not scale logarithmically for almost all the multicritical models considered in this work (that is, for all ϕ^{2n} with $3 \leq n < \infty$) they do, however, show a logarithmic behavior if there are values of k for which $d = 2 + k$, which corresponds to poles of the gamma function. In this case, one sees the failure of capturing the logarithmic behavior explicitly through the $\epsilon \to 0$ limit of (F.12) which is not regular even outside $x \sim x'$.

For this reason, when $d = 2 + k$ one needs to subtract an ϵ-pole to (F.12) for the results to be valid *at and close to* the dimension d. For example, close to $d = 2$ the leading propagator is already logarithmic and we subtract

$$
G_0^{2-\epsilon}(x,x') = \frac{\Delta^{1/2}}{(4\pi)^{d/2}}\frac{\Gamma\left(d/2 - 1\right)}{(2\sigma)^{d/2-1}} + \mu^{-\epsilon}\,\frac{\Delta^{1/2}}{2\pi\epsilon}\,. \tag{F.13}
$$

As desired, the above expression is valid for d close to two dimensions and is regular in the limit $\epsilon \to 0$ for $d = 2 - \epsilon$. We report here the subleading part of the Green function for $d = 4 - \epsilon$, which is also needed in the main text

$$
G_1^{4-\epsilon}(x,x') = \frac{\Delta^{1/2}}{(4\pi)^{d/2}}\frac{\Gamma\left(d/2 - 2\right)}{(2\sigma)^{d/2-2}} + \mu^{-\epsilon}\,\frac{\Delta^{1/2}}{8\pi^2\epsilon}\,. \tag{F.14}
$$

General expressions for leading and subleading parts can be found in [115, 118]. The generalization to $d = 6 - \epsilon$, which requires subtractions starting from $k = 2$, can be found in [116, 117].

The need for a correct subtraction of the $\epsilon \to 0$ limit can be seen in the general counterterms (5.12). In fact the $(n-2)$ pole of the third counterterm is a symptom of the fact that the general $n \geq 3$ results cannot be straightforwardly continued because in $d = 4$ because the first subleading propagator becomes logarithmic. This is the main reason why the renormalization of the ϕ^4 universality class is an outlier.

Appendix G.

ϵ-poles in curved space

Covariant Feynman diagrams are constructed as products of propagators and hence of Green functions. Taking advantage of the Seeley-de Witt representation given in Appendix F, we notice that diagrams are generally written as products of powers of $\sigma(x,x')$, $\Delta(x,x')$, heat kernel coefficients and eventually other bilocal operators which could be introduced by the theory's vertices. In the case of a simple scalar field with a canonical kinetic term and no derivative interactions the leading Feynman diagrams only have local vertices and can be represented by products of "bundles" of propagators. In the applications investigated in this work, there is always just one bundle of propagators attached to the same two spacetime points as seen in (5.12).

Generally, we want to obtain the dimensionally regulated divergent parts of covariant structures of the form

$$Q(x,x')\Delta(x,x')^a \frac{1}{\sigma(x,x')^b} \tag{G.1}$$

in which $Q(x,x')$ is an arbitrary bilocal operator, coming from the Seeley-de Witt coefficients, or the vertices, or other parts of the diagram. The constants a and b are arbitrary powers that depend by the details of the diagram itself (for example by the number of propagators and the value of the critical dimension). In the following, we shall briefly describe an algorithm due to Jack and Osborn which was developed to treat this kind of structures in dimensional regularization [115].

One starts with the basic relation

$$\frac{1}{\sigma(x,x')^{\frac{d}{2}-c\epsilon}} \sim \frac{(2\pi)^{\frac{d}{2}}}{c\;\epsilon\;\Gamma(d/2)}\;\mu^{-2c\epsilon}\;\delta^{(d)}(x,x')\,, \tag{G.2}$$

in which we introduced the symbol $\sim$ to establish equivalence of the divergent parts of both sides of the equation and a reference scale μ to preserve the dimensionality of the right hand side. It is easy to prove the above relation in flat space for which it is sufficient to perform a Fourier transform and use the fact that $\sigma(x,x') = |x-x'|^2/2$; it is then sufficient to argue that divergences are local and there cannot be curvature corrections on the right hand side because of dimensional reasons. More generally, this relation can be proven using Riemann normal coordinates in curved space [115].

