Some set intersection theorems of extremal type

Thesis

Some set intersection theorems of extremal type

by

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Abstract

For a family $\mathcal{F}$ of sets, let

$$\mathrm{ex}(\mathcal{F}) := \{\mathcal{A} \colon \mathcal{A} \text{ is an } \textit{extremal intersecting} \text{ sub-family of } \mathcal{F}\}.$$

The Erdős-Ko-Rado (EKR) Theorem states that $\{A \in \binom{[n]}{r} \colon 1 \in A\} \in \mathrm{ex}(\binom{[n]}{r})$ if $r \leq n/2$. The Hilton-Milner (HM) Theorem states that if $r \leq n/2$ and $\mathcal{A}$ is a *non-trivial* intersecting sub-family of $\binom{[n]}{r}$ then $|\mathcal{A}| \leq |\{A \in \binom{[n]}{r} \colon 1 \in A, A \cap [2, r+1] \neq \emptyset\} \cup \{[2, r+1]\}|$; hence $\{\{A \in \binom{[n]}{r} \colon j \in A\} \colon j \in [n]\} = \mathrm{ex}(\binom{[n]}{r})$ if $r < n/2$. Thus we say that a family $\mathcal{F}$ is *(strictly) EKR* if $\mathrm{ex}(\mathcal{F})$ contains (only) *trivial* intersecting families.

We obtain a partial solution to the following problem: for $r \leq n/2$, which sets $Z \subseteq [n]$ have the property that $|\{A \in \mathcal{A} \colon A \cap Z \neq \emptyset\}| \leq |\{A \in \binom{[n]}{r} \colon 1 \in A, A \cap Z \neq \emptyset\}|$ for all *compressed* intersecting sub-families of $\binom{[n]}{r}$? Using the idea of this problem, we generalise the HM Theorem to a setting of compressed *hereditary* families.

For a set $X := \{x_1, ..., x_{|X|}\}$, we define the family $\mathcal{S}_{X,k}$ of *signed sets* by

$$\mathcal{S}_{X,k} := \{\{(x_1, a_1), ..., (x_{|X|}, a_{|X|})\} \colon a_1, ..., a_{|X|} \in [k]\}$$

and the sub-family $\mathcal{S}^*_{X,k}$ by

$$\mathcal{S}^*_{X,k} := \{\{(x_1, a_1), ..., (x_{|X|}, a_{|X|})\} \colon \{a_1, ..., a_{|X|}\} \in \binom{[k]}{|X|}\}.$$

For a family $\mathcal{F}$, let

$$\mathcal{S}_{\mathcal{F},k} := \bigcup_{F \in \mathcal{F}} \mathcal{S}_{F,k}, \quad \mathcal{S}^*_{\mathcal{F},k} := \bigcup_{F \in \mathcal{F}} \mathcal{S}^*_{F,k}.$$

$\mathcal{S}^*_{\binom{[n]}{r},n}$ describes *r-partial permutations* of $[n]$.

We conjecture that for any $\mathcal{F}$ and $k \geq 2$, $\mathcal{S}_{\mathcal{F},k}$ is EKR, and strictly so if $k > 2$. We prove this conjecture for families $\mathcal{F}$ that are *compressed with respect to an element* $f^* \in \bigcup_{F \in \mathcal{F}} F$ (i.e. $f \in F \in \mathcal{F}, f^* \notin F \Rightarrow (F \backslash \{f\}) \cup \{f^*\} \in \mathcal{F}$). We then prove an analogue of the HM Theorem for $\mathcal{S}_{\binom{[n]}{r},k}$, and we show that the case $r = n$ of the result implies the truth of the conjecture for $k \geq k_0(\mathcal{F})$. We go on to prove much more: for any $r \geq t$ there exists $k_0(r,t)$ such that for any $k \geq k_0(r,t)$ and any $\mathcal{F}$ with $\max\{|F| \colon F \in \mathcal{F}\} \leq r$, the largest *t-intersecting* sub-families of $\mathcal{S}_{\mathcal{F},k}$ are trivial. We also provide an analogue of this result for $\mathcal{S}^*_{\mathcal{F},k}$.

The work on signed sets is followed by other EKR-type results for a setting that strongly generalises that given by $\mathcal{S}_{2^{[n]},k}$.

For a *monotonic non-decreasing sequence* $\{d_i\}_{i \in \mathbb{N}}$ of non-negative integers, let

$$\mathcal{P}(\{d_i\}_{i \in \mathbb{N}}) := \{\{a_1, ..., a_r\} \subset \mathbb{N} \colon r \in \mathbb{N}, a_{i+1} > a_i + d_{a_i} \text{ for } i = 1, ..., r-1\},$$
$$\mathcal{P}_n(\{d_i\}_{i \in \mathbb{N}}) := \mathcal{P}(\{d_i\}_{i \in \mathbb{N}}) \cap 2^{[n]}.$$

Let $\mathcal{P}_n := \mathcal{P}_n(\{d_i\}_{i \in \mathbb{N}})$ and $\mathcal{P}_n^{(r)} := \{A \in \mathcal{P}_n \colon |A| = r\}$. We determine $\mathrm{ex}(\mathcal{P}_n^{(r)})$ for $d_1 > 0$ and any r, and for $d_1 = 0$ and $r \leq \frac{1}{2}\max\{|A| \colon A \in \mathcal{P}_n\}$.

We finally provide a graph-theoretical re-formulation to a number of results in this thesis and in the EKR literature in general, and, using the work for $\mathcal{P}(\{d_i\}_{i \in \mathbb{N}})$, we show that an interesting EKR-type conjecture of Holroyd and Talbot indeed holds for a class of graphs studied by Holroyd, Spencer and Talbot, and much larger classes.

I hereby declare that the work in this thesis is the account of my own research.

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The various milestones in my path of education, first in Malta (my country of origin) and then in the UK, were reached with the help of many other people, some of whom I am compelled to mention.

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... search, and you will find ...

Luke 11:9

This thesis is dedicated to my parents,
as I would not have come this far without
their unfaltering love and sacrifices.

Contents

Chapter 1

Introduction

1.1 The basic terminology and an outline of the thesis

Before giving a gentle description of the work in this thesis, we shall first set up some basic notation and terminology that will be used throughout.

We shall use small letters such as x to denote elements of a set or integers, capital letters such as X to denote sets, and 'calligraphic' letters such as $\mathcal{X}$ to denote *families* (i.e. sets whose members are sets themselves). Unless otherwise stated, it is to be assumed that sets and families represented in this way are *finite*.

We refer to a set of size r as an *r-set*. A family whose members are all r-sets is said to be *r-uniform*, or simply *uniform* if the size r needs not be specified.

$\mathbb{N}$ is the set of positive integers $\{1, 2, ...\}$. For $m, n \in \mathbb{N}$, $m \leq n$, we denote the set $\{i \in \mathbb{N} \colon m \leq i \leq n\}$ by $[m, n]$, and if $m = 1$ then we also write $[n]$. $[0]$ is taken to be the *empty set* $\emptyset$.

The family of all subsets of a set X, called the *power set of* X, is denoted by 2^X. We denote the r-uniform sub-family $\{Y \subseteq X \colon |Y| = r\}$ of 2^X by $\binom{X}{r}$. For a family $\mathcal{F}$ and an integer r, we set $\mathcal{F}^{(r)} := \{A \in \mathcal{F} \colon |A| = r\}$.

A family $\mathcal{F}$ is said be *centred* if the sets in $\mathcal{F}$ have a common member c, i.e. $c \in A$ for all $A \in \mathcal{F}$; c is called a *centre of* $\mathcal{A}$, and the family of all sets in $\mathcal{F}$ that own c is called a *star of* $\mathcal{F}$. If $\mathcal{F}$ is not centred then $\mathcal{F}$ is said to be *non-centred.*

A family $\mathcal{A}$ is said to be *intersecting* if any two sets in $\mathcal{A}$ have a non-empty intersection. More generally, $\mathcal{A}$ is said to be *t-intersecting* if the size of the intersection of any two sets in $\mathcal{A}$ is not smaller than t. A t-intersecting family $\mathcal{A}$ is said to be *trivial* if the sets in $\mathcal{A}$ have a common t-subset; otherwise, $\mathcal{A}$ is said to be *non-trivial.* Note that a non-trivial 1-intersecting family is a non-centred intersecting family.

We are now able to state two fundamental results in extremal set theory, known as the Erdős-Ko-Rado (EKR) Theorem [25] and the Hilton-Milner (HM) Theorem [38], that inspired much of the work in this thesis and also many results in the literature. The subsequent sections of this introductory chapter provide a review of some important or well-known results in the literature that were primarily inspired by the EKR Theorem and that are directly relevant to this thesis.

It is trivial that if $n/2 < r \leq n$ then $\binom{[n]}{r}$ is a non-centred intersecting family. The EKR Theorem states that if $r \leq n/2$ and $\mathcal{A}$ is an intersecting sub-family of $\binom{[n]}{r}$, then the size of $\mathcal{A}$ is at most $\binom{n-1}{r-1}$; thus, if $r \leq n/2$ then a star of $\binom{[n]}{r}$ is a largest intersecting sub-family of $\binom{[n]}{r}$. The HM Theorem goes one step ahead of the EKR Theorem and states that if $r \leq n/2$ and $\mathcal{A}$ is a *non-centred* intersecting sub-family of $\binom{[n]}{r}$, then the size of $\mathcal{A}$ is at most $\binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$; so the union of the one-member family $\{[2, r+1]\}$ and the family of sets in the star of $\binom{[n]}{r}$ with centre 1 that intersect $[2, r+1]$ (i.e. $\{A \in \binom{[n]}{r} : 1 \in A,\ A \cap [2, r+1] \neq \emptyset\} \cup \{[2, r+1]\}$) is a largest non-centred intersecting sub-family of $\binom{[n]}{r}$. Consequently, the stars of $\binom{[n]}{r}$ constitute the set of largest intersecting sub-families of $\binom{[n]}{r}$ if $r < n/2$. In view of these facts, we say that a family $\mathcal{F}$ is *EKR* if the set of largest intersecting sub-families of $\mathcal{F}$ contains a star, and *strictly EKR* if the set of largest intersecting sub-families of $\mathcal{F}$ contains *only* stars.

Suppose we want to investigate the EKR and strict EKR properties of a certain family. Normally, as is often the case in this thesis, the important step, or rather the step involving the important ideas, is that of proving that the family is EKR, and proving that the family is strictly EKR would require a refinement of the main ideas employed for proving that it is EKR. However, this is not always the case in general. For example, when the families in concern are families of *permutations*, which we describe later, extending the EKR part to the strict EKR part normally turns out to be a significant jump with new ideas and an even harder step than proving the EKR part.

One of the most powerful techniques - and probably the most commonly used - in extremal set theory is that of *compression*, also known as *shifting*. This technique surfaced in the original proof [25] of the EKR Theorem. **Chapter 2** mainly gives a description of this technique and provides generalisations of certain established fundamental properties of compressions; these generalisations have crucial applications in various parts of the thesis. Chapter 2 also sets up some notation, mainly for certain sets and families that are defined on any given family, that is employed in the majority of main proofs in this thesis.

A family $\mathcal{A} \subseteq 2^{[n]}$ is said to be *compressed* if for any set A in $\mathcal{A}$, replacing any element in A by a smaller element in $[n] \backslash A$ (the complement of A relative to $[n]$) gives another set in $\mathcal{A}$, i.e. $\mathcal{A} \ni A \ni j > i \notin A$ implies $A \backslash \{j\} \cup \{i\} \in \mathcal{A}$. A family $\mathcal{F}$ is said to be *compressed with respect to an element* u^* (of the union of all sets in $\mathcal{F}$) if replacing by u^* an element of a set in $\mathcal{F}$ not owning u^* gives another set in $\mathcal{F}$, i.e. $u \in A \in \mathcal{F}$ and $u^* \notin A$ implies $(A \backslash \{u\}) \cup \{u^*\} \in \mathcal{F}$.

Let $\mathcal{S}_{n,r}$ be the star of $\binom{[n]}{r}$ with centre 1; so $\mathcal{S}_{n,r}$ is compressed. In **Chapter 3**, we deal with the problem of establishing which subsets Z of $[2, n]$ have the property that $|\{A \in \mathcal{A} \colon A \cap Z \neq \emptyset\}| \leq |\{A \in \mathcal{S}_{n,r} \colon A \cap Z \neq \emptyset\}|$ for all compressed intersecting subfamilies of $\binom{[n]}{r}$, where $r \leq n/2$. Note that if we instead have $1 \in Z$ then the answer is simply given by the EKR Theorem. We solve all the cases $|Z| \geq r$ and obtain a partial solution to the problem with $|Z| < r$. This work was motivated mainly by two observations. The first is that the solution for the extreme case where Z is an *initial segment* $[2, l]$ $(r + 1 \leq l \leq n)$ of $[2, n]$ leads to a proof of the HM Theorem, and the second is that the solution for the other extreme case where Z is a final segment $[m, n]$ $(m \geq 2)$ of $[2, n]$ leads to a short proof of an extension of the EKR Theorem due to Holroyd and Talbot; the former assertion is proved in a more general setting in Chapter 4, whereas the latter assertion is proved in Chapter 3 itself.

A set M in a family $\mathcal{F}$ is said to be *maximal in* $\mathcal{F}$ if M is not a subset of any other set in $\mathcal{F}$. The size of a smallest maximal set in $\mathcal{F}$ will be denoted by $\mu(\mathcal{F})$, and the size of a largest (maximal) set in $\mathcal{F}$ will be denoted by $\alpha(\mathcal{F})$.

A family is said to be a *hereditary family* (or an *ideal* or a *downset*) if any subset of any set in the family is also in the family.

The main result of **Chapter 4** is a generalisation of the HM Theorem to a setting of compressed hereditary families. It says that if $\mathcal{H}$ is a compressed hereditary family with $\mu(\mathcal{H}) \geq 2r$ and $\mathcal{A}$ is a non-centred intersecting sub-family of $\mathcal{H}^{(r)}$ (the family of r-sets in $\mathcal{H}$), then $\mathcal{A}$ is at most as large as the 'HM-type' family $\{A \in \mathcal{H}^{(r)}: 1 \in A,\ A \cap [2, r+1] \neq \emptyset\} \cup \{[2, r+1]\}$. Note that the HM Theorem is the case $\mathcal{H} = 2^{[n]}$. A question that arises immediately is whether we can do without the condition that $\mathcal{H}$ is compressed. The answer is 'no'. As we show in the same chapter, we cannot even relax the condition of having $\mathcal{H}$ compressed to having $\mathcal{H}$ compressed with respect to an element.

Families $\mathcal{A}_1, ..., \mathcal{A}_k$ of sets are said to be *cross-intersecting* if the intersection of any set in any of the k families with that of any other set in any other family is non-empty, i.e. $A_i \cap A_j \neq \emptyset$ for any $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$, $i \neq j$.

Sometimes a result for a pair of cross-intersecting families is needed as a stepping stone to a result for intersecting families. For example, in order to obtain the HM Theorem, Hilton and Milner [38] proved that if $r \leq n/2$ and $\mathcal{A}, \mathcal{B}$ are non-empty cross-intersecting sub-families of $\binom{[n]}{r}$, then $|\mathcal{A}|+|\mathcal{B}| \leq 1+\binom{n}{r}-\binom{n-r}{r} = |\mathcal{A}_0|+|\mathcal{B}_0|$ where $\mathcal{A}_0$ is the one-member family $\{[r]\}$ and $\mathcal{B}_0$ is the family of all sets in $\binom{[n]}{r}$ that intersect $[r]$. Frankl and Tokushige [32] extended this result by showing that if $r \leq s \leq n-r$ and $\mathcal{B}$ is taken to be a sub-family of $\binom{[n]}{s}$, then $|\mathcal{A}|+|\mathcal{B}| \leq 1+\binom{n}{s}-\binom{n-r}{s} = |\mathcal{A}_0|+|\mathcal{B}'_0|$ where $\mathcal{B}'_0$ is the family of all sets in $\binom{[n]}{s}$ that intersect $[r]$. In Chapter 4, we also generalise this result by showing that if $\mathcal{A}$ and $\mathcal{B}$ are taken to be a sub-family of $\mathcal{H}^{(r)}$ and a sub-family of $\mathcal{H}^{(s)}$ respectively, where $\mathcal{H}$ is a compressed hereditary sub-family of $2^{[n]}$ with $\mu(\mathcal{H}) \geq r+s$, then $|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{A}_0| + |\mathcal{B}''_0|$ where $\mathcal{B}''_0$ is the family of all sets in $\mathcal{H}^{(s)}$ that intersect $[r]$. The case $r = s$ in this generalisation is used a stepping stone to the main result of Chapter 4 that we mentioned above.

Chapter 4 is followed by three chapters dedicated to intersecting families of *signed sets*. The 'signed sets' terminology was introduced in [6], where a signed set on $[n]$ is defined as a pair (A, f) with A being a subset of $[n]$ and f being a function mapping A

to $\{1,-1\}$; informally, each element of A is given a sign, $+$ or $-$. Also in [6], a *k-signed r-set* on $[n]$ is then defined to be a pair (A, f) where A is an r-subset of $[n]$ and f maps A to $[k]$, $k \geq 2$; thus, instead of having just two signs, we have k points to choose from for labeling any element in A. Here we represent a k-signed r-set differently, and the formulation that we are about to present is intended for a very general purpose, as can be seen from the definition of a family $\mathcal{S}_{\mathcal{F},k}$ below.

Let X be an r-set $\{x_1, ..., x_r\}$, and let $y_1, ..., y_r \in \mathbb{N}$. We call the set $\{(x_1, y_1), ..., (x_r, y_r)\}$ a *k-signed r-set* if $|\{y_1, ..., y_r\}| \leq k$. For $k \geq 2$, we define $\mathcal{S}_{X,k}$ to be the family of k-signed r-sets given by

$$\begin{aligned}\mathcal{S}_{X,k} &:= \{\{(x_1, a_1), ..., (x_r, a_r)\} \colon a_1, ..., a_r \in [k]\} \\ &= \{A \in \binom{X \times [k]}{r} \colon |A \cap (\{x\} \times [k])| = 1 \text{ for all } x \in X\}\end{aligned}$$

(recall that the *Cartesian product* $A \times B$ of sets A and B is the set $\{(a, b) \colon a \in A, b \in B\}$). Thus a set in $\mathcal{S}_{X,k}$ is obtained by giving each point in X a label from $[k]$. We shall set $\mathcal{S}_{\emptyset,k} := \emptyset$.

For a family $\mathcal{F}$ of sets, we define

$$\mathcal{S}_{\mathcal{F},k} := \bigcup_{F \in \mathcal{F}} \mathcal{S}_{F,k}.$$

In **Chapter 5**, it is conjectured that for any $\mathcal{F}$ and $k \geq 2$, $\mathcal{S}_{\mathcal{F},k}$ is EKR, and strictly so unless $k = 2$ and $\mathcal{F}$ has a particular structure. The main result is that if $\mathcal{F}$ is compressed with respect to an element then the conjecture is true. This generalises a well-known result supporting the conjecture for $\mathcal{F} = \binom{[n]}{r}$ that was first stated by Meyer [52] and then proved in different ways by Deza and Frankl [22], Bollobás and Leader [6], Engel [23] and Erdős et al. [24]. By strengthening a result of Holroyd and Talbot, we also verify the conjecture for families $\mathcal{F}$ that are uniform and EKR.

The main result of **Chapter 6** characterises the extremal non-centred intersecting sub-families of $\mathcal{S}_{\binom{[n]}{r},k}$, hence providing an analogue of the HM Theorem for signed sets. In order to achieve this, we prove a cross-intersection result for sub-families of $\mathcal{S}_{\binom{[n]}{r},k}$. At the end of this chapter, we first prove directly that there exists an integer $k_0(\mathcal{F})$

such that the conjecture in the preceding chapter is true if $k \geq k_0(\mathcal{F})$, and then we show that by applying the main result (of this chapter) with $r = n$ we obtain a much better value of $k_0(\mathcal{F})$.

For an r-set $X := \{x_1, ..., x_r\}$, we define $\mathcal{S}^*_{X,k}$ to be the special sub-family of $\mathcal{S}_{X,k}$ given by

$$\begin{aligned}\mathcal{S}^*_{X,k} &:= \{\{(x_1, a_1), ..., (x_r, a_r)\} \colon a_1, ..., a_r \in [k],\ |\{a_1, ..., a_r\}| = r\} \\ &= \{\{(x_1, a_1), ..., (x_r, a_r)\} \colon \{a_1, ..., a_r\} \in \binom{[k]}{r}\}.\end{aligned}$$

Thus a set in $\mathcal{S}^*_{X,k}$ is obtained by giving points in X *distinct* labels from $[k]$. So $\mathcal{S}^*_{X,k} \neq \emptyset$ iff $r \leq k$.

An *r-partial permutation of a set* N is a pair (A, f) where $A \in \binom{N}{r}$ and $f \colon A \rightarrow N$ is an injection. An $|N|$-partial permutation of N is simply called a *permutation of* N. Clearly, the family of permutations of $[n]$ can be re-formulated as $\mathcal{S}^*_{[n],n}$, and the family of r-partial permutations of $[n]$ can be re-formulated as $\mathcal{S}^*_{\binom{[n]}{r},n}$.

Let X be as above. $\mathcal{S}^*_{X,k}$ can be interpreted as the family of permutations of sets in $\binom{[k]}{r}$: consider the bijection $\beta \colon \mathcal{S}^*_{X,k} \rightarrow \{(A, f) \colon A \in \binom{[k]}{r}, f \colon A \rightarrow A \text{ is a bijection}\}$ defined by $\beta(\{(x_1, a_1), ..., (x_r, a_r)\}) := (\{a_1, ..., a_r\}, f)$ where, for $b_1 < ... < b_r$ such that $\{b_1, ..., b_r\} = \{a_1, ..., a_r\}$, $f(b_i) := a_i$ for $i = 1, ..., r$. $\mathcal{S}^*_{X,k}$ can also be interpreted as the sub-family $\mathcal{X} := \{(A, f) \colon A \in \binom{[k]}{r}, f \colon A \rightarrow [r] \text{ is an injection}\}$ of the family of r-partial permutations of $[k]$: consider an obvious bijection from $\mathcal{S}^*_{X,k}$ to $\mathcal{S}^*_{\binom{[k]}{r},r}$ and another one from $\mathcal{S}^*_{\binom{[k]}{r},r}$ to $\mathcal{X}$.

For a family $\mathcal{F}$, we define $\mathcal{S}^*_{\mathcal{F},k}$ to be the special sub-family of $\mathcal{S}_{\mathcal{F},k}$ given by

$$\mathcal{S}^*_{\mathcal{F},k} := \bigcup_{F \in \mathcal{F}} \mathcal{S}^*_{F,k}.$$

Chapter 7 features two t-intersection theorems of a very general nature; one for signed sets and another one for partial permutations. The first one is that for any $r \geq t$ there exists $k_0(r, t)$ such that for any $k \geq k_0(r, t)$ and any family $\mathcal{F}$ such that the maximum size of a set in $\mathcal{F}$ is not smaller than t and not larger than r, the largest t-intersecting sub-families of $\mathcal{S}_{\mathcal{F},k}$ are trivial. The second one is an analogous version for t-intersecting sub-families of $\mathcal{S}^*_{\mathcal{F},k}$.

Before describing the content of Chapter 8, we explain the meaning of the term *isomorphic*. Let $\mathcal{I}_1, \mathcal{I}_2$ be two families. For $j = 1, 2$, let U_j be the union of all sets in $\mathcal{I}_j$. Then $\mathcal{I}_2$ is said to be *isomorphic to* $\mathcal{I}_1$ or a *copy of* $\mathcal{I}_1$ if there exists a bijection $\beta \colon U_2 \to U_1$ such that for any subset I_2 of U_2, I_2 is a member of $\mathcal{I}_2$ iff the set $\{\beta(i) \colon i \in I_2\}$ is a member of $\mathcal{I}_1$; we write $\mathcal{I}_2 \cong \mathcal{I}_1$. Note that $\mathcal{I}_2 \cong \mathcal{I}_1$ iff $\mathcal{I}_1 \cong \mathcal{I}_2$. Loosely speaking, $\mathcal{I}_2 \cong \mathcal{I}_1$ if $\mathcal{I}_2$ is simply the result of fixing $\mathcal{I}_1$ and "labeling" U_1 differently.

In **Chapter 8**, we generalise the notion of a family $\mathcal{S}_{2^{[n]},k}$ of signed sets. We define a *double partition* P of a set V to be a partition of V into *large sets* V_i $(0 \leq i \leq n)$ that are in turn partitioned into k_i *small sets* $V_{i1}, ..., V_{ik_i}$. Given such a partition, the family $\mathcal{V}(\mathsf{P})$ *induced by* P is the family of subsets of V whose intersection with each large set is a subset of just one small set or empty. $\mathcal{S}_{2^{[n]},k}$ is isomorphic to $\mathcal{V}(\mathsf{P})$ with P given by the double partition of $[kn]$ with large sets $[(i-1)k+1, ik]$, $i = 1, ..., n$, and small sets $\{j\}$, $j = 1, ..., kn$. Our main result is that if $2r$ is no larger than $\mu(\mathcal{V}(\mathsf{P})) = \sum_{i=0}^{n} \min\{|V_{i,j}| \colon j \in [k_i]\}$ and at least one of the large sets is partitioned into just one small set, then $\mathcal{V}(\mathsf{P})^{(r)}$ is EKR, and strictly so if $2r < \mu(\mathcal{V}(\mathsf{P}))$. As explained in Chapter 11, this result can be interpreted as saying that if $\mathcal{I}_G$ denotes the family of *independent sets* of a *graph* G given by a disjoint union of *complete multipartite graphs* and *singletons*, then $\mathcal{I}_G^{(r)}$ (the family of r-sets in $\mathcal{I}_G$) is EKR if $2r \leq \mu(\mathcal{I}_G)$, and strictly EKR if $2r < \mu(\mathcal{I}_G)$. This extension of the EKR Theorem will be used as a foundation for a much more general result in Chapter 11.

Chapter 9 concerns a discovery of a significant and important extension of the EKR Theorem. For a sub-family $\mathcal{A}$ of $\binom{[n]}{r}$, let $\mathcal{A}^*$ be the family of sets in $\mathcal{A}$ that intersect every set in $\mathcal{A}$, and let $\mathcal{A}'$ be the family of sets in $\mathcal{A}$ that are not in $\mathcal{A}^*$ (so a set in $\mathcal{A}$ is in $\mathcal{A}'$ iff it is disjoint from some set in $\mathcal{A}$). We prove that if $r \leq n/2$ then

$$|\mathcal{A}^*| + \frac{r}{n}|\mathcal{A}'| \leq \binom{n-1}{r-1}.$$

We also prove that if $r < n/2$ then the bound is attained iff either $|\mathcal{A}^*| = \binom{n-1}{r-1}$ and $\mathcal{A}' = \emptyset$ or $\mathcal{A}^* = \emptyset$ and $\mathcal{A}' = \binom{[n]}{r}$. Note that the EKR Theorem is the special case

$\mathcal{A} = \mathcal{A}^*$. Using the above result, we provide a very short proof of a beautiful theorem of Hilton [37] that gives a sharp upper bound for the sum of the sizes of an arbitrary number of cross-intersecting sub-families of $\binom{[n]}{r}$, $r \leq n/2$. A slight extension of this cross-intersection result, which we also prove in Chapter 9, will have an application in the subsequent chapter.

For a *monotonic non-decreasing sequence* $\{d_i\}_{i \in \mathbb{N}}$ (i.e. $d_1 \leq d_2 \leq ...$) of non-negative integers, let $\mathcal{P}_n := \mathcal{P}_n(\{d_i\}_{i \in \mathbb{N}})$ be the family of all subsets $\{a_1, ..., a_m\}$, $a_1 < ... < a_m$, of $[n]$ such that for all $i \in [m-1]$, the difference between a_{i+1} and a_i is greater than d_i. For example, suppose $n = 8$ and $d_1 = 1$, $d_2 = 1$, $d_3 = 3$, $d_4 = 5$. If a set A in $\mathcal{P}_8$ has an element $a \in [4, 8]$ then, since $d_i \geq d_4$ for all $i \geq 4$, a is the unique element of A that is in $[4, 8]$; thus, if $2 \in B \in \mathcal{P}_8$ and $2 < b \in B$ then, since $1 \notin B$ (as $1 + d_1 > 2 \in B$) and $b > 2 + d_1 = 3$ (i.e. $b \in [4, 8]$), we have $B = \{2, b\}$. If $3 \in C \in \mathcal{P}_8$ and $3 < c \in C$ then $c > 3 + d_3 = 6$ and hence c is 7 or 8. So $\mathcal{P}_8^{(3)}$ consists of the sets $\{1, 3, 7\}$ and $\{1, 3, 8\}$.

In **Chapter 10**, we obtain another generalisation of the EKR Theorem by characterising the extremal intersecting sub-families of $\mathcal{P}_n^{(r)}$ for $d_1 > 0$ and any r, and for $d_1 = 0$ and r no larger than half the size of a largest set in $\mathcal{P}_n$ (i.e. $r \leq \alpha(\mathcal{P}_n)/2$). The definition of the family $\mathcal{P}_n$ and the study of its uniform intersecting sub-families are crucial for the proof of the main result in the subsequent chapter.

Finally, in **Chapter 11**, we start by providing a graph-theoretical re-formulation to a number of results in preceding chapters (namely Chapters 5, 8 and 10) and also in the literature in general, and then we prove that an interesting EKR-type conjecture of Holroyd and Talbot indeed holds for a class of graphs studied by Holroyd, Spencer and Talbot, and much larger classes. Most of the arguments in this chapter are of a graph-theoretical nature.

The work in Chapter 5 has been published in [9]. Chapters 3, 4, 7, 8, 9 and 10 have been submitted for publication, and they correspond to [12], [13], [14], [15], [8] and [9] respectively.

1.2 Intersecting families: the Erdős-Ko-Rado Theorem and beyond

Perhaps the simplest result in extremal set combinatorics is that $2^{[n]}$ is EKR, i.e. if $\mathcal{A}$ is an *extremal* (i.e. largest) intersecting sub-family of $2^{[n]}$ then the size of $\mathcal{A}$ is 2^{n-1}, the size of a star of $2^{[n]}$. The lower bound on $|\mathcal{A}|$ follows from the fact that $\mathcal{A}$ must be at least as large as a star of $2^{[n]}$, and the upper bound follows from the fact that, since $\mathcal{A}$ is intersecting, the complement (relative to $[n]$) of any set in $\mathcal{A}$ is not in $\mathcal{A}$. For $n \geq 3$, the set of extremal intersecting sub-families of $2^{[n]}$ does not consist solely of stars of $2^{[n]}$; for example, the non-centred intersecting family $\{A \in 2^{[n]} \colon |A \cap [3]| \geq 2\}$ has size 2^{n-1} and is therefore extremal.

Let us next consider the uniform sub-families $\binom{[n]}{r}$ of $2^{[n]}$. As we mentioned in Section 1.1, it is trivial that if $n/2 < r \leq n$ and $\mathcal{A}$ is an extremal intersecting sub-family of $\binom{[n]}{r}$, then $\mathcal{A}$ is $\binom{[n]}{r}$ itself. It is also straightforward that if $r = n/2$ then $\mathcal{A}$ is an extremal intersecting sub-family of $\binom{[n]}{r}$) iff for any set A in $\binom{[n]}{r}$, exactly one of A and its complement is in $\mathcal{A}$. However, for $r < n/2$, the problem of determining the set of extremal intersecting sub-families of $\binom{[n]}{r}$, or even just the size of a family in this set, proved to be far from trivial, and this brings us to the classical EKR Theorem that we mentioned in Section 1.1 and that we now state formally.

Theorem 1.2.1 (Erdős-Ko-Rado Theorem [25]) *Let $r \leq n/2$. Let $\mathcal{A}$ be an intersecting sub-family of $\binom{[n]}{r}$, and let $\mathcal{C}$ be a maximal centred sub-family of $\binom{[n]}{r}$ (i.e. a star of $\binom{[n]}{r}$). Then*

$$|\mathcal{A}| \leq |\mathcal{C}| = \binom{n-1}{r-1}.$$

There are various proofs of this theorem, two of which are particularly short and beautiful: Katona's [42] using the *cycle method* and Daykin's [19] using another fundamental result known as the Kruskal-Katona Theorem [43, 46].

Getting back to our original discussion, we see that Theorem 1.2.1 does not give a complete characterisation of the set of extremal intersecting sub-families of $\binom{[n]}{r}$ for $r < n/2$. Erdős, Ko and Rado [25] conjectured that if $r \leq n/2$ and $\mathcal{A}$ is a non-centred intersecting sub-family of $\binom{[n]}{r}$ then $|\mathcal{A}| \leq |\{A \in \binom{[n]}{r} : |A \cap [3]| \geq 2\}|$, which would imply that $\binom{[n]}{r}$ is strictly EKR for $r < n/2$. Hilton and Milner disproved the conjecture and solved the whole problem with the following fundamental theorem.

Theorem 1.2.2 (Hilton-Milner Theorem [38]) *Let $r \leq n/2$. Let $\mathcal{A}$ be a non-centred intersecting sub-family of $\binom{[n]}{r}$, and let $\mathcal{N}$ be the non-centred sub-family $\{A \in \binom{[n]}{r} : 1 \in A,\ A \cap [2, r+1] \neq \emptyset\} \cup \{[2, r+1]\}$ of $\binom{[n]}{r}$. Then*

$$|\mathcal{A}| \leq |\mathcal{N}| = \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1.$$

Actually, this theorem is part of a much more general result in [38], the proof of which is long and complicated. Consequently, shorter and simpler proofs were obtained by other authors; see, for example, [30, 32].

By Theorem 1.2.2, if $r < n/2$ and $\mathcal{A}$ is a non-centred sub-family of $\binom{[n]}{r}$ or a *proper* sub-family of a star of $\binom{[n]}{r}$, then $\mathcal{A}$ is smaller than the stars of $\binom{[n]}{r}$. This confirms that $\binom{[n]}{r}$ is strictly EKR for $r < n/2$.

Also in [25], Erdős, Ko and Rado initiated the study of extremal t-intersecting families for $t \geq 2$. They posed the following question: What is the size of an extremal t-intersecting sub-family of $2^{[n]}$? The answer in a complete form was given by Katona.

Theorem 1.2.3 (Katona [44]) *Let $t \geq 2$, and let $\mathcal{A}$ be an extremal t-intersecting sub-family of $2^{[n]}$.*

(i) If $n + t = 2l$ then $\mathcal{A} = \{A \subseteq [n] : |A| \geq l\}$.

(ii) If $n + t = 2l + 1$ then $\mathcal{A}$ is isomorphic to $\{A \subseteq [n] : |A \cap ([n-1])| \geq l\}$.

For the uniform case, Erdős, Ko and Rado [25] proved the following.

Theorem 1.2.4 (Erdős, Ko and Rado [25]) *For $t \leq r$ there exists $n_0(r, t) \in \mathbb{N}$ such that for all $n \geq n_0(r, t)$, the extremal t-intersecting sub-families of $\binom{[n]}{r}$ are trivial.*

For $t \geq 15$, Frankl [28] showed that the smallest $n_0(r, t)$ for which their result holds is $(r - t + 1)(t + 1) + 1$, and that if $n = (r - t + 1)(t + 1)$ then the maximal trivial t-intersecting sub-families of $\binom{[n]}{r}$ are also extremal but not uniquely so. Subsequently, Wilson [59] proved the sharp upper bound $\binom{n-t}{r-t}$ for the size of any t-intersecting sub-family of $\binom{[n]}{r}$ for $n \geq (r-t+1)(t+1)$ and any t. Frankl [28] conjectured that if $\mathcal{A}$ is an extremal t-intersecting sub-family of $\binom{[n]}{r}$ then $|\mathcal{A}| = \max\{|\{A \in \binom{[n]}{r} \colon |A \cap [t+2i]| \geq t+i\}| \colon i \in \{0\} \cup [r-t]\}$. A proof of this long-standing conjecture together with a complete characterisation of the extremal structures was finally obtained by Ahlswede and Khachatrian, and this may be regarded as one of the major and most remarkable breakthroughs in combinatorics.

Theorem 1.2.5 (Ahlswede and Khachatrian [1]) *Let $1 \leq t \leq r \leq n$, and let $\mathcal{A}$ be an extremal t-intersecting sub-family of $\binom{[n]}{r}$.*
(i) If $(r-t+1)(2+\frac{t-1}{m+1}) < n < (r-t+1)(2+\frac{t-1}{m})$ for some $m \in \{0\} \cup \mathbb{N}$ - where, by convention, $(t-1)/m = \infty$ if $m = 0$ - then $\mathcal{A}$ is isomorphic to $\{A \in \binom{[n]}{r} \colon |A \cap [t+2m]| \geq t+m\}$.
(ii) If $t \geq 2$ and $(r-t+1)(2+\frac{t-1}{m+1}) = n$ for some $m \in \{0\} \cup \mathbb{N}$ then $\mathcal{A}$ is isomorphic to $\{A \in \binom{[n]}{r} \colon |A \cap [t+2m]| \geq t+m\}$ or $\{A \in \binom{[n]}{r} \colon |A \cap [t+2m+2]| \geq t+m+1\}$.

We conclude this section by mentioning that a vast amount of research stemmed out of the seminal Erdős-Ko-Rado paper [25], and this field is now rich in beautiful results and still very active; we have only outlined the central results. The survey papers [22] and [29] are recommended.

In the rest of this chapter, we discuss some of the EKR-type problems that have attracted most attention and that will be treated in this thesis.

1.3 Intersecting sub-families of hereditary families

One of the central problems in extremal combinatorics is the following well-known old conjecture.

Conjecture 1.3.1 (Chvátal [17]) *If $\mathcal{H}$ is a hereditary family then $\mathcal{H}$ is EKR.*

Note that this is true if $\mathcal{H} = 2^{[n]}$ (see the beginning of Section 1.2). Chvátal [18] made the first significant step towards his conjecture.

Theorem 1.3.2 (Chvátal [18]) *Conjecture 1.3.1 is true if $\mathcal{H}$ is compressed.*

Snevily [56] took the above result a big step forward.

Theorem 1.3.3 (Snevily [56]) *Conjecture 1.3.1 is true if $\mathcal{H}$ is compressed with respect to an element.*

Many other results have been inspired by Conjecture 1.3.1; for example, the Ph.D. dissertation [53] is dedicated to it. The above two results are perhaps the most well-known in this area, and the only two that we need to refer to later on.

Before turning our attention to uniform sub-families of hereditary families, we recall the following. A *graph* G is a pair (V, E) with $E \subseteq \binom{V}{2}$, and a set $I \subseteq V$ is said to be an *independent set of* G if $\{i, j\} \notin E$ for any $i, j \in I$.

Let $\mathcal{I}_G$ denote the family of all independent sets of a graph G. Holroyd and Talbot [41] made the following interesting but also very difficult conjecture.

Conjecture 1.3.4 (Holroyd and Talbot [41]) *If G is a graph with $\mu(\mathcal{I}_G) \geq 2r$, then $\mathcal{I}_G^{(r)}$ is EKR, and strictly so if $\mu(\mathcal{I}_G) > 2r$.*

Clearly, the family $\mathcal{I}_G$ is a hereditary family. The author suggested the following generalisation of Conjecture 1.3.4.

Conjecture 1.3.5 (Borg [9]) *If $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) \geq 2r$, then $\mathcal{H}^{(r)}$ is EKR, and strictly so if $\mu(\mathcal{H}) > 2r$.*

Note that Theorems 1.2.1 and 1.2.2 solve the special case $\mathcal{H} = 2^{[n]}$.

Theorem 1.2.3 tells us that if $t \geq 2$ then for all $n > t$, the extremal t-intersecting sub-families of the hereditary family $2^{[n]}$ are non-trivial. Thus Conjecture 1.3.1 does not have an obvious extension for t-intersecting sub-families of hereditary families. It is therefore natural to question whether a t-intersection version of Conjecture 1.3.5 may hold, or more precisely, whether there exists an integer $n_0(r, t)$ such that for any hereditary family $\mathcal{H}$ with $\mu(\mathcal{H}) \geq n_0(r, t)$, the extremal t-intersecting sub-families of $\mathcal{H}^{(r)}$ are trivial. Only very recently, the author [10] proved that such an integer $n_0(r, t)$ exists indeed; the proof is based on the fundamental fact established in this thesis as Lemma 4.3.1. So Conjecture 1.3.5 is true if $\mu(\mathcal{H}) \geq n_0(r, 1)$.

1.4 Intersecting families of signed sets

For a signed set A and integers q and k, let $\theta_k^q(A)$ be the *translation operation* defined by

$$\theta_k^q(A) := \{(a, b+q \text{ modulo } k) \colon (a,b) \in A\}.$$

For $q = 1$, we also write $\theta_k(A)$.

Trivially, if $k = 2$ then $\theta_2(A)$ is the unique set in $\mathcal{S}_{X,2}$ that does not intersect A. Thus, for $\mathcal{A} \subset \mathcal{S}_{X,2}$,

$$\begin{gathered}\mathcal{A} \text{ is an extremal intersecting sub-family of } \mathcal{S}_{X,2} \text{ iff} \\ \text{for all } A \in \mathcal{S}_{X,2}, \text{ exactly one of } A \text{ and } \theta_2(A) \text{ is in } \mathcal{A}.\end{gathered} \tag{1.1}$$

Note that stars of $\mathcal{S}_{X,2}$ are extremal intersecting sub-families of $\mathcal{S}_{X,2}$, and not uniquely so unless $|X| \leq 2$. In other words, $\mathcal{S}_{X,2}$ is EKR, and strictly so iff $|X| \leq 2$.

Berge [3] showed that $\mathcal{S}_{[n],k}$ is EKR; the proof of this result is simply that if $\mathcal{A} \subset \mathcal{S}_{[n],k}$ is intersecting and $A \in \mathcal{A}$ then $\theta_k^q(A) \notin \mathcal{A}$ for $q = 1, ..., k-1$, and hence $|\mathcal{A}| \leq |\mathcal{S}_{[n],k}|/k = |\{A \in \mathcal{S}_{[n],k} \colon (1,1) \in A\}|$. Livingston [51] made a significant step forward by establishing the strict EKR property of $\mathcal{S}_{[n],k}$ for $k > 2$.

Theorem 1.4.1 (Berge [3], Livingston [51]) *(i) $\mathcal{S}_{[n],k}$ is EKR, and (ii) strictly so unless $n \geq 3$ and $k = 2$.*

Other proofs of this result were given by Gronau [34] and Moon [54].

Holroyd and Talbot [41] recently showed that if $\mathcal{F}$ is an EKR family of independent r-sets of a graph then $\mathcal{S}_{\mathcal{F},k}$ is EKR; however, their proof carries forward to the following generalisation of Theorem 1.4.1(i).

Theorem 1.4.2 (Holroyd and Talbot [41]) *If $\mathcal{F}$ is r-uniform and EKR then $\mathcal{S}_{\mathcal{F},k}$ is EKR.*

This result follows by a slight extension of the proof given above for Berge's result; see Proof of Theorem 1.4.2 in Section 5.5.

The next generalisation of Theorem 1.4.1 is a well-known result that was first stated by Meyer [52] and proved in different ways by Deza and Frankl [22] and Bollobás and Leader [6].

Theorem 1.4.3 (Meyer [52], Deza, Frankl [22], Bollobás, Leader [6]) *For $r \leq n$ and $k \geq 2$,*
(i) $\mathcal{S}_{\binom{[n]}{r},k}$ is EKR, and
(ii) strictly so unless $r = n \geq 3$ and $k = 2$.

The proof of Deza and Frankl is based on the well-known compression technique (see Section 2.2), whereas the proof of Bollobás and Leader is based on the idea of the cycle method used by Katona [42] in his alternative proof of the EKR Theorem. Engel [23] and Erdős et al. [24] gave other proofs that are also based on variants of the cycle method.

Note that Theorem 1.4.3(i) with $r \leq n/2$ follows from Theorem 1.4.2 and the EKR Theorem. Also note that the case $r > n/2$ in Theorem 1.4.3 provides an example of a family $\mathcal{F}$ such that $\mathcal{S}_{\mathcal{F},k}$ is EKR but $\mathcal{F}$ is not.

Frankl and Füredi [31] conjectured that if $\mathcal{A}$ is an extremal t-intersecting sub-family of $\mathcal{S}_{[n],k}$ then $|\mathcal{A}| = \max\{|\{A \in \mathcal{S}_{[n],k} \colon |A \cap ([t+2m] \times [1])| \geq t+m\}| \colon m \in \{0\} \cup \mathbb{N}\}$. If $k \geq t+1$ then the conjecture claims that $|\mathcal{A}| = k^{n-t}$, the size of a maximal trivial t-intersecting sub-family of $\mathcal{S}_{[n],k}$. They showed that this is true if $t \geq 15$. A result of Kleitman [45], which was shown to be equivalent to Theorem 1.2.3 via the compression technique (described in Section 2.2), had long established the truth of the conjecture for the special case $k = 2$. After Theorem 1.2.5 was established, Ahlswede and Khachatrian [2] and Frankl and Tokushige [33] were able to solve this conjecture independently and by different methods; Ahlswede and Khachatrian also determined the set of extremal structures.

Theorem 1.4.4 (Ahlswede, Khachatrian [2]; Frankl, Tokushige [33]) *Let $t \leq n$ and $k \geq 2$. Let m be the largest integer such that $t + 2m < \min\{n+1, t + 2\frac{t-1}{k-2}\}$ where, by convention, $\frac{t-1}{k-2} = \infty$ if $k = 2$. Let $\mathcal{A}$ be an extremal t-intersecting sub-family of $\mathcal{S}_{[n],k}$.*

(i) If $(k,t) \neq (2,1)$ *and* $\frac{t-1}{k-2}$ *is not integral then* $\mathcal{A}$ *is isomorphic to* $\{A \in \mathcal{S}_{[n],k} \colon |A \cap ([t+2m] \times [1])| \geq t+m\}$.

(ii) If $(k,t) \neq (2,1)$ *and* $\frac{t-1}{k-2}$ *is integral then* $\mathcal{A}$ *is isomorphic to* $\{A \in \mathcal{S}_{[n],k} \colon |A \cap ([t+2m] \times [1])| \geq t+m\}$ *or* $\{A \in \mathcal{S}_{[n],k} \colon |A \cap ([t+2m+2] \times [1])| \geq t+m+1\}$.

(iii) If $(k,t) = (2,1)$ *then the result is given by (1.1).*

To the best of the author's knowledge, no analogous results for t-intersecting sub-families of $\mathcal{S}_{\mathcal{F},k}$ with $|\mathcal{F}| \geq 2$ have been established (excluding the one we present in Chapter 7). However, some very important results have been obtained for a modification of the problem, which we describe next.

1.5 Intersecting families of permutations and partial permutations

In [20, 21], the study of intersecting permutations was initiated. Deza and Frankl [21] showed that $\mathcal{S}^*_{[n],n}$ is EKR (so $(n-1)!$ is a sharp upper bound for the size of an intersecting sub-family of $\mathcal{S}^*_{[n],n}$); the proof follows by the same translation argument given in the preceding section for Berge's result. However, Deza and Frankl did not proceed further to determine the extremal structures; this was accomplished only a few years ago by Cameron and Ku [16].

Theorem 1.5.1 (Cameron and Ku [16]) $\mathcal{S}^*_{[n],n}$ *is strictly EKR.*

This result was also deduced from a more general result on certain vertex transitive graphs in [49].

Ku and Leader [48] established the EKR property of $\mathcal{S}^*_{\binom{[n]}{r},n}$ for all $r \in [n]$, and they proved that $\mathcal{S}^*_{\binom{[n]}{r},n}$ is strictly EKR for all $r \in [8, n-3]$. Naturally, they conjectured that $\mathcal{S}^*_{\binom{[n]}{r},n}$ is also strictly EKR for the few remaining values of r. This was settled by Li and Wang using tools forged by Ku and Leader.

Theorem 1.5.2 (Ku, Leader [48]; Li, Wang [50]) $\mathcal{S}^*_{\binom{[n]}{r},n}$ *is strictly EKR for all* $r \in [n]$.

When it comes to t-intersecting families of permutations, things are of course much harder, and the most interesting challenge comes from the following conjecture.

Conjecture 1.5.3 (Deza and Frankl [21]) *For any t there exists $n_0(t)$ such that for any $n \geq n_0(t)$, the size of a t-intersecting sub-family of $S^*_{[n],n}$ is at most that of a maximal trivial t-intersecting sub-family of $S^*_{[n],n}$, i.e. $(n-t)!$.*

This conjecture suggests an obvious extension for the extremal case. It is worth pointing out that the condition $n \geq n_0(t)$ is necessary; [47, Example 3.1.1] is a simple illustration of this fact. An analogue of the statement of the conjecture for partial permutations has been proved by Ku.

Theorem 1.5.4 (Ku [47, Theorem 6.6.6]) *For any $r, t \in \mathbb{N}$ there exists $n_0(r,t)$ such that for any $n \geq n_0(r,t)$, the size of a t-intersecting sub-family of $S^*_{\binom{[n]}{r},n}$ is at most that of a maximal trivial t-intersecting sub-family of $S^*_{\binom{[n]}{r},n}$, i.e. $\binom{n-t}{r-t}\frac{(n-t)!}{(n-r)!}$.*

For further reading on problems and results of this kind, Ku's Ph.D. thesis [47] (dedicated precisely to intersecting families of permutations and partial permutations) is recommended.

1.6 Cross-intersecting families

As we mentioned in Section 1.1, in order to obtain Theorem 1.2.2, Hilton and Milner proved the following result.

Theorem 1.6.1 (Hilton and Milner [38]) *Let $r \leq n/2$. If $\mathcal{A}$ and $\mathcal{B}$ are non-empty cross-intersecting sub-families of $\binom{[n]}{r}$ then*

$$|\mathcal{A}| + |\mathcal{B}| \leq 1 + \binom{n}{r} - \binom{n-r}{r} = |\mathcal{A}_0| + |\mathcal{B}_0|,$$

where $\mathcal{A}_0 := \{[r]\}$ and $\mathcal{B}_0 := \{B \in \binom{[n]}{r} : B \cap [r] \neq \emptyset\}$.

Similarly to the case of Theorem 1.2.2, the proof was long and complicated due to the result being part of a more general one. A streamlined proof was later obtained by Simpson [55] by means of the compression technique (see Section 2.2). Frankl and Tokushige instead used the Kruskal-Katona Theorem [46, 43] to establish the following extension.

Theorem 1.6.2 (Frankl and Tokushige [32]) *Let $r \le s \le n-r$. If $\mathcal{A} \subseteq \binom{[n]}{r}$ and $\mathcal{B} \subseteq \binom{[n]}{s}$ are non-empty and cross-intersecting then*

$$|\mathcal{A}| + |\mathcal{B}| \le 1 + \binom{n}{s} - \binom{n-r}{s} = |\mathcal{A}_0| + |\mathcal{B}_0|,$$

where $\mathcal{A}_0 := \{[r]\}$ and $\mathcal{B}_0 := \{B \in \binom{[n]}{s} \colon B \cap [r] \neq \emptyset\}$.

The obvious EKR-type problem for multiple cross-intersecting families was addressed by Hilton [37]. Suppose we want to construct k cross-intersecting sub-families $\mathcal{A}_1, ..., \mathcal{A}_k$ of $\binom{[n]}{r}$, $r \le n/2$, such that the sum of sizes of these families is a maximum. The simplest configuration one can think of is where one family is the whole of $\binom{[n]}{r}$ and hence, by the cross-intersection condition and the $r \le n/2$ condition, all the other families are empty. The second simplest configuration is where the k families are the same and hence intersecting; an obvious example is where each of the k families is the star of $\binom{[n]}{r}$ with centre 1. Using the Kruskal-Katona Theorem [46, 43], Hilton [37] showed that at least one of the two configurations we mentioned is optimal. More precisely, he proved the following beautiful generalisation of the EKR Theorem.

Theorem 1.6.3 (Hilton [37]) *If $r \le n/2$, $k \ge 2$, and $\mathcal{A}_1$, ..., $\mathcal{A}_k$ are cross-intersecting sub-families of $\binom{[n]}{r}$ then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \le \begin{cases} \binom{n}{r} & \text{if } k \le \frac{n}{r}; \\ k\binom{n-1}{r-1} & \text{if } k \ge \frac{n}{r}. \end{cases}$$

Suppose equality holds and $\mathcal{A}_1 \neq \emptyset$:

(i) if $k < n/r$ then $\mathcal{A}_1 = \binom{[n]}{r}$ and $\mathcal{A}_i = \emptyset$ for $i = 2, ..., k$;

(ii) if $k > n/r$ then $|\mathcal{A}_i| = \binom{n-1}{r-1}$ for $i = 1, ..., k$;

(iii) if $k = n/r > 2$ then $\mathcal{A}_1$, ..., $\mathcal{A}_k$ are as in (i) or (ii).

Note that if the $\mathcal{A}_i$'s are intersecting and equal to each other then $\mathcal{A}_1, ..., A_k$ are cross-intersecting. Thus setting $k > n/r$ and $\mathcal{A}_1 = ... = \mathcal{A}_k$ clarifies why the EKR Theorem follows from the above theorem.

There are many other cross-intersection results in the literature, some of which are mentioned in Frankl's survey paper [29]; however, those mentioned above are the ones that are relevant to this thesis.

Chapter 2

Notation and compression tools for proofs

2.1 Some notation for sets and families

The scope of this section is to develop some notation for certain sets and families defined on an arbitrary family $\mathcal{F}$. This notation will be used mainly in the proofs.

Let $U(\mathcal{F}) := \bigcup_{A \in \mathcal{F}} A$. For a set V, let

$$\mathcal{F}[V] := \{A \in \mathcal{F}\colon V \subseteq A\},$$

$$\mathcal{F}]V[:= \{A \in \mathcal{F}\colon A \cap V = \emptyset\}.$$

$$\mathcal{F}\langle V\rangle := \{A \backslash V\colon A \in \mathcal{F}[V]\} = \{B\colon B \cap V = \emptyset,\ B \cup V \in \mathcal{F}\},$$

$$\mathcal{F}(V) := \{A \in \mathcal{F}\colon A \cap V \neq \emptyset\}.$$

For $u \in U(\mathcal{F})$, we abbreviate $\mathcal{F}[\{u\}]$, $\mathcal{F}]\{u\}[$, $\mathcal{F}\langle\{u\}\rangle$ and $\mathcal{F}(\{u\})$ to $\mathcal{F}[u]$, $\mathcal{F}]u[$, $\mathcal{F}\langle u\rangle$ and $\mathcal{F}(u)$ respectively. Note that $\mathcal{F}[u] = \mathcal{F}(u)$.

To illustrate an example of a family that can be defined on $\mathcal{F}$ with the above notation, we have

$$\begin{aligned}\mathcal{F}(z)\langle X\rangle]Y[&= \{A \in \mathcal{F}(z)\langle X\rangle \colon A \cap Y = \emptyset\}\\ &= \{A \colon A \cap X = \emptyset,\ A \cup X \in \mathcal{F}(z),\ A \cap Y = \emptyset\}\\ &= \{A \colon z \in A \cup X \in \mathcal{F},\ A \cap X = \emptyset,\ A \cap Y = \emptyset\}.\end{aligned}$$

We point out that the family $\mathcal{F}(z)\langle X\rangle]Y[$ has only been considered for the purpose of making the reader familiar with the use of this notation and that no such somewhat complicated family will arise in any other part of the thesis.

Note that a star of a family $\mathcal{F}$ is $\mathcal{F}(u)$ for some $u \in U(\mathcal{F})$, and u is a centre of $\mathcal{F}$ iff $\mathcal{F}(u) = \mathcal{F}$ (which implies $u \in \bigcap_{A \in \mathcal{F}} A$).

We set

$$\mathrm{ex}(\mathcal{F}) := \{\mathcal{A} \colon \mathcal{A} \text{ is an \textit{extremal} intersecting sub-family of } \mathcal{F}\},$$

and we define the subset $L(\mathcal{F})$ of $U(\mathcal{F})$ by

$$L(\mathcal{F}) := \{u \in U(\mathcal{F}) \colon \mathcal{F}(u) \text{ is a largest star of } \mathcal{F}\}.$$

So $\mathcal{F}$ is EKR iff $\{\mathcal{F}(u) \colon u \in L(\mathcal{F})\} \subseteq \mathrm{ex}(\mathcal{F})$, and $\mathcal{F}$ is strictly EKR iff $\{\mathcal{F}(u) \colon u \in L(\mathcal{F})\} = \mathrm{ex}(\mathcal{F})$.

2.2 The compression operation and compressed families

As we mentioned in Section 1.1, the *compression* (or *shifting*) technique is one of the most powerful tools in extremal set theory. The survey paper [29] gives an excellent account of many applications of this technique. The idea surfaced in the original proof [25] of the EKR Theorem, and Theorems 1.2.3, 1.2.5, 1.4.3, 1.4.4 are also among the many results that were proved by means of this technique. It must be mentioned, though, that it fails to work for certain interesting EKR-type problems, particularly the one for intersecting sub-families of $\mathcal{S}^*_{[n],k}$.

A *compression operation*, or simply a *compression*, is a function that maps a family to another family while retaining some important properties of the original family, such as its size or t-intersection of its set members. The idea is that a family resulting from a compression or a sequence of compressions has key structural properties that the original family might not have.

Various forms of compression have been invented for specific problems. For example, the recent publications [40] and [41], which motivated a number of results in this thesis, feature compressions defined in a graph-theoretical setting that are, however, widely applicable. We now present a form of compression that is general enough for the purposes of this thesis and that particularly generalises the compression defined in [40].

For a family $\mathcal{F}$ and $u, v \in U(\mathcal{F})$, $u \neq v$, let $\Delta_{u,v}\colon 2^{\mathcal{F}} \to 2^{\mathcal{F}}$ be defined by

$$\Delta_{u,v}(\mathcal{A}) := \{\delta_{u,v}(A)\colon A \in \mathcal{A}\} \cup \{A \in \mathcal{A}\colon \delta_{u,v}(A) \in \mathcal{A}\},$$

where $\delta_{u,v}\colon \mathcal{F} \to \mathcal{F}$ is defined by

$$\delta_{u,v}(F) := \begin{cases} (F\backslash\{v\}) \cup \{u\} & \text{if } u \notin F,\ v \in F,\ (F\backslash\{v\}) \cup \{u\} \in \mathcal{F}; \\ F & \text{otherwise.} \end{cases}$$

The function $\Delta_{u,v}$ is a compression operation; it is also commonly referred to as a *shift operation*. The very first thing to be noted is that

$$|\Delta_{u,v}(\mathcal{A})| = |\mathcal{A}|.$$

We now prove a number of properties, given by Proposition 2.2.1, of the compression operation defined above. These properties have a fundamental role in the work of this thesis. Parts (i) and (ii) are well known. Parts (iii) and (iv), which will have applications in Chapters 10 and 11, may be regarded as new although they arise as a generalisation of properties - mostly discovered in [40] - of compressions on intersecting families of *independent sets* of graphs (see Chapter 11).

In the following, we make use of the notation introduced in the preceding section.

We say that $\mathcal{F}$ is (u,v)-*compressed* if for any $F \in \mathcal{F}]u[(v)$, $(F\backslash\{v\}) \cup \{u\} \in \mathcal{F}$.

Proposition 2.2.1 *Let $\mathcal{F}$ be a family, and let $u, v \in U(\mathcal{F})$, $u \neq v$. Let $\mathcal{A}^*$ be a t-intersecting sub-family of $\mathcal{F}$, and let $\mathcal{A} := \Delta_{u,v}(\mathcal{A}^*)$.*

(i) $\mathcal{A}]v[$ is t-intersecting.

(ii) If $\mathcal{F}$ is (u,v)-compressed then $\mathcal{A}$ is t-intersecting.

(iii) If $t = 1$, $\mathcal{F}[\{u,v\}] = \emptyset$ and $\mathcal{F}$ is (u,v)-compressed then $\mathcal{A}\langle v\rangle \cup \mathcal{A}]v[$ is intersecting.

(iv) If $t = 1$, $\mathcal{F}[\{u,v\}] = \emptyset$ and there exists $w \in U(\mathcal{F})\backslash\{u,v\}$ such that $\mathcal{F}]w[$ is (u,v)-compressed then $\mathcal{A}\langle v\rangle$ is intersecting.

Proof. Let $B_1, B_2 \in \mathcal{A}$. Then, for each $p \in [2]$, $B_p = A_p$ or $B_p = \delta_{u,v}(A_p)$ for some $A_p \in \mathcal{A}^*$.

Suppose $B_1, B_2 \in \mathcal{A}]v[(u)$. It is straightforward that $|B_1 \cap B_2| \geq t$ if $B_p = A_p$, $p = 1, 2$, or $B_p = \delta_{u,v}(A_p) \neq A_p$, $p = 1, 2$. So suppose without loss of generality that $B_1 = A_1$ and $B_2 = \delta_{u,v}(A_2) \neq A_2$ (hence $u \notin A_2$). Then $|B_1 \cap B_2| = |(A_1 \cap A_2) \cup \{u\}| = |A_1 \cap A_2| + 1 \geq t + 1$. So $\mathcal{A}]v[(u)$ is t-intersecting. Now, it clearly holds that $\mathcal{A}]v[]u[= \mathcal{A}^*]v[]u[$, and hence $|A \cap A'| \geq t$ for any $A \in \mathcal{A}]v[]u[$ and $A' \in \mathcal{A}$. Hence (i).

Suppose $\mathcal{F}$ is (u,v)-compressed. As mentioned above, $|B_1 \cap B_2| \geq t$ if $B_p = A_p$, $p = 1, 2$, or $B_p = \delta_{u,v}(A_p)$, $p = 1, 2$. So suppose $B_1 = A_1$, $B_2 = \delta_{u,v}(A_2) \neq A_2$ (hence $u \notin A_2$) and $|B_1 \cap B_2| < t$. Then $|(A_1 \cap A_2)\backslash\{v\}| = t - 1$ (since $\mathcal{A}^*$ is t-intersecting), $u \notin A_1$ (otherwise $|B_1 \cap B_2| = |A_1 \cap A_2| \geq t$) and $A_1 \neq \delta_{u,v}(A_1) \in \mathcal{A}$ (since $\mathcal{F}$ is (u,v)-compressed). But then $|\delta_{u,v}(A_1) \cap A_2| = t - 1$, contradicting $\mathcal{A}^*$ t-intersecting. Hence (ii).

Suppose $t = 1$, $\mathcal{F}[\{u,v\}] = \emptyset$ and $\mathcal{F}$ is (u,v)-compressed. By (i) and (ii), $\mathcal{A}]v[$ is intersecting and $A \cap B \neq \emptyset$ for any $A \in \mathcal{A}]v[$ and $B \in \mathcal{A}\langle v\rangle$. So (iii) follows if we show that $\mathcal{A}\langle v\rangle$ is intersecting. So suppose $B_1, B_2 \in \mathcal{A}(v)$. Then, for each $p \in [2]$, $B_p \in \mathcal{A}^*(v) \subseteq \mathcal{F}(v)$, and $u \notin B_p$ since $\mathcal{F}[\{u,v\}] = \emptyset$. Since $\mathcal{F}$ is (u,v)-compressed, we must have $B_p \neq \delta_{u,v}(B_p) \in \mathcal{A}$, which implies $\delta_{u,v}(B_p) \in \mathcal{A}^*$ (since $B_p \in \mathcal{A}$). So $(B_1 \cap B_2)\backslash\{v\} = B_1 \cap \delta_{u,v}(B_2) \neq \emptyset$ since $u \notin B_1$, $B_1, \delta_{u,v}(B_2) \in \mathcal{A}^*$ and $\mathcal{A}^*$ is intersecting. Hence (iii).

Suppose $t = 1$, $\mathcal{F}[\{u,v\}] = \emptyset$ and there exists $w \in U(\mathcal{F})\backslash\{u,v\}$ such that $\mathcal{F}]w[$ is (u,v)-compressed. Suppose $B_1, B_2 \in \mathcal{A}(v)$. Then, for each $p \in [2]$, $B_p \in \mathcal{A}^*(v) \subseteq \mathcal{F}(v)$, and $u \notin B_p$ since $\mathcal{F}[\{u,v\}] = \emptyset$. Thus, if $w \notin B_p$ for some $p \in [2]$ then, since $\mathcal{F}]w[$ is (u,v)-compressed, we must have $B_p \neq \delta_{u,v}(B_p) \in \mathcal{A}$, which implies $\delta_{u,v}(B_p) \in \mathcal{A}^*$ (since $B_p \in \mathcal{A}$) and hence $(B_1 \cap B_2)\backslash\{v\} = B_{3-p} \cap \delta_{u,v}(B_p) \neq \emptyset$ (since $u \notin B_{3-p}$, $B_{3-p}, \delta_{u,v}(A_p) \in \mathcal{A}^*$ and $\mathcal{A}^*$ is intersecting). If on the contrary $w \notin B_p$ for each $p \in [2]$ then trivially $w \in (B_1 \cap B_2)\backslash\{v\}$. Hence (iv). □

Note that $\mathcal{F}$ is compressed with respect to u (see definition in Section 1.1) if $\mathcal{F}$ is (u,v)-compressed for all $v \in U(\mathcal{F})\backslash\{u\}$, and that $\mathcal{F} \subseteq 2^{[n]}$ is compressed if $\Delta_{i,j}(\mathcal{F}) = \mathcal{F}$ for any $i, j \in [n]$ such that $i < j$.

If $a_1 < ... < a_r$, $b_1 < ... < b_r$, $a_1 \leq b_1$, ..., $a_r \leq b_r$, $A := \{a_1, ..., a_r\}$ and $B := \{b_1, ..., b_r\}$ then we write $A \leq B$, and if also $a_j < b_j$ for some $j \in [r]$ then we write $A < B$. It is easy to see that

$$\mathcal{A} \text{ is compressed } \Leftrightarrow \text{ for any } A \in \mathcal{A} \text{ and } A' < A,\ A' \in \mathcal{A}.$$

If $i, j \in [n]$ and $i < j$ then $\Delta_{i,j}$ is said to be a *left-compression.* It only takes a finite number of left-compressions to obtain a compressed family from a sub-family of $2^{[n]}$. This is because the positive quantity $\sum_{A \in \mathcal{A}} |A|$ decreases with a left-compression that changes $\mathcal{A}$. In fact, there are compositions of all the $\binom{n}{2}$ left-compressions, and in which each left-compression appears exactly once, that always yield a compressed family when applied to a sub-family of $2^{[n]}$. A set of such compositions is determined in [29], and the following demonstrates another composition.

Proposition 2.2.2 *For $\mathcal{A} \subseteq 2^{[n]}$, let*

$$\mathcal{A}' := \Delta_{n-1,n} \circ \Delta_{n-2,n} \circ ... \circ \Delta_{1,n} \circ ... \circ \Delta_{2,3} \circ \Delta_{1,3} \circ \Delta_{1,2}(\mathcal{A}).$$

Then $\mathcal{A}'$ is compressed.

Proof. Let $N := \{(a,b) \in [n] \times [n] \colon a < b\}$. We define the partial order relation $\prec$

on members of N by $(a_1, b_1) \prec (a_2, b_2)$ iff $b_1 < b_2$ or $a_1 < a_2 < b_1 = b_2$. Suppose that for some $(s,t) \in N$, $\Delta_{p,q}(\mathcal{A}) = \mathcal{A}$ for all $(p,q) \prec (s,t)$. Let $\mathcal{B} := \Delta_{s,t}(\mathcal{A})$. Clearly, $\Delta_{s,t}(\mathcal{B}) = \mathcal{B}$. We claim that, moreover, $\Delta_{p,q}(\mathcal{B}) = \mathcal{B}$ for all $(p,q) \prec (s,t)$. The claim clearly implies that $\Delta_{i,j}(\mathcal{A}') = \mathcal{A}'$ for all $(i,j) \prec (n-1,n)$, which in turn implies the required result.

We now prove the claim. Let $B \in \mathcal{B}$, and fix p and q such that $(p,q) \prec (s,t)$. We first consider the case $B \in \mathcal{A}$, i.e. $B \in \mathcal{A} \cap \mathcal{B} = \{A \in \mathcal{A} : \delta_{s,t}(A) \in \mathcal{A}\}$. So $E := \delta_{s,t}(B) \in \mathcal{A} \cap \mathcal{B}$ and $C, F \in \mathcal{A}$, where $C := \delta_{p,q}(B)$ and $F := \delta_{p,q}(E)$. Suppose $B \neq C \in \mathcal{A} \backslash \mathcal{B} = \{A \in \mathcal{A} : \delta_{s,t}(A) \notin \mathcal{A}\}$. Therefore $C \neq D := \delta_{s,t}(C) \in \mathcal{B} \backslash \mathcal{A} = \{\delta_{s,t}(A) : A \in \mathcal{A} \backslash \mathcal{B}\}$. If $\{p,q\} \cap \{s,t\} = \emptyset$ then $F = \delta_{p,q} \circ \delta_{s,t}(B) = \delta_{s,t} \circ \delta_{p,q}(B) = D \in \mathcal{B} \backslash \mathcal{A}$, a contradiction. Now suppose $|\{p,q\} \cap \{s,t\}| = 1$. There are three possible cases:

(i) $p = s < q < t$: $D = \delta_{s,t} \circ \delta_{s,q}(B) = \delta_{s,q}(B) = C$, a contradiction.

(ii) $p < q = s < t$: $D = \delta_{s,t} \circ \delta_{p,q}(B) = \delta_{s,t} \circ \delta_{p,s}(B) = \delta_{p,t}(B) \in \mathcal{A}$, a contradiction.

(iii) $p < s < q = t$: $D = \delta_{s,t} \circ \delta_{p,t}(B) = \delta_{p,t}(B) = C$, a contradiction.

Therefore, if $B \in \mathcal{A} \cap \mathcal{B}$ then $\delta_{p,q}(B) \in \mathcal{B}$.

We must now consider the case $B \notin \mathcal{A}$, in which there exists $A \in \mathcal{A}$ such that $B = \delta_{s,t}(A) \neq A$. Again, suppose $C := \delta_{p,q}(B) \neq B$. Since $s \in B$ and $t \notin B$, we have $p \neq s$ and $q < t$. So we are left with the following two cases:

(i) $\{p,q\} \cap \{s,t\} = \emptyset$: $C = \delta_{p,q} \circ \delta_{s,t}(A) = \delta_{s,t} \circ \delta_{p,q}(A) \in \Delta_{s,t}(\mathcal{A}) = \mathcal{B}$ as $\delta_{p,q}(A) \in \mathcal{A}$.

(ii) $p < q = s < t$: $C = \delta_{p,s} \circ \delta_{s,t}(A) = \delta_{p,t}(A) = \delta_{s,t}(\delta_{p,t}(A)) \in \Delta_{s,t}(\mathcal{A}) = \mathcal{B}$ as $\delta_{p,t}(A) \in \mathcal{A}$. $\square$

We now illustrate the fact that if all left-compressions are applied on a family of sets exactly once (as above) but in an arbitrary order then the resulting family is not necessarily compressed. Consider $\mathcal{A} := \{\{2,3\}, \{2,4\}, \{3,4\}\} \subset 2^{[4]}$ (note that $\mathcal{A}$ is intersecting). If the left-compressions $\Delta_{2,4}$, $\Delta_{2,3}$, $\Delta_{3,4}$, $\Delta_{1,2}$, $\Delta_{1,3}$, $\Delta_{1,4}$ are applied in the given order then the resulting family is $\{\{1,3\}, \{1,4\}, \{3,4\}\}$, which is not compressed.

Chapter 3

Maximum hitting of a segment by sets in compressed intersecting families

3.1 Problem specification and results

For the purpose of this chapter, let us denote the compressed star $\{A \in \binom{[n]}{r} : 1 \in A\}$ and the compressed non-centred intersecting family $\{A \in \binom{[n]}{r} : 1 \in A, A \cap [2, r+1] \neq \emptyset\} \cup \{[2, r+1]\}$ by $\mathcal{S}_{n,r}$ and $\mathcal{N}_{n,r}$ respectively. We use the abbreviations $\mathcal{S}$ and $\mathcal{N}$ when n and r are clear from the context.

We shall make frequent use of the notation in Section 2.1. We stick to the definition of the relations $\leq$ and $<$ for sets in $2^{\mathbb{N}}$ given in Section 2.2. Recall (from Section 2.2) that $\mathcal{A}$ is compressed iff for any $A \in \mathcal{A}$ and $A' < A$, $A' \in \mathcal{A}$.

In this chapter, we are concerned with the following problem: Given $r \leq n/2$, which segments (i.e. non-empty sets) $Z \subseteq [2, n]$ obey the condition - call it (*) - that $|\mathcal{A}(Z)| \leq |\mathcal{S}(Z)|$ for any compressed intersecting family $\mathcal{A} \subset \binom{[n]}{r}$? Note that if we allow $1 \in Z$ then $\mathcal{S}(Z) = \mathcal{S}$, and hence (*) follows immediately from Theorem 1.2.1. As the following examples show, not all segments Z obey (*):

1. $Z \subseteq [2, r+1]$, $n \geq 2r$:
   If $\mathcal{A} = \mathcal{N}$ then $|\mathcal{A}(Z)| = |\mathcal{S}(Z)| + 1$.

2. $Z \subseteq [2, 2r-1]$, $|Z| \leq r$, $n = 2r$:
   Let $\mathcal{A} := \binom{[2r-1]}{r} = \mathcal{S}]n[\cup\mathcal{B}$. So $\mathcal{B} = \binom{[2,2r-1]}{r}$, and therefore $|\mathcal{A}(Z)| - |\mathcal{S}(Z)| =$

$|\mathcal{B}(Z)| - |\mathcal{S}(n)(Z)| = \left(\binom{2r-2}{r} - \binom{2r-2-|Z|}{r}\right) - \left(\binom{2r-2}{r-2} - \binom{2r-2-|Z|}{r-2}\right) = \binom{2r-2-|Z|}{r-2} - \binom{2r-2-|Z|}{r} > 0.$

3. $2r \in Z$, $\emptyset \neq Z \backslash \{2r\} \subseteq [2, r]$, $n = 2r$:

   For each $i \in [r]$, let $A_i := [2, r] \cup \{r + i\}$ and $A'_i := [2r] \backslash A_i$. Let $\mathcal{A} := (\mathcal{S} \backslash \{A'_1, ..., A'_r\}) \cup \{A_1, ..., A_r\}$. Note that A'_r is the unique set in $\{A_1, ..., A_r\} \cup \{A'_1, ..., A'_r\}$ that does not intersect Z. So $|\mathcal{A}(Z)| = |\mathcal{S}(Z)| + 1$.

Note that in each of the three examples above, $\mathcal{A}$ is non-centred, compressed and intersecting.

By Theorem 1.2.2, if $Z \subseteq \binom{[2,n]}{>r}$ then $|\mathcal{A}(Z)| \leq |\mathcal{A}| \leq |\mathcal{N}| < |\mathcal{S}(Z)|$ for any non-centred intersecting family $\mathcal{A} \subset \binom{[n]}{r}$. So this settles the case $|Z| > r$; however, we will also prove this directly. We will also settle the special case $|Z| = r$. The case $|Z| < r$ is far more challenging, and we will not determine fully which of these segments obey or disobey (*); however, many such segments obeying (*) are captured by the following result.

Theorem 3.1.1 *Let $\mathcal{A} \subset \binom{[n]}{r}$ be a compressed intersecting family, $2 \leq r \leq n/2$. Let $\emptyset \neq Z \subseteq [2, n]$ and $Y := Z \cap [2r]$. Suppose*

(a) $Y = \emptyset$, or

(b) $|Z| \leq r$ and $Y > W := [2r] \backslash ([2r - 2|Y|] \cup Y)$, or

(c) $|Z| > r$.