Notice that if both sides of (G.2) are multiplied by the same bilocal operator, then the Dirac delta on the right hand side allows for the substitution of its coincidence limit $Q(x,x')\delta^{(d)}(x,x') = [Q]\delta^{(d)}(x,x')$. The core of the algorithm is thus to transform all possible inverse powers of the world function into those of the left-hand-side of the basic relation, substitute them with the right-hand-side, and then sort all bilocal operators at the numerator so that they enter in contact with the Delta function.

Higher inverse powers of the world function can be manipulated inverting

$$\begin{aligned} &(\nabla^2 - Y)\frac{\Delta^{1/2}}{\sigma^b} = b(2(b+1)-d)\frac{\Delta^{1/2}}{\sigma^{b+1}}\,, \\ &Y(x,x') \equiv \Delta^{-1/2}\nabla^2\Delta^{1/2}\,, \end{aligned} \tag{G.3}$$

which can be proven easily using (F.8). For the purpose of this thesis we just need

$$\frac{\Delta^{1/2}}{\sigma(x,x')^{\frac{d}{2}+1-c\epsilon}} \sim \frac{(2\pi)^{\frac{d}{2}}\mu^{-2c\epsilon}}{c\;\epsilon\;d\;\Gamma(d/2)}\left(\nabla^2 - \frac{R}{6}\right)\delta^{(d)}(x,x')\,, \tag{G.4}$$

which is obtained inverting (G.3) for $b = d/2$ and using the coincidence limits of the biscalars $[\Delta^{1/2}] = 1$ and $[Y] = R/6$.

Generalizations of (G.4) including higher inverse powers can also be easily obtained, however further iterations of (G.3) typically exhibit bilocal operators which are separated from the Dirac delta by the presence of covariant derivatives (imagine, for example, placing $Q(x,x')$ on both sides of (G.4)) and hence their coincidence limit cannot be taken. In such cases, it is necessary to integrate by parts all covariant derivatives so that all bilocal operators come in contact with the Dirac delta. For example, if one covariant derivative is located between the bilocal operator and the Delta we manipulate as follows

$$\begin{aligned} &Q(x,x')\nabla_\mu\delta^{(d)}(x,x') \\ &\quad= \nabla_\mu\left(Q(x,x')\delta^{(d)}(x,x')\right) - \nabla_\mu Q(x,x')\delta^{(d)}(x,x') \\ &\quad\sim \nabla_\mu\left([Q]\delta^{(d)}(x,x')\right) - [\nabla_\mu Q]\delta^{(d)}(x,x')\,. \end{aligned} \tag{G.5}$$

In the second line we have exploited the Delta to take the coincidence limit of the neighboring operators. A similar manipulation can be performed for the case of two derivatives and results in

$$\begin{aligned} &Q(x,x')\nabla_\mu\nabla_\nu\delta^{(d)}(x,x') \\ &\quad\sim \nabla_\mu\nabla_\nu\left([Q]\delta^{(d)}(x,x')\right) + [\nabla_\nu\nabla_\mu Q]\delta^{(d)}(x,x') \\ &\quad- \nabla_\nu\left([\nabla_\mu Q]\delta^{(d)}(x,x')\right) - \nabla_\mu\left([\nabla_\mu Q]\delta^{(d)}(x,x')\right)\,. \end{aligned} \tag{G.6}$$

In order to obtain further generalizations, one has to integrate by parts all derivatives one-by-one, and take the coincidence limits only of operators which are in direct contact with the Dirac delta. Generalizations of (G.5) are thus straightforward but rather lenghty.

Systematic applications of (G.3), to manipulate the inverse powers of the world function, and of (G.5), to take the local parts of the biscalars multiplying the divergences, can reduce the divergence part of the arbitrary expression (G.1) into a simple sum of dimensionally regulated poles.

We illustrate the use of the formulas derived in the appendix for the process of dimensional regularization showing the basics steps involved in explicitly isolating the diverging part of the first diagram in (5.12). We recall that the leading order renormalization comes from n propagators and that the upper critical dimension of the model ϕ^{2n} is $d_n = 2n/(n-1)$. The diagram is thus given by the nth power of the leading term of the covariant Green function (F.11). The integrand is proportional to

$$\frac{1}{\sigma(x,x')^{n\left(\frac{d}{2}-1\right)}} = \frac{1}{\sigma(x,x')^{n\frac{d_n}{2}-n-\frac{n}{2}\epsilon}} \,. \tag{G.7}$$