Then $|\mathcal{A}(Z)| \leq |\mathcal{S}(Z)|$.

Moreover, if $\mathcal{A}$ is non-centred then $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$ if and only if

(i) $r = 2$, and $Z = Y \neq \{4\}$ or $\{2, 3\} \subset Z \in \binom{[2,n]}{3}$, or

(ii) $r > 2$, $n = 2r$, $Z \cap [2, r + 1] \neq \emptyset$, $|\mathcal{A}| = |\mathcal{S}|$ and $\mathcal{A}]Z[= \mathcal{S}]Z[$ (such a family $\mathcal{A}$ exists).

We shall make two remarks that should make the statement of Theorem 3.1.1 easier to grasp:

- Consider (b). Let $U := [2r] \backslash [2r - 2|Y|]$. Clearly, $W \subset U$. By definition of $<$ on members of $\binom{[n]}{r}$, we must have $Y \subset U$, otherwise $|W| = 2r - (2r - 2|Y|) - |Y \cap U| = 2|Y| - |Y \cap U| > |Y|$, i.e. Y and W are incomparable. So $|W| = |Y| = |U|/2$, $W \cap Y = \emptyset$, and hence

$$Y \cup W = U. \tag{3.1}$$

- Clearly, if a compressed family is centred then it must be a sub-family of $\mathcal{S}$. Therefore, $\mathcal{A}$ is non-centred iff $\mathcal{A} \not\subseteq \mathcal{S}$. We now show that in some cases, $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$ holds for proper sub-families $\mathcal{A}$ of $\mathcal{S}$, and we determine these cases exactly. Let $m := \max\{z \in Z\}$ and $\mathcal{S}^* := \mathcal{S}\backslash\{A \in \mathcal{S}\colon A\backslash\{1\} \subset [n]\backslash[m]\}$. $\mathcal{S}^* \neq \mathcal{S}$ iff $m \leq n - r + 1$; also, $\mathcal{S}^*$ is compressed and $\mathcal{S}^*(Z) = \mathcal{S}(Z)$. It is easy to check that for any $A \in \mathcal{S}^*\backslash\mathcal{S}(Z)$ there exists $B \in \mathcal{S}(Z)$ such that $A < B$. This implies that if $\mathcal{A} \subset \mathcal{S}$ and $\mathcal{A}(Z) = \mathcal{S}(Z)$ then $\mathcal{S}^* \subseteq \mathcal{A}$.

For $x_1 < ... < x_n$, $X := \{x_i\colon i \in [n]\}$, $m \leq n$, we call the set $\{x_i\colon i \in [m, n]\}$ a *final $(n-m+1)$-segment of X*. The following is an immediate consequence of Theorem 3.1.1.

Corollary 3.1.2 *Let $\mathcal{A} \subset \binom{[n]}{r}$ be as in Theorem 3.1.1, and let Z be a final segment of $[n]$. Then $|\mathcal{A}(Z)| \leq |\mathcal{S}(Z)|$, and if $\mathcal{A} \neq \mathcal{S}$ then $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$ only if $n = 2r$ and $|Z| \geq r$.*

The next theorem settles our problem for the special case $|Z| = r$.

Theorem 3.1.3 *Let $Z \in \binom{[2,n]}{r}$, $2 \leq r \leq n/2$. Let $\mathcal{A}$ be a compressed intersecting sub-family of $\binom{[n]}{r}$ such that $\mathcal{A}(Z)$ is of largest size. If*
(a) $\{2, 3\} \subseteq Z$ and $r \leq 3$, or
(b) $n = 2r$ and $[n]\backslash Z \not< Z$, or
(c) $Z = [2, r + 1]$
then $|\mathcal{A}(Z)| = |\mathcal{S}(Z)| + 1$, otherwise $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$.

We now present an application. In Section 3, we show that the following extension of the EKR Theorem follows from Corollary 3.1.2 [1].

Theorem 3.1.4 (Holroyd, Talbot [41]) *Let $X_1, ..., X_p$ be distinct sets such that $\bigcap_{i=1}^{p} X_i \neq \emptyset$ and $X_j \cap X_k = \bigcap_{i=1}^{p} X_i$ for any $j, k \in [p]$. Let $\mathcal{H} := \bigcup_{k=1}^{p} 2^{X_k}$. Suppose*

[1]Theorems 3.1.4 and 3.1.5 are proved in [7] by a method that 'extends' the original proof of the EKR Theorem and that is yet different from both the method used in this chapter and the one used in [41]; however, the material in [7] is not included in this thesis.

$4 \leq 2r \leq \mu(\mathcal{H})$. *Then:*

(i) $\mathcal{H}^{(r)}$ *is EKR, and*

(ii) strictly so if $2r < \mu$.

We remark that, in the literature, a family consisting of sets X_i as in Theorem 3.1.4 is called a *sunflower* or *delta-system.* The Erdős-Rado Theorem [26] is an example of a well-known result about sunflowers. Sunflowers are used in the *kernel method* introduced by Hajnal and Rothschild [35]; a brief review together with another application of this method is given in [27]. The *maximal independent sets* of a union of a *complete multipartite graph* and an *empty graph* form a sunflower; Holroyd and Talbot expressed Theorem 3.1.4 in these graph-theoretical terms (see Chapter 11).

In Section 3.3, we also apply Corollary 3.1.2 to sharpen Theorem 3.1.4 with $2r = \mu$.

Theorem 3.1.5 (Extension of Theorem 3.1.4) *Suppose that in Theorem 3.1.4 we have* $2r = \mu(\mathcal{H})$ *and* $p > 1$. *Then* $\mathcal{H}^{(r)}$ *is not strictly EKR if and only if* $\mu(\mathcal{H}) = \alpha(\mathcal{H})$ *and* $3 \leq |\bigcap_{i=1}^{p} X_i| \leq r$.

We conclude this section by mentioning that in the next chapter we obtain a generalisation of the Hilton-Milner Theorem by employing the idea of the problem we have presented here; see, for example, Proposition 4.7.2.

3.2 Proof of main results

We begin with the key lemma concerning ordered pairs of subsets of $\binom{[n]}{r}$.

Lemma 3.2.1 *Let* $A, B \in \binom{[n]}{r}$, $A \neq B$, *and let* $C \subseteq A \cap B$. *Then*

$$A \backslash C < B \backslash C \Leftrightarrow A < B.$$

Proof. Let $D := A\backslash C, E := B\backslash C$. We have $D < E$ and must prove $D \cup C < E \cup C$.

Suppose $C = \{c\}$. Let $D := \{d_1, ..., d_s\}$, $E := \{e_1, ..., e_s\}$, each set listed in increasing order. If $c < d_1$ or $c > e_s$ then the result is immediate; so we may assume $c \in [d_1 + 1, e_s - 1]$. Let $j := \max\{i\colon d_i < c\}$, $k := \min\{i\colon c < e_i\}$. Then

$D \cup \{c\} = \{d_1^*, ..., d_{s+1}^*\}$ and $E \cup \{c\} = \{e_1^*, ..., e_{s+1}^*\}$, where $d_i^* := d_i$ for $i = 1, ..., j$, $d_{j+1}^* := c$, $d_i^* := d_{i-1}$ for $i = j+2, ..., s+1$, $e_i^* := e_i$ for $i = 1, ..., k-1$, $e_k^* := c$, $e_i^* := e_{i-1}$ for $i = k+1, ..., s+1$. Note that $k \leq j+1$ since $D < E$. It is straightforward that if $k = j+1$ then $d_i^* \leq e_i^*$, $i = 1, ..., s+1$, with at least one strict inequality. If $k < j+1$ then $d_i^* = d_i \leq e_i = e_i^*$ for $i = 1, ..., k-1$, $d_i^* = d_i < c = e_k^* \leq e_i^*$ for $i = k, ..., j$, $d_{j+1}^* = c < e_{j+1}^*$, and $d_i^* = d_{i-1} \leq e_{i-1} = e_i^*$ for $i = j+2, ..., s+1$. So $D \cup C < E \cup C$ as required.

The result for general C follows by a simple inductive argument.

Conversely, we have $A < B$ and must prove $A \backslash C < B \backslash C$.

Suppose $C = \{c\}$. Let $A := \{a_1, ..., a_r\}$, $B := \{b_1, ..., b_r\}$, each set listed in increasing order. Since $A < B$, we have $c = a_p = b_q$ for some $p \geq q$. If $p = q$ then the result is immediate. Suppose $p > q$. Then $A \backslash \{c\} = \{a_1^*, ..., a_{r-1}^*\}$ and $B \backslash \{c\} = \{b_1^*, ..., b_{r-1}^*\}$, where $a_i^* := a_i \leq b_i =: b_i^*$ for $i = 1, ..., q-1$, $a_i^* := a_i \leq b_i < b_{i+1} =: b_i^*$ for $i = q, ..., p-1$, and $a_i^* := a_{i+1} \leq b_{i+1} =: b_i^*$ for $i = p, ..., r-1$. So $A \backslash C < B \backslash C$ as required.

The result for general C again follows by a simple inductive argument. □

Lemma 3.2.2 *If $\mathcal{A} \subseteq 2^{[n]}$ is compressed and $Z, \{a, b\} \subset [n]$, $a < b$, then $|\mathcal{A}(Z)| \leq |\mathcal{A}(\delta_{a,b}(Z))|$.*

Proof. Suppose $Z' := \delta_{a,b}(Z) \neq Z$. Letting $Z'' := Z \cap Z'$, we then have $Z = Z'' \cup \{b\} \neq Z''$ and $Z' = Z'' \cup \{a\} \neq Z''$. Since $\mathcal{A}$ is compressed, $\Delta_{a,b}(\mathcal{A}]Z''[(b)]a[) \subseteq \mathcal{A}]Z''[(a)]b[$. So $|\mathcal{A}]Z''[(a)]b[| \geq |\mathcal{A}]Z''[(b)]a[|$. We therefore have

$$|\mathcal{A}(Z')| - |\mathcal{A}(Z)| = (|\mathcal{A}(Z'')| + |\mathcal{A}]Z''[(a)|) - (|\mathcal{A}(Z'')| + |\mathcal{A}]Z''[(b)|)$$
$$= (|\mathcal{A}]Z''[(a)(b)| + |\mathcal{A}]Z''[(a)]b[|) - (|\mathcal{A}]Z''[(b)(a)| + |\mathcal{A}]Z''[(b)]a[|) \geq 0,$$

and hence result. □

Proof of Theorem 3.1.1. By induction on n. It is easy to check the result for $r = 2$ because $\binom{[3]}{2}$ is the only non-centred compressed intersecting sub-family of $\binom{[n]}{2}$.

We shall now assume that $r > 2$. Thus, for the remainder of the proof, we are concerned with condition (ii) in the statement of the theorem. If this condition holds then $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$ trivially, so we now prove the converse. We may assume $\mathcal{A}$ to be such that $|\mathcal{A}(Z)|$ is maximised; that is

$$|\mathcal{A}'(Z)| \leq |\mathcal{A}(Z)| \text{ for any compressed intersecting } \mathcal{A}' \subset \binom{[n]}{r}. \tag{3.2}$$

Case 1: $n = 2r$. So $Y = Z \neq \emptyset$.

Let $A_1 := [2, r+1]$ and $A_2 := \{1\} \cup [r+2, ..., 2r]$. Then $\mathcal{N} := (\mathcal{S}\backslash\{A_2\}) \cup \{A_1\}$ is non-centred, compressed, intersecting, and has size equal to $|\mathcal{S}|$. If $|Z| > r$ then $|\mathcal{N}(Z)| = |\mathcal{N}| = |\mathcal{S}| = |\mathcal{S}(Z)|$ trivially. Suppose $|Z| \leq r$ and $Z \cap [2, r+1] \neq \emptyset$. So $Z \cap A_1 \neq \emptyset$. By (b), we have $2r \in Z$, and hence $A_2 \cap Z \neq \emptyset$. So $|\mathcal{N}(Z)| = |\mathcal{S}(Z)|$. This proves the existence of a family for which (ii) holds.

Sub-case 1.1: $|Z| > r$. Since $\mathcal{A}$ is intersecting, if $A \in \mathcal{A}$ then $[2r]\backslash A \notin \mathcal{A}$; hence $|\mathcal{A}| \leq \frac{1}{2}\binom{2r}{r} = \binom{2r-1}{r-1} = |\mathcal{S}|$. So the result is straightforward since here $\mathcal{A}(Z) = \mathcal{A}$.

Sub-case 1.2: $|Z| = r$. Clearly, $Z \cap [2, r+1] \neq \emptyset$. If $Z \in \mathcal{A}$ then, since $[2r]\backslash Z = W < Z$ (by (b)) and $\mathcal{A}$ is compressed, we have $[2r]\backslash Z \in \mathcal{A}$, which contradicts $\mathcal{A}$ intersecting. So $Z \notin \mathcal{A}$. Since $Z, [2r]\backslash Z \notin \mathcal{A}(Z)$, it clearly follows that $|\mathcal{A}(Z)| \leq \frac{1}{2}\binom{2r}{r} - 1 = |\mathcal{S}(Z)|$ (note that $Z > [2r]\backslash Z \Rightarrow 1 \in [2r]\backslash Z \Rightarrow \{[2r]\backslash Z\} = \mathcal{S}]Z[)$. Thus, by (3.2), $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$.

Since $W = [2r]\backslash Z$, $\mathcal{A}]Z[\subseteq \{W\}$. Suppose $W \notin \mathcal{A}$. Let $w_1 < ... < w_r$ and $z_1 < ... < z_r$ such that $W = \{w_1, ..., w_r\}$ and $Z = \{z_1, ..., z_r\}$. Since $[2r]\backslash Z = W < Z$, we have $w_1 = 1$, $z_r = 2r$ and $w_i < z_i$, $i = 1, ..., r$. Let $W' := (W\backslash\{w_r\}) \cup \{z_r\}$. Since $W' > W \notin \mathcal{A}$ and $\mathcal{A}$ is compressed, $W' \notin \mathcal{A}$. Similarly, $[2r]\backslash W' \notin \mathcal{A}$ since $[2r]\backslash W' = (Z\backslash\{z_r\}) \cup \{w_r\} > W$. Thus, since $W, [2r]\backslash W, W', [2r]\backslash W' \notin \mathcal{A}$ (recall that $[2r]\backslash W = Z \notin \mathcal{A}$), it clearly follows that $|\mathcal{A}(Z)| \leq \frac{1}{2}\binom{2r}{r} - 2 < |\mathcal{S}(Z)|$, a contradiction. So $\{W\} = \mathcal{A}]Z[= \mathcal{S}]Z[$. Thus, since $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$, $|\mathcal{A}| = |\mathcal{S}|$.

Sub-case 1.3: $|Z| < r$. Let $\mathcal{A}_1 \cup \mathcal{A}_2$ be the partition of $\mathcal{A}(Z)$ defined by $\mathcal{A}_1 := \{A \in \mathcal{A}(Z) \colon Z\backslash A \neq \emptyset\}$ and $\mathcal{A}_2 := \{A \in \mathcal{A}(Z) \colon Z \subset A\}$. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be defined similarly. Let $f \colon \mathcal{A}_1 \to \mathcal{S}_1$ be defined by $f(A) = A$ if $1 \in A$ and $f(A) = [2r]\backslash A$ if $1 \notin A$ $(A \in \mathcal{A}_1)$. So f is injective because if $1 \in C \in \mathcal{A}_1$, $1 \notin D \in \mathcal{A}_1$, and $f(D) = f(C)$ then $[n]\backslash D = C \in \mathcal{A}$, contradicting $\mathcal{A}$ intersecting. Hence $|\mathcal{A}_1| \leq |\mathcal{S}_1|$.

Now consider $\mathcal{A}_2$, and suppose there exist $C, D \in \mathcal{A}_2$ such that $(C \cap D)\backslash Z = \emptyset$. Thus, taking $E := [2r]\backslash D$ and $F := E\backslash C$, we have $C\backslash E = Z$ and $F \subset [2r]\backslash Z$. Note that $|F| = |([2r]\backslash D)\backslash C| = |[2r]\backslash(C \cup D)| = 2r - (|C| + |D| - |C \cap D|) = |Z|$. Since $Y = Z$, we have $W\backslash F \subset W \subset [2r - 2|Z| + 1, 2r]$ by (b), and $F\backslash W \subset [2r]\backslash(Z \cup W) = [2r - 2|Z|]$ by (3.1). So $F\backslash W \leq W\backslash F$. If $F\backslash W = W\backslash F$ then $F = W$, and if $F\backslash W < W\backslash F$ then $F < W$ by Lemma 3.2.1; hence $F \leq W$. So $E\backslash C \leq W < Z = C\backslash E$, and hence Lemma 3.2.1 gives us $E < C$. Thus, since $\mathcal{A} \ni C$ is compressed, we get $E \in \mathcal{A}$, which contradicts $\mathcal{A}$ intersecting as $D \in \mathcal{A}$ and $E \cap D = \emptyset$. So

$$(C \cap D)\backslash Z \neq \emptyset \text{ for all } C, D \in \mathcal{A}_2. \tag{3.3}$$

Next, define $\mathcal{X} := \{A\backslash Z \colon A \in \mathcal{A}_2\} \subset \binom{[n']}{r'}$, where $n' = 2r - |Z|$ and $r' = r - |Z|$. Since $|Z| < r < n/2$, $1 \leq r' < n'/2$. By (3.3), $\mathcal{X}$ is intersecting. $\mathcal{X}$ is also compressed because $A < B \in \mathcal{X} \Rightarrow (A \cup Z)\backslash Z < (B \cup Z)\backslash Z \in \mathcal{X} \Rightarrow A \cup Z < B \cup Z \in \mathcal{A}_2$ (by Lemma 3.2.1) $\Rightarrow A \cup Z \in \mathcal{A}_2$ (since $\mathcal{A}$ is compressed) $\Rightarrow A \in \mathcal{X}$. Let $\mathcal{Y} := \{A\backslash Z \colon A \in \mathcal{S}_2\}$. If $r' = 1$ then $\mathcal{X} \subseteq \mathcal{Y}$ trivially. If $r' > 1$ then we take $Z' := [2, 2r]\backslash Z$ and, since $\mathcal{X} = \mathcal{X}(Z')$ and $\mathcal{Y} = \mathcal{Y}(Z')$, we apply the inductive hypothesis to get $|\mathcal{X}| \leq |\mathcal{Y}|$ with equality only if $\mathcal{X} = \mathcal{Y}$. So $|\mathcal{A}_2| \leq |\mathcal{S}_2|$ with equality only if $\mathcal{A}_2 = \mathcal{S}_2$. Thus, since $|\mathcal{A}_1| \leq |\mathcal{S}_1|$, $|\mathcal{A}(Z)| \leq |\mathcal{S}(Z)|$. By (3.2), $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$. So $|\mathcal{A}_1| = |\mathcal{S}_1|$, $|\mathcal{A}_2| = |\mathcal{S}_2|$, and hence $\mathcal{A}_2 = \mathcal{S}_2$.

Suppose $Z \cap [2, r+1] \neq \emptyset$. Take any $K \in \mathcal{S}]Z[$. Let $k_1 < \dots < k_r$ such that $K = \{k_1, \dots, k_r\}$, and let $L := \{k_{r-|Z|+1}, \dots, k_r\}$, $K' := (K\backslash L) \cup Z$. Similarly to F above, $L \leq W$. Given that $W < Z$, it follows that $L < Z$, and hence $K < K'$. So $K \in \mathcal{A}]Z[$ because $K' \in \mathcal{S}_2 = \mathcal{A}_2$, $\mathcal{A}$ is compressed, and $K \cap Z = \emptyset$. We have therefore shown that $\mathcal{S}]Z[\subseteq \mathcal{A}]Z[$. So $|\mathcal{A}| \geq |\mathcal{S}|$ as $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$. But $|\mathcal{A}| \leq \frac{1}{2}\binom{2r}{r} = |\mathcal{S}|$ (see Sub-case 1.1). So $|\mathcal{A}| = |\mathcal{S}|$ and $\mathcal{A}]Z[= \mathcal{S}]Z[$.

Now suppose $Z \cap [2, r+1] = \emptyset$. So $Z \subseteq [r+2, 2r]$, and hence $A^* := \{1\} \cup [r+2, 2r] \in \mathcal{A}_2$ as $A^* \in \mathcal{S}_2 = \mathcal{A}_2$. Thus, since $\mathcal{A}$ is compressed and $A < A^*$ for all $A \in \mathcal{S}\backslash\{A^*\}$, we have $\mathcal{S} \subseteq \mathcal{A}$. Together with $|\mathcal{A}| \leq |\mathcal{S}|$ (see Sub-case 1.1), this gives us $\mathcal{A} = \mathcal{S}$.

Case 2: $n > 2r$. Let $n' := n - 1$. We have $\mathcal{A}]n[, \mathcal{S}]n[\subset \binom{[n']}{r}$ and $\mathcal{A}\langle n\rangle, \mathcal{S}\langle n\rangle \subset \binom{[n']}{r'}$,

$r' = r-1$. Note that $r \le n'/2$ and $r' < n'/2$ (as we now have $r < n/2$). Also note that $\mathcal{A}\langle n\rangle$ and $\mathcal{A}]n[$ are compressed. We now show that $\mathcal{A}\langle n\rangle \cup \mathcal{A}]n[$ is intersecting.

Suppose $A \cap B \cap [n'] = \emptyset$ for some $A, B \in \mathcal{A}$. So $A \cap B = \{n\}$ (as $\mathcal{A}$ is intersecting). Since $|A \cup B| \le 2r-1 < n'$, $[n]\backslash(A \cup B) \ne \emptyset$. Let $a \in [n]\backslash(A \cup B)$. Since $A' := (A\backslash\{n\}) \cup \{a\} < A$ and $\mathcal{A}$ is compressed, $A' \in \mathcal{A}$. But $A' \cap B = \emptyset$, a contradiction. So $A \cap B \cap [n'] \ne \emptyset$ for any $A, B \in \mathcal{A}$, and hence $\mathcal{A}\langle n\rangle \cup \mathcal{A}]n[$ is intersecting as required.

Sub-case 2.1: $n \notin Z$. It is immediate from the inductive hypothesis that $|\mathcal{A}]n[(Z)| \le |\mathcal{S}]n[(Z)|$ and $|\mathcal{A}\langle n\rangle(Z)| \le |\mathcal{S}\langle n\rangle(Z)|$. Since $|\mathcal{A}(Z)| = |\mathcal{A}]n[(Z)| + |\mathcal{A}\langle n\rangle(Z)|$, it follows that $|\mathcal{A}(Z)| \le |\mathcal{S}(Z)|$. By (3.2), we actually have $|\mathcal{A}(Z)| = |\mathcal{S}(Z)|$, $|\mathcal{A}]n[(Z)| = |\mathcal{S}]n[(Z)|$ and $|\mathcal{A}\langle n\rangle(Z)| = |\mathcal{S}\langle n\rangle(Z)|$. It remains to show that $\mathcal{A}$ is centred.

Consider the equality $|\mathcal{A}\langle n\rangle(Z)| = |\mathcal{S}\langle n\rangle(Z)|$. Since $r' < n'/2$, it follows by the inductive hypothesis that $\mathcal{A}\langle n\rangle$ is centred, and hence $\mathcal{A}\langle n\rangle \subseteq \mathcal{S}\langle n\rangle$ as $\mathcal{A}\langle n\rangle$ is compressed. This gives us the stronger equality $\mathcal{A}(n)(Z) = \mathcal{S}(n)(Z)$.

Let $m := \max\{z \in Z\}$. If $r > 3$ then we take F_1 to be a final $(r-3)$-segment for $[n]\backslash\{1, m, n\}$; otherwise, we take F_1 to be $\emptyset$. Let $S_1 := \{1, m, n\} \cup F_1$ (recall that we are dealing with $r \ge 3$). Since $Z \subseteq [2, n]$, if $|Z| \ge r+1$ then $m \ge r+2$. Suppose $|Z| \le r$. If $Y = \emptyset$ then $m > 2r$, and if $Y \ne \emptyset$ then, by (b) and (3.1), we have $2r \in Y$, and hence $m \ge 2r$. So we have $m \ge r+2$. Suppose that $\mathcal{A}$ is non-centred. Given that $\mathcal{A}$ is compressed, we then have $[2, r+1] \in \mathcal{A}$, which is a contradiction because $[2, r+1] \cap S_1 = \emptyset$, $S_1 \in \mathcal{S}(n)(Z) = \mathcal{A}(n)(Z)$ and $\mathcal{A}$ is intersecting. So $\mathcal{A}$ is centred.

Sub-case 2.2: $n \in Z$. Suppose $Z \ne [2, n]$. Let $m' := \max\{a \in [n]\backslash Z\}$ and $Z' := \delta_{m',n}(Z)$. So $n \notin Z'$. It is easy to check that Z' also satisfies one of (a), (b), (c). Therefore, as in Sub-case 2.1, $|\mathcal{A}(Z')| \le |\mathcal{S}(Z')|$ with equality only if $\mathcal{A}$ is centred. Now $|\mathcal{S}(Z)| = |\mathcal{S}(Z')|$ and, by Lemma 3.2.2, $|\mathcal{A}(Z)| \le |\mathcal{A}(Z')|$. Thus, by (3.2), $|\mathcal{A}(Z)| = |\mathcal{A}(Z')| = |\mathcal{S}(Z')|$, and hence $\mathcal{A}$ is centred.

Now suppose $Z = [2, n]$. Then, taking $Z'' := Z\backslash\{n\}$ and applying the inductive hypothesis, we have $|\mathcal{A}]n[| = |\mathcal{A}]n[(Z'')| \le |\mathcal{S}]n[(Z'')| = |\mathcal{S}]n[|$ and $|\mathcal{A}\langle n\rangle| = |\mathcal{A}\langle n\rangle(Z'')| \le |\mathcal{S}\langle n\rangle(Z'')| = |\mathcal{S}\langle n\rangle|$, and (since $r' < n'/2$) the latter inequality is an equality only if $\mathcal{A}\langle n\rangle(Z'')$ is centred and hence $\mathcal{A}\langle n\rangle = \mathcal{S}\langle n\rangle$ (as $\mathcal{A}$ is compressed and $\mathcal{A}\langle n\rangle = \mathcal{A}\langle n\rangle(Z'')$). It follows by (3.2) that $|\mathcal{A}]n[| = |\mathcal{S}]n[|$ and $|\mathcal{A}\langle n\rangle| = |\mathcal{S}\langle n\rangle|$, and hence $\mathcal{A}\langle n\rangle = \mathcal{S}\langle n\rangle$. Now, since $r' < n'/2$, for any $A \in \binom{[2,n']}{r}$ there exists $B \in \mathcal{S}\langle n\rangle$ such that $A \cap B = \emptyset$. Since $\mathcal{A}\langle n\rangle \cup \mathcal{A}]n[$ is intersecting and $\mathcal{A}\langle n\rangle = \mathcal{S}\langle n\rangle$, it follows that $\mathcal{A}]n[\subset \mathcal{S}$ and hence $\mathcal{A} \subseteq \mathcal{S}$. □

We now come to the proof of Theorem 3.1.3, for which we need the following second lemma concerning ordered pairs of subsets of $\binom{[n]}{r}$.

Lemma 3.2.3 *Let* $A, B \in \binom{[n]}{r}$, $1 \leq r \leq n-1$. *Then*

$$A < B \Leftrightarrow [n]\backslash B < [n]\backslash A.$$

Proof. By induction on n. The case $n = 2$ is trivial. Consider $n > 2$.

Suppose $C := A \cap B \neq \emptyset$. Let $X := [n]\backslash C$. Let $D := A\backslash C, E := B\backslash C \in \binom{X}{r-|C|}$. By Lemma 3.2.1, $D < E$. By the inductive hypothesis, $F := X\backslash E < G := X\backslash D$. The result follows since $F = [n]\backslash A$ and $G = [n]\backslash B$.

Now suppose $A \cap B = \emptyset$. If $n = 2r$ then $[n]\backslash B = A < B = [n]\backslash A$. Suppose $n > 2r$. Let $c \in [n]\backslash(A \cup B)$ and $Y := [n]\backslash\{c\}$. By the inductive hypothesis, $H := Y\backslash B < I := Y\backslash A$. By Lemma 3.2.1, $[n]\backslash B = H \cup \{c\} < I \cup \{c\} = [n]\backslash A$. □

Proof of Theorem 3.1.3. For the same reason specified in the proof of Theorem 3.1.1, the case $r = 2$ is straightforward. So we consider $r \geq 3$.

We start by demonstrating the lower bound $|\mathcal{S}(Z)| + 1 \leq |\mathcal{A}(Z)|$ for each of the cases (a), (b), (c). For case (a) (with $r = 3$), take $\mathcal{A}_{(a)} := \{A \in \binom{n}{r} : |A \cap [3]| \geq 2\}$. For case (b), take $\mathcal{A}_{(b)}$ to be the union of $\mathcal{A}'_{(b)} := \{A \in \binom{[2r]}{r} : A \leq Z\}$ and $\mathcal{A}''_{(b)} := \mathcal{S}\backslash\{A \in \mathcal{S} : [2r]\backslash A \in \mathcal{A}'_{(b)}\}$. For case (c), take $\mathcal{A}_{(c)} := \mathcal{N}$. It is easy to check that $\mathcal{A}_{(a)}$, $\mathcal{A}_{(b)}$ and $\mathcal{A}_{(c)}$ attain the required lower bound and that $\mathcal{A}_{(a)}$ and $\mathcal{A}_{(c)}$ are compressed and intersecting. We now show the less straightforward fact that $\mathcal{A}_{(b)}$ is compressed and intersecting.

By definition of $\mathcal{A}'_{(b)}$, if $A < B \in \mathcal{A}'_{(b)}$ then $A < B \leq Z$, and hence $A \in \mathcal{A}'_{(b)}$; so $\mathcal{A}'_{(b)}$ is compressed. Now suppose $A < B \in \mathcal{A}''_{(b)}$. Then, by Lemma 3.2.3 and the definition of $\mathcal{A}''_{(b)}$, we have $[2r]\backslash A > [2r]\backslash B \notin \mathcal{A}'_{(b)}$, and hence, since $\mathcal{A}'_{(b)}$ is compressed,

$[2r]\backslash A \notin \mathcal{A}'_{(b)}$. Also, $A \in \mathcal{S}$ since $A < B \in \mathcal{A}''_{(b)} \subset \mathcal{S}$. So $A \in \mathcal{A}''_{(b)}$, which proves that $\mathcal{A}''_{(b)}$ is compressed. Thus, as required, $\mathcal{A}_{(b)}$ is compressed because clearly, in general, the union of two compressed families is compressed.

Suppose $A, B \in \mathcal{A}_{(b)}$. It is straightforward that if $A \notin \mathcal{A}'_{(b)}$ or $B \notin \mathcal{A}'_{(b)}$ then $A \cap B \neq \emptyset$. Now suppose $A, B \in \mathcal{A}'_{(b)}$ and $A \cap B = \emptyset$. Since therefore $A \leq Z$ and $B = [2r]\backslash A$, we have $[2r]\backslash Z \leq B$ by Lemma 3.2.3. Since $B \leq Z$ (by definition of $\mathcal{A}'_{(b)}$), we then have $[2r]\backslash Z \leq Z$, a contradiction. So $\mathcal{A}'_{(b)}$ is intersecting, and hence $\mathcal{A}_{(b)}$ is intersecting.

The result now follows if we prove the upper bound $|\mathcal{A}(Z)| \leq |\mathcal{S}(Z)| + 1$ and that equality holds only if one of (a), (b), (c) holds.

Case 1: $n = 2r$. It is immediate that therefore $|\mathcal{A}(Z)| \leq |\mathcal{S}| = |\mathcal{S}(Z)| + 1$ because $|\mathcal{A}| \leq \frac{1}{2}\binom{2r}{r} = |\mathcal{S}|$ (see proof of Theorem 3.1.1). Suppose (b) does not hold, i.e. $[n]\backslash Z < Z$. Then, by Theorem 3.1.1, $|\mathcal{A}(Z)| \leq |\mathcal{S}(Z)|$. So $|\mathcal{A}(Z)| = |\mathcal{S}(Z)| + 1$ only if (b) holds.

Case 2: $n > 2r$. As in the proof of Theorem 3.1.1, $\mathcal{A}]n[\subset \binom{[n']}{r}$ $(n' = n-1)$ and $\mathcal{A}\langle n\rangle \subset \binom{[n']}{r'}$ $(r' = r-1)$ are compressed and intersecting; moreover, $\mathcal{A}\langle n\rangle \cup \mathcal{A}]n[$ is intersecting. Also recall that $r \leq n'/2$ and $r' < n'/2$.

Sub-case 2.1: $n \notin Z$. By the inductive hypothesis, we have $|\mathcal{A}]n[(Z)| \leq |\mathcal{S}]n[(Z)| + 1$. By Theorem 3.1.1, $|\mathcal{A}\langle n\rangle(Z)| \leq |\mathcal{S}\langle n\rangle(Z)|$ with equality only if $\mathcal{A}\langle n\rangle$ is centred or $r' = 2$ and $\{2,3\} \subset Z$. So $|\mathcal{A}(Z)| \leq |\mathcal{S}(Z)| + 1$ with equality only if $\mathcal{A}\langle n\rangle \subseteq \mathcal{S}\langle n\rangle$ (as $\mathcal{A}\langle n\rangle$ is compressed) or (a) holds. Suppose $|\mathcal{A}(Z)| = |\mathcal{S}(Z)| + 1$ and (a) is not the case. So $|\mathcal{A}]n[(Z)| = |\mathcal{S}]n[(Z)| + 1$, $|\mathcal{A}\langle n\rangle(Z)| = |\mathcal{S}\langle n\rangle(Z)|$ and $\mathcal{A}\langle n\rangle \subseteq \mathcal{S}\langle n\rangle$. The last two relations yield $\mathcal{A}\langle n\rangle(Z) = \mathcal{S}\langle n\rangle(Z)$, and the first relation yields $\mathcal{A}]n[\not\subseteq \mathcal{S}]n[$, implying that $A^* := [2, r+1] \in \mathcal{A}]n[$ as $\mathcal{A}]n[$ is compressed. Suppose $Z \neq A^*$. Then, since $\mathcal{A}\langle n\rangle(Z) = \mathcal{S}\langle n\rangle(Z)$, we clearly have $\mathcal{A}' := \mathcal{A}\langle n\rangle(Z)]A^*[\neq \emptyset$. Let $A' \in \mathcal{A}'$. So $A' \cap A^* = \emptyset$, but this is a contradiction because $\mathcal{A}\langle n\rangle \cup \mathcal{A}]n[$ is intersecting. So $Z = A^*$, i.e. (c) holds. Hence we are done.

Sub-case 2.2: $n \in Z$. Let $m := \max\{a \colon a \in [n]\backslash Z\}$ and $Z' := \delta_{m,n}(Z)$. So $n \notin Z'$ and clearly Z' does not satisfy (c). Thus, according to what we have shown in Sub-case 2.1, we have $|\mathcal{A}(Z')| \leq |\mathcal{S}(Z')| + 1$, and equality holds only if Z' satisfies (a), in which case Z satisfies (a) too. The result follows since $|\mathcal{A}(Z)| \leq |\mathcal{A}(Z')|$ by Lemma 3.2.2. □

3.3 An application: the EKR properties of the sunflower

We now start working towards the proofs of Theorems 3.1.4 and 3.1.5. We shall first develop some further notation.

Let $W := \bigcap_{i=1}^{p} X_i$. Let $a := |W|$, and let w_1, ..., w_a be the elements of W. For $i \in [p]$, let $V_i := X_i \backslash W$ and $b_i := |V_i|$, and let v_{i1}, ..., v_{ib_i} be the elements of V_i; for the purpose of the left-compression operation, we put the elements of X_i in the order $w_1 < ... < w_a < v_{i1} < ... < v_{ib_i}$.

Let $\mu := \mu(\mathcal{H})$ and $\alpha := \alpha(\mathcal{H})$. Fix $r \leq \mu$, and let $\mathcal{U} := \mathcal{H}^{(r)} = \bigcup_{i=1}^{p} \binom{X_i}{r}$. For $\mathcal{A} \subset \mathcal{U}$, let $\mathcal{A}_{(i)} := \{A \in \mathcal{A} \colon A \subset X_i\}$ and $\mathcal{A}_i := \mathcal{A}_{(i)}(V_i)$, $i = 1, ..., p$.

We will use the following lemma when dealing with the extremal cases of Theorems 3.1.4 and 3.1.5; we will prove this lemma later.

Lemma 3.3.1 *Let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{U}$. Suppose $p > 1$ or $2r < \mu$, and $\Delta_{x,y}(\mathcal{A}) = \mathcal{U}(x)$ for some $x, y \in X_i$, $x < y$, $i \in [p]$. Then $\mathcal{A} = \mathcal{U}(x)$ or $\mathcal{A} = \mathcal{U}(y)$ or $\mathcal{A} = \mathcal{A}_{(i)}$.*

We will often also use the following fact.

Lemma 3.3.2 *Let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{U}$. Suppose $\mathcal{U}_{(j)}(x) \subseteq \mathcal{A}_{(j)}$ for some $x \in \bigcup_{i \in [p]} X_i$ and $j \in [p]$. Then $\mathcal{A} \subseteq \mathcal{U}(x)$.*

Proof. We have $\mathcal{U}_{(j)}(x) \subseteq \mathcal{A}_{(j)}$ and $|X_j| \geq \mu \geq 2r$. Thus, for all $B \in \mathcal{U} \backslash \mathcal{U}(x)$, we can find $A \in \mathcal{A}_{(j)}$ such that $A \cap B = \emptyset$. Since $\mathcal{A}$ is intersecting, the result follows. □

Proof of Theorem 3.1.4. We apply compressions $\Delta_{x,y}$, $x, y \in X_1$, $x < y$, to $\mathcal{A}$ until $\mathcal{A}_{(1)}$ is compressed (see Section 2.2). We then repeat this procedure for $\mathcal{A}_{(2)}$, ...,

$\mathcal{A}_{(p)}$ in the given order, and we observe that after the i'th procedure we get $\mathcal{A}_{(j)}$ compressed for all $j \in [i]$ $(i = 1, ..., p)$. Clearly, $\mathcal{A}$ remains intersecting, and $\mathcal{A}_{(i)}$ becomes compressed, $i = 1, ..., p$.

Clearly, $\mathcal{A}_{(1)} \cup \mathcal{A}_2 \cup ... \cup \mathcal{A}_p$ is a partition of $\mathcal{A}$. Let $\mathcal{J} := \mathcal{U}(w_1)$. Since $2r \leq \mu$, by taking X_1 and $X_1' := X_1 \backslash \{w_1\}$ to represent $[n]$ and Z respectively in Corollary 3.1.2, we get $|\mathcal{A}_{(1)}| = |\mathcal{A}_{(1)}(X_1')| \leq |\mathcal{J}_{(1)}(X_1')| = |\mathcal{J}_{(1)}|$. Similarly, for $i = 2, ..., p$, by taking X_i and V_i to represent $[n]$ and Z respectively in Corollary 3.1.2, we get $|\mathcal{A}_i| \leq |\mathcal{J}_i|$. So $|\mathcal{A}| \leq |\mathcal{J}|$. Therefore $\mathcal{J} \in \mathrm{ex}(\mathcal{U})$ and hence (i).

Suppose $2r < \mu$ and $\mathcal{A}_{(j)} \neq \mathcal{J}_{(j)}$ for some $j \in [p]$. Taking Z' to be X_1' or V_j depending on whether $j = 1$ or $j > 1$ respectively, Corollary 3.1.2 gives us $|\mathcal{A}_{(j)}(Z')| < |\mathcal{J}_{(j)}(Z')|$, and hence $|\mathcal{A}| < |\mathcal{J}|$. Lemma 3.3.1 ensures that if $\mathcal{A}$ is initially non-centred then the compressions mentioned above do not change $\mathcal{A}$ to $\mathcal{J}$. Hence (ii). □

Proof of Theorem 3.1.5. We now have $2r = \mu$ and $p > 1$. We base this proof on the proof of Theorem 3.1.4.

Suppose $\mu = \alpha$ and $3 \leq |W| \leq r$. Note that $|X_1| = ... = |X_p| = 2r$. It is easy to check that therefore $(\mathcal{J} \backslash \{A \in \mathcal{U} \colon A \cap W = \{w_1\}\} \cup \{A \in \mathcal{U} \colon A \cap W = W \backslash \{w_1\}\}$ is a non-centred intersecting family that is as large as $\mathcal{J}$. Since $\mathcal{J} \in \mathrm{ex}(\mathcal{U})$, the sufficiency condition follows.

We now prove the necessary condition. So suppose $\mathcal{A}$ is a non-centred intersecting family in $\mathrm{ex}(\mathcal{U})$. Then $|\mathcal{A}| = |\mathcal{J}|$. Let $\mathcal{A}'$ be the resulting family after applying compressions as in the proof of Theorem 3.1.4. So $|\mathcal{A}'| = |\mathcal{A}| = |\mathcal{J}|$. Suppose $\mathcal{A}'$ is centred. Then, by Lemma 3.3.1, either $\mathcal{A}'$ is a proper sub-family of a star of $\mathcal{U}$ or $\mathcal{A}'$ is a star of $\mathcal{U}$ and $\mathcal{A}' = \mathcal{A}'_{(i)}$ for some $i \in [p]$. Since $\mathcal{J}$ is a star of largest size, the former case immediately gives us the contradiction that $|\mathcal{A}'| < |\mathcal{J}|$, and the latter case clearly gives us $\mathcal{A}' = \mathcal{U}(v_{ij})$ for some $j \in [b_i]$, which again results in the contradiction that $|\mathcal{A}'| < |\mathcal{J}|$. So $\mathcal{A}'$ is non-centred, and we may therefore assume that $\mathcal{A} = \mathcal{A}'$. Since $|\mathcal{A}| = |\mathcal{J}|$, we have $|\mathcal{A}_i| = |\mathcal{J}_i|$ for all $i \in [2, p]$ (see the proof of Theorem 3.1.4). An argument similar to that for $\mathcal{A}_{(1)}$ (in the proof of Theorem 3.1.4) yields $|\mathcal{A}_{(i)}| \leq |\mathcal{J}_{(i)}|$ for all $i \in [p]$.

Suppose $2r = \mu < \alpha$. So there exists $j \in [p]$ such that $|X_j| = \alpha > 2r$. The proof of Theorem 3.1.4(ii) for $2r < \mu$ shows us that we must then have $\mathcal{A}_{(j)} = \mathcal{J}_{(j)}$. By Lemma 3.3.2, $\mathcal{A} \subseteq \mathcal{J}$, which contradicts $\mathcal{A}$ non-centred. So $2r = \mu = \alpha$.

Next, suppose $r < |W|$. Let $i \in [2, p]$. Since $2r = \mu = \alpha = |W| + |V_i|$, it follows that $|V_i| < r$. Since $|\mathcal{A}_i| = |\mathcal{J}_i|$, it follows by Corollary 3.1.2 (with $2r = n = |X_i|$ and $|Z| = |V_i| < r$) that $\mathcal{A}_{(i)} = \mathcal{J}_{(i)}$. By Lemma 3.3.2, $\mathcal{A} \subseteq \mathcal{J}$, which contradicts $\mathcal{A}$ non-centred. So $r \geq |W|$.

Finally, suppose $1 \leq |W| \leq 2$. If $A \cap W = \emptyset$ for some $A \in \mathcal{A}$ then, since $\mathcal{A}$ is intersecting, $\mathcal{A} \subset \binom{X_j}{r}$ for some $j \in [p]$, and hence $|\mathcal{A}| = |\mathcal{A}_{(j)}| \leq |\mathcal{J}_{(j)}| < |\mathcal{J}|$. Suppose instead $\mathcal{A} = \mathcal{A}(W)$. If $W = \{w_1\}$ then $\mathcal{A} \subseteq \mathcal{J}$, which contradicts $\mathcal{A}$ non-centred. So $W = \{w_1, w_2\}$, and hence $\mathcal{A}(w_1)]w_2[\cup \mathcal{A}]w_1[(w_2) \cup \mathcal{A}(w_1)(w_2)$ is a partition of $\mathcal{A}$. Since $\mathcal{A}$ is non-centred, we have $\mathcal{A}(w_1)]w_2[\ \neq \emptyset$ and $\mathcal{A}]w_1[(w_2) \neq \emptyset$. Thus, since $\mathcal{A}$ is intersecting, $\mathcal{A}(w_1)]w_2[\cup \mathcal{A}]w_1[(w_2) \subset \binom{X_j}{r}$ for some $j \in [p]$. So $\mathcal{A}(w_1)]w_2[\cup \mathcal{A}]w_1[(w_2) \cup \mathcal{A}(w_1)(w_2)(V_j) = \mathcal{A}_{(j)}$, and we know that $|\mathcal{A}_{(j)}| \leq |\mathcal{J}_{(j)}|$. It remains to consider $\mathcal{A}_i = \mathcal{A}(w_1)(w_2)(V_i)$ for each $i \in [p]\backslash\{j\}$, for which we clearly have $|\mathcal{A}_i| < |\mathcal{J}_i|$. Thus, since $\mathcal{A}_{(j)} \cup \bigcup_{i \in [p]\backslash j} \mathcal{A}_i$ is a partition for $\mathcal{A}$, we get $|\mathcal{A}| < |\mathcal{J}|$, a contradiction. So $|W| \geq 3$. Hence result. □

We now come to the proof of Lemma 3.3.1, for which we need the lemma below that is often useful for determining the structure of extremal intersecting families.

Lemma 3.3.3 *Suppose $\emptyset \neq \mathcal{A} \subseteq \binom{X}{r}$, $2r < n := |X|$, such that if $A \in \mathcal{A}$ and $B \in \binom{X \backslash A}{r}$ then $B \in \mathcal{A}$. Then $\mathcal{A} = \binom{X}{r}$.*

Proof. Let $A_0 \in \mathcal{A}$ and $B \in \binom{X}{r}$ such that $1 \leq q_0 := |A_0 \cap B| \leq r - 1$, i.e. $B \neq A_0$ and $B \notin \binom{X \backslash A}{r}$. It is required to show that $B \in \mathcal{A}$. We claim that the following procedure takes a finite number of steps k, and we first assume the claim is true. For $i = 1, 2, ..., k$, choose $A_i \in \binom{X \backslash A_{i-1}}{r}$ such that $|A_i \cap B|$ is a minimum if i is odd, and $|A_i \cap B|$ is a maximum if i is even, where k is the first even integer that gives $A_k = B$. So $A_i \in \mathcal{A}$ for all $i \in [k]$, and hence we are done.

We now prove the claim. Let $q_i := |A_i \cap B|$ if i is even, and $q_i := r - |A_i \cap B|$ if i is odd. If i is even then $q_i = r - |A_{i-1} \cap B| = q_{i-1}$. If i is odd then $q_i = r - \max\{0, r -$

$((n - |A_{i-1} \cup B|))\} = \min\{r, n - (2r - |A_{i-1} \cap B|)\} = \min\{r, (n-2r) + q_{i-1}\} > q_{i-1}$. So the claim holds. □

Proof of Lemma 3.3.1. Clearly, if $x \in V_i$ then $\mathcal{A} = \mathcal{A}_{(i)}$.

Now consider $x \in W$. Suppose $y \notin X_i$, $i \in [p]$. Then, since $\Delta_{x,y}(\mathcal{A}) = \mathcal{J}$, we clearly have $\mathcal{A}_{(i)} = \mathcal{J}_{(i)}$, and hence, by Lemma 3.3.2 and $|\mathcal{A}| = |\Delta_{x,y}(\mathcal{A})| = |\mathcal{J}|$, $\mathcal{A} = \mathcal{J}$. Therefore, we now assume that $y \in X_i$ for all $i \in [p]$. So $p = 1$ or $y \in W$. We also assume that $\mathcal{A} \neq \mathcal{J}$. Let $\mathcal{K} := \mathcal{U}(y)$. Our main observation is that

$$A_1, A_2 \in \mathcal{K} \backslash \mathcal{J},\ A_1 \in \mathcal{A},\ A_1 \cap A_2 = \{y\} \ \Rightarrow\ A_2 \in \mathcal{A} \tag{3.4}$$

because otherwise, since $\Delta_{x,y}(\mathcal{A}) = \mathcal{J}$, we get $\delta_{x,y}(A_2) \in \mathcal{A}$ and $A_1 \cap \delta_{x,y}(A_2) = \emptyset$, contradicting $\mathcal{A}$ intersecting.

Suppose $2r < \alpha$. Let $j \in [p]$ such that $|X_j| = \alpha$. If $A \in \mathcal{J} \cap \mathcal{K}$ then $A = \delta_{x,y}(A)$. So $\mathcal{J} \cap \mathcal{K} \subset \mathcal{A}$. Since $\mathcal{A} \neq \mathcal{J}$ and $\Delta_{x,y}(\mathcal{A}) = \mathcal{J}$, there exists $B \in \mathcal{A}$ such that $\delta_{xy}(B) \neq B$. So $B \in \mathcal{K} \backslash \mathcal{J}$. Let $Y := X_j \backslash \{x\}$ and $\mathcal{Y} := \{A \in \binom{Y}{r} \colon y \in A\}$. Let $Z := Y \backslash \{y\}$ and $\mathcal{B} := \{A \backslash \{y\} \colon A \in \mathcal{A} \cap \mathcal{Y}\} \subseteq \binom{Z}{r-1}$. Since $B \backslash \{y\} \in \mathcal{B}$ and $\mathcal{A} \cap \mathcal{Y} \subseteq \mathcal{K} \backslash \mathcal{J}$, it follows by (3.4) and Lemma 3.3.3 that $\mathcal{B} = \binom{Z}{r-1}$. So $\mathcal{Y} \subset \mathcal{A}$, and hence $\mathcal{K}_{(j)} \subseteq \mathcal{A}_{(j)}$ as $\mathcal{J} \cap \mathcal{K} \subset \mathcal{A}$. By Lemma 3.3.2, $\mathcal{A} \subseteq \mathcal{K}$. Since $|\mathcal{K}| \leq |\mathcal{J}| = |\mathcal{A}|$, $\mathcal{A} = \mathcal{K}$.

Finally, suppose $2r = \mu = \alpha$. So $p > 1$ and $y \in W$. In this case, $2r = |X_i|$ and $b := b_1 = \ldots = b_p = 2r - a$. Let $C \in \mathcal{K}$. We show that $C \in \mathcal{A}$. As above, $\mathcal{J} \cap \mathcal{K} \subset \mathcal{A}$. So suppose $C \in \mathcal{K} \backslash \mathcal{J}$. We have $C \subset X_j$ for some $j \in [p]$. Since $\Delta_{xy}(\mathcal{A}) = \mathcal{J} \neq \mathcal{A}$, there exists $B_0 \in \mathcal{K} \backslash \mathcal{J}$ such that $B_0 \in \mathcal{A}$. Let $Y_i = X_i \backslash \{x\}$, $i = 1, \ldots, p$. We can assume that $B_0 \subset Y_j$ because otherwise we can choose $B_0' \subset Y_j$ such that $B_0' \cap B_0 = \{y\}$, and $B_0' \in \mathcal{A}$ by (3.4). Let $j' \in [p] \backslash \{j\}$. Take $B_1 \in \binom{Y_{j'}}{r}$ such that $B_0 \cap B_1 = \{y\}$ and $|B_1 \cap V_{j'}| = \min\{r-1, b\}$. By (3.4), $B_1 \in \mathcal{A}$. If $b \geq r-1$ then $B_1 \cap C = \{y\}$, and hence $C \in \mathcal{A}$ by (3.4). Suppose $b \leq r-2$. Let $U := W \backslash \{x, y\}$ and $\mathcal{D} := \{(A \cap W) \backslash \{y\} \colon A \in \mathcal{A} \cap \mathcal{K} \backslash \mathcal{J}, |A \cap (V_j \cup V_{j'})| = b\} \subseteq \binom{U}{s}$, where $s = r - b - 1 = a - r - 1 \leq (a-1)/2 - 1 = (|U|+1)/2 - 1 < |U|/2$. $\mathcal{D} \neq \emptyset$ as $(B_1 \cap W) \backslash \{y\} \in \mathcal{D}$. Moreover, $D \in \mathcal{D} \Rightarrow D \cup V_{j^*} \cup \{y\} \in \mathcal{A}$ for some $j^* \in \{j, j'\} \Rightarrow D' \cup V_{j^\sim} \cup \{y\} \in \mathcal{A}$

for any $D' \in \binom{U \setminus D}{s}, j^{\sim} \in \{j, j'\} \setminus \{j^*\}$ (by (3.4)) $\Rightarrow D' \in \mathcal{D}$ for any $D' \in \binom{U \setminus D}{s}$. Thus, by Lemma 3.3.3, $\mathcal{D} = \binom{U}{s}$. Since $|U| - |C \cap U| \geq (a-2) - (r-1) = s$, there exists $D \in \mathcal{D}$ such that $D \cap C = \emptyset$. Let $B_2 := D \cup V_{j'} \cup \{y\}$, $B_3 := D \cup V_j \cup \{y\}$. So $B_2 \in \mathcal{A}$ or $B_3 \in \mathcal{A}$. Since $B_2 \cap C = \{y\}$, if $B_2 \in \mathcal{A}$ then $C \in \mathcal{A}$ by (3.4). Suppose $B_3 \in \mathcal{A}$. We can take $D' \in \binom{U \setminus D}{s}$ such that $D' \subset C$. Let $B_4 := D' \cup V_{j'} \cup \{y\}$, $B_5 := (Y_j \setminus C) \cup \{y\}$. Clearly, $B_3 \cap B_4 = B_4 \cap B_5 = B_5 \cap C = \{y\}$. So $B_4, B_5, C \in \mathcal{A}$ by (3.4). □

Chapter 4

Non-centred intersecting sub-families of compressed hereditary families

4.1 Introduction

For any pair of families $\mathcal{A}$ and $\mathcal{B}$, let

$$\partial_{\mathcal{B}}^{(s)}\mathcal{A} := \{B \in \mathcal{B}^{(s)}\colon \text{ there exists } A \in \mathcal{A} \text{ such that } A \subseteq B \text{ or } B \subseteq A\}.$$

The following is a well-known result due to Sperner [57]:

$$|\partial_{2^{[n]}}^{(r+1)}\mathcal{A}| > |\mathcal{A}| \text{ for any } \mathcal{A} \subseteq \binom{[n]}{r} \text{ and } r < \lfloor n/2 \rfloor. \tag{4.1}$$

A proof of this inequality is also found in [25, 38].

A family $\mathcal{A}$ is said to be an *antichain* or a *Sperner family* if all sets in $\mathcal{A}$ are maximal in $\mathcal{A}$, i.e. $B \subsetneq A \in \mathcal{A}$ implies $B \notin \mathcal{A}$.

Let $\sim$ be any of the relations $<, \leq, \geq, >$ for numbers. We denote the sub-family $\{Y \subseteq X\colon |Y| \sim r\}$ of 2^X by $\binom{X}{\sim r}$. For a family $\mathcal{F}$, we denote the sub-family $\{A \in \mathcal{F}\colon |A| \sim r\}$ of $\mathcal{F}$ by $\mathcal{F}^{(\sim r)}$.

Erdős, Ko and Rado actually proved the following stronger version of Theorem 1.2.1.

Theorem 4.1.1 (Erdős, Ko, Rado [25]) *If $r \leq n/2$ and $\mathcal{A} \subset \binom{[n]}{\leq r}$ be an intersecting antichain, then $|\mathcal{A}| \leq |\{A \in \binom{[n]}{r}\colon 1 \in A\}| = \binom{n-1}{r-1}$, and strict inequality holds if*

$\mathcal{A} \cap \binom{[n]}{<r} \neq \emptyset$.

Similarly, Hilton and Milner actually proved the following stronger version of Theorem 1.2.2.

Theorem 4.1.2 (Hilton and Milner [38]) *If $r \leq n/2$ and $\mathcal{A} \subset \binom{[n]}{\leq r}$ is a non-centred intersecting antichain, then $|\mathcal{A}| \leq |\{A \in \binom{[n]}{r} : 1 \in A, A \cap [2, r+1] \neq \emptyset\} \cup \{[2, r+1]\}|$, and strict inequality holds if $\mathcal{A} \cap \binom{[n]}{<r} \neq \emptyset$.*

These two versions were easily obtained from the respective weaker versions by applying (4.1).

In this chapter, we obtain generalisations of Theorems 1.6.2 and 4.1.2 to sub-families of compressed hereditary families using the compression method, exploiting the fact that if $\mathcal{F} \subseteq 2^{[n]}$ is a compressed family and $\mathcal{A} \subseteq \mathcal{F}$ then $\Delta_{i,j}(\mathcal{A}) \subseteq \mathcal{F}$ for any $i, j \in [n]$, $i < j$. We also determine extremal structures. More precisely, we prove the following two results, which are stated with some light notation from Section 2.1.

Theorem 4.1.3 *Let $\mathcal{H} \subseteq 2^{[n]}$ be hereditary and compressed, $\mathcal{H}(n) \neq \emptyset$. Let $1 \leq r \leq s \leq \mu(\mathcal{H}) - r$, and let $\emptyset \neq \mathcal{A} \subset \mathcal{H}^{(r)}$ and $\emptyset \neq \mathcal{B} \subset \mathcal{H}^{(s)}$ such that $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting, and $|\mathcal{A}| \leq |\mathcal{B}|$ if $r = s$. Then:*
(i) $|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{A}_0| + |\mathcal{B}_0|$ where $\mathcal{A}_0 := \{[r]\} \subset \mathcal{H}^{(r)}$ and $\mathcal{B}_0 := \mathcal{H}^{(s)}([r])$;
(ii) if $s < n - r$ then equality in (i) holds only if either $\mathcal{A} = \{A\}$ and $\mathcal{B} = \mathcal{H}^{(s)}(A)$ for some $A \in \mathcal{H}^{(r)}$ or $r = s = 2$ and $\mathcal{A} = \mathcal{B} = \mathcal{H}^{(2)}(a)$ for some $a \in [n]$.

Theorem 4.1.4 *Let $\mathcal{H} \subseteq 2^{[n]}$ be hereditary and compressed, $\mathcal{H}(n) \neq \emptyset$. Suppose $2 \leq r \leq \mu(\mathcal{H})/2$ and $\mathcal{A} \subseteq \mathcal{H}^{(\leq r)}$ is a non-centred intersecting antichain. Then:*
(i) $|\mathcal{A}| \leq |\mathcal{N}|$ where $\mathcal{N}$ is a non-centred intersecting sub-family of $\mathcal{H}^{(r)}$ given by $\{A \in \mathcal{H}^{(r)} : 1 \in A, A \cap [2, r+1] \neq \emptyset\} \cup \{[2, r+1]\}$;
(ii) if $r < n/2$ then equality in (i) holds only if either $\mathcal{A} = \{A \in \mathcal{H}^{(r)} : a \in A, A \cap B \neq \emptyset\} \cup \{B\}$ for some $B \in \mathcal{H}^{(r)}$ and $a \in [n] \backslash B$ or $\mathcal{A} = \{A \in \mathcal{H}^{(3)} : |A \cap C| \geq 2\}$ for some $C \in \mathcal{H}^{(3)}$.

Theorem 4.1.4 is proved using Theorem 4.1.3 with $r = s$. The two theorems make distinct use of a generalisation of (4.1) that is given in Section 4.3.

We alert the reader to the fact that, in the subsequent sections of this chapter, heavy use is made of the notation in Section 2.1, especially in the proofs of the main results.

4.2 A consequence of Theorem 4.1.4 and a counterexample

An immediate consequence of Theorem 4.1.4 is the following analogue of Theorem 1.3.2.

Theorem 4.2.1 *Conjecture 1.3.5 is true if $\mathcal{H}$ is compressed.*

Proof. Since $\mathcal{H}$ is compressed, $[\mu(\mathcal{H})] \in \mathcal{H}$. Therefore,

$$\mathcal{H} \text{ compressed and hereditary} \Rightarrow 2^{[\mu(\mathcal{H})]} \subseteq \mathcal{H}. \tag{4.2}$$

Let $\mathcal{A} \subset \mathcal{H}^{(r)}$ be non-centred and intersecting, and let $\mathcal{N}$ be as in Theorem 4.1.4. By Theorem 4.1.4, $|\mathcal{A}| \leq |\mathcal{N}|$. Let $\mathcal{B} := \{B \in \binom{[\mu(\mathcal{H})]\backslash[2,r+1]}{r} : 1 \in B\}$. Clearly, $\mathcal{B} \cap \mathcal{N} = \emptyset$. By (4.2), $\mathcal{B} \subset \mathcal{H}^{(r)}(1)$. So

$$|\mathcal{H}^{(r)}(1)| \geq |(\mathcal{N}\backslash\{[2,r+1]\}) \cup \mathcal{B}| = |\mathcal{N}| - 1 + \binom{\mu(\mathcal{H}) - r - 1}{r-1}.$$

Thus, since $r \leq \mu(\mathcal{H})/2$, $|\mathcal{A}| \leq |\mathcal{H}^{(r)}(1)|$ with strict inequality if $r < \mu(\mathcal{H})/2$. □

Of course, this result can be proved directly; the 'non-strict' part can be obtained by employing Lemmas 4.4.1 and 4.4.2 in an inductive argument (based on the compression technique) similar to that in the original proof of the EKR Theorem. An improvement of Theorem 4.2.1 similar to that given by Theorem 1.3.3 over Theorem 1.3.2 may be regarded as a step worth attempting next towards Conjecture 1.3.5.

A counterexample. In view of the above, it is natural to ask the following question: If $\mathcal{H}$ is taken to be any hereditary sub-family of $2^{[n]}$ or to be at least compressed with respect to 1, does Theorem 4.1.4 still hold in the sense that for any $r \leq \mu(\mathcal{H})/2$

there exist $B \in \mathcal{H}^{(r)}$ and $a \in [n]\backslash B$ such that $\mathcal{H}^{(r)}(a)(B) \cup \{B\}$ is an extremal non-centred intersecting sub-family of $\mathcal{H}^{(r)}$? We now use a sunflower setting to provide a counterexample; note that this contrasts Conjecture 1.3.5.

Let $m, l, p, r \in \mathbb{N}$ such that $3 \leq l \leq m$, $l \leq r \leq (m+l)/2$ and $p\binom{\mu-l}{r-l+1} > \binom{\mu-l}{r-1} - \binom{\mu-r-1}{r-1} + 1$. Let $H_i := [l] \cup [(i-1)m+l+1, im+l]$, $i = 1, ..., p$. So $H_{i_1} \cap H_{i_2} = [l]$ for any $i_1, i_2 \in [p]$ (and hence $\{H_i : i \in [p]\}$ is a sunflower). Let $\mathcal{H}$ be the hereditary family $\bigcup_{i=1}^{p} 2^{H_i}$. Note that $\mathcal{H}$ is compressed with respect to 1. We have $\mu := \mu(\mathcal{H}) = |H_1| = ... = |H_p| = m + l \geq 2r$. Fix $B \in \mathcal{H}^{(r)}$ and $a \in [pm+l]\backslash B$, and let $\mathcal{A}_1 := \mathcal{H}^{(r)}(a)(B) \cup \{B\}$. Let $\mathcal{N}$ be as in Theorem 4.1.4.

We first show that $|\mathcal{A}_1| \leq |\mathcal{N}|$. This is straightforward if $a \notin [l]$ because then $\mathcal{H}^{(r)}(a) \subset \binom{H_{i'}}{r}$ for some $i' \in [p]$. Suppose $a \in [l]$ instead, and let $j \in [p]$ such that $B \subset H_j$. Then

$$\begin{aligned} |\mathcal{A}_1| = & \binom{|H_j|-1}{r-1} - \binom{|H_j|-r-1}{r-1} + 1 + \textstyle\sum_{i\in[p]\backslash\{j\}} \left(\binom{|H_i|-1}{r-1} - \binom{|H_i|-|B\cap([l]\backslash\{a\})|-1}{r-1}\right) \\ \leq & \; p\binom{\mu-1}{r-1} - \binom{\mu-r-1}{r-1} + 1 - (p-1)\binom{\mu-l}{r-1} = |\mathcal{N}|. \end{aligned}$$

Now let $\mathcal{A}_2 := \mathcal{H}^{(r)}(1)([l]\backslash\{1\}) \cup \{([l]\backslash\{1\}) \cup C : C \in \binom{H_i\backslash[l]}{r-l+1} \text{ for some } i \in [p]\} \subset \mathcal{H}^{(r)}$. So $|\mathcal{A}_2| = p\left(\binom{\mu-1}{r-1} - \binom{\mu-l}{r-1} + \binom{\mu-l}{r-l+1}\right)$. Our aim is to show that $|\mathcal{A}_2| > |\mathcal{A}_1|$. Indeed, $|\mathcal{A}_2| - |\mathcal{A}_1| \geq |\mathcal{A}_2| - |\mathcal{N}| = p\binom{\mu-l}{r-l+1} - \binom{\mu-l}{r-1} + \binom{\mu-r-1}{r-1} - 1 > 0$ (by choice of p).

4.3 A Sperner-type lemma for hereditary families and some corollaries

The first important tool that we forge is the generalisation of (4.1) given by Lemma 4.3.1 below. This lemma is a discovery of a very fundamental property of hereditary families. We prove it using the double-counting method.

Lemma 4.3.1 *If $\mathcal{H}$ is hereditary, $r < s \leq \mu(\mathcal{H}) - r$ and $\mathcal{A} \subseteq \mathcal{H}^{(r)}$ then*

$$|\partial_{\mathcal{H}}^{(s)} \mathcal{A}| \geq \frac{\binom{\mu(\mathcal{H})-r}{s-r}}{\binom{s}{s-r}} |\mathcal{A}|.$$

Proof. For $A \in \mathcal{A}$, let M_A be some maximal set of $\mathcal{H}$ such that $A \subset M_A$. Then

$$\begin{aligned}\binom{\mu(\mathcal{H})-r}{s-r}|\mathcal{A}| &\le \sum_{A\in\mathcal{A}}\binom{|M_A|-r}{s-r} = \sum_{A\in\mathcal{A}}|(\partial_{\mathcal{H}}^{(s)}\{A\})\cap\binom{M_A}{s}| \\ &\le \sum_{A\in\mathcal{A}}|\partial_{\mathcal{H}}^{(s)}\{A\}| = \sum_{B\in\partial_{\mathcal{H}}^{(s)}\mathcal{A}}|\partial_{\mathcal{A}}^{(r)}\{B\}| \le \sum_{B\in\partial_{\mathcal{H}}^{(s)}\mathcal{A}}\binom{s}{r} \\ &= \binom{s}{s-r}|\partial_{\mathcal{H}}^{(s)}\mathcal{A}|.\end{aligned}$$

Hence result. □

Corollary 4.3.2 *If $\mathcal{H}$ is hereditary and $r < s \le \mu(\mathcal{H}) - r$ then $|\mathcal{H}^{(r)}| \le |\mathcal{H}^{(s)}|$, and strict inequality holds if $s < \mu(\mathcal{H}) - r$.*

Proof. Since $s \le \mu(\mathcal{H}) - r$, we have $\binom{s}{s-r} \le \binom{\mu(\mathcal{H})-r}{s-r}$ with strict inequality if $s < \mu(\mathcal{H}) - r$. This result now follows immediately from Lemma 4.3.1 as $\partial_{\mathcal{H}}^{(s)}\mathcal{H}^{(r)} \subseteq \mathcal{H}^{(s)}$. □

Corollary 4.3.3 *Let $\mathcal{H}$ be hereditary. Let $\mathcal{A} \subset \mathcal{H}^{(\le r)}$ be an antichain such that $\mathcal{A} \cap \mathcal{H}^{(<r)} \ne \emptyset$, where $r \le \mu(\mathcal{H})/2$. Then $|\partial_{\mathcal{H}}^{(r)}\mathcal{A}| > |\mathcal{A}|$.*

Proof. Set $m := \min\{|A| : A \in \mathcal{A}\}$. So $\bigcup_{s=m}^{r} \mathcal{A}^{(s)}$ is a partition for $\mathcal{A}$. Since $\mathcal{A} \cap \mathcal{H}^{(<r)} \ne \emptyset$, $m < r$. Take ${}_1\mathcal{A} := (\mathcal{A}\backslash\mathcal{A}^{(m)}) \cup \partial_{\mathcal{H}}^{(m+1)}\mathcal{A}^{(m)}$. Since $\mathcal{A}$ is an antichain, we have $(\partial_{\mathcal{H}}^{(m+1)}\mathcal{A}^{(m)}) \cap \mathcal{A} = \emptyset$, and hence $|{}_1\mathcal{A}| > |\mathcal{A}|$ since $|\partial_{\mathcal{H}}^{(m+1)}\mathcal{A}^{(m)}| > |\mathcal{A}^{(m)}|$ by Lemma 4.3.1. Also note that ${}_1\mathcal{A}$ is an antichain. Repeating the same procedure $r - m$ times, we obtain a family ${}_q\mathcal{A} \in \mathcal{H}^{(r)}$, $q = r - m + 1$, such that $|{}_q\mathcal{A}| > |\mathcal{A}|$. Clearly, ${}_q\mathcal{A} = \partial_{\mathcal{H}}^{(r)}\mathcal{A}$. □

Corollary 4.3.4 *Let $\mathcal{H}$ be hereditary. If $r \le \mu(\mathcal{H})/2$ and $\mathcal{A}$ is a largest intersecting antichain sub-family of $\mathcal{H}^{(\le r)}$ then $\mathcal{A} \subset \mathcal{H}^{(r)}$.*

Proof. Suppose $\mathcal{A} \cap \mathcal{H}^{(<r)} \ne \emptyset$. Trivially, $\partial_{\mathcal{H}}^{(r)}\mathcal{A}$ is an intersecting antichain sub-family of $\mathcal{H}^{(r)}$. By Corollary 4.3.3, $|\partial_{\mathcal{H}}^{(r)}\mathcal{A}| > |\mathcal{A}|$, a contradiction. □

By the above corollary, if Conjecture 1.3.5 is true then for any intersecting antichain $\mathcal{A} \subset \mathcal{H}^{(\le r)}$ such that $\mathcal{H}$ is hereditary and $r \le \mu(\mathcal{H})/2$, $|\mathcal{A}| \le |\mathcal{H}^{(r)}(h)|$ for some $h \in U(\mathcal{H})$.

Corollary 4.3.5 *Let $\mathcal{H}$ be hereditary. If $r \leq \mu(\mathcal{H})/2$ and $\mathcal{A}$ is a largest non-centred intersecting antichain sub-family of $\mathcal{H}^{(\leq r)}$ then $\mathcal{A} \subset \mathcal{H}^{(r)}$.*

Proof. Suppose $\mathcal{A} \cap \mathcal{H}^{(<r)} \neq \emptyset$. Since $\mathcal{A}^* := \partial_{\mathcal{H}}^{(r)} \mathcal{A}$ is an intersecting antichain sub-family of $\mathcal{H}^{(r)}$ and $|\mathcal{A}^*| > |\mathcal{A}|$ by Corollary 4.3.3, $\mathcal{A}^*$ must be centred. Let $a \in \bigcap_{A \in \mathcal{A}^*} A$. Since $\mathcal{A}$ is non-centred, $a \notin A'$ for some $A' \in \mathcal{A}$. Suppose $|A'| = r$. Then $A' \in \mathcal{A}^*$, but this contradicts $\mathcal{A}^* = \mathcal{A}^*(a)$. So $|A'| < r$. Let M be some maximal set in $\mathcal{H}$ such that $A' \subset M$. Since $|A'| < r \leq \mu(\mathcal{H})/2 \leq |M|/2 \leq |M \backslash \{a\}|$ and $\mathcal{H}$ is hereditary, there exists $A'' \in \mathcal{H}$ such that $A' \subset A'' \subseteq M \backslash \{a\}$ and $|A''| = r$. So $a \notin A'' \in \mathcal{A}^*$, contradicting $\mathcal{A}^* = \mathcal{A}^*(a)$. Therefore $\mathcal{A} \cap \mathcal{H}^{(<r)} = \emptyset$, and hence result. □

We point out that the following corollary of Lemma 4.3.1 is much stronger than Corollary 4.3.2; however, unlike all the preceding corollaries, we will not need to refer to it.

Corollary 4.3.6 *If $\mathcal{H}$ is hereditary and $r < s \leq \mu(\mathcal{H}) - r$ then there exists an injection $f \colon \mathcal{H}^{(r)} \to \mathcal{H}^{(s)}$ such that $A \subset f(A)$ for all $A \in \mathcal{H}^{(r)}$. If $s < \mu(\mathcal{H}) - r$ then f is not a bijection.*

Proof. The result follows immediately from Lemma 4.3.1 and Hall's Marriage Theorem [36]. □

4.4 Further tools for proofs

The following is the key lemma for the main results in this chapter.

Lemma 4.4.1 *Let $\emptyset \neq \mathcal{F} \subseteq 2^{[n]}$ and $a \in [n]$.*
(i) If $\mathcal{F}(a) \neq \emptyset$ then $\mu(\mathcal{F}\langle a \rangle) \geq \mu(\mathcal{F}) - 1$.
(ii) If $\mathcal{F}$ is hereditary then $\mu(\mathcal{F}]a[) \geq \mu(\mathcal{F}) - 1$.
(iii) If $\mathcal{F}$ is compressed and $[n] \notin \mathcal{F}$ then $\mu(\mathcal{F}]n[) \geq \mu(\mathcal{F})$.

Proof. Suppose $\mathcal{F}(a) \neq \emptyset$. Let $M \in \mathcal{F}\langle a \rangle$ be maximal in $\mathcal{F}\langle a \rangle$. Then $M' := M \cup \{a\}$ is maximal in $\mathcal{F}$. So $|M| = |M'| - 1 \geq \mu(\mathcal{F}) - 1$. Hence (i).

Suppose $\mathcal{F}$ is hereditary. Then, since $\mathcal{F} \neq \emptyset$, $\emptyset \in \mathcal{F}$. So $\mathcal{F}]a[\neq \emptyset$. Suppose $M \in \mathcal{F}]a[$ is maximal in $\mathcal{F}]a[$. Suppose also that $|M| < \mu(\mathcal{F})$. So M is not maximal in $\mathcal{F}$, and hence there exists $M' \in \mathcal{F}(a)$ such that $M \subset M'$ and M' is maximal in $\mathcal{F}$. Since $\mathcal{F}$ is hereditary, $M'' := M' \backslash \{a\} \in \mathcal{F}$. Since M is maximal in $\mathcal{F}]a[$ and $M \subseteq M'' \in \mathcal{F}]a[$, $M = M''$. So $M' = M \cup \{a\}$. Therefore $|M| = |M'| - 1 \geq \mu(\mathcal{F}) - 1$. Hence (ii).

Suppose $\mathcal{F}$ is compressed and $[n] \notin \mathcal{F}$. Let $M \in \mathcal{F}]n[$ be maximal in $\mathcal{F}]n[$. Suppose $|M| < \mu(\mathcal{F})$. Then there exists $M' \in \mathcal{F}(n)$ such that $M \subset M'$. Since $[n] \notin \mathcal{F}$, $X := [n] \backslash M' \neq \emptyset$. Let $x \in X$ and $M'' := \delta_{x,n}(M') = (M' \backslash \{n\}) \cup \{x\}$. Since $\mathcal{F}$ is compressed, $M'' \in \mathcal{F}$. But $M \subsetneq M'' \in \mathcal{F}]n[$, which is a contradiction to the maximality of M in $\mathcal{F}]n[$. So $|M| \geq \mu(\mathcal{F})$. Hence (iii). □

We remark that the inequalities above cannot be replaced by equalities. An example for (iii) is that if $n \geq 3$ and $\mathcal{F}$ is the compressed (hereditary) family $2^{[n-1]} \cup 2^{[n-3] \cup \{n\}}$ then $\mu(\mathcal{F}]n[) = n - 1 > n - 2 = \mu(\mathcal{F})$.

We shall say that a family $\mathcal{F} \subseteq 2^{[n]}$ is *quasi-compressed* if $\delta_{i,j}(F) \in \mathcal{F}$ for any $i, j \in U(\mathcal{F})$ such that $i < j$. Therefore a quasi-compressed family $\mathcal{F} \subseteq 2^{[n]}$ is isomorphic to a compressed sub-family of $2^{[|U(\mathcal{F})|]}$, and the isomorphism is induced by the bijection $\beta \colon U(\mathcal{F}) \to [|U(\mathcal{F})|]$ defined by $\beta(u_i) := i$, $i = 1, ..., |U(\mathcal{F})|$, where $\{u_1, ..., u_{|U(\mathcal{F})|}\} = U(\mathcal{F})$ and $u_1 < ... < u_{|U(\mathcal{F})|}$.

The next lemma is straightforward, so we omit its proof.

Lemma 4.4.2 *If $\mathcal{H} \subseteq 2^{[n]}$ is hereditary/quasi-compressed and $a \in [n]$ then $\mathcal{H}]a[$ and $\mathcal{H}\langle a \rangle$ are hereditary/quasi-compressed.*

We shall frequently use the following generalisation of Lemma 3.2.2.

Lemma 4.4.3 *Let $\mathcal{F} \subseteq 2^{[n]}$ be a quasi-compressed family such that $|U(\mathcal{F})| \geq 2$. Let $Z \subseteq [n]$ and $\{a, b\} \subset U(\mathcal{F})$, $a < b$. Then:*

(i) $|\mathcal{F}[Z]| \leq |\mathcal{F}[\delta_{a,b}(Z)]|$;

(ii) $|\mathcal{F}(Z)| \leq |\mathcal{F}(\delta_{a,b}(Z))|$.

Proof. Let $Z' := \delta_{a,b}(Z)$. Suppose $Z' \neq Z$. Setting $Z'' := Z \cap Z'$, we therefore have $Z = Z'' \cup \{b\} \neq Z''$ and $Z' = Z'' \cup \{a\} \neq Z''$. Since $\mathcal{F}$ is quasi-compressed and

$\{a,b\} \subset U(\mathcal{F})$, we have $\Delta_{a,b}(\mathcal{F}[Z]]a[) \subseteq \mathcal{F}[Z']]b[$ and $\Delta_{a,b}(\mathcal{F}]Z''[(b)]a[) \subseteq \mathcal{F}]Z''[(a)]b[$, and hence $|\mathcal{F}[Z']]b[|) \geq |\mathcal{F}[Z]]a[|$ and $|\mathcal{F}]Z''[(a)]b[| \geq |\mathcal{F}]Z''[(b)]a[|$. So

$$|\mathcal{F}[Z']| - |\mathcal{F}[Z]| = (|\mathcal{F}[Z'' \cup \{a,b\}]| + |\mathcal{F}[Z']]b[|)$$
$$-(|\mathcal{F}[Z'' \cup \{a,b\}]| + |\mathcal{F}[Z]]a[|) \geq 0,$$

which proves (i), and

$$|\mathcal{F}(Z')| - |\mathcal{F}(Z)| = (|\mathcal{F}(Z'')| + |\mathcal{F}]Z''[(a)|) - (|\mathcal{F}(Z'')| + |\mathcal{F}]Z''[(b)|)$$
$$= (|\mathcal{F}]Z''[(a)(b)| + |\mathcal{F}]Z''[(a)]b[|) - (|\mathcal{F}]Z''[(b)(a)| + |\mathcal{F}]Z''[(b)]a[|) \geq 0,$$

which proves (ii). □

For a set $X := \{x_1, ..., x_n\} \subset \mathbb{N}$, $x_1 < ... < x_n$, and $r \leq n$, call $\{x_1, ..., x_r\}$ an *initial r-segment of X*.

Corollary 4.4.4 *Let $\mathcal{F} \subseteq 2^{[n]}$ be quasi-compressed. Let $\emptyset \neq Z \subseteq [n]$, and let $Y \in \binom{[n]}{|Z|}$ such that if $Z \cap U(\mathcal{F}) \neq \emptyset$ then Y contains an initial $|Z \cap U(\mathcal{F})|$-segment Y' of $U(\mathcal{F})$. Then:*
(i) $|\mathcal{F}[Z]| \leq |\mathcal{F}[Y]|$;
(ii) $|\mathcal{F}(Z)| \leq |\mathcal{F}(Y)|$.

Proof. Let $Z' := Z \cap U(\mathcal{F})$. Clearly, $|\mathcal{F}(Z)| = |\mathcal{F}(Z')|$, and $\mathcal{F}[Z] = \emptyset$ if $Z \neq Z'$. So the result is trivial if $Z' = \emptyset$. Suppose $Z' \neq \emptyset$. Since $\mathcal{F}$ is quasi-compressed and $Z' \subseteq U(\mathcal{F})$, we can construct a composition of compressions $\delta_{a,b}$, $a < b$, $a, b \in U(\mathcal{F})$, that yields Y' when applied on Z'. By repeated application of Lemma 4.4.3, we therefore get $|\mathcal{F}[Z']| \leq |\mathcal{F}[Y']|$ and $|\mathcal{F}(Z')| \leq |\mathcal{F}(Y')|$. Hence result. □

The following is a well-known result that surfaced in the proof of the original EKR Theorem [25].

Lemma 4.4.5 (Erdős, Ko, Rado [25]) *If $\mathcal{A} \subset 2^{[n]}$ is intersecting and $p, q \in [n]$ then $\Delta_{p,q}(\mathcal{A})$ is intersecting.*

Proof. Since $2^{[n]}$ is (p, q)-compressed, the result follows by Proposition 2.2.1(ii).

4.5 Proof of Theorem 4.1.3

In order to prove Theorem 4.1.3, we need the lemma below. The first two parts of the lemma are the important ones, and parts (iii) and (iv) are only needed for obtaining Theorem 4.1.3(ii). Surprisingly, we need to do much more work than expected to prove part (iv), hence the length of the whole proof.

Lemma 4.5.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be as in Theorem 4.1.3. Let $1 \le i < j \le n$.*

(i) $\Delta_{i,j}(\mathcal{A})$ and $\Delta_{i,j}(\mathcal{B})$ are cross-intersecting.

(ii) If $\Delta_{m,n}(\mathcal{A}) = \mathcal{A}$ and $\Delta_{m,n}(\mathcal{B}) = \mathcal{B}$ for all $m \in [n-1]$ then $(A \cap B)\backslash\{n\} \neq \emptyset$ for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

(iii) If $\Delta_{i,j}(\mathcal{A}) = \{A\}$ and $\Delta_{i,j}(\mathcal{B}) = \mathcal{H}^{(s)}(A)$ for some $A \in \mathcal{H}^{(r)}$ then $\mathcal{A} = \{A'\}$ and $\mathcal{B} = \mathcal{H}^{(s)}(A')$ for some $A' \in \mathcal{H}^{(r)}$.

(iv) If $\Delta_{i,j}(\mathcal{A}) = \Delta_{i,j}(\mathcal{B}) = \mathcal{H}^{(2)}(a)$ for some $a \in [n]$ then $\mathcal{A} = \mathcal{B} = \mathcal{H}^{(2)}(a')$ for some $a' \in [n]$.

Proof. Let $\mathcal{A}' := \{A \cup \{n+1\} \colon A \in \mathcal{A}\}$, $\mathcal{A}'' := \{A^* \cup \{n+1\} \colon A^* \in \Delta_{i,j}(\mathcal{A})\}$, $\mathcal{B}' := \{B \cup \{n+2\} \colon B \in \mathcal{B}\}$, $\mathcal{B}'' := \{B^* \cup \{n+2\} \colon B^* \in \Delta_{i,j}(\mathcal{B})\}$. Clearly, $\mathcal{C} := \mathcal{A}' \cup \mathcal{B}'$ is intersecting, and hence $\Delta_{i,j}(\mathcal{C})$ is intersecting by Lemma 4.4.5. Since $\Delta_{i,j}(\mathcal{C}) = \mathcal{A}'' \cup \mathcal{B}''$, (i) clearly follows.

Suppose $\Delta_{m,n}(\mathcal{A}) = \mathcal{A}$ and $\Delta_{m,n}(\mathcal{B}) = \mathcal{B}$ for all $m \in [n-1]$. Suppose $A \cap B = \{n\}$ for some $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then, since $|(A \cup B)\backslash\{n\}| = r + s - 2 < n - 1$, $X := [n-1]\backslash(A \cup B) \neq \emptyset$. Let $x \in X$. Since $\Delta_{x,n}(\mathcal{A}) = \mathcal{A}$, $\delta_{x,n}(A) \in \mathcal{A}$. But $\delta_{x,n}(A) \cap B = \emptyset$, a contradiction. Hence (ii).

Suppose $\Delta_{i,j}(\mathcal{A}) = \{A\}$ and $\Delta_{i,j}(\mathcal{B}) = \mathcal{H}^{(s)}(A)$ for some $A \in \mathcal{H}^{(r)}$. Then $\mathcal{A} = \{A\}$ or $\mathcal{A} = \{\delta_{j,i}(A)\}$. If $\mathcal{A} = \{A\}$ then $\mathcal{B} = \mathcal{H}^{(s)}(A)$ since $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting and $\Delta_{i,j}(\mathcal{B}) = \mathcal{H}^{(s)}(A)$. Suppose $\mathcal{A} = \{\delta_{j,i}(A)\}$. Then $\mathcal{B} \subseteq \mathcal{C} := \mathcal{H}^{(s)}(\delta_{j,i}(A))$. By Lemma 4.4.3(ii), $|\mathcal{C}| \le |\mathcal{H}^{(s)}(A)|$. So $\mathcal{B} = \mathcal{C}$ since $|\mathcal{C}| \le |\mathcal{H}^{(s)}(A)| = |\Delta_{i,j}(\mathcal{B})| = |\mathcal{B}| \le |\mathcal{C}|$. Hence (iii).

We now prove (iv). Suppose $r = 2$ and $\Delta_{i,j}(\mathcal{A}) = \Delta_{i,j}(\mathcal{B}) = \mathcal{H}^{(2)}(a)$ for some $a \in [n]$. Without loss of generality, suppose $\mathcal{A} \neq \Delta_{i,j}(\mathcal{A})$. Let $A_1 \in \mathcal{A}\backslash\Delta_{i,j}(\mathcal{A})$. So $a = i$ and $A_1 = \delta_{j,i}(A_1') = \{j, a_1\}$ for some $A_1' = \{i, a_1\} \in \mathcal{H}^{(2)}\backslash\mathcal{A}$, $a_1 \in [n]\backslash\{i,j\}$. From $\mathcal{A}$ and $\mathcal{B}$ cross-intersecting and $\Delta_{i,j}(\mathcal{B}) = \mathcal{H}^{(2)}(i)$, we get $\mathcal{B} \subset \mathcal{H}^{(2)}(A_1) \cap \mathcal{H}^{(2)}(\{i,j\}) = \mathcal{H}^{(2)}(j) \cup \{A_1'\}$. Let us first assume that $A_1' \notin \mathcal{B}$, i.e. $\mathcal{B} \subseteq \mathcal{H}^{(2)}(j)$. Since $|\mathcal{B}| = |\Delta_{i,j}(\mathcal{B})| = |\mathcal{H}^{(2)}(i)|$ and Lemma 4.4.3 gives us $|\mathcal{H}^{(2)}(j)| \leq |\mathcal{H}^{(2)}(i)|$, we then have $\mathcal{B} = \mathcal{H}^{(2)}(j)$. So $A_1 \in \mathcal{B}\backslash\Delta_{i,j}(\mathcal{B})$, and hence, by a similar argument, $\mathcal{A} = \mathcal{H}^{(2)}(j)$. We now verify (iv) by showing that $A_1' \notin \mathcal{B}$ indeed.

We first show that $|\mathcal{H}^{(2)}(i)]j[| \geq 3$. Let M_i be a maximal set in $\mathcal{H}$ such that $i \in M_i$; moreover, if $\{i,j\} \in \mathcal{H}^{(2)}$ then take $M_i \supset \{i,j\}$. Let $\mathcal{M}_i := \binom{M_i}{2}$; so $\mathcal{M}_i \subseteq \mathcal{H}^{(2)}$ as $\mathcal{H}$ is hereditary. Since $2 = r \leq \mu(\mathcal{H})/2 \leq |M_i|/2$, $|M_i| \geq 4$. Thus, if $\{i,j\} \notin \mathcal{H}$ then $\mathcal{H}^{(2)}(i)]j[\geq 3$ is immediate, and if $\{i,j\} \in \mathcal{H}$ then $|\mathcal{H}^{(2)}(i)]j[| \geq 2$. Suppose $|\mathcal{H}^{(2)}(i)]j[| = 2$. So $\{i,j\} \subset M_i$, $|M_i| = 4 = \mu(\mathcal{H})$ and $\mathcal{H}^{(2)}(i) = \mathcal{M}_i(i)$. Suppose $M_i \neq [4]$. Let $x := \min\{l \colon l \in [4]\backslash M_i\}$ and $y := \max\{m \colon m \in M_i\}$. So $x < y$. Since $i < j \in M_i$, $i \neq y$. So $\{i,x\} \in \mathcal{H}^{(2)}(i)$ because $\delta_{x,y}(\{i,y\}) = \{i,x\}$, $\{i,y\} \subset M_i$ and $\mathcal{H}$ is compressed and hereditary. But $\{i,x\} \notin \mathcal{M}_i(i)$, which contradicts $\mathcal{H}^{(2)}(i) = \mathcal{M}_i(i)$. So $M_i = [4]$. Thus, since $2 = r < n/2$, we have $5 \leq n \notin M_i$. Given that $\mathcal{H}(n) \neq \emptyset$, we have $\mu(\mathcal{H}(n)) \geq \mu(\mathcal{H})$ by Lemma 4.4.1(i). So $\mu(\mathcal{H}(n)) \geq 4$. Thus, taking M_n to be defined similarly to M_i, we have $|M_n| \geq 4$. Since $\mathcal{H}$ is compressed, $M_n^* := [|M_n|-1] \cup \{n\} \in \mathcal{H}$. We have $i \leq 3$ as $i < j \in M_i = [4]$. So $\{i,n\} \subset M_n^*$, and hence $\{i,n\} \in \mathcal{H}^{(2)}(i)\backslash\mathcal{M}_i(i)$, another contradiction. So $|\mathcal{H}^{(2)}(i)]j[| \geq 3$ indeed.

Therefore $|\mathcal{H}^{(2)}(i)]j[]a_1[| \geq 2$. Let $H_1 := \{i, h_1\}$ and $H_2 := \{i, h_2\}$ be two distinct sets in $\mathcal{H}^{(2)}(i)]j[]a_1[$. We have $H_1, H_2 \in \Delta_{i,j}(\mathcal{A}) \cap \Delta_{i,j}(\mathcal{B})$ since $\Delta_{i,j}(\mathcal{A}) = \Delta_{i,j}(\mathcal{B}) = \mathcal{H}^{(2)}(i)$. So $H_1' := \delta_{j,i}(H_1) = \{j, h_1\} \in \mathcal{B}$ since $H_1 \cap A_1 = \emptyset$ and $\mathcal{A}$ $(\ni A_1)$ and $\mathcal{B}$ are cross-intersecting. Similarly, $H_2' := \delta_{j,i}(H_2) = \{j, h_2\} \in \mathcal{A}$ since $H_2 \cap H_1' = \emptyset$. Thus, since $A_1' \cap H_2' = \emptyset$, we obtain $A_1' \notin \mathcal{B}$ as desired. □

Proof of Theorem 4.1.3. Consider first $r = 1$. Let $Z := \{z \in [n] \colon \{z\} \in \mathcal{A}\}$; so $|Z| = |\mathcal{A}|$. Clearly, $\mathcal{B} \subseteq \mathcal{H}^{(s)}[Z]$. By Corollary 4.4.4(i), $|\mathcal{B}| \leq |\mathcal{H}^{(s)}[[|Z|]]|$. Thus, if $|Z| = 1$ then $|\mathcal{A}| + |\mathcal{B}| \leq 1 + \mathcal{H}^{(s)}(1) = |\mathcal{A}_0| + |\mathcal{B}_0|$ and the result is straightforward.

Suppose instead $|Z| > 1$. We are given that $s \leq \mu(\mathcal{H}) - 1$. By (4.2), $\binom{[\mu(\mathcal{H})]}{s} \subseteq \mathcal{H}$. Since $\emptyset \neq \mathcal{B} \subseteq \mathcal{H}^{(s)}[Z]$, $|Z| \leq s$. Therefore

$$\begin{aligned}
|\mathcal{A}_0| + |\mathcal{B}_0| &= 1 + |\mathcal{H}^{(s)}(1)| \\
&\geq 1 + |\mathcal{H}^{(s)}([|Z|])| \\
&\quad + |\{A \in \binom{[\mu(\mathcal{H}]}{s} : 1 \in A, |A \cap Z| = |Z| - 1\} \cup \mathcal{H}^{(s)}(1)(n)| \\
&\geq 1 + |\mathcal{H}^{(s)}([|Z|])| + \binom{|Z|-1}{|Z|-2}\binom{\mu(\mathcal{H}) - |Z|}{s - (|Z|-1)} + x \\
&\geq |Z| + |\mathcal{H}^{(s)}([|Z|])| + x \geq |\mathcal{A}| + |\mathcal{B}| + x \geq |\mathcal{A}| + |\mathcal{B}|, \qquad (4.3)
\end{aligned}$$

where

$$x := \begin{cases} |\mathcal{H}^{(s)}(1)(n)| & \text{if } \mu(\mathcal{H}) < n; \\ 0 & \text{if } \mu(\mathcal{H}) = n. \end{cases}$$

Suppose equality holds throughout (4.3). Then $s = \mu(\mathcal{H}) - 1$ and $x = 0$. We are given that $\mathcal{H}(n) \neq \emptyset$. By Lemma 4.4.1(i), $\mu(\mathcal{H}(n)) \geq \mu(\mathcal{H})$. Thus, since $\mathcal{H}$ is hereditary and $s < \mu(\mathcal{H})$, $\mathcal{H}^{(s)}(n) \neq \emptyset$. Let $A \in \mathcal{H}^{(s)}(n)$. Since $s \geq |Z| > 1$, $A \backslash \{n\} \neq \emptyset$. Let $a \in A \backslash \{n\}$ and $A' := \delta_{1,a}(A)$. So $A' \in \mathcal{H}^{(s)}(1)(n)$ as $\mathcal{H}$ is compressed. Since $x = 0$, it follows by definition of x that $\mu(\mathcal{H}) = n$. Thus, since $s = \mu(\mathcal{H}) - 1$, we have $s = n - 1$, which settles the case $r = 1$.

Next, suppose $s = n - r$. So $\mu(\mathcal{H}) = n$, and hence $[n] \in \mathcal{H}$. So $\mathcal{H}^{(p)} = \binom{[n]}{p}$, $p = 1, ..., n$, as $\mathcal{H}$ is hereditary. The result now follows easily from the fact that for every $A \in \binom{[n]}{r}$ there is only one set $B \in \binom{[n]}{s}$ such that $A \cap B = \emptyset$.

We now need to consider $r \geq 2$ and $s \leq n' - r$, $n' := n - 1$, and we proceed by induction on n.

In view of Lemma 4.5.1 (parts (i), (iii), (iv)) and the assumption that $\mathcal{H}$ is compressed, if $\Delta_{m,n}(\mathcal{A}) \neq \mathcal{A}$ or $\Delta_{m,n}(\mathcal{B}) \neq \mathcal{B}$ for some $m \in [n-1]$ then we can replace $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{A}' := \Delta_{m,n}(\mathcal{A})$ and $\mathcal{B}' := \Delta_{m,n}(\mathcal{B})$ respectively, and repeat the procedure until we obtain families $\mathcal{A}^* \subset \mathcal{H}^{(r)}$ and $\mathcal{B}^* \subset \mathcal{H}^{(s)}$ such that $\Delta_{m,n}(\mathcal{A}^*) = \mathcal{A}^*$ and $\Delta_{m,n}(\mathcal{B}^*) = \mathcal{B}^*$ for all $m \in [n-1]$ (it is well-known and easy to see that such a procedure indeed takes a finite number of steps). We can therefore assume that

$$\Delta_{m,n}(\mathcal{A}) = \mathcal{A} \text{ and } \Delta_{m,n}(\mathcal{B}) = \mathcal{B} \text{ for all } m \in [n-1]. \tag{4.4}$$

Thus, by Lemma 4.5.1(ii),

$$(A \cap B)\backslash\{n\} \neq \emptyset \text{ for any } A \in \mathcal{A} \text{ and } B \in \mathcal{B}. \tag{4.5}$$

Note that $\mathcal{A}]n[\ \neq \emptyset$ and $\mathcal{B}]n[\neq \emptyset$ by (4.4). Since $\mathcal{H}$ is hereditary, if $[n] \in \mathcal{H}$ then $\mu(\mathcal{H}]n[) = n-1$. Thus, if $[n] \in \mathcal{H}$ then $\mu(\mathcal{H}]n[) - r = n' - r \geq s$, and if $[n] \notin \mathcal{H}$ then, since $s \leq \mu(\mathcal{H}) - r$, it follows by Lemma 4.4.1(iii) that $s \leq \mu(\mathcal{H}]n[) - r$. Clearly, $\mathcal{H}]n[$ is hereditary and compressed. Therefore, by the inductive hypothesis,

$$|\mathcal{A}]n[| + |\mathcal{B}]n[| \leq |\mathcal{A}_0| + |\mathcal{B}_0]n[|. \tag{4.6}$$

Let $\mathcal{J} := \mathcal{H}\langle n\rangle$. Clearly, $\mathcal{J}$ is hereditary and compressed, and $\mu(\mathcal{J}) \geq \mu(\mathcal{H}) - 1$ by Lemma 4.4.1(i). Setting $r' := r-1$ and $s' := s-1$, we have $\mathcal{A}\langle n\rangle \subset \mathcal{J}^{(r')}$, $\mathcal{B}\langle n\rangle \subset \mathcal{J}^{(s')}$, and

$$r' \leq s' \leq (\mu(\mathcal{H}) - 1) - r \leq \mu(\mathcal{J}) - r < \mu(\mathcal{J}) - r'. \tag{4.7}$$

By (4.5), $\mathcal{A}\langle n\rangle$ and $\mathcal{B}\langle n\rangle$ are cross-intersecting.

Suppose $\mathcal{A}\langle n\rangle \neq \emptyset$, $\mathcal{B}\langle n\rangle \neq \emptyset$. By the inductive hypothesis, $|\mathcal{A}\langle n\rangle| + |\mathcal{B}\langle n\rangle| \leq |\{[r']\}| + |\mathcal{J}^{(s')}([r'])|$ with equality only if $|\mathcal{A}\langle n\rangle| = 1$. Since $2^{[\mu(\mathcal{J})]} \subseteq \mathcal{J}$ (by (4.2)) and $\mathcal{B}_0\langle n\rangle = \mathcal{J}^{(s')}([r])$, we have

$$\begin{aligned} |\mathcal{B}_0\langle n\rangle| - |\mathcal{J}^{(s')}([r'])| = |\mathcal{J}^{(s')}]\{1, ..., r'\}[(r)| &\geq |\{B \in \binom{[\mu(\mathcal{J})]\backslash[r']}{s'} : r \in B\}| \\ &= \binom{\mu(\mathcal{J}) - r' - 1}{s' - 1}, \end{aligned}$$

and hence, by (4.7), $|\mathcal{B}_0\langle n\rangle| \geq |\mathcal{J}^{(s')}([r'])| + 1 \geq |\mathcal{A}\langle n\rangle| + |\mathcal{B}\langle n\rangle|$ with equality only if $s' = 1$ (thus $r' = 1$) and $\mathcal{A}\langle n\rangle = \mathcal{B}\langle n\rangle = \{\{a\}\}$ for some $a \in [n']$. By (4.5) and (4.6), it follows that $|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{A}_0| + |\mathcal{B}_0|$ with equality only if $\mathcal{A} = \mathcal{B} = \mathcal{H}^{(2)}(a)$.

Suppose $\mathcal{A}\langle n\rangle = \emptyset$. Let $A \in \mathcal{A}]n[$. By (4.5), $|\mathcal{B}\langle n\rangle| \leq |\mathcal{J}^{(s')}(A)|$ with equality only if $\mathcal{A}]n[= \{A\}$. Since $\mathcal{J}$ is compressed, we clearly have $U(\mathcal{J}) = [l]$ for some $l \in [n']$, and hence $|\mathcal{J}^{(s')}(A)| \leq |\mathcal{J}^{(s')}([r])|$ by Corollary 4.4.4(ii). Since $\mathcal{B}_0\langle n\rangle = \mathcal{J}^{(s')}([r])$, it follows by (4.5) and (4.6) that $|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{A}_0| + |\mathcal{B}_0|$ with equality only if $\mathcal{A} = \{A\}$ and $\mathcal{B} = \mathcal{H}^{(r)}(A)$.

We finally consider $\mathcal{B}\langle n\rangle = \emptyset$. Suppose $r' = s'$, i.e. $r = s$. Then, by an argument similar to the one for the previous case $\mathcal{A}\langle n\rangle = \emptyset$, $|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{A}_0| + |\mathcal{B}_0|$ with equality only if $|\mathcal{B}| = 1$. Suppose equality holds. Since for $r = s$ we require $|\mathcal{A}| \leq |\mathcal{B}|$, then $|\mathcal{A}| = |\mathcal{B}| = 1$. So $|\mathcal{A}_0| + |\mathcal{B}_0| = 2$, but this is not true for $r > 1$. So $|\mathcal{A}| + |\mathcal{B}| < |\mathcal{A}_0| + |\mathcal{B}_0|$.

Now suppose $r' < s'$. By Lemma 4.4.2, $\mathcal{J}]1[$ and $\mathcal{J}\langle 1\rangle$ are hereditary and quasi-compressed. By (i) and (ii) of Lemma 4.4.1, $\mu(\mathcal{J}]1[) \geq \mu(\mathcal{J}) - 1$ and $\mu(\mathcal{J}\langle 1\rangle) \geq \mu(\mathcal{J}) - 1$. Thus, by (4.7), $r' \leq s' \leq \mu(\mathcal{J}) - r' - 1 \leq \mu(\mathcal{J}]1[) - r'$. By the inductive hypothesis,

$$|\mathcal{J}]1[^{(r')}([2,s])| + |\{[2,s]\}| \leq |\mathcal{J}]1[^{(s')}([2,r])| + |\{[2,r]\}|.$$

Thus, since $\mathcal{J}_1 := \mathcal{J}^{(r')}([s])]1[= \mathcal{J}]1[^{(r')}([2,s])$ and $\mathcal{B}_0\langle n\rangle]1[= \mathcal{J}]1[^{(s')}([2,r])$, $|\mathcal{J}_1| \leq |\mathcal{B}_0\langle n\rangle]1[|$. Similarly to (4.7), $r'' := r' - 1 < s'' := s' - 1 < \mu(\mathcal{J}\langle 1\rangle) - r''$. By Corollary 4.3.2, $|\mathcal{J}\langle 1\rangle^{(r'')}| < |\mathcal{J}\langle 1\rangle^{(s'')}|$. Thus, since $\mathcal{J}_2 := \mathcal{J}^{(r')}([s])\langle 1\rangle = \mathcal{J}\langle 1\rangle^{(r'')}$ and $\mathcal{B}_0\langle n\rangle\langle 1\rangle = \mathcal{J}\langle 1\rangle^{(s'')}$, $|\mathcal{J}_2| < |\mathcal{B}_0\langle n\rangle\langle 1\rangle|$. We therefore have

$$|\mathcal{J}^{(r')}([s])| = |\mathcal{J}_1| + |\mathcal{J}_2| < |\mathcal{B}_0\langle n\rangle]1[| + |\mathcal{B}_0\langle n\rangle\langle 1\rangle| = |\mathcal{B}_0\langle n\rangle|. \tag{4.8}$$

Now let $B \in \mathcal{B}]n[$. By (4.5), $|\mathcal{A}\langle n\rangle| \leq |\mathcal{J}^{(r')}(B)|$. Since $U(\mathcal{J}) = [l]$ for some $l \in [n']$ (see above), $|\mathcal{J}^{(r')}(B)| \leq |\mathcal{J}^{(r')}([s])|$ by Corollary 4.4.4(ii). Thus, by (4.8), $|\mathcal{A}\langle n\rangle| < |\mathcal{B}_0\langle n\rangle|$. Together with (4.6) and $\mathcal{B}\langle n\rangle = \emptyset$, this gives us $|\mathcal{A}| + |\mathcal{B}| < |\mathcal{A}_0| + |\mathcal{B}_0|$. □

4.6 Consequences of Theorem 4.1.3 with $r = s$

The scope of the main results in this section (i.e. Propositions 4.6.2, 4.6.5 and 4.6.6) is to ensure that in the proof of Theorem 4.1.4 we can work with a non-centred intersecting family $\mathcal{A}$ that is compressed. This will become clear in the proof itself.

Lemma 4.6.1 *Let $\mathcal{H} \subseteq 2^{[n]}$ be hereditary and compressed, $\mathcal{H}(n) \neq \emptyset$, $\mu(\mathcal{H}) > 2$. Let $z_1 < z_2 < ... < z_{n-2}$ such that $Z := \{z_1, ..., z_{n-2}\} = [n]\backslash\{p, q\}$. Let $1 \leq p < q \leq n$ and $\mathcal{Z} := \mathcal{H}\langle p\rangle]q[$.*
(i) If $p > \mu(\mathcal{H}) - 1$ then $\mathcal{Z}(z_{\mu(\mathcal{H})-1}) \neq \emptyset$.
(ii) If $p \leq \mu(\mathcal{H}) - 1$ then $\mathcal{Z}(z_{n-2}) \neq \emptyset$.

Proof. By Lemma 4.4.2, $\mathcal{H}\langle n\rangle$ is hereditary and compressed. Thus, by (4.2) and Lemma 4.4.1(i), $[\mu(\mathcal{H}) - 1] \in \mathcal{H}\langle n\rangle$. So $M := [\mu(\mathcal{H}) - 1] \cup \{n\} \in \mathcal{H}(n)$.

Suppose $p > \mu(\mathcal{H}) - 1$. Then $z_{\mu(\mathcal{H})-1} = \mu(\mathcal{H}) - 1$. Also, since $p < q < n$ and $\mathcal{H}$ is compressed, $\delta_{p,n}(M) \in \mathcal{H}(p)]q[(\mu(\mathcal{H}) - 1)$. Hence (i).

Now consider (ii). If $[n] \in \mathcal{H}$ then $\mathcal{H} = 2^{[n]}$ (as $\mathcal{H}$ is hereditary), and hence $Z \in \mathcal{Z}(z_{n-2})$. Suppose $[n] \notin \mathcal{H}$ instead. So $M \neq [n]$, and hence $n - 1 \notin M$ and $\mu(\mathcal{H}) - 1 < n - 1$. Also, $p \in M$ as $p \leq \mu(\mathcal{H}) - 1$. Suppose $q = n$. Since $p < n - 1$, we then have $z_{n-2} = n - 1$, and hence $\delta_{n-1,n}(M) = \delta_{z_{n-2},q}(M) \in \mathcal{H}(p)]q[(z_{n-2})$ as $\mathcal{H}$ is compressed and $\{n - 1, n\} \cap M = \{n\}$. Now suppose $q < n$. So $z_{n-2} = n$, and hence $z_{n-2} \in M' := M\backslash\{q\}$. Since $\mathcal{H}$ is hereditary, $M' \in \mathcal{H}$. So $M' \in \mathcal{H}(p)]q[(z_{n-2})$. Hence (ii). □

Proposition 4.6.2 *Let $\mathcal{H} \subseteq 2^{[n]}$ be hereditary and compressed, $\mathcal{H}(n) \neq \emptyset$. Suppose $1 \leq p < q \leq n$, $2 \leq r \leq \mu(\mathcal{H})/2$, $r < n/2$. Let $\mathcal{A}$ be a non-centred intersecting sub-family of $\mathcal{H}^{(r)}$ such that $\Delta_{p,q}(\mathcal{A})$ is centred. Then $|\mathcal{A}| \leq |\mathcal{H}^{(r)}(p)(I \cup \{q\})|$, where I is an initial $(r - 1)$-segment of $[n]\backslash\{p, q\}$.*

Proof. We are given that $\Delta_{p,q}(\mathcal{A}) \subseteq \mathcal{H}^{(r)}(a)$ for some $a \in [n]$. If $a \neq p$ then $\mathcal{A} \subseteq \mathcal{H}^{(r)}(a)$, contradicting $\mathcal{A}$ non-centred. So

$$\Delta_{p,q}(\mathcal{A}) \subseteq \mathcal{H}^{(r)}(p), \tag{4.9}$$

and hence $\mathcal{A} = \mathcal{A}(\{p,q\})$. So

$$|\mathcal{A}| = |\mathcal{A}(p)(q)| + |\mathcal{A}\langle p\rangle]q[| + |\mathcal{A}]p[\langle q\rangle|. \tag{4.10}$$

$\mathcal{A}\langle p\rangle]q[$ and $\mathcal{A}]p[\langle q\rangle$ are cross-intersecting. $\mathcal{A}\langle p\rangle]q[$ and $\mathcal{A}]p[\langle q\rangle$ are also non-empty because otherwise $\mathcal{A} \subseteq \mathcal{H}^{(r)}(p)$ or $\mathcal{A} \subseteq \mathcal{H}^{(r)}(q)$ (contradicting $\mathcal{A}$ non-centred). Since $\mathcal{H}$ is compressed and $p < q$, $\mathcal{H}]p[\langle q\rangle \subseteq \mathcal{H}\langle p\rangle]q[=: \mathcal{Z}$. So $\mathcal{A}\langle p\rangle]q[, \mathcal{A}]p[\langle q\rangle \subset \mathcal{Z}^{(r')}$, $r' := r-1$. $\mathcal{Z}$ is hereditary and quasi-compressed by Lemma 4.4.2, and $\mu(\mathcal{Z}) \geq \mu(\mathcal{H})-2$ by (i) and (ii) of Lemma 4.4.1. Thus, since $r \leq \mu(\mathcal{H})/2$, we have $r' \leq (\mu(\mathcal{H})-2)/2 \leq \mu(\mathcal{Z})/2$. Let $Z := \{z_1, ..., z_{n-2}\}$ as in Lemma 4.6.1. So $\mathcal{Z} \subseteq 2^Z$. Let $n'' := n-2$ and $\mu' := \mu(\mathcal{H}) - 1$. By Lemma 4.6.1, if $p > \mu'$ then $\mathcal{Z}(z_{n''}) \neq \emptyset$, and if $p \leq \mu'$ then $\mathcal{Z}(z_{\mu'}) \neq \emptyset$. Note that $r' < (n'')/2$ (as $r < n/2$) and $r' < \mu'/2$ (as $r \leq \mu(\mathcal{H})/2$). Thus, by Theorem 4.1.3,

$$|\mathcal{A}\langle p\rangle]q[| + |\mathcal{A}]p[\langle q\rangle| \leq |\mathcal{Z}^{(r')}(I)| + 1 \tag{4.11}$$

with equality only if

(a) $\mathcal{A}\langle p\rangle]q[= \{A'\}$ and $\mathcal{A}']p[\langle q\rangle = \mathcal{Z}^{(r')}(A')$ for some $A' \in \mathcal{Z}^{(r')}$, or

(b) $\mathcal{A}\langle p\rangle]q[= \mathcal{Z}^{(r')}(B')$ and $\mathcal{A}]p[\langle q\rangle = \{B'\}$ for some $B' \in \mathcal{Z}^{(r')}$, or

(c) $\mathcal{A}\langle p\rangle]q[= \mathcal{A}]p[\langle q\rangle = \mathcal{Z}^{(r')}(z)$ for some $z \in Z$ if $r' = 2$.

Suppose (a) holds. Then $A' \in \mathcal{A}\langle p\rangle]q[\cap\mathcal{A}]p[\langle q\rangle$, and hence $A' \cup \{p\}, A' \cup \{q\} \in \mathcal{A}$. But this gives us $A' \cup \{q\} \in \Delta_{p,q}(\mathcal{A})\backslash\mathcal{H}^{(r)}(p)$, which contradicts (4.9). So (a) does not hold. (b) and (c) do not hold for a similar reason. We therefore have strict inequality in (4.11). Thus, by (4.10),

$$\begin{aligned}|\mathcal{A}| &\leq |\mathcal{H}^{(r)}(p)(q)| + |\mathcal{Z}^{(r')}(I)| = |\mathcal{H}^{(r)}(p)(q)| + |\mathcal{H}^{(r)}(p)]q[(I)| \\ &= |\mathcal{H}^{(r)}(p)(I \cup \{q\})|. \end{aligned} \tag{4.12}$$

□

Lemma 4.6.3 *Let $\mathcal{F} \subseteq 2^{[n]}$ be compressed. If $\mathcal{A} \subseteq \mathcal{F}$, $1 \leq p < q \leq n$, and $\Delta_{q,p}(\Delta_{p,q}(\mathcal{A})) \cap \mathcal{F} \subseteq \Delta_{p,q}(\mathcal{A})$ then $\Delta_{p,q}(\mathcal{A}) = \mathcal{A}$.*

Proof. Suppose instead $\mathcal{A}\backslash\Delta_{p,q}(\mathcal{A}) \neq \emptyset$. Let $A \in \mathcal{A}\backslash\Delta_{p,q}(\mathcal{A})$. So $\delta_{p,q}(A) \in \Delta_{p,q}(\mathcal{A})\backslash\mathcal{A}$ and $A = \delta_{q,p}(\delta_{p,q}(A)) \in \Delta_{q,p}(\Delta_{p,q}(\mathcal{A}))\backslash\Delta_{p,q}(\mathcal{A})$. Also, $A \in \mathcal{F}$ as $\mathcal{A} \subseteq \mathcal{F}$. So $A \in (\Delta_{q,p}(\Delta_{p,q}(\mathcal{A})) \cap \mathcal{F})\backslash\Delta_{p,q}(\mathcal{A})$, a contradiction. □

Lemma 4.6.4 *Let $\mathcal{F} \subseteq 2^{[n]}$ be compressed. Suppose $B \in \mathcal{F}^{(r)} \neq \emptyset$, $2 \leq r < n$, and $a \in [n]\backslash B$. Then $|\mathcal{F}^{(r)}(a)(B)| \leq |\mathcal{F}^{(r)}(1)([2, r+1])|$.*

Proof. Let $\mathcal{B} := \mathcal{F}^{(r)}(a)(B)$. Since $\mathcal{F}$ is compressed, we clearly have $\Delta_{1,a}(\mathcal{B}) \subseteq \mathcal{F}^{(r)}(1)(C)$ where

$$C = \begin{cases} (B\backslash\{1\}) \cup \{a\} & \text{if } a \neq 1 \in B; \\ B & \text{if } a = 1 \text{ or } 1 \notin B, \end{cases}$$

It is easy to see that having $\mathcal{F}$ compressed and $\mathcal{F}^{(r)} \neq \emptyset$ for $2 \leq r < n$ implies that $\mathcal{F}^{(r)}\langle 1\rangle$ is quasi-compressed and $U(\mathcal{F}^{(r)}\langle 1\rangle) = [2, m]$, $m = \min\{k \in [2, n] \colon \mathcal{F}^{(r)} \subseteq 2^{[k]}]\}$. So $|\mathcal{F}^{(r)}\langle 1\rangle(C)| \leq |\mathcal{F}^{(r)}\langle 1\rangle([2, r+1])|$ by Corollary 4.4.4(ii). Since $1 \notin C$, $|\mathcal{F}^{(r)}\langle 1\rangle(C)| = |\mathcal{F}^{(r)}(1)(C)|$. So we have $|\mathcal{B}| = |\Delta_{1,a}(\mathcal{B})| \leq |\mathcal{F}^{(r)}(1)(C)| \leq |\mathcal{F}^{(r)}(1)([2, r+1])|$. □

Proposition 4.6.5 *Let $\mathcal{H}, n, p, q, r$ be as in Proposition 4.6.2. Let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{H}^{(r)}$ such that*
(i) $|\mathcal{A}| = |\mathcal{H}^{(r)}(1)([2, r+1]) \cup \{[2, r+1]\}|$ and
(ii) $\mathcal{A} \neq \Delta_{p,q}(\mathcal{A}) = \mathcal{H}^{(r)}(a)(B) \cup \{B\}$ for some $B \in \mathcal{H}^{(r)}$ and $a \in [n]\backslash B$.
Then $\mathcal{A} = \mathcal{H}^{(r)}(c)(D) \cup \{D\}$ for some $D \in \mathcal{H}^{(r)}$ and $c \in [n]\backslash D$.

Proof. Let $E := \{a\} \cup B$. Let $\mathcal{B} := \mathcal{H}^{(r)}(a)(B) \cup \{B\}$. Let $A_1 \in \mathcal{B}\backslash\mathcal{A}$. Since $\mathcal{B} = \Delta_{p,q}(\mathcal{A})$, we have

$$p \in A_1, q \notin A_1, \tag{4.13}$$

$$A_1 \neq A_2 := \delta_{q,p}(A_1) \in \mathcal{A}\backslash\mathcal{B}. \tag{4.14}$$

Since $B \in \Delta_{p,q}(\mathcal{A})$,

$$\delta_{p,q}(B) \in \Delta_{p,q}(\mathcal{A}). \tag{4.15}$$

We now start treating all the possible cases of p and q.

Suppose $p \notin E$. If $q \in B$ then $\delta_{p,q}(B) \notin \mathcal{B}$, but this contradicts (4.15) since $\mathcal{B} = \Delta_{p,q}(\mathcal{A})$. So $q \notin B$. Suppose $q = a$; then, by (4.13), we get $A_1 \notin \mathcal{B}$, a contradiction. So $q \notin E$, and hence $\Delta_{q,p}(\mathcal{B}) \cap \mathcal{H} \subseteq \mathcal{B}$. Since $\mathcal{B} = \Delta_{p,q}(\mathcal{A})$, it follows by Lemma 4.6.3 that $\mathcal{B} = \mathcal{A}$, a contradiction. So $p \in E$.

Suppose $p \in B$. Then $\Delta_{b,p}(\mathcal{B}) \cap \mathcal{H} \subseteq \mathcal{B}$ for any $b \in B \backslash \{p\}$, and hence, by Lemma 4.6.3, $q \notin B$ since $\mathcal{B} = \Delta_{p,q}(\mathcal{A}) \neq \mathcal{A}$. Suppose $q = a$. Then, by (4.13) and $A_1 \in \mathcal{B}$, $A_1 = B$. Hence $A_2 \in \mathcal{H}^{(r)}(a)(B) \subset \mathcal{B}$, contradicting (4.14). So $q \notin E$. Since $p \in B$ and $\Delta_{p,q}(\mathcal{A}) = \mathcal{B}$ and $\mathcal{A} \subseteq \mathcal{H}^{(r)}$ is intersecting, it follows that $B^* \in \mathcal{A} \subseteq \mathcal{B}^* := \mathcal{H}^{(r)}(a)(B^*) \cup \{B^*\}$ for some $B^* \in \{B, \delta_{q,p}(B)\}$. By (i) and Lemma 4.6.4, we must have $\mathcal{A} = \mathcal{B}^*$ (and hence $B^* = \delta_{q,p}(B) \neq B$ since $\mathcal{A} \neq \mathcal{B}$), and this proves the result for this case.

Finally, suppose $p = a$. So $q \in B$ or $q \notin E$. Consider first $q \in B$. So $\delta_{p,q}(B) \neq B$. Thus, since $B, \delta_{p,q}(B) \in \Delta_{p,q}(\mathcal{A}) = \mathcal{B}$, we have $B \in \mathcal{A}(q)]p[$, $\delta_{p,q}(B) \in \mathcal{A}(p)]q[$, and $\mathcal{A} = \mathcal{A}(\{p,q\})$; moreover,

$$\mathcal{A}\langle p\rangle]q[\cap \mathcal{A}\langle q\rangle]p[= \{L\}, \quad L = B\backslash\{q\} = \delta_{p,q}(B)\backslash\{p\} \tag{4.16}$$

(which again follows from $\Delta_{p,q}(\mathcal{A}) = \mathcal{B}$, $p = a \notin B$, and $q \in B$). As in the proof of Proposition 4.6.2, (4.11) holds, and equality in (4.11) holds only if one of (a), (b) and (c) holds. Suppose we have strict inequality in (4.11). Then (4.12) also holds, and hence, together with Lemma 4.6.4, this gives us $|\mathcal{A}| \leq |\mathcal{H}^{(r)}(1)([2, r+1])|$, a contradiction to (i). So equality in (4.11) holds, and hence one of (a), (b) and (c) holds. However, by (4.16), (c) cannot be the case. Suppose (b) holds; so $\mathcal{A}]p[\langle q\rangle = \{B'\}$. By (4.16), we have $B' = L$, and hence $\mathcal{A}]p[(q) = \{B\}$. Since $\mathcal{A} = \mathcal{A}(\{p,q\})$ and $\mathcal{A}$ is intersecting, it follows that $\mathcal{A} \subseteq \mathcal{H}^{(r)}(p)(B) \cup \{B\}$. Since $p = a$, $\mathcal{A} \subseteq \mathcal{B}$. Since $\mathcal{B} = \Delta_{p,q}(\mathcal{A})$, we get the contradiction that $\mathcal{A} = \Delta_{p,q}(\mathcal{A})$. So (a) holds, and hence $\mathcal{A} \subseteq \mathcal{H}^{(r)}(q)(L \cup \{p\}) \cup \{L \cup \{p\}\}$ (by an argument similar to the one for the supposition that (b) holds); by (i) and Lemma 4.6.4, we have equality. So the result for $q \in B$ is verified.

Suppose $q \notin E$ instead. Since $\Delta_{p,q}(\mathcal{A}) = \mathcal{B}$, in this case we have $\mathcal{A} = \mathcal{A}(\{p,q\}) \cup \{B\}$ and

$$\mathcal{A}\langle p\rangle]q[\cap\mathcal{A}\langle q\rangle]p[= \emptyset. \tag{4.17}$$

Suppose $\mathcal{A}\langle p\rangle]q[$ and $\mathcal{A}\langle q\rangle]p[$ are both non-empty. Again, as in the proof of Proposition 4.6.2, (4.11) follows; however, since (4.17) implies that none of (a), (b) and (c) hold, strict inequality in (4.11) holds here too, i.e. $|\mathcal{A}\langle p\rangle]q[|+|\mathcal{A}]p[\langle q\rangle| \leq |\mathcal{H}^{(r)}\langle p\rangle]q[(I)|$, where I is also as in Proposition 4.6.2. Since $\{p,q\} \cap B = \emptyset$ and $\mathcal{A}$ is intersecting, $|\mathcal{A}\langle p\rangle\langle q\rangle| \leq |\mathcal{H}^{(r)}\langle p\rangle\langle q\rangle(B)|$. Let J be an initial r-segment for $[n]\backslash\{p,q\}$; so $J = I \cup \{j\}$ for some $j \in [n]\backslash I$. Note that having $\mathcal{H} \subseteq 2^{[n]}$ compressed implies that $\mathcal{H}^{(r)}\langle p\rangle\langle q\rangle$ is quasi-compressed and that if $\mathcal{H}^{(r)}\langle p\rangle\langle q\rangle \neq \emptyset$ then $U(\mathcal{H}^{(r)}\langle p\rangle\langle q\rangle)$ is an initial segment of $[n]\backslash\{p,q\}$. By Corollary 4.4.4(ii), $|\mathcal{H}^{(r)}\langle p\rangle\langle q\rangle(B)| \leq |\mathcal{H}^{(r)}\langle p\rangle\langle q\rangle(J)|$. So $|\mathcal{A}\langle p\rangle\langle q\rangle| \leq |\mathcal{H}^{(r)}\langle p\rangle\langle q\rangle(J)|$. Therefore,

$$\begin{aligned}|\mathcal{A}| = |\mathcal{A}(\{p,q\}) \cup \{B\}| &= |\mathcal{A}\langle p\rangle\langle q\rangle| + (|\mathcal{A}\langle p\rangle]q[| + |\mathcal{A}]p[\langle q\rangle|) + 1\\ &\leq |\mathcal{H}^{(r)}\langle p\rangle\langle q\rangle(J)| + |\mathcal{H}^{(r)}\langle p\rangle]q[(I)| + 1\\ &= |\mathcal{H}^{(r)}(p)(J)| - |\mathcal{H}^{(r)}(p)(j)]I \cup \{q\}[|) + 1.\end{aligned} \tag{4.18}$$

Since $A_1 \in \mathcal{H}(p)$ and $r \leq \mu(\mathcal{H})/2$, we have $\mu(\mathcal{H}\langle p\rangle) \geq 2r-1$ by Lemma 4.4.1(i). Let M be the initial $\mu(\mathcal{H}\langle p\rangle)$-segment of $[n]\backslash\{p\}$. Since $\mathcal{H}$ is compressed, $M \in \mathcal{H}\langle p\rangle$. So $2^M \subseteq \mathcal{H}\langle p\rangle$ as $\mathcal{H}$ is hereditary. Since $|M\backslash\{q\}| \geq 2r-2 \geq r = |J|$, $J \subset M$. So $j \in M$. Since $|I \cup \{q\}| = r$ and $j \notin I \cup \{q\}$, there exists $S \in \binom{M}{r-1}$ such that $S \cap (I \cup \{q\}) = \emptyset$ and $j \in S$. So $S \cup \{p\} \in \mathcal{H}^{(r)}(p)(j)]I \cup \{q\}[$. Thus, by (4.18) and Lemma 4.6.4, $|\mathcal{A}| \leq |\mathcal{H}^{(r)}(p)(J)| \leq |\mathcal{H}^{(r)}(1)([2,r+1])|$, a contradiction to (ii). Therefore $\mathcal{A}\langle p\rangle]q[= \emptyset$ or $\mathcal{A}\langle q\rangle]p[= \emptyset$. If $\mathcal{A}\langle q\rangle]p[= \emptyset$ then $\Delta_{p,q}(\mathcal{A}) = \mathcal{A}$, a contradiction. So $\mathcal{A}\langle p\rangle]q[= \emptyset$, and hence $\mathcal{A} \subseteq \mathcal{C} := \mathcal{H}^{(r)}(q)(B) \cup \{B\}$ (as $\mathcal{A} = \mathcal{A}(\{p,q\}) \cup \{B\}$ and $\mathcal{A}$ is intersecting). By Lemma 4.6.4, $|\mathcal{C}| \leq |\mathcal{H}^{(r)}(1)([2,r+1]) \cup \{[2,r+1]\}|$. So, by (i), $\mathcal{A} = \mathcal{C}$. □

Proposition 4.6.6 *Let $\mathcal{H}, n, p, q, r$ be as in Proposition 4.6.2, except that $r = 3$. Let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{H}^{(3)}$ such that* *(i)* $|\mathcal{A}| = |\mathcal{T}|$, $\mathcal{T} := \{A \in \mathcal{H}^{(3)} \colon |A \cap [3]| \geq 2\}$, *and*

(ii) $\mathcal{A} \neq \Delta_{p,q}(\mathcal{A}) = \{A \in \mathcal{H}^{(3)} \colon |A \cap C| \geq 2\}$ *for some* $C \in \mathcal{H}^{(3)}$.

Then $\mathcal{A} = \{A \in \mathcal{H}^{(3)} \colon |A \cap B| \geq 2\}$ *for some* $B \in \mathcal{H}^{(3)}$.

Proof. Let $\mathcal{C} := \{A \in \mathcal{H}^{(3)} \colon |A \cap C| \geq 2\}$. Let $A^* \in \mathcal{C} \backslash \mathcal{A}$. Since $\mathcal{C} = \Delta_{p,q}(\mathcal{A})$, we have

$$p \in A^*, q \notin A^* \quad \text{and} \quad A^* \neq A' := \delta_{q,p}(A^*) \in \mathcal{A} \backslash \mathcal{C}. \tag{4.19}$$

Suppose $q \in C$. Then, by (4.19), $q \in A' \cap C$. Since $C \in \mathcal{C}$, $A' \neq C$. So $|A' \cap C| \leq 2$. Therefore, if $p \notin C$ then

$$|A^* \cap C| = |\delta_{p,q}(A') \cap C| = |(A' \backslash \{q\}) \cap C| = |(A' \cap C) \backslash \{q\}| = |A' \cap C| - 1 \leq 1,$$

contradicting $A^* \in \mathcal{C}$. So $p \in C$, and hence $|A' \cap C| = |A^* \cap C|$. Since $A^* \in \mathcal{C}$, $|A^* \cap C| \geq 2$. So $|A' \cap C| \geq 2$, and hence $A' \in \mathcal{C}$, contradicting (4.19). So $q \notin C$.

Suppose $p \notin C$. Then $\Delta_{q,p}(\mathcal{C}) \cap \mathcal{H} \subseteq \mathcal{C}$, and hence, since $\mathcal{C} = \Delta_{p,q}(\mathcal{A})$, Lemma 4.6.3 gives us $\Delta_{p,q}(\mathcal{A}) = \mathcal{A}$, a contradiction. So $p \in C$.

Let $C' := C \backslash \{p\} \subset [n] \backslash \{p, q\}$, and let I be an initial 2-segment of $[n] \backslash \{p, q\}$. Since $\Delta_{p,q}(\mathcal{A}) = \mathcal{C}$, $\mathcal{A}]\{p,q\}[\subseteq \mathcal{H}^{(3)}]\{p,q\}[[C']$. Note that having $\mathcal{H} \subseteq 2^{[n]}$ compressed implies that $\mathcal{H}^{(3)}]\{p,q\}[$ is quasi-compressed and that if $\mathcal{H}^{(3)}]\{p,q\}[\neq \emptyset$ then $U(\mathcal{H}^{(3)}]\{p,q\}[)$ is an initial segment of $[n] \backslash \{p,q\}$. So $|\mathcal{H}^{(3)}]\{p,q\}[[C']| \leq |\mathcal{H}^{(3)}]\{p,q\}[[I]|$ by Lemma 4.4.4(i). Therefore

$$|\mathcal{A}]\{p,q\}[| \leq |\mathcal{H}^{(3)}]\{p,q\}[[I]|. \tag{4.20}$$

Since $\Delta_{p,q}(\mathcal{A}) = \mathcal{C}$ and $\{p,q\} \cap C = \{p\}$, $\mathcal{A}(p)(q) \subseteq \mathcal{H}^{(3)}(p)(q)(C')$. By the same argument for obtaining $|\mathcal{H}^{(r)}\langle p\rangle\langle q\rangle(B)| \leq |\mathcal{H}^{(r)}\langle p\rangle\langle q\rangle(J)|$ in the proof of Proposition 4.6.5, $|\mathcal{H}^{(3)}\langle p\rangle\langle q\rangle(C')| \leq |\mathcal{H}^{(3)}\langle p\rangle\langle q\rangle(I)|$. Therefore

$$|\mathcal{A}(p)(q)| \leq |\mathcal{H}^{(3)}(p)(q)(I)|. \tag{4.21}$$

By (4.19), $\mathcal{A}]p[\langle q\rangle \neq \emptyset$. We now show that we also have $\mathcal{A}\langle p\rangle]q[\neq \emptyset$ by establishing that

$$C' \in \mathcal{A}\langle p\rangle]q[. \tag{4.22}$$

Suppose $\delta_{q,p}(C) \notin \mathcal{H}$. Then $\delta_{q,p}(C) \notin \mathcal{A}$ (as $\mathcal{A} \subset \mathcal{H}$), and hence $C \in \mathcal{A}$ as $C \in \Delta_{p,q}(\mathcal{A})$ (by (ii)). Since $C \cap \{p,q\} = \{p\}$, (4.22) follows. Suppose $\delta_{q,p}(C) \in \mathcal{H}$ instead. Then $\delta_{q,p}(C) \in \mathcal{C}$ as $|\delta_{p,q}(C) \cap C| = 2$. So $\delta_{q,p}(C) \in \Delta_{p,q}(\mathcal{A})$ (as $\mathcal{C} = \Delta_{p,q}(\mathcal{A})$). Also, $C \in \Delta_{p,q}(\mathcal{A})$ by (ii). Therefore $C, \delta_{q,p}(C) \in \mathcal{A}$, and (4.22) follows again.

Let $\mathcal{Z} := \mathcal{H}\langle p\rangle]q[$. As in the proof of Proposition 4.6.2, we have

$$|\mathcal{A}\langle p\rangle]q[| + |\mathcal{A}]p[\langle q\rangle| \leq |\mathcal{Z}^{(2)}(I) \cup \{I\}| \tag{4.23}$$

with equality only if

(d) $\mathcal{A}\langle p\rangle]q[= \{D\}$ and $\mathcal{A}]p[\langle q\rangle = \mathcal{Z}^{(2)}(D)$ for some $D \in \mathcal{Z}^{(2)}$, or

(e) $\mathcal{A}\langle p\rangle]q[= \mathcal{Z}^{(2)}(E)$ and $\mathcal{A}]p[\langle q\rangle = \{E\}$ for some $E \in \mathcal{Z}^{(2)}$, or

(f) $\mathcal{A}\langle p\rangle]q[= \mathcal{A}]p[\langle q\rangle = \mathcal{Z}^{(2)}(z)$ for some $z \in [n]\backslash\{p,q\}$.

Let $\mathcal{I} := \{A \in \mathcal{H}^{(3)} \colon |A \cap \{I \cup \{p\}| \geq 2\}$. Since $|\mathcal{A}| = |\mathcal{A}]\{p,q\}[| + |\mathcal{A}(p)(q)| + |\mathcal{A}\langle p\rangle]q[| + |\mathcal{A}]p[\langle q\rangle|$, we have

$$|\mathcal{A}| \leq |\mathcal{H}^{(3)}]\{p,q\}[[I]| + |\mathcal{H}^{(3)}(p)(q)(I)| + |\mathcal{H}^{(3)}(p)]q[(I) \cup \{I \cup \{p\}\}| = |\mathcal{I}| \tag{4.24}$$

by (4.20), (4.21) and (4.23).

Given that $\mu(\mathcal{H}) \geq 2r \geq 4$ and $\mathcal{H}(n) \neq \emptyset$, we have $\mu(\mathcal{H}\langle n\rangle) \geq 3$ by Lemma 4.4.1(i). Since $\mathcal{H}$ is hereditary and compressed, $\mathcal{H}\langle n\rangle$ is hereditary and compressed, and hence $2^{[\mu(\mathcal{H}\langle n\rangle)]} \subseteq \mathcal{H}\langle n\rangle$ (see (4.2)). So $T := [2,3] \cup \{n\} \in \mathcal{H}^{(3)}(n)$. Clearly, for any $F \in \mathcal{T}' := \{A \in \binom{[n]}{3} \colon |A \cap [3]| \geq 2\}$ there exists a composition of compressions $\delta_{i,j}$, $i < j$, that yields F when applied on T. Since $\mathcal{H}$ is compressed and $T \in \mathcal{T}$, we therefore have $\mathcal{T}' \subseteq \mathcal{T}$. Trivially, $\mathcal{T} \subseteq \mathcal{T}'$. Therefore

$$\mathcal{T} = \{A \in \binom{[n]}{3} \colon |A \cap [3]| \geq 2\}. \tag{4.25}$$

This gives us $|\mathcal{I}| \leq |\mathcal{T}|$. Thus, by (i), we have equality in (4.24), and hence we also have equalities in (4.20), (4.21) and (4.23), the first two yielding

$$\mathcal{A}]\{p,q\}[= \mathcal{H}^{(3)}]\{p,q\}[[C'] \quad \text{and} \quad \mathcal{A}(p)(q) = \mathcal{H}^{(3)}(p)(q)(C') \tag{4.26}$$

respectively, and the third requiring one of (d), (e) and (f) to hold. By (4.19),

$$A'' := A' \backslash \{q\} \in \mathcal{A}]p[\langle q\rangle. \tag{4.27}$$

Since $A'' \cup \{p\} = A^* \notin \mathcal{A}$, $A'' \notin \mathcal{A}\langle p\rangle]q[$. So $\mathcal{A}\langle p\rangle]q[\neq \mathcal{A}]p[\langle q\rangle$, implying (f) does not hold. Suppose (e) holds. So $\mathcal{A}]p[\langle q\rangle = \{E\} \subset \mathcal{A}\langle p\rangle]q[$. By (4.27), $E = A''$. So $A'' \cup \{p\} \in \mathcal{A}(p)]q[$, which is a contradiction since $A'' \cup \{p\} = A^* \notin \mathcal{A}$. So (d) holds. By (4.22), $D = C'$. So we have $\mathcal{A}\langle p\rangle]q[= \{C'\}$, $\mathcal{A}]p[\langle q\rangle = \mathcal{Z}^{(2)}(C')$. It follows by (4.26) that $\mathcal{A} = \{A \in \mathcal{H}^{(3)} \colon |A \cap (C' \cup \{q\})| \geq 2\}$. By (i) and (4.25), we must have $\mathcal{A} = \{A \in \binom{[n]}{3} \colon |A \cap (C' \cup \{q\})| \geq 2\}$, and hence $C' \cup \{q\} \in \mathcal{H}^{(3)}$. □

4.7 Proof of Theorem 4.1.4

We finally come to the proof of the main result in this chapter, i.e. Theorem 4.1.4. We start by stating another well-known fact that emerged in [25], the proof of which is similar to that of Lemma 4.5.1(ii).

Lemma 4.7.1 (Erdős, Ko, Rado [25]) *If $\mathcal{A} \subseteq \binom{[n]}{r}$ is intersecting, $r \leq n/2$, and $\Delta_{a,n}(\mathcal{A}) = \mathcal{A}$ for all $a \in [n-1]$ then $(A \cap B)\backslash\{n\} \neq \emptyset$ for any $A, B \in \mathcal{A}$.*

Proposition 4.7.2 *Let $\mathcal{H} \subseteq 2^{[n]}$ be hereditary and compressed, $\mathcal{H}(n) \neq \emptyset$. Suppose $2 \leq r \leq \mu(\mathcal{H})/2$. Let $\mathcal{A} \subseteq \mathcal{H}^{(r)}$ be compressed and intersecting, and let $k \geq r+1$, $K := [2,k]$, ${}_1\mathcal{K} := \mathcal{H}^{(r)}(1)(K) \cup \{K\}$ and ${}_2\mathcal{K} := \mathcal{H}^{(r)}(1)$.*
(i) If $k = r+1$ then $|\mathcal{A}(K)| \leq |{}_1\mathcal{K}(K)|$ $(= |{}_1\mathcal{K}|)$.
(ii) If $k \geq r+2$ then $|\mathcal{A}(K)| \leq |{}_2\mathcal{K}(K)|$.
If moreover $r < n/2$ and $\mathcal{A}$ is such that equality in (i) or (ii) holds then:
(iii) if $k = r+1$ then either $\mathcal{A} = {}_1\mathcal{K}$ or $\mathcal{A} = \{A \in \mathcal{H}^{(3)} \colon |A \cap [3]| \geq 2\}$;
(iv) if $k \geq r+2$ then either ${}_2\mathcal{K}(K) \subseteq \mathcal{A} \subseteq {}_2\mathcal{K}$ or $k = 3$ and $\mathcal{A} = \binom{[3]}{2}$.

Proof. Suppose $r = n/2$. Then $\mu(\mathcal{H}) = 2r$, and hence $[2r] \in \mathcal{H}$. So $\mathcal{H}^{(r)} = \binom{[2r]}{r}$ (as $\mathcal{H}$ is hereditary). For every $A \in \binom{[2r]}{r}$, $A' := [2r]\backslash A$ is the unique set in $\binom{[2r]}{r}$ such that $A \cap A' = \emptyset$. So $|\mathcal{A}| \leq \frac{1}{2}\binom{2r}{r} = |{}_1\mathcal{K}(K)|$. Since $|{}_1\mathcal{K}(K)| = |{}_2\mathcal{K}(K)|$ for $k \geq r+2$, (i) and (ii) follow.

Given that $\mathcal{A}$ is a compressed intersecting family, it is trivial that if $r = 2$ then $\mathcal{A} = \binom{[3]}{2}$ or $\mathcal{A} \subseteq \mathcal{H}^{(2)}(1)$, depending on whether $\mathcal{A}$ is non-centred or centred respectively. So the result for $r = 2$ is easy to check.

We now consider $3 \leq r < n/2$, and we proceed by induction on n. Let $n' := n - 1$. Clearly, $\mathcal{H}]n[$ and $\mathcal{H}\langle n\rangle$ are compressed hereditary sub-families of $2^{[n']}$. We have $\mathcal{A}]n[\subset \mathcal{H}]n[^{(r)}$ and $\mathcal{A}\langle n\rangle \subset \mathcal{H}\langle n\rangle^{(r')}$, $r' = r - 1$. Since $\mathcal{H}(n) \neq \emptyset$ and $r \leq \mu(\mathcal{H})/2$, it follows by Lemma 4.4.1(i) that $r' < \mu(\mathcal{H}\langle n\rangle)/2$. If $[n] \in \mathcal{H}$ then $n = \mu(\mathcal{H}) = \mu(\mathcal{H}]n[) + 1$, and hence $r \leq \mu(\mathcal{H}]n[)/2$ as $r < n/2$. If instead $[n] \notin \mathcal{H}$ then $r \leq \mu(\mathcal{H}]n[)/2$ follows from Lemma 4.4.1(iii) and $r \leq \mu(\mathcal{H})/2$. Thus, by the inductive hypothesis,

$$|\mathcal{A}]n[(K)| \leq |\mathcal{K}]n[(K)| \text{ and } |\mathcal{A}\langle n\rangle(K)| \leq |{}_2\mathcal{K}\langle n\rangle(K)|,$$

where

$$\mathcal{K} = \begin{cases} {}_1\mathcal{K} & \text{if } k = r+1; \\ {}_2\mathcal{K} & \text{if } k \geq r+2. \end{cases}$$

It is clear that we therefore have

$$|\mathcal{A}(K)| = |\mathcal{A}]n[(K)| + |\mathcal{A}\langle n\rangle(K)| \leq |\mathcal{K}]n[(K)| + |{}_2\mathcal{K}\langle n\rangle(K)| = |\mathcal{K}(K)|,$$

and hence (i) and (ii).

We now prove (iii) and (iv). So consider $|\mathcal{A}| = |\mathcal{K}(K)|$. Then $|\mathcal{A}]n[(K)| = |\mathcal{K}]n[(K)|$ and $|\mathcal{A}\langle n\rangle(K)| = |{}_2\mathcal{K}\langle n\rangle(K)|$. It follows by the inductive hypothesis that

(g) ${}_2\mathcal{K}\langle n\rangle(K) \subseteq \mathcal{A}\langle n\rangle$ or

(h) $k = 3$ and $\mathcal{A}\langle n\rangle = \binom{[3]}{2}$.

Suppose (g) holds. Since $r' < \mu(\mathcal{H}\langle n\rangle)/2$, $\mu(\mathcal{H}\langle n\rangle) \geq 2r - 1$. Thus, by (4.2), we have $2^{[2r-1]} \subset \mathcal{H}\langle n\rangle$, which gives us $\binom{[2r-1]}{r'}(1) \subseteq {}_2\mathcal{K}\langle n\rangle$. Since $\mathcal{A}\langle n\rangle \supseteq {}_2\mathcal{K}\langle n\rangle(K)$ (by (g)) and $2r-1 = r+r'$, it follows that

$$\mathcal{A}\langle n\rangle \supseteq \begin{cases} \binom{[2r-1]}{r'}(1)\backslash\{[2r-1]\backslash K\} & \text{if } k = r+1; \\ \binom{[2r-1]}{r'}(1) & \text{if } k \geq r+2, \end{cases}$$

Therefore, if $H \in \mathcal{H}^{(r)}\backslash({}_2\mathcal{K} \cup \{K\})$ then there exists $L \in \binom{[2r-1]}{r'}(1) \cap \mathcal{A}\langle n\rangle$ such that $L \cap H = \emptyset$, and hence $H \notin \mathcal{A}$ by Lemma 4.7.1. So

$$\mathcal{A} \subseteq \begin{cases} {}_2\mathcal{K}(K) \cup \{K\} = {}_1\mathcal{K} & \text{if } k = r+1 \text{ and } K \in \mathcal{A}; \\ {}_2\mathcal{K} & \text{if } k \geq r+2 \text{ or } K \notin \mathcal{A}, \end{cases}$$

i.e. $\mathcal{A} \subseteq \mathcal{K}$. Therefore, since $|\mathcal{A}(K)| = |\mathcal{K}(K)|$, if $k = r+1$ then $\mathcal{A} = {}_1\mathcal{K}$, and if $k \geq r+2$ then ${}_2\mathcal{K}(K) \subseteq \mathcal{A} \subseteq {}_2\mathcal{K}$.

Now suppose (h) holds. So $r = k = 3$. If $H \in \mathcal{H}$ and $|H \cap [3]| \leq 1$ then, since $\mathcal{A}\langle n\rangle = \binom{[3]}{2}$, there exists $L \in \mathcal{A}\langle n\rangle$ such that $H \cap L = \emptyset$, and hence $H \notin \mathcal{A}$ by Lemma 4.7.1. So $\mathcal{A} \subseteq \mathcal{T} := \{A \in \mathcal{H}^{(3)} \colon |A \cap [3]| \geq 2\}$. Note that $\mathcal{T}$ is compressed since $\mathcal{H}$ is compressed. Since $|\mathcal{T}| = |\mathcal{T}(K)| \leq |\mathcal{K}(K)| = |\mathcal{A}(K)| = |\mathcal{A}|$ (where the inequality is given by (i) and (ii)), we actually have $\mathcal{A} = \mathcal{T}$. □

Proof of Theorem 4.1.4. By Corollary 4.3.5, we need only consider $\mathcal{A} \subset \mathcal{H}^{(r)}$. The case $r = n/2$ follows by an argument similar to the one given in the proof of Proposition 4.7.2 for the same case. So we assume $r < n/2$.

By (4.2) and the given condition that $\mu(\mathcal{H}) \geq 2r \geq 4$, we have $[3], [2, r+1] \in 2^{[2r]} \subset \mathcal{H}$. So $[3] \in \mathcal{T} := \{A \in \mathcal{H}^{(3)} \colon |A \cap [3]| \geq 2\}$ and $\mathcal{N}$ is a non-centred intersecting sub-family of $\mathcal{H}^{(r)}$.

We may assume that $\mathcal{A}$ is a non-centred intersecting sub-family of largest size. So

$$|\mathcal{A}| \geq |\mathcal{N}|. \tag{4.28}$$

If $\mathcal{A}$ is not compressed then, similarly to the proof of Theorem 4.1.3, we apply compressions $\Delta_{p,q}$, $p < q$, until a compressed family $\mathcal{A}^*$ is obtained. $\mathcal{A}^*$ is intersecting by Lemma 4.4.5, and $\mathcal{A}^* \subset \mathcal{H}^{(r)}$ as $\mathcal{H}$ is compressed.

Suppose $\mathcal{A}^*$ is centred. By Proposition 4.6.2, $|\mathcal{A}^*| \leq |\mathcal{H}^{(r)}(p^*)(I \cup \{q^*\})|$ for some $p^*, q^* \in [n]$, $p^* < q^*$, where I is an initial $(r-1)$-segment of $[n]\backslash\{p^*, q^*\}$. Thus, by Lemma 4.6.4, $|\mathcal{A}^*| \leq |\mathcal{H}^{(r)}(1)([2, r+1])| = |\mathcal{N}| - 1$, a contradiction to (4.28). So $\mathcal{A}^*$ is non-centred, and hence $[2, r+1] \in \mathcal{A}^*$ as $\mathcal{A}^*$ is compressed. Thus, since $\mathcal{A}^*$ is intersecting, we have $|\mathcal{A}^*| = |\mathcal{A}^*([2, r+1])|$ and, by Proposition 4.7.2(i),

$$|\mathcal{A}^*([2, r+1])| \leq |\mathcal{N}([2, r+1])|. \tag{4.29}$$

Since $|\mathcal{A}| = |\mathcal{A}^*|$ and $\mathcal{N}([2, r+1]) = \mathcal{N}$, part (i) of our result follows. By (4.28) and (4.29), $|\mathcal{A}^*([2, r+1])| = |\mathcal{N}([2, r+1])|$. By Proposition 4.7.2(iii), $\mathcal{A}^* = \mathcal{N}$ or $\mathcal{A}^* = \mathcal{T}$. By Propositions 4.6.5 and 4.6.6, part (ii) of our result follows. □

Chapter 5

Intersecting systems of signed sets

5.1 The motivating conjecture

As was mentioned in Section 1.4, Theorem 1.4.3 with $r > n/2$ provides an example of a family $\mathcal{F}$ such that $\mathcal{S}_{\mathcal{F},k}$ is EKR but $\mathcal{F}$ is not. This chapter is motivated by the question "Do families $\mathcal{F}$ such that $\mathcal{S}_{\mathcal{F},k}$ is not EKR for some $k \geq 2$ exist after all?". We conjecture that the answer is "no", and, as is described below, we will present some strong evidence for this conjecture. This conjecture has some resemblance with the famous Chvátal Conjecture, i.e. Conjecture 1.3.1. Indeed, let $\mathcal{F}$ be a family, and let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{S}_{\mathcal{F},k}$. In view of a result we present later, stated as Corollary 5.3.2, we may assume that the any two sets in $\mathcal{A}$ intersect on $X := (\bigcup_{G \in \mathcal{F}} G) \times [1]$. Thus, by defining $\mathcal{A}' := \{A \cap (X \times [1]) \colon A \in \mathcal{A}\}$, we may further assume that $\mathcal{A}$ is the family of all sets in $\mathcal{S}_{\mathcal{F},k}$ that contain a set that is a member of $\mathcal{A}'$. Note that $\mathcal{A}'$ is a sub-family of the hereditary family $\{H \times [1] \colon H \subseteq G \text{ for some } G \in \mathcal{F}\}$. So our conjecture can be vaguely described as a "weighted" version of the Chvátal Conjecture, where, however, the distribution of the "weights" does not seem to exhibit any characteristic by which we can reduce one conjecture to the other.

We are also concerned with strict and non-strict EKR cases. For this reason, it is convenient to introduce the following notation, using some notation from Section 2.1, before stating the conjecture fully and formally.

For any $u \in U(\mathcal{F})$, set

$$\mathrm{O}_{\mathcal{F}}(u) := \{u' \in U(\mathcal{F})\colon \mathcal{F}(u') = \mathcal{F}(u)\}.$$

We call $\mathrm{O}_{\mathcal{F}}(u)$ the *$\mathcal{F}$-orbit of u*. Thus, two elements $u, u' \in U(\mathcal{F})$ are in the same $\mathcal{F}$-orbit iff they belong to exactly the same members of $\mathcal{F}$. Recall from Section 2.1 that $L(\mathcal{F})$ is the set $\{u \in U(\mathcal{F})\colon \mathcal{F}(u)$ is a largest star of $\mathcal{F}\}$.

Conjecture 5.1.1 *Let $\mathcal{F}$ be any family of sets, and let $k \geq 2$. Then*
(i) $\mathcal{S}_{\mathcal{F},k}$ is EKR;
(ii) $\mathcal{S}_{\mathcal{F},k}$ is strictly EKR unless $k = 2$ and $|\mathrm{O}_{\mathcal{F}}(u)| \geq 3$ for some $(u, 1) \in L(\mathcal{S}_{\mathcal{F},2})$.

As we show in Chapter 6, it is not very difficult to prove that for an integer $k_0(\mathcal{F})$, the above conjecture indeed holds if $k \geq k_0(\mathcal{F})$. In Chapter 7, we go beyond this by showing that for k sufficiently large, an even stronger statement for t-intersecting families of signed sets is true. The main result here provides a strong generalisation of Theorem 1.4.3 by establishing the truth of our conjecture for families $\mathcal{F}$ that are compressed with respect to an element.

Theorem 5.1.2 *Conjecture 5.1.1 is true if $\mathcal{F} \subseteq 2^{[n]}$ is compressed with respect to* 1.

We also confirm Conjecture 5.1.1(ii) for families $\mathcal{F}$ as in Theorem 1.4.2.

Theorem 5.1.3 *Conjecture 5.1.1 is true if $\mathcal{F}$ is r-uniform and EKR.*

We now proceed to the proofs, employing the notation in Section 2.1 and the notation $\theta_k(A)$ and $\theta_k^q(A)$ introduced in Section 1.4 as we go along.

5.2 An auxiliary result for the special case $k = 2$

We shall generalise (1.1) in the following direction.

Theorem 5.2.1 *Let $I \subseteq \mathbb{N}$. For each $i \in I$, let X_i be a finite set and let $\mathcal{A}_i \subset \mathcal{S}_{X_i,2}$. Suppose $\bigcup_{i \in I} \mathcal{A}_i$ is intersecting. Then $\mathcal{A}_i \in \mathrm{ex}(\mathcal{S}_{X_i,2})$ holds for all $i \in I$ iff $X := \bigcap_{i \in I} X_i \neq \emptyset$ and, for some $\mathcal{X} \in \mathrm{ex}(\mathcal{S}_{X,2})$ and for all $i \in I$, $\mathcal{A}_i = \{A \in \mathcal{S}_{X_i,2}\colon A \cap (X \times [2]) \in \mathcal{X}\}$.*

Proof. The sufficiency condition is straightforward since then, for all $i \in I$, $\mathcal{A}_i$ satisfies (1.1). Now assume that for all $i \in I$, $\mathcal{A}_i \in \mathrm{ex}(\mathcal{S}_{X_i,2})$. We prove the necessary condition by induction on $m := \min\{|X_i| \colon i \in I\}$.

Suppose there exist $i_1, i_2, l \in I$ such that $A_{i_1} \cap A_{i_2} \cap (X_l \times [2]) = \emptyset$ for some $A_{i_1} \in \mathcal{A}_{i_1}$ and $A_{i_2} \in \mathcal{A}_{i_2}$. Let $B_p := A_{i_p} \cap (X_l \times [2])$, $p = 1, 2$. Let $Y := X_l \backslash (X_{i_1} \cup X_{i_2})$. If $Y \neq \emptyset$ then choose $C \in \mathcal{S}_{Y,2}$, otherwise take $C := \emptyset$. Let $D := B_1 \backslash (X_{i_2} \times [2])$ and $E := B_2 \cup \theta_2(D) \cup C \in \mathcal{S}_{X_l,2}$. So $E \cap A_{i_1} = \emptyset$ and $\theta_2(E) \cap A_{i_2} = \emptyset$. Therefore $E, \theta_2(E) \notin \mathcal{A}_l$, and hence, by (1.1), we have $\mathcal{A}_l \notin \mathrm{ex}(\mathcal{S}_{X_l,2})$, a contradiction. Thus,

$$\text{for any } A, B \in \textstyle\bigcup_{i \in I} \mathcal{A}_i \text{ and } l \in I,\ A \cap B \cap (X_l \times [2]) \neq \emptyset. \tag{5.1}$$

Let $\mathcal{X} := \{A \cap (X \times [2]) \colon A \in \bigcup_{i \in I} \mathcal{A}_i\}$ $(X := \bigcap_{i \in I} X_i)$. Let $j \in I$ such that $|X_j| = m$.

Suppose $X = X_j$. So $\mathcal{X} \supseteq \mathcal{A}_j$ and, by (5.1) (with $l = j$), $\mathcal{X}$ is intersecting. Since $\mathcal{A}_j \in \mathrm{ex}(\mathcal{S}_{X_j,2})$, it follows that $\mathcal{X} = \mathcal{A}_j$, and hence result. It will be clear from the following that we have also just covered the basis of induction $m = 1$.

Now suppose $X \neq X_j$. So there exists $h \in I$ such that $X_j \backslash X_h \neq \emptyset$. Let $x_j \in X_j \backslash X_h$. Let $X_j' := X_j \backslash \{x_j\}$, and for each $i \in I \backslash \{j\}$, let $X_i' := X_i$. So $\bigcap_{i \in I} X_i' = X$. If $X_j' = \emptyset$ then $X_j \cap X_h = \emptyset$, and hence $\mathcal{A}_j = \emptyset$ or $\mathcal{A}_h = \emptyset$, a contradiction. Therefore $m' := \min\{|X_i'| \colon i \in I\} = |X_j'| = m - 1 \geq 1$. For each $i \in I \backslash \{j\}$, we have $\mathcal{A}_i' := \mathcal{A}_i \in \mathrm{ex}(\mathcal{S}_{X_i',2})$. Let $\mathcal{A}_j' := \{A \backslash \{(x_j, 1), (x_j, 2)\} \colon A \in \mathcal{A}_j\} \subseteq \mathcal{S}_{X_j',2}$. By (5.1), for any $A, B \in \bigcup_{i \in I} \mathcal{A}_i$, $A \cap B \cap (X_h \times [2]) \neq \emptyset$; hence $\bigcup_{i \in I} \mathcal{A}_i'$ is intersecting. Suppose $\mathcal{A}_j' \notin \mathrm{ex}(\mathcal{S}_{X_j',2})$. Then, by (1.1), $A_j', \theta_2(A_j') \notin \mathcal{A}_j'$ for some $A_j' \in \mathcal{S}_{X_j',2}$. So $A_j := A_j' \cup \{x_j, 1\} \notin \mathcal{A}_j$ and $\theta_2(A_j) \notin \mathcal{A}_j$, which, by (1.1), contradicts $\mathcal{A}_j \notin \mathrm{ex}(\mathcal{S}_{X_j,2})$. Hence $\mathcal{A}_j' \in \mathrm{ex}(\mathcal{S}_{X_j',2})$. Clearly, the result follows immediately after applying the inductive hypothesis for the families $\mathcal{A}_i'$, $i \in I$. □

Corollary 5.2.2 *If $f \in U(\mathcal{F})$ and $\mathcal{A} \in \mathrm{ex}(\mathcal{S}_{\mathcal{F}(f),2})$ then:*
(i) $\mathcal{A} = \{F \in \mathcal{F} \colon F \cap (\mathrm{O}_{\mathcal{F}(f)}(f) \times [2]) \in \mathcal{X}\}$ for some $\mathcal{X} \in \mathrm{ex}(\mathcal{S}_{\mathrm{O}_{\mathcal{F}(f)}(f),2})$;
(ii) if $|\mathrm{O}_{\mathcal{F}(f)}(f)| \leq 2$ then $\mathcal{A} = \mathcal{S}_{\mathcal{F},2}((f', b))$ for some $(f', b) \in \mathrm{O}_{\mathcal{F}(f)}(f) \times [2]$.

Proof. Let $\mathcal{B} := \mathcal{S}_{\mathcal{F},2}((f,1))$. Set $n := |\mathcal{F}(f)|$, and let $X_1, ..., X_n$ be the sets in $\mathcal{F}(f)$. For each $i \in [n]$, let $\mathcal{A}_i := \mathcal{A} \cap \mathcal{S}_{X_i,2}$, $\mathcal{B}_i := \mathcal{B} \cap \mathcal{S}_{X_i,2}$. So $\bigcup_{i=1}^n \mathcal{A}_i$ and $\bigcup_{i=1}^n \mathcal{B}_i$ are partitions of $\mathcal{A}$ and $\mathcal{B}$ respectively. By (1.1), $\mathcal{B}_i \in \mathrm{ex}(\mathcal{S}_{X_i,2})$, $i = 1, ..., n$, and hence $\mathcal{B} \in \mathrm{ex}(\mathcal{S}_{\mathcal{F}(f),2})$. Thus, if $\mathcal{A}_i \notin \mathrm{ex}(\mathcal{S}_{X_i,2})$ for some $i \in [n]$ then $|\mathcal{A}| < |\mathcal{B}|$, but this contradicts $\mathcal{A} \in \mathrm{ex}(\mathcal{S}_{\mathcal{F}(f),2})$. So $\mathcal{A}_i \in \mathrm{ex}(\mathcal{S}_{X_i,2})$, $i = 1, ..., n$. By Theorem 5.2.1, for some $\mathcal{X} \in \mathrm{ex}(\mathcal{S}_{X,2})$ and for all $i \in [n]$, $\mathcal{A}_i = \{A \in \mathcal{S}_{X_i,2} \colon A \cap (X \times 2) \in \mathcal{X})\}$, where $X = \bigcap_{i=1}^n X_i \ni f$. So (i) follows if $X = \mathrm{O}_{\mathcal{F}(f)}(f)$. Let $x \in U(\mathcal{F}(f))$. If $x \notin X$ then there exists $j \in [n]$ such that $x \notin X_j$, and hence $x \notin \mathrm{O}_{\mathcal{F}(f)}(f)$; contrapositively, if $x \in \mathrm{O}_{\mathcal{F}(f)}(f)$ then $x \in X$. So $\mathrm{O}_{\mathcal{F}(f)}(f) \subseteq X$. If $x \in X$ then $\mathcal{F}(f)(x) = \mathcal{F}(f)$. So $X \subseteq \mathrm{O}_{\mathcal{F}(f)}(f)$, and hence $X = \mathrm{O}_{\mathcal{F}(f)}(f)$ indeed.

Suppose $|\mathrm{O}_{\mathcal{F}(f)}(f)| \leq 2$. So $1 \leq |\mathrm{O}_{\mathcal{F}(f)}(f)| = |X| \leq 2$, and it is trivial that $\mathcal{X}$ can only be a star in this case. Hence (ii). □

The strict and non-strict EKR cases for $k = 2$ in each of Theorems 5.1.2 and 5.1.3 will be determined using Theorem 5.2.1, Corollary 5.2.2 and the following fact.

Proposition 5.2.3 *If $|\mathrm{O}_{\mathcal{F}}(f)| \geq 3$ for some $(f,1) \in L(\mathcal{S}_{\mathcal{F},2})$ then $\mathcal{S}_{\mathcal{F},2}$ is not strictly EKR.*

Proof. We have $\mathcal{F}(f_1) = \mathcal{F}(f_2) = \mathcal{F}(f_3) = \mathcal{F}(f)$ for some distinct $f_1, f_2, f_3 \in \mathrm{O}_{\mathcal{F}}(f)$ (possibly, $f \in \{f_1, f_2, f_3\}$). It follows that for all $F \in \mathcal{F}(f)$, $f_1, f_2, f_3 \in F$. Define $Y_1 := \{(f_1,1), (f_2,1), (f_3,1)\}$, $Y_2 := \{(f_1,1), (f_2,1), (f_3,2)\}$, $Y_3 := \{(f_1,1), (f_2,2), (f_3,1)\}$, $Y_4 := \{(f_1,2), (f_2,1), (f_3,1)\}$. Then $\bigcup_{F \in \mathcal{F}} \{Y_i \cup Z \colon i \in [4], Z \in \mathcal{S}_{F \setminus \{f_1,f_2,f_3\},2}\}$ is a non-trivial intersecting family that is as large as $\mathcal{S}_{\mathcal{F},2}((f,1))$, a largest star of $\mathcal{S}_{\mathcal{F},2}$. □

5.3 Two compression methods

The proof of Theorem 5.1.2 will be based on two different compression methods.

The first compression operation was used in [22] for the proof of Theorem 1.4.3. For $(a,b) \in [n] \times [2,k]$, instead of writing $\delta_{(a,1),(a,b)}$ and $\Delta_{(a,1),(a,b)}$ we will write $\delta_{a,b}$ and

$\Delta_{a,b}$ respectively; so $\Delta_{a,b}\colon 2^{\mathcal{S}_{2[n],k}} \to 2^{\mathcal{S}_{2[n],k}}$ is defined by

$$\Delta_{a,b}(\mathcal{A}) := \{\delta_{a,b}(A)\colon A \in \mathcal{A}\} \cup \{A \in \mathcal{A}\colon \delta_{a,b}(A) \in \mathcal{A}\},$$

where $\delta_{a,b}\colon \mathcal{S}_{2[n],k} \to \mathcal{S}_{2[n],k}$ is defined by

$$\delta_{a,b}(A) := \begin{cases} A\backslash\{(a,b)\} \cup \{(a,1)\} & \text{if } (a,b) \in A; \\ A & \text{otherwise} \end{cases}$$

Clearly, $\mathcal{S}_{2[n],k}$ is $((a,1),(a,b))$-compressed. So Proposition 2.2.1(ii) tells us that if $\mathcal{A} \subset \mathcal{S}_{2[n],k}$ is t-intersecting then $\Delta_{a,b}(\mathcal{A})$ (as defined here) is t-intersecting. We now prove a bit more than this, and we shall also use the following result in the next two chapters.

Lemma 5.3.1 *Let $\mathcal{A} \subset \mathcal{S}_{2[n],k}$ and $V \subseteq [n] \times [2,k]$ such that $|(A \cap B)\backslash V| \geq t$ for any $A, B \in \mathcal{A}$. Then $|(C \cap D)\backslash(V \cup \{(a,b)\})| \geq t$ for any $C, D \in \Delta_{a,b}(\mathcal{A})$.*

Proof. Let $C, D \in \Delta_{a,b}(\mathcal{A})$. Let $C' := (C\backslash\{(a,1)\}) \cup \{(a,b)\}$, $D' := (D\backslash\{(a,1)\}) \cup \{(a,b)\}$. Suppose $|(C \cap D)\backslash V| < t$. So C and D cannot both be in $\mathcal{A}$. Suppose $C, D \notin \mathcal{A}$; then $C', D' \in \mathcal{A}$ and $|(C' \cap D')\backslash V| \leq |(C \cap D)\backslash V| < t$, a contradiction. Thus, without loss of generality, $C \notin \mathcal{A}$ and $D \in \mathcal{A}$. So $C' \in \mathcal{A}$. If $(a,b) \notin D$ then $|(C' \cap D)\backslash V| = |(C \cap D)\backslash V| < t$, contradicting $C', D \in \mathcal{A}$. So $(a,b) \in D$, and hence $\delta_{a,b}(D) \in \mathcal{A}$ (otherwise $D \notin \Delta_{a,b}(\mathcal{A})$). But then $|(C' \cap \delta_{a,b}(D))\backslash V| = |(C \cap D)\backslash V| < t$, contradicting $C', \delta_{a,b}(D) \in \mathcal{A}$. We therefore conclude that $|(C \cap D)\backslash V| \geq t$.

Now suppose $|(C \cap D)\backslash(V \cup \{(a,b)\})| < t$. Since $|(C \cap D)\backslash V| \geq t$, $(a,b) \in C \cap D$. Hence $C, \delta_{a,b}(C), D, \delta_{a,b}(D) \in \mathcal{A}$ and $(C \cap \delta_{a,b}(D))\backslash V = \emptyset$, a contradiction. □

Corollary 5.3.2 *Let $\mathcal{A}^*$ be a t-intersecting sub-family of $\mathcal{S}_{2[n],k}$. Let*

$$\mathcal{A} := \Delta_{n,k} \circ \dots \circ \Delta_{n,2} \circ \dots \circ \Delta_{1,k} \circ \dots \circ \Delta_{1,2}(\mathcal{A}^*).$$

Then $|A \cap B \cap ([n] \times [1])| \geq t$ for any $A, B \in \mathcal{A}$.

Proof. By repeated application of Lemma 5.3.1, $|(A \cap B)\backslash([n] \times [2,k])| \geq t$ for any $A, B \in \mathcal{A}$. The result follows since $(A \cap B)\backslash([n] \times [2,k]) = A \cap B \cap ([n] \times [1])$. □

We next introduce our second compression operation. We set $K := ([1] \times [k]) \cup \{(n,1)\}$ and define $\lambda \colon \mathcal{S}_{2^{[n]},k} \longrightarrow \mathcal{S}_{2^{[n]},k}$ by

$$\lambda(A) := \begin{cases} A\backslash\{(n,1)\} \cup \{(1,1)\} & \text{if } A \cap K = \{(n,1)\}; \\ A\backslash\{(1,b),(n,1)\} \cup \{(1,1),(n,b)\} & \text{if } A \cap K = \{(1,b),(n,1)\}; \\ A & \text{otherwise.} \end{cases}$$

Similarly to $\Delta_{a,b}$, we define $\Lambda \colon 2^{\mathcal{S}_{2^{[n]},k}} \longrightarrow 2^{\mathcal{S}_{2^{[n]},k}}$ by

$$\Lambda(\mathcal{A}) := \{\lambda(A) \colon A \in \mathcal{A}\} \cup \{A \in \mathcal{A} \colon \lambda(A) \in \mathcal{A}\}.$$

At this point, we need to introduce some further notation. For $A \in \mathcal{S}_{F,k}$, let

$$\gamma(A) := F.$$

For $\mathcal{A} \subseteq \mathcal{S}_{\mathcal{F},k}$, let $\Gamma(\mathcal{A})$ be the sub-family of $\mathcal{F}$ given by

$$\Gamma(\mathcal{A}) := \{\gamma(A) \colon A \in \mathcal{A}\} = \{F \in \mathcal{F} \colon \mathcal{A} \cap \mathcal{S}_{F,k} \neq \emptyset\}.$$

Lemma 5.3.3 *Let $\mathcal{F} \subseteq 2^{[n]}$ be compressed with respect to 1. Let $\mathcal{A} \subset \mathcal{S}_{\mathcal{F},k}$. Then:*
(i) $|\Lambda(\mathcal{A})| = |\mathcal{A}|$;
(ii) $\Lambda(\mathcal{A}) \subset \mathcal{S}_{\mathcal{F},k}$;
(iii) if $A \cap A' \cap ([n] \times [1]) \neq \emptyset$ for any $A, A' \in \mathcal{A}$ then $B \cap B' \cap ([n] \times [1]) \neq \emptyset$ for any $B, B' \in \Lambda(\mathcal{A})$.

Proof. (i) is straightforward.

If $A \in \mathcal{S}_{\mathcal{F},k}$ and $\gamma(\lambda(A)) \neq \gamma(A)$ then $n \in \gamma(A)$, $1 \notin \gamma(A)$ and $\gamma(\lambda(A)) = (\gamma(A)\backslash\{n\}) \cup \{1\}$. Since $\mathcal{F}$ is compressed with respect to 1, it follows that $\Gamma(\Lambda(\mathcal{A})) \subseteq \mathcal{F}$. Hence (ii).

Suppose $A \cap A' \cap ([n] \times [1]) \neq \emptyset$ for any $A, A' \in \mathcal{A}$. Let $B_1, B_2 \in \Lambda(\mathcal{A})$. Then, for each $p \in [2]$, $B_p = A_p$ or $B_p = \lambda(A_p)$ for some $A_p \in \mathcal{A}$. It is straightforward that (iii) holds if $B_p = A_p$, $p = 1, 2$, or $B_p = \lambda(A_p)$, $p = 1, 2$. Without loss of generality, suppose $B_1 = A_1$, $B_2 = \lambda(A_2) \neq A_2$ and $B_1 \cap B_2 \cap ([n] \times [1]) = \emptyset$. It follows that $A_1 \cap A_2 \cap ([n] \times [1]) = \{(n, 1)\}$ and $A_1 \neq \lambda(A_1) \in \mathcal{A}$. But then $\lambda(A_1) \cap A_2 \cap ([n] \times [1]) = \emptyset$, a contradiction. Hence (iii). □

5.4 Proof of main result

We here prove Theorem 5.1.2. In the process of doing so, we also determine the extremal intersecting sub-families of $\mathcal{S}_{\mathcal{F},k}$ for $\mathcal{F}$ as in the theorem.

Theorem 5.4.1 *Let $1 \in J \subseteq [n]$. Let $\mathcal{F} \subseteq 2^{[n]}$ such that $\mathcal{F}$ is compressed with respect to $j \in [n]$ iff $j \in J$. Let $\mathcal{A}^* \subset \mathcal{S}_{\mathcal{F},k}$ be intersecting. Then*
(i) $|\mathcal{A}^*| \leq |\mathcal{S}_{\mathcal{F},k}((1, 1))|$, and
(ii) equality holds iff $\mathcal{A}^* = \mathcal{S}_{\mathcal{F},k}((j, b))$, $(j, b) \in J \times [k]$, or $k = 2$, $|\mathrm{O}_{\mathcal{F}}(1)| > 1$ and $\mathcal{A}^* = \{F \in \mathcal{F} \colon F \cap (\mathrm{O}_{\mathcal{F}}(1) \times [2]) \in \mathcal{X}\}$, $\mathcal{X} \in \mathrm{ex}(\mathcal{S}_{\mathrm{O}_{\mathcal{F}}(1),2})$.

Proof of Theorem 5.4.1(i). The case $n = 2$ is trivial, so we assume $n > 2$. Let $\mathcal{A}' := \Delta_{n,k} \circ \ldots \circ \Delta_{n,2} \circ \ldots \circ \Delta_{1,k} \circ \ldots \circ \Delta_{1,2}(\mathcal{A}^*)$, and let $\mathcal{A} := \Lambda(\mathcal{A}')$. So $|\mathcal{A}| = |\mathcal{A}^*|$ and, by Corollary 5.3.2 and Lemma 5.3.3,

$$A \cap B \cap Z \neq \emptyset \text{ for any } A, B \in \mathcal{A}, \tag{5.2}$$

where $Z := [n] \times [1]$.

Let $\mathcal{B} := \{A \in \mathcal{A}((n, 1))((1, 1)) \colon A \cap B \cap Z = \{(n, 1)\} \text{ for some } B \in \mathcal{A}((n, 1))\}$. Let $\mathcal{A}_1 := \mathcal{A}((n, 1)) \backslash \mathcal{B}$. For $l \in [2, k]$, let $\mathcal{B}_l := \{(A \backslash \{(n, 1)\}) \cup \{n, l\} \colon A \in \mathcal{B}\}$ and $\mathcal{A}_l := \mathcal{A}((n, l)) \cup \mathcal{B}_l$. If $\mathcal{A}((n, l)) \cap \mathcal{B}_l \neq \emptyset$ and $A \in \mathcal{A}((n, l)) \cap \mathcal{B}_l$ then $\delta_{n,l}(A) \cap B \cap Z = \{(n, 1)\}$ for some $B \in \mathcal{A}((n, 1))$, and hence $A \cap B \cap Z = \emptyset$, a contradiction to (5.2). So $\mathcal{A}((n, l)) \cap \mathcal{B}_l = \emptyset$. Therefore

$$\sum_{i=1}^{k} |\mathcal{A}((n, i))| \leq (|\mathcal{A}((n, 1))| - |\mathcal{B}|) + \sum_{l=2}^{k} (|\mathcal{A}((n, l))| + |\mathcal{B}|) = \sum_{i=1}^{k} |\mathcal{A}_i|. \tag{5.3}$$

Let $Z' := [n-1] \times [1]$. Suppose that for some $i \in [k]$ and $A, B \in \mathcal{A}_i$, $A \cap B \cap Z' = \emptyset$. It is immediate by (5.2) that $i \notin [2, k]$. So $A, B \in \mathcal{A}_1$ and $A \cap B = \{(n, 1)\}$. By definition of $\mathcal{A}_1$, $(1, 1) \notin A \cup B$. But $\lambda(A) \in \mathcal{A}$ and $\lambda(A) \cap B \cap Z = \emptyset$, a contradiction to (5.2). Thus,

$$\text{for any } i \in [k] \text{ and } A, B \in \mathcal{A}_i,\ A \cap B \cap Z' \neq \emptyset. \tag{5.4}$$

Let $\mathcal{F}_0 := \mathcal{F} \backslash \mathcal{F}(n)$ and $\mathcal{F}_1 := \mathcal{F}\langle n \rangle$. Clearly, $\mathcal{F}_0$ and $\mathcal{F}_1$ are compressed with respect to 1. Let $\mathcal{A}_0 := \mathcal{A} \cap \mathcal{S}_{\mathcal{F}_0,k}$. By (5.4), $\mathcal{A}_i\langle(n,i)\rangle$ is an intersecting sub-family of $\mathcal{S}_{\mathcal{F}_1,k}$, $i = 1, ..., k$. The result now follows by induction on n since

$$\begin{aligned} |\mathcal{A}^*| = |\mathcal{A}| = |\mathcal{A}_0| + \sum_{i=1}^{k} |\mathcal{A}((n,i))| &\leq |\mathcal{A}_0| + \sum_{i=1}^{k} |\mathcal{A}_i\langle(n,i)\rangle| \\ \leq |\mathcal{S}_{\mathcal{F}_0,k}((1,1))| + k|\mathcal{S}_{\mathcal{F}_1,k}((1,1))| &= |\mathcal{S}_{\mathcal{F},k}((1,1))|, \end{aligned} \tag{5.5}$$

where the first inequality is obtained from (5.3). □

We need to do more work to prove the extremal structures given in Theorem 5.4.1(ii). We start with a simple lemma that we will use often.

Lemma 5.4.2 *Let $\mathcal{A} \subseteq \mathcal{S}_{\mathcal{F},k}$ be intersecting. Suppose that for some $(a, b) \in U(\mathcal{F})$ and $F \in \mathcal{F}$, $\mathcal{S}_{F,k}((a, b)) \subseteq \mathcal{A}$. Then $\mathcal{A} \subseteq \mathcal{S}_{\mathcal{F},k}((a, b))$.*

Proof. It suffices to show that if $B \in \mathcal{S}_{\mathcal{F},k} \backslash \mathcal{S}_{\mathcal{F},k}((a, b))$ then $A \cap B = \emptyset$ for some $A \in \mathcal{S}_{F,k}((a, b))$. Let $C \in \mathcal{S}_{F,k} \backslash \mathcal{S}_{F,k}((a, b))$ such that $B \cap (F \times [k]) \subseteq C$. Clearly, for some $q \in [k-1]$, $\theta_k^q(C) \in \mathcal{S}_{F,k}((a, b))$ and $B \cap \theta_k^q(C) = \emptyset$. □

Lemma 5.4.3 *Let $\mathcal{F} \subseteq 2^{[n]}$. Let $\mathcal{A} \subset \mathcal{S}_{\mathcal{F},k}$ be intersecting, where $k \geq 3$. If $\mathcal{A} \neq \Delta_{a,b}(\mathcal{A}) = \mathcal{S}_{\mathcal{F},k}((a', b'))$ then $\mathcal{A} = \mathcal{S}_{\mathcal{F},k}((a, b))$.*

Proof. Since $\mathcal{A} \neq \Delta_{a,b}(\mathcal{A}) = \mathcal{S}_{\mathcal{F},k}((a', b'))$, there exists $A \in \mathcal{A}$ such that $A \notin \mathcal{S}_{\mathcal{F},k}((a', b'))$ and $\delta_{a,b}(A) \in \mathcal{S}_{\mathcal{F},k}((a', b'))$. This implies that $(a', b') = (a, 1)$. Let $F := \gamma(A)$. Let $\mathcal{A}_F := \mathcal{A} \cap \mathcal{S}_{F,k}$. Clearly, $\Delta_{a,b}(\mathcal{A}_F) = \Delta_{a,b}(\mathcal{A}) \cap \mathcal{S}_{F,k} = \mathcal{S}_{F,k}((a, 1))$ and $|\mathcal{A}_F| = |\Delta_{a,b}(\mathcal{A}_F)| = |\mathcal{S}_{F,k}((a, 1))|$. Thus, by Theorem 1.4.1, $\mathcal{A}_F = \mathcal{S}_{F,k}((c, d))$

for some $(c,d) \in F \times [k]$. Since $A \in \mathcal{A}_F \backslash \mathcal{S}_{F,k}((a,1))$ and $\Delta_{a,b}(\mathcal{A}_F) = \mathcal{S}_{F,k}((a,1))$, it follows that $(c,d) = (a,b)$. By Lemma 5.4.2, $\mathcal{A} \subseteq \mathcal{S}_{\mathcal{F},k}((a,b))$. Since $|\mathcal{A}| = |\Delta_{a,b}(\mathcal{A})| = |\mathcal{S}_{\mathcal{F},k}((a,b))|$, $\mathcal{A} = \mathcal{S}_{\mathcal{F},k}((a,b))$. □

For $F \in \mathcal{F}$ and $f, g \in U(\mathcal{F})$, let $\psi_{f,g}(F) := (F\backslash\{g\}) \cup \{f\}$.

Proposition 5.4.4 *Let $\mathcal{F}$ and J be as in Theorem 5.4.1. Then $L(\mathcal{S}_{\mathcal{F},k}) = J \times [k]$.*

Proof. Let $j \in J$ and $h \in [n]$. Then, since $\mathcal{F}$ is compressed with respect to j,

$$\begin{aligned} |\mathcal{S}_{\mathcal{F},k}((h,b))| &= \sum_{H \in \mathcal{F}(h)(j)} |\mathcal{S}_{H,k}((h,b))| + \sum_{G \in \mathcal{F}(h)\backslash\mathcal{F}(j)} |\mathcal{S}_{G,k}((h,b))| \\ &= \sum_{H \in \mathcal{F}(j)(h)} |\mathcal{S}_{H,k}((j,b))| + \sum_{G \in \mathcal{F}(h)\backslash\mathcal{F}(j)} |\mathcal{S}_{\psi_{j,h}(G),k}((j,b))| \\ &\leq |\mathcal{S}_{\mathcal{F},k}((j,b))| \end{aligned} \tag{5.6}$$

with equality iff for all $F \in \mathcal{F}(j)\backslash\mathcal{F}(h)$, $\psi_{h,j}(F) \in \mathcal{F}(h)$. This already gives $J \times [k] \subseteq L(\mathcal{F})$. Now consider equality in (5.6). Suppose $h \notin J$. So there exist $F \in \mathcal{F}\backslash\mathcal{F}(h)$ and $f \in F$ such that $F_1 := \psi_{h,f}(F) \notin \mathcal{F}(h)$. Therefore $f \neq j$. If $j \notin F$ then $F_2 := \psi_{j,f}(F) \in \mathcal{F}(j)\backslash\mathcal{F}(h)$, $\psi_{h,j}(F_2) = F_1$, and hence $\psi_{h,j}(F_2) \notin \mathcal{F}(h)$; but this is a contradiction as it yields a strict inequality in (5.6). So $F \in \mathcal{F}(j)\backslash\mathcal{F}(h)$, and hence $F_3 := \psi_{h,j}(F) \in \mathcal{F}(h)\backslash\mathcal{F}(j)$. Thus, $F_4 := \psi_{j,f}(F_3) \in \mathcal{F}(j)$. But $F_4 = F_1 \notin \mathcal{F}(h)$, which is a contradiction because $h \in F_4$. So $h \in J$. Therefore $L(\mathcal{F}) \subseteq J \times [k]$, and hence result. □

Lemma 5.4.5 *Let $\mathcal{A}'$ be as in the Proof of Theorem 5.4.1(i). Suppose $\mathcal{A}' \neq \Lambda(\mathcal{A}') = \mathcal{S}_{\mathcal{F},k}((a,1))$. Then $n \in J$ and $\mathcal{A}' = \mathcal{S}_{\mathcal{F},k}((n,1))$.*

Proof. Since $\mathcal{A}' \neq \Lambda(\mathcal{A}') = \mathcal{S}_{\mathcal{F},k}((a,1))$, there exists $A' \in \mathcal{A}'$ such that $A' \notin \mathcal{S}_{\mathcal{F},k}((a,1))$ and $\lambda(A') \in \mathcal{S}_{\mathcal{F},k}((a,1))$, and hence $a = 1$. Let $A := \{(n,1)\} \cup \{(a,2) \colon a \in \gamma(A')\backslash\{n-1\}\}$. Since $\Lambda(\mathcal{A}') = \mathcal{S}_{\mathcal{F},k}((1,1))$, exactly one of A and $\lambda(A) \neq A$ is in $\mathcal{A}'$. Recall that we arrived at (5.2) using the fact that, by Corollary 5.3.2, $A_1 \cap A_2 \cap ([n] \times [1]) \neq \emptyset$ for any $A_1, A_2 \in \mathcal{A}'$. Since $\lambda(A) \cap A' \cap ([n] \times [1]) = \emptyset$, it follows that $A \in \mathcal{A}'$ and, since $A \cap ([n] \times [1]) = \{(n,1)\}$, $\mathcal{A}' \subseteq \mathcal{S}_{\mathcal{F},k}((n,1))$. Since $1 \in J$

and $|\mathcal{A}'| = |\Lambda(\mathcal{A}')| = |\mathcal{S}_{\mathcal{F},k}((1,1))|$, it follows by Proposition 5.4.4 that $n \in J$ and $\mathcal{A}' = \mathcal{S}_{\mathcal{F},k}((n,1))$. □

Before coming to the proof of Theorem 5.4.1(ii), we finally determine two nice properties of $\mathcal{F}$-orbits of elements $j \in U(\mathcal{F})$ such that $\mathcal{F}$ is compressed with respect to j. This will be very useful when dealing with the case $k = 2$ of Theorem 5.4.1(ii).

Proposition 5.4.6 *Let $\mathcal{F}$ and J be as in Theorem 5.4.1. Let $j^* \in J$.*
(i) If $\mathrm{O}_{\mathcal{F}}(j^*)\backslash\{j^*\} \neq \emptyset$ then $\mathcal{F}(j^*) = \mathcal{F}$ and $\mathrm{O}_{\mathcal{F}}(j^*) = J$.
(ii) If $\mathrm{O}_{\mathcal{F}}(j^*) = \{j^*\}$ then $\mathrm{O}_{\mathcal{F}}(j) = \{j\}$ for all $j \in J$.

Proof. Suppose $\mathrm{O}_{\mathcal{F}}(j^*)\backslash\{j^*\} \neq \emptyset$. Suppose $\mathcal{F}(j^*) \neq \mathcal{F}$. Let $F \in \mathcal{F}\backslash\mathcal{F}(j^*)$ and $j' \in \mathrm{O}_{\mathcal{F}}(j^*)\backslash\{j^*\}$. So $j' \notin F$ since $F \notin \mathcal{F}(j^*) = \mathcal{F}(j')$. But then, since $\mathcal{F}$ is compressed with respect to j^*, for any $f \in F$ we have $(F\backslash\{f\}) \cup \{j^*\} \in \mathcal{F}(j^*)\backslash\mathcal{F}(j')$, which contradicts $\mathcal{F}(j^*) = \mathcal{F}(j')$. So $\mathcal{F}(j^*) = \mathcal{F}$.

Let $j \in J$. Suppose $j \notin \mathrm{O}_{\mathcal{F}}(j^*)$. So $\mathcal{F}(j) \subsetneq \mathcal{F}(j^*)$ as $\mathcal{F}(j^*) = \mathcal{F}$. Let $F^* \in \mathcal{F}(j^*)\backslash\mathcal{F}(j)$. Since $\mathcal{F}$ is compressed with respect to j, we have $(F^*\backslash\{j^*\}) \cup \{j\} \in \mathcal{F}\backslash\mathcal{F}(j^*)$, which contradicts $\mathcal{F}(j^*) = \mathcal{F}$. Therefore $J \subseteq \mathrm{O}_{\mathcal{F}}(j^*)$. Also, $\mathrm{O}_{\mathcal{F}}(j^*) \subseteq J$ because if $j \in \mathrm{O}_{\mathcal{F}}(j^*)$ then $\mathcal{F}(j) = \mathcal{F}(j^*) = \mathcal{F}$. Hence (i).

Now suppose $\mathrm{O}_{\mathcal{F}}(j^*) = \{j^*\}$ and $\mathrm{O}_{\mathcal{F}}(j)\backslash\{j\} \neq \emptyset$ for some $j \in J$. By (i), $\mathrm{O}_{\mathcal{F}}(j) = J$. So $j^* \in \mathrm{O}_{\mathcal{F}}(j)$, but this implies $\mathrm{O}_{\mathcal{F}}(j^*) = \mathrm{O}_{\mathcal{F}}(j)$, a contradiction. Hence (ii). □

Proof of Theorem 5.4.1(ii). By Proposition 5.4.4, $|\mathcal{S}_{\mathcal{F},k}((j,b))| = |\mathcal{S}_{\mathcal{F},k}((1,1))|$ iff $(j,b) \in J \times [k]$. Also, if $k = 2$ and $\mathcal{A}^* = \{F \in \mathcal{F} \colon F \cap (\mathrm{O}_{\mathcal{F}}(1) \times [2]) \in \mathcal{X}\}$ for some $\mathcal{X} \in \mathrm{ex}(\mathcal{S}_{\mathrm{O}_{\mathcal{F}}(1),2})$ then $|\mathcal{A}^*| = |\mathcal{S}_{\mathcal{F},2}((1,1))|$ as $\mathcal{S}_{\mathcal{F},2}((1,1)) = \{F \in \mathcal{F} \colon F \cap (\mathrm{O}_{\mathcal{F}}(1) \times [2]) \in \mathcal{X}^*\}$, where $\mathcal{X}^* = \mathcal{S}_{\mathrm{O}_{\mathcal{F}}(1),2}((1,1)) \in \mathrm{ex}(\mathcal{S}_{\mathrm{O}_{\mathcal{F}}(1),2})$. By Theorem 5.4.1(i), the sufficiency condition follows.

We now continue on the Proof of Theorem 5.4.1(i) to prove the necessary condition. Therefore, we now consider equality in (5.5). This gives us equality in (5.3) together with $|\mathcal{A}_0| = |\mathcal{S}_{\mathcal{F}_0,k}((1,1))|$ and $|\mathcal{A}_i\langle(n,i)\rangle| = |\mathcal{S}_{\mathcal{F}_1,k}((1,1))|$, $i = 1, ..., k$.

Since we are proving the result by induction on n, we may assume that $\mathcal{F}(n) \neq \emptyset$ and $n \in \gamma(A^*)$ for some $A^* \in \mathcal{A}^*$. If $|U(\mathcal{F}(n))| = 1$ then $\gamma(A^*) = \{n\}$ and $\mathcal{A}^* = \{A^*\}$.

So we assume that $|U(\mathcal{F}(n))| \geq 2$, which implies $|U(\mathcal{F}\langle n\rangle)| \geq 1$. Thus, for each $i \in [k]$, we have $|\mathcal{A}_i\langle(n,i)\rangle| = |\mathcal{S}_{\mathcal{F}_1,k}((1,1))| > 0$.

Let $J_0 := \{j_0 \in [n-1]\colon \mathcal{F}_0$ is compressed with respect to $j_0\}$ and $J_1 := \{j_1 \in [n-1]\colon \mathcal{F}_1$ is compressed with respect to $j_1\}$. Clearly, $1 \in J_0 \cap J_1$.

Consider first $k \geq 3$. Since we have equality in (5.3), it follows that $\mathcal{B} = \emptyset$ and $\mathcal{A}_i = \mathcal{A}((n,i))$, $i = 1,...,k$. Thus, by (5.2) and (5.4), $\mathcal{A}^- := \mathcal{A}_0 \cup \bigcup_{i=1}^{k} \mathcal{A}\langle(n,i)\rangle$ is an intersecting sub-family of $\mathcal{S}_{\mathcal{F}_0\cup\mathcal{F}_1,k}$. Now, by the inductive hypothesis and (5.4), $\mathcal{A}_1\langle(n,1)\rangle = \mathcal{S}_{\mathcal{F}_1,k}((j_1,1))$ for some $j_1 \in J_1$. Since $\mathcal{A}((n,1)) = \mathcal{A}_1$, we have $\mathcal{A}\langle(n,1)\rangle = \mathcal{S}_{\mathcal{F}_1,k}((j_1,1))$, and hence $\mathcal{A}^- \subseteq \mathcal{S}_{\mathcal{F}_0\cup\mathcal{F}_1,k}((j_1,1))$ by Lemma 5.4.2. So $\mathcal{A} \subseteq \mathcal{S}_{\mathcal{F},k}((j_1,1))$. Since we have equality in (5.5), Proposition 5.4.4 gives us $j_1 \in J$ and $\mathcal{A} = \mathcal{S}_{\mathcal{F},k}((j_1,1)$. Therefore, by Lemmas 5.4.3 and 5.4.5, $\mathcal{A}^*$ is as required.

Next, consider $k = 2$ and $|\mathrm{O}_{\mathcal{F}}(1)| > 1$. By Proposition 5.4.6(i), $\mathcal{F} = \mathcal{F}(1)$. By Corollary 5.2.2(i), $\mathcal{A}^* = \{F \in \mathcal{F}\colon F \cap (\mathrm{O}_{\mathcal{F}}(1) \times [2]) \in \mathcal{X}\}$ for some $\mathcal{X} \in \mathrm{ex}(\mathcal{S}_{\mathrm{O}_{\mathcal{F}}(1),2})$.

Finally, consider $k = 2$ and $\mathrm{O}_{\mathcal{F}}(1) = \{1\}$. Suppose $\mathcal{F}_0 = \emptyset$. Then $\mathcal{F} = \mathcal{F}(n)$. If $\mathcal{F}(n)\backslash\mathcal{F}(1) \neq \emptyset$ and $F \in \mathcal{F}(n)\backslash\mathcal{F}(1)$ then, given that $1 \in J$, we have $(F\backslash\{n\}) \cup \{1\} \in \mathcal{F}_0$, a contradiction. So $\mathcal{F}(n)\backslash\mathcal{F}(1) = \emptyset$, and hence, since $\mathcal{F}(n) = \mathcal{F}$, $\mathcal{F}(n) = \mathcal{F}(1)$; but this contradicts $\mathrm{O}_{\mathcal{F}}(1) = \{1\}$. So $\mathcal{F}_0 \neq \emptyset$, and hence $|\mathcal{A}_0| > 0$ as $|\mathcal{A}_0| = |\mathcal{S}_{\mathcal{F}_0,2}((1,1))|$ and $1 \in J_0$. It remains to consider the following three cases.

Case 1: $|\mathrm{O}_{\mathcal{F}_0}(1)| = 1$. By the inductive hypothesis and (5.2), $\mathcal{A}_0 = \mathcal{S}_{\mathcal{F}_0,2}((j_0,1))$ for some $j_0 \in J_0$. Clearly, $\mathcal{A}_0 \cup \mathcal{A}_1$ and $\mathcal{A}_0 \cup \mathcal{A}_2$ are intersecting. By Lemma 5.4.2, we therefore have $\mathcal{A}_0 \cup \mathcal{A}_1, \mathcal{A}_0 \cup \mathcal{A}_2 \subseteq \mathcal{S}_{\mathcal{F},2}((j_0,1))$, and hence $\mathcal{A} \subseteq \mathcal{S}_{\mathcal{F},2}((j_0,1))$. Since we have equality in (5.5), Proposition 5.4.4 gives us $j_0 \in J$ and $\mathcal{A} = \mathcal{S}_{\mathcal{F},k}((j_0,1)$. By Lemma 5.4.5, $\mathcal{A}' = \mathcal{S}_{\mathcal{F},k}((j,1))$, where $j \in \{j_0,n\} \cap J$. Since $1 \in J$ and $\mathrm{O}_{\mathcal{F}}(1) = \{1\}$, Proposition 5.4.6(ii) gives us $\mathrm{O}_{\mathcal{F}}(j) = \{j\}$. Since $\Gamma(\mathcal{A}^*) = \Gamma(\mathcal{A}') = \mathcal{F}(j)$, it follows by Corollary 5.2.2(ii) that $\mathcal{A}^* = \mathcal{S}_{\mathcal{F},k}(j,b)$ for some $b \in [2]$.

Case 2: $|\mathrm{O}_{\mathcal{F}_1}(1)| = 1$. By the inductive hypothesis and (5.4), we have $\mathcal{A}_1\langle(n,1)\rangle = \mathcal{S}_{\mathcal{F}_1,2}((j_1,1))$ for some $j_1 \in J_1$. Since $\mathcal{A}_0 \cup \mathcal{A}_1\langle(n,1)\rangle$ is an intersecting sub-family of $\mathcal{S}_{\mathcal{F}_0\cup\mathcal{F}_1,k}$, Lemma 5.4.2 gives us $\mathcal{A}_0 \cup \mathcal{A}_1\langle(n,1)\rangle \subseteq \mathcal{S}_{\mathcal{F}_0\cup\mathcal{F}_1,2}((j_1,1))$. So $\mathcal{A}_0 \subseteq \mathcal{S}_{\mathcal{F}_0,2}((j_1,1))$. Since $|\mathcal{A}_0| = |\mathcal{S}_{\mathcal{F}_0,2}((1,1))|$, it follows by Proposition 5.4.4 that $j_1 \in J_0$ and $\mathcal{A}_0 = \mathcal{S}_{\mathcal{F}_0,2}((j_1,1))$. As in Case 1, this leads us to the desired result.

Case 3: $|O_{\mathcal{F}_0}(1)| > 1$, $|O_{\mathcal{F}_1}(1)| > 1$. By Proposition 5.4.6(i), $\mathcal{F}_0(1) = \mathcal{F}_0$ and $\mathcal{F}_1(1) = \mathcal{F}_1$. So $\mathcal{F}(1) = \mathcal{F}$. Thus, by Corollary 5.2.2(ii), $\mathcal{A}^* = \mathcal{S}_{\mathcal{F},2}((1,b))$ for some $b \in [2]$. □

Proof of Theorem 5.1.2. By Theorem 5.4.1, $(1,1) \in L(\mathcal{S}_{\mathcal{F},k})$ and, furthermore, $\mathcal{S}_{\mathcal{F},k}((1,1)) \in \mathrm{ex}(\mathcal{S}_{\mathcal{F},k})$; hence $\mathcal{S}_{\mathcal{F},k}$ is EKR. By Theorem 5.4.1(ii), if $k \geq 3$ then $\mathcal{S}_{\mathcal{F},k}$ is strictly EKR. Now consider $k = 2$. If $\mathcal{S}_{\mathcal{F},2}$ is not strictly EKR then, by Theorem 5.4.1(ii) and Corollary 5.2.2(ii), $|O_{\mathcal{F}}(1)| \geq 3$; the converse holds by Proposition 5.2.3 with $f = 1$. □

5.5 Proof of Theorem 5.1.3

We now give a simple proof of Theorem 1.4.2 and prove the stronger Theorem 5.1.3. Thus, in the following, $\mathcal{F}$ is taken to be r-uniform. Our first simple observation is that

$$f^* \in L(\mathcal{F}) \;\Leftrightarrow\; (f^*, 1) \in L(\mathcal{S}_{\mathcal{F},k}) \tag{5.7}$$

since for all $f \in U(\mathcal{F})$, $k^{r-1}|\mathcal{F}(f)| = k^{r-1}|\Gamma(\mathcal{S}_{\mathcal{F},k}((f,1)))| = |\mathcal{S}_{\mathcal{F},k}((f,1))|$.

For $\mathcal{A} \subseteq \mathcal{S}_{\mathcal{F},k}$, let $\Theta^q(\mathcal{A}) := \{\theta_k^q(A) \colon A \in \mathcal{A}\}$. Suppose $\mathcal{F} = \{F\}$ and $\mathcal{A}$ is intersecting. If $A \in \mathcal{A}$ and $q \in [k-1]$ then $\theta_k^q(A) \notin \mathcal{A}$ as $\theta_k^q(A) \cap A = \emptyset$. So $\mathcal{A}, \Theta^1(\mathcal{A}), ..., \Theta^{k-1}(\mathcal{A})$ are k disjoint copies of the same intersecting family, and hence $k|\mathcal{A}| \leq |\mathcal{S}_{F,k}| = k^r$. Therefore

$$\mathcal{A} \subset \mathcal{S}_{F,k} \text{ intersecting } \Rightarrow |\mathcal{A}| \leq k^{r-1}, \tag{5.8}$$

which proves Theorem 1.4.1(i) since $|\mathcal{S}_{F,k}((f,1))| = k^{r-1}$.

Proof of Theorem 1.4.2. Let $\mathcal{F}$ be EKR, and let $\mathcal{A} \subset \mathcal{S}_{\mathcal{F},k}$ be intersecting. Clearly, $\Gamma(\mathcal{A})$ is intersecting. Thus, for $f^* \in L(\mathcal{F})$, $|\Gamma(\mathcal{A})| \leq |\mathcal{F}(f^*)|$. For any $F \in \Gamma(\mathcal{A})$, let

$\mathcal{A}_F := \mathcal{A} \cap \mathcal{S}_{F,k}$. By (5.8), $|\mathcal{A}_F| \leq k^{r-1}$. Thus, for any $b \in [k]$,

$$|\mathcal{A}| = \sum_{F \in \Gamma(\mathcal{A})} |\mathcal{A}_F| \leq k^{r-1}|\Gamma(\mathcal{A})| \leq k^{r-1}|\mathcal{F}(f^*)| = |\mathcal{S}_{\mathcal{F},k}((f^*, b))|, \tag{5.9}$$

and hence result. □

Proof of Theorem 5.1.3. We continue on the proof of Theorem 1.4.2, and we now consider equality in (5.9). So $\Gamma(\mathcal{A}) = \mathcal{F}(f^*)$ and for all $F \in \Gamma(\mathcal{A})$, $|\mathcal{A}_F| = k^{r-1}$. Thus, by (5.8),

$$\text{for all } F \in \Gamma(\mathcal{A}),\ \mathcal{A}_F \in \text{ex}(\mathcal{S}_{F,k}). \tag{5.10}$$

Case 1: $k \geq 3$. Let $F^* \in \Gamma(\mathcal{A})$. By (5.10) and Theorem 1.4.1(ii), $\mathcal{A}_F = \mathcal{S}_{F,k}((a^*, b^*))$ for some $(a^*, b^*) \in F^* \times [k]$. By Lemma 5.4.2, $\mathcal{A} \subseteq \mathcal{S}_{\mathcal{F},k}((a^*, b^*))$. So $\mathcal{S}_{\mathcal{F},k}$ is strictly EKR.

Case 2: $k = 2$ and for all $f \in L(\mathcal{F})$, $|\mathrm{O}_{\mathcal{F}}(f)| \leq 2$. Let $X := \bigcap_{F \in \Gamma(\mathcal{A})} F$. By (5.10) and Theorem 5.2.1, $|X| \geq 1$. Let $f' \in X$; so $\Gamma(\mathcal{A}) \subseteq \mathcal{F}(f')$. Since $|\Gamma(\mathcal{A})| = |\mathcal{F}(f^*)|$ and $f^* \in L(\mathcal{F})$, $\Gamma(\mathcal{A}) = \mathcal{F}(f')$ and $f' \in \mathrm{L}(\mathcal{F})$. So $|\mathrm{O}_{\mathcal{F}}(f')| \leq 2$. Since $\mathcal{A} \in \text{ex}(\mathcal{S}_{\mathcal{F}(f'),2})$ (by (5.10)), it follows by Corollary 5.2.2(ii) that $\mathcal{A}$ is a star of $\mathcal{S}_{\mathcal{F},2}$. So $\mathcal{S}_{\mathcal{F},2}$ is strictly EKR.

Case 3: $k = 2$ and $|\mathrm{O}_{\mathcal{F}}(f)| \geq 3$ for some $f \in L(\mathcal{F})$. By (5.7), $(f, 1) \in L(\mathcal{S}_{\mathcal{F},2})$. Thus, by Proposition 5.2.3, $\mathcal{S}_{\mathcal{F},2}$ is not strictly EKR. □

Chapter 6

Non-centred intersecting families of signed sets

6.1 Introduction

For $r \in [n]$, let $\mathcal{S}_{n,r,k} := \mathcal{S}_{\binom{[n]}{r},k}$. The main objective of this chapter is to establish a characterisation of the extremal non-centred intersecting sub-families of $\mathcal{S}_{n,r,k}$.

In Section 1.6, we alluded to the fact that in [38], Theorem 1.6.1 was used as a stepping stone to Theorem 1.2.2. Similarly, in Section 6.2, we prove the following signed sets analogue of Theorem 1.6.1 in order to arrive at our main result.

Theorem 6.1.1 *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are non-empty cross-intersecting sub-families of $\mathcal{S}_{n,r,k}$ then $|\mathcal{A}_1| + |\mathcal{A}_2| \leq |\{A \in \mathcal{S}_{n,r,k} \colon A \cap ([r] \times [1]) \neq \emptyset\}| + 1$. Unless $r = n$ and $k = 2$, equality holds iff either $\mathcal{A}_1 = \mathcal{A}_2 \cong \{A \in \mathcal{S}_{n,2,k} \colon (1,1) \in A\}$ or $\mathcal{A}_i = \{A^*\}$ and $\mathcal{A}_{3-i} = \{A \in \mathcal{S}_{n,r,k} \colon A \cap A^* \neq \emptyset\}$ for some $i \in [2]$ and $A^* \in \mathcal{S}_{n,r,k}$.*

For $2 \leq r \leq n$, let $\mathcal{N}_{n,r,k} := \{A \in \mathcal{S}_{n,r,k} \colon (1,1) \in A, A \cap N_1 \neq \emptyset\} \cup \{N_1, ..., N_{k-1}\}$ where

$$N_i := \begin{cases} [2, r+1] \times [1] & \text{if } r < n; \\ \{(1, i+1)\} \cup ([2, n] \times [1]) & \text{if } r = n, \end{cases} \qquad i = 1, ..., k-1.$$

Note that if $r < n$ then $N_i = [2, r+1] \times [1]$, $i = 1, ..., k-1$, and hence $\mathcal{N}_{n,r,k}$ is the 'Hilton-Milner-type' family $\{A \in \mathcal{S}_{n,r,k} \colon (1,1) \in A, A \cap ([2, r+1] \times [1]) \neq \emptyset\} \cup \{[2, r+1] \times [1]\}$.

For $3 \leq r \leq n$, let $\mathcal{T}_{n,r,k}$ be the *triangle family* $\{A \in \mathcal{S}_{n,r,k} \colon |A \cap ([3] \times [1])| \geq 2\}$.

In Section 6.3, using Theorem 6.1.1, we prove our desired 'Hilton-Milner-type' extension of Theorem 1.4.3.

Theorem 6.1.2 *If $\mathcal{A}$ is a non-centred intersecting sub-family of $\mathcal{S}_{n,r,k}$ then $|\mathcal{A}| \leq |\mathcal{N}_{n,r,k}|$. Unless $r = n$ and $k = 2$, equality holds iff $\mathcal{A} \cong \mathcal{N}_{n,r,k}$ or $\mathcal{A} \cong \mathcal{T}_{n,3,k}$ or $\mathcal{A} \cong \mathcal{T}_{4,4,k}$.*

In Section 6.4, we give two proofs of the fact that there exists an integer $k_0(\mathcal{F})$ such that Conjecture 5.1.1 is true if $k \geq k_0(\mathcal{F})$. The first proof is a direct proof based on an argument that Erdős, Ko and Rado used for proving Theorem 1.2.4, and the second proof is based on Theorem 6.1.2 with $r = n$. Comparing the two proofs, we see that the latter yields a much better value of $k_0(\mathcal{F})$.

Before starting the proofs, we point out to the reader that we will be making use of the notation in Section 2.1, especially in Section 6.3.

6.2 Non-empty cross-intersecting families of signed sets

This section is dedicated to the proof of Theorem 6.1.1, which, like the proof of Theorem 6.1.2, will be based on the compression $\Delta_{a,b}$ as defined in Section 5.4 (and hence not as defined in Section 2.2).

Lemma 6.2.1 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be non-empty cross-intersecting sub-families of $\mathcal{S}_{n,2,k}$, where $(2,k) \neq (n,2)$. Suppose $\mathcal{A}_i \neq \Delta_{a,b}(\mathcal{A}_1) = \Delta_{a,b}(\mathcal{A}_2) = \mathcal{S}_{n,2,k}((c,d))$ for some $i \in [2]$ and $(c,d) \in [n] \times [k]$. Then $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{S}_{n,2,k}((a,b))$.*

Proof. We may assume that $i = 1$. So there exists $A_1 \in \mathcal{A}_1 \backslash \Delta_{a,b}(\mathcal{A}_1)$ such that $\delta_{a,b}(A_1) \in \Delta_{a,b}(\mathcal{A}_1) \backslash \mathcal{A}_1$; let $A_1' := \delta_{a,b}(A_1)$. Thus, for some $(a_1, b_1) \in ([n] \backslash \{a\}) \times [k]$, $A_1 = \{(a,b), (a_1,b_1)\}$ and $(c,d) \in A_1' = \{(a,1), (a_1,b_1)\}$. If $(c,d) = (a_1,b_1)$ then $A_1 \in \mathcal{S}_{n,2,k}((c,d))$, and hence $A_1 \in \Delta_{a,b}(\mathcal{A}_1)$, a contradiction. So $(c,d) = (a,1)$ and hence, by the assumptions of the lemma, $\Delta_{a,b}(\mathcal{A}_1) = \Delta_{a,b}(\mathcal{A}_2) = \mathcal{S}_{n,2,k}((a,1))$.

Note that this implies that $|A \cap \{(a,1),(a,b)\}| = 1$ for all $A \in \mathcal{A}_1 \cup \mathcal{A}_2$. If there exists $A_2 \in \mathcal{A}_2$ such that $(a,1) \in A_2$, then, since $A_1 \cap A_2 \neq \emptyset$ (as $\mathcal{A}_1, \mathcal{A}_2$ are cross-intersecting), A_2 can only be A_1'. Together with $\Delta_{a,b}(\mathcal{A}_2) = \mathcal{S}_{n,2,k}((a,1))$, this implies that $\mathcal{A}_2$ contains $\mathcal{B} := \mathcal{S}_{n,2,k}((a,b)) \backslash \{A_1\}$. Given that $(2,k) \neq (n,2)$ (i.e. $k \geq 3$ if $n = 2$), for any $A \in \mathcal{S}_{n,2,k}((a,1))$ there exists $B \in \mathcal{B}$ such that $A \cap B = \emptyset$. By the above, it follows that $\mathcal{A}_1 = \mathcal{S}_{n,2,k}((a,b))$, which in turn forces $\mathcal{A}_2$ to be $\mathcal{S}_{n,2,k}((a,b))$. $\square$

Proof of Theorem 6.1.1. The result is trivial for $r = 1$. If $r = n$ and $k = 2$ then the result follows from the fact that for any $A := \{(a_1,k_1),...,(a_n,k_n)\} \in \mathcal{S}_{n,n,2}$, the unique set in $\mathcal{S}_{n,n,2}$ that does not intersect A is $\{(a_1, 3-k_1),...,(a_n, 3-k_n)\}$. We will therefore assume that $r \geq 2$ and $(r,n) \neq (n,2)$.

Let $\mathcal{C} := \mathcal{A}_1' \cup \mathcal{A}_2' \subset \mathcal{S}_{n+1,r+1,k}$, where $\mathcal{A}_1' := \{A_1 \cup \{(n+1,1)\} \colon A_1 \in \mathcal{A}_1\}$ and $\mathcal{A}_2' := \{A_2 \cup \{(n+1,2)\} \colon A_2 \in \mathcal{A}_2\}$. So $\mathcal{C}$ is intersecting. Let

$$\mathcal{D} := \Delta_{n,k} \circ ... \circ \Delta_{n,2} \circ ... \circ \Delta_{1,k} \circ ... \circ \Delta_{1,2}(\mathcal{C}).$$

Let $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{S}_{n,r,k}$ be given by $\mathcal{B}_1 := \{D \backslash \{(n+1,1)\} \colon (n+1,1) \in D \in \mathcal{D}\}$ and $\mathcal{B}_2 := \{D \backslash \{(n+1,2)\} \colon (n+1,2) \in D \in \mathcal{D}\}$. By Corollary 5.3.2, we have

$$D_1 \cap D_2 \cap (([n+1] \times [1]) \cup \{(n+1,2)\}) \neq \emptyset \text{ for any } D_1, D_2 \in \mathcal{D},$$

and hence

$$B_1 \cap B_2 \cap ([n] \times [1]) \neq \emptyset \text{ for any } B_1 \in \mathcal{B}_1 \text{ and } B_2 \in \mathcal{B}_2. \tag{6.1}$$

For each $i \in [2]$, let $A_i^* \in \mathcal{A}_i$ and $B_i^* := \{(a,1) \colon A_i^* \cap (\{a\} \times [k]) \neq \emptyset\}$. It is easy to see that $B_1^* \in \mathcal{B}_1$ and $B_2^* \in \mathcal{B}_2$.

Let $X := [n] \times [1]$. For each $i \in [2]$, let $\mathcal{B}_i^{(q)} := \{B \in \mathcal{B}_i \colon |B \cap X| = q\}$, $\mathcal{X}_i := \{B \cap X \colon B \in \mathcal{B}_i\}$, $\mathcal{X}_i^{(q)} := \{A \in \mathcal{X}_i \colon |A| = q\}$. For each $q \in [r]$, let $\mathcal{E}^{(q)} := \{A \in \binom{X}{q} \colon A \cap ([r] \times [1]) \neq \emptyset\}$ and $w_q := |\mathcal{S}_{n-q,r-q,k-1}| = \binom{n-q}{r-q}(k-1)^{r-q}$. For each $i \in [2]$,

$\bigcup_{q=1}^{r} \mathcal{B}_i^{(q)}$ is a partition for $\mathcal{B}_i$ by (6.1). So we have

$$\begin{aligned} |\mathcal{A}_1| + |\mathcal{A}_2| = |\mathcal{C}| = |\mathcal{D}| = |\mathcal{B}_1| + |\mathcal{B}_2| &= \sum_{q=1}^{r} (|\mathcal{B}_1^{(q)}| + |\mathcal{B}_2^{(q)}|) \\ &\leq \sum_{q=1}^{r} (|\mathcal{X}_1^{(q)}| + |\mathcal{X}_2^{(q)}|) w_q \end{aligned} \tag{6.2}$$

and

$$1 + \sum_{q=1}^{r} |\mathcal{E}^{(q)}| w_q = |\mathcal{S}_{n,r,k}([r] \times [1])| + 1. \tag{6.3}$$

Let $1 \leq p \leq \min\{r, n/2\}$. If $\mathcal{X}_1^{(p)} \neq \emptyset$ and $\mathcal{X}_2^{(p)} \neq \emptyset$ then, by Theorem 1.6.1, we have $|\mathcal{X}_1^{(p)}| + |\mathcal{X}_2^{(p)}| \leq \binom{n}{p} - \binom{n-p}{p} + 1$, and hence $|\mathcal{X}_1^{(p)}| + |\mathcal{X}_2^{(p)}| \leq |\mathcal{E}^{(p)}| + 1$ with equality only if $p = r$. Now, without loss of generality, suppose $\mathcal{X}_2^{(p)} = \emptyset$. Then $|\mathcal{X}_1^{(p)}| + |\mathcal{X}_2^{(p)}| = |\mathcal{X}_1^{(p)}| \leq |\{A \in \binom{X}{p} : A \cap B_2^* \neq \emptyset\}| \leq |\mathcal{E}^{(p)}|$, where the first inequality follows by (6.1).

Therefore, we have just shown that

$$1 \leq p \leq \min\{r, n/2\} \;\Rightarrow\; |\mathcal{X}_1^{(p)}| + |\mathcal{X}_2^{(p)}| \leq \begin{cases} |\mathcal{E}^{(p)}| & \text{if } p < r; \\ |\mathcal{E}^{(p)}| + 1 & \text{if } p = r. \end{cases} \tag{6.4}$$

If $r \leq n/2$ then the upper bound in the theorem is immediate from (6.2), (6.3) and (6.4). So suppose $r > n/2$. Set $w_0 := 0$, $\mathcal{E}^{(0)} := \mathcal{X}_1^{(0)} := \mathcal{X}_2^{(0)} := \emptyset$. Let $n - r \leq p \leq \lfloor n/2 \rfloor$. Then,

$$|\mathcal{E}^{(n-p)}| = \binom{n}{n-p}, \quad |\mathcal{E}^{(p)}| = \begin{cases} \binom{n}{p} & \text{if } p \geq n - r + 1; \\ \binom{n}{p} - 1 & \text{if } p = n - r. \end{cases} \tag{6.5}$$

Also, since $(r, n) \neq (n, 2)$, an easy calculation yields

$$w_p \geq w_{n-p} \text{ with strict inequality if } p < n/2. \tag{6.6}$$

By (6.1), for any $A \in \mathcal{X}_i^{(p)}$ and $B \in \mathcal{X}_{3-i}^{(n-p)}$, we cannot have $A = X \backslash B$; hence

$$|\mathcal{X}_i^{(p)}| + |\mathcal{X}_{3-i}^{(n-p)}| \leq \binom{n}{n-p}. \tag{6.7}$$

Therefore,

$$\sum_{q=n-r}^{r} (|\mathcal{X}_1^{(q)}| + |\mathcal{X}_2^{(q)}|)w_q = \sum_{p=n-r}^{\lfloor n/2 \rfloor} \left((|\mathcal{X}_1^{(p)}| + |\mathcal{X}_2^{(p)}|)w_p + (|\mathcal{X}_1^{(n-p)}| + |\mathcal{X}_2^{(n-p)}|)w_{n-p}\right)$$

$$\begin{aligned}
&\leq \sum_{p=n-r}^{\lfloor n/2 \rfloor} \left((|\mathcal{X}_1^{(p)}| + |\mathcal{X}_2^{(p)}|)w_p + \left(2\binom{n}{n-p} - (|\mathcal{X}_1^{(p)}| + |\mathcal{X}_2^{(p)}|)\right) w_{n-p}\right) \quad \text{(by (6.7))} \\
&\leq \sum_{p=n-r}^{\lfloor n/2 \rfloor} \left(|\mathcal{E}^{(p)}|w_p + \left(2\binom{n}{n-p} - |\mathcal{E}^{(p)}|\right) w_{n-p}\right) \quad \text{(by (6.4), (6.6))} \\
&= \sum_{p=n-r}^{\lfloor n \rfloor/2} \left(|\mathcal{E}^{(p)}|w_p + \begin{cases} |\mathcal{E}^{(n-p)}|w_{n-p} & \text{if } p \geq n-r+1; \\ (|\mathcal{E}^{(n-p)}| + 1)w_{n-p} & \text{if } p = n-r. \end{cases}\right) \quad \text{(by (6.5))} \\
&= w_r + \sum_{q=n-r}^{r} |\mathcal{E}^{(q)}|w_q = 1 + \sum_{q=n-r}^{r} |\mathcal{E}^{(q)}|w_q
\end{aligned} \tag{6.8}$$

We know that if $n - r \geq 2$ then (6.4) holds for $p = 1, ..., n-r-1$. Together with (6.2), (6.3) and (6.8) (recall that (6.8) holds for $p = n-r, ..., \lfloor n/2 \rfloor$), this gives us the desired upper bound for $|\mathcal{A}_1| + |\mathcal{A}_2|$.

Now suppose the upper bound is attained. Then $|\mathcal{X}_1^{(1)}| + |\mathcal{X}_2^{(1)}| = |\mathcal{E}^{(1)}| = r$ if $n - r \geq 2$ (by (6.4)), and the same holds by (6.6) and (6.8) if $n - r \leq 1$. For each $i \in [2]$, since (6.1) tells us that each set in $\mathcal{X}_i$ intersects each set in $\mathcal{X}_{3-i}$, it is clear that each single-element set in $\mathcal{X}_i^{(1)}$ must be contained in the intersection of sets in $\mathcal{X}_{3-i}$. Assuming without loss of generality that $|\mathcal{X}_1^{(1)}| \geq |\mathcal{X}_2^{(1)}|$, it follows by the equality $|\mathcal{X}_1^{(1)}| + |\mathcal{X}_2^{(1)}| = r$ above that if $\mathcal{X}_2^{(1)} = \emptyset$ then $|\mathcal{X}_1^{(1)}| = r$ and hence $\mathcal{X}_2$ owns only the set B_2^*, and if instead $\mathcal{X}_2^{(1)} \neq \emptyset$ then $\mathcal{X}_1^{(1)} = \mathcal{X}_2^{(1)} = \{x\}$ $(x \in X)$ and hence $r = 2$.

Suppose $\mathcal{X}_2 = \{B_2^*\}$. Clearly, this implies $\mathcal{B}_2 = \{B_2^*\}$ and hence $\mathcal{A}_2 = \{A_2^*\}$. So $\mathcal{A}_1 \subseteq \mathcal{S}_{n,r,k}(A_2^*)$. Since $|\mathcal{A}_1| + |\mathcal{A}_2| = |\mathcal{S}_{n,r,k}([r] \times [1])| + 1$, $\mathcal{A}_1 = \mathcal{S}_{n,r,k}(A_2^*)$.

Suppose instead $\mathcal{X}_1^{(1)} = \mathcal{X}_2^{(1)} = \{x\}$ ($x \in X$) and $r = 2$. Then, by (6.1), $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{S}_{n,2,k}(x)$. Since $|\mathcal{B}_1| = |\mathcal{B}_2| = |\mathcal{A}_1| + |\mathcal{A}_2| = |\mathcal{S}_{n,2,k}([2] \times [1])| + 1 = 2|\mathcal{S}_{n,2,k}(x)|$, we actually have $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{S}_{n,2,k}(x)$. It follows by Lemma 6.2.1 that $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{S}_{n,2,k}((a,b))$ for some $(a,b) \in [n] \times [k]$. □

6.3 Non-centred intersecting families of signed sets

This section is dedicated to the proof of the main result of this chapter, i.e. Theorem 6.1.2. We shall first provide a set of lemmas that ensure that in the proof we may work with a non-centred intersecting family $\mathcal{A} \subset \mathcal{S}_{n,r,k}$ that is invariant under any compression $\Delta_{a,b}$; this will become clear in the proof itself.

Lemma 6.3.1 *Let $a \in [n]$, $b \in [2,k]$ and $(r,k) \neq (n,2)$. Suppose $\mathcal{A}$ is a non-centred intersecting sub-family of $\mathcal{S}_{n,r,k}$ and $\Delta_{a,b}(\mathcal{A})$ is centred. Then $|\mathcal{A}| < |\mathcal{N}_{n,r,k}|$.*

Proof. Since $\mathcal{A}$ is non-centred and $\Delta_{a,b}(\mathcal{A})$ is centred, we have $\mathcal{A} = \mathcal{A}(\{(a,1),(a,b)\})$, $\mathcal{A}((a,1))](a,b)[\neq \emptyset$ and $\mathcal{A}](a,1)[((a,b)) \neq \emptyset$. So $\mathcal{A}_1 := \mathcal{A}\langle(a,1)\rangle](a,b)[$ and $\mathcal{A}_2 := \mathcal{A}](a,1)[\langle(a,b)\rangle$ are non-empty cross-intersecting sub-families of $\mathcal{S}_{\binom{[n]\setminus\{a\}}{r-1},k}$. Thus, by Theorem 6.1.1,

$$|\mathcal{A}_1| + |\mathcal{A}_2| \leq |\mathcal{S}_{n-1,r-1,k}([r-1] \times [1])| + 1$$
$$< |\mathcal{S}_{\binom{[2,n]}{r-1},k}([2,r+1] \times [1])| + 1 = |\mathcal{N}_{n,r,k}|.$$

Since $|\mathcal{A}| = |\mathcal{A}(\{a,b\})| = |\mathcal{A}_1| + |\mathcal{A}_2|$, the result follows. □

Lemma 6.3.2 *Let $a \in [n]$, $b \in [2,k]$ and $(r,k) \neq (n,2)$. Suppose $\mathcal{A}$ is an intersecting sub-family of $\mathcal{S}_{n,r,k}$ and $\mathcal{A} \neq \Delta_{a,b}(\mathcal{A}) \cong \mathcal{N}_{n,r,k}$. Then $\mathcal{A} \cong \mathcal{N}_{n,r,k}$.*

Proof. We may assume without loss of generality that

$$\Delta_{a,b}(\mathcal{A}) = \mathcal{N}' := \mathcal{S}_{n,r,k}((1,k_1))([2,\min\{r+1,n\}] \times \{k_2\}) \cup \{N_i' : i \in [k]\},\ k_1, k_2 \in [k],$$

where $N_i' := [2,r+1] \times \{k_2\}$ if $r < n$, and $N_i' := \{(1,i)\} \cup ([2,n] \times \{k_2\})$ if $r = n$. Let $N := [2,\min\{r+1,n\}] \times \{k_2\}$.

Since $\Delta_{a,b}(\mathcal{A}) \neq \mathcal{A}$, there exists $A^* \in \mathcal{A}\backslash\Delta_{a,b}(\mathcal{A})$ such that $A' := \delta_{a,b}(A^*) \in \Delta_{a,b}(\mathcal{A})\backslash\mathcal{A}$, and hence $(a,1) \in A' \in \mathcal{N}'$. Suppose $r+1 < a \leq n$. Then, by definition of $\mathcal{N}'$, we must have $(A'\backslash(a,1)) \cup (a,b) \in \mathcal{N}'$ (i.e. $A^* \in \mathcal{N}'$), but this contradicts $A^* \notin \Delta_{a,b}(\mathcal{A}) = \mathcal{N}'$. So $a \leq \min\{r+1, n\}$.

Let $\mathcal{A}_0 := \mathcal{A}\backslash(\mathcal{A}_1 \cup \mathcal{A}_2)$, where $\mathcal{A}_1 := \mathcal{A}((1,1))$ and $\mathcal{A}_2 := \mathcal{A}((1,b))$. Let $\mathcal{A}'_1 := \mathcal{A}\langle(1,1)\rangle$ and $\mathcal{A}'_2 := \mathcal{A}\langle(1,b)\rangle$.

Case I: $r < n$.

Consider first $a = 1$. By $(a,1) \in A' \in \mathcal{N}'$ and the definition of $\mathcal{N}'$, we then have $k_1 = 1$; also, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \{N\}$ and $A^* \in \mathcal{A}_2$. Suppose $\mathcal{A}_1 \neq \emptyset$. Then, $\mathcal{A}'_1$ and $\mathcal{A}'_2$ are non-empty and cross-intersecting sub-families of $\mathcal{S}_{\binom{[2,n]}{r-1},k}$. By Theorem 6.1.1, we obtain $|\mathcal{A}'_1| + |\mathcal{A}'_2| \leq |\{A \in \mathcal{S}_{n-1,r-1,k} \colon A \cap ([r-1] \times [1]) \neq \emptyset\}| + 1$, and hence $|\mathcal{A}| < |\mathcal{N}_{n,r,k}|$, which is a contradiction as $|\mathcal{A}| = |\Delta_{a,b}(\mathcal{A})| = |\mathcal{N}'| = |\mathcal{N}_{n,r,k}|$. So $\mathcal{A}((1,1)) = \emptyset$, and hence $\mathcal{A} \subseteq \mathcal{N}'' := \mathcal{S}_{n,r,k}((1,b))(N) \cup \{N\} \cong \mathcal{N}'$. Since $|\mathcal{A}| = |\mathcal{N}'|$, $\mathcal{A} = \mathcal{N}''$.

Now consider $2 \leq a \leq r+1$. Suppose $k_2 \neq 1$. Since $(a,1) \in A' \in \mathcal{N}'$, we then get $A' \neq N$, $(1,k_1) \in A' \cap A^*$, $|A^* \cap N| \geq |A' \cap N| > 0$, and hence $A^* \in \mathcal{N}'$, a contradiction. So $k_2 = 1$. Let $N' := (N\backslash\{(a,1)\}) \cup \{(a,b)\}$. Since $\Delta_{a,b}(\mathcal{A}) = \mathcal{N}'$, we clearly have $\mathcal{A} \subset \mathcal{S}_{n,r,k}((1,k_1)) \cup \{M\}$, where $M \in \{N, N'\}$ and $M \in \mathcal{A}$. Since $\mathcal{A}$ is intersecting and $|\mathcal{A}| = |\Delta_{a,b}(\mathcal{A})| = |\mathcal{N}_{n,r,k}|$, $\mathcal{A} = \mathcal{S}_{n,r,k}((1,k_1))(M) \cup \{M\}$. Since $\mathcal{A} \neq \mathcal{N}'$, $M = N