Our task is to cast the inverse power of the Synge function on the right hand side to match either $\frac{d}{2}$ or any integer displacement of the latter. Using again $d = d_n - \epsilon$ and the explicit form of the upper critical dimension we find

$$\begin{aligned} n\frac{d_n}{2} - n - \frac{n}{2}\epsilon &= \frac{n^2 - n^2 + n}{n-1} - \frac{n}{2}\epsilon \\ &= \frac{d_n}{2} - \frac{n}{2}\epsilon = \frac{d}{2} - \frac{n-1}{2}\epsilon \,. \end{aligned} \tag{G.8}$$

As anticipated we could identify the leading part of the exponent to be $\frac{d}{2}$, which allows us to use (G.2). This is not a coincidence as it is related to the superficial degree of divergence of the diagram under consideration; in practice we find a pole because $n\left(\frac{d}{2}-1\right) \sim \frac{d}{2}$ for $\epsilon \sim 0$. We are finally lead to

$$\begin{aligned} \frac{1}{\sigma(x,x')^{n\frac{d_n}{2}-n-\frac{n}{2}\epsilon}} &= \frac{1}{\sigma(x,x')^{\frac{d}{2}-\frac{n-1}{2}\epsilon}} \\ &\underset{\epsilon\to 0}{\sim} \frac{(2\pi)^{\frac{d}{2}}}{(n-1)\,\epsilon\,\Gamma\left(d/2\right)}\mu^{(1-n)\epsilon}\delta^{(d)}(x,x')\,, \end{aligned} \tag{G.9}$$

which was used to evaluate the right hand side of (5.12). Similar steps can be followed to evaluate the other two diagrams of (5.12) which exhibit the pole given by (G.4) because their leading power is $\frac{d}{2}+1$.

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Die geltende Promotionsordnung der Physikalisch-Astronomischen Fakultät ist mir bekannt.

Ich versichere ehrenwörtlich, dass ich nach bestem Wissen die reine Wahrheit gesagt und nichts verschwiegen habe.

Jena, 26/05/2019

Riccardo Martini

Curriculum Vitae

PERSONAL DETAILS

First Name	Riccardo
Last Name	Martini
Birth	December 17, 1990
Address	Magdelstieg 58, 07745, Jena, Germany
Phone	(+49) 152 05125564 (+39 349 3342235)
Mail	`riccardo90martini@gmail.com` `riccardo.martini@uni-jena.de`
Nationality	Italian

EDUCATION

PhD position in Physics 2016-ongoing
Friedrich-Schiller Universität, Jena

Laurea Magistrale in Physics, theoretical specialization 2012-2015
Alma Mater Studiorum, University of Bologna
110/110 cum Laude

Laurea in Physics 2009-2012
Alma Mater Studiorum, University of Bologna
109/110

Matriculation 2004-2009
Istituto Statale di Istruzione Superiore, Archimede
81/100

WORK EXPERIENCE

Softwer developer 2015-2016
CSE-Consorzio Servizi Bancari

TEACHING EXPERIENCE

Employed as Wissenschaft Mitarbeiter with teaching duties 2016-ongoing
Friedrich-Schiller Universität, Jena

- exercise/seminar: Particles and Fields, WiSe 2016/2017,
- exercise/seminar: Quantum Theory, SoSe 2017,
- exercise/seminar: Quantum Field Theory, SoSe 2018,
- exercise/seminar/lecturer substitute: Effective Action in Statistical and Quantum Field Theory, WiSe 2018/2019,
- exercise/seminar: Quantum Field Theory, SoSe 2019.

SKILLS

Languages	English
	Italian
	German
Software	Mathematica, LaTeX, Java EE

TALKS AND SCHOOLS

"Functional RG Methods in Group Field theories":

- "LOOPS 15", Erlangen, 2015,
- "Current Problems in Theoretical Physics", Vietri, 2016,
- GRK "Monitoring Workshop Graz-Jena-Wien", Wien, 2016.

"A Curvature Bound from Gravitational Catalysis":

- "DPG Spring meeting", Würzburg, 2018,
- "Asymptotic Safety online seminars", 2018,
- (Poster) "Quantum Spacetime and Renormalization Group", Bad Honnef, 2018
- (Poster) "Exact Renormalization Group 2018", Paris, 2018.

Conferences attended without contribution:

- "Exact Renormalization Group 2018", Trieste, 2016.

Schools:

- Winter school on "Theory of Fundamental Interactions", Galileo Galilei Institute, Florence, 2017,
- Saalburg Summer school on "Foundations and new Methods in Theoretical Physics", W.E. Heraeus, 2017.

SUPERVISOR

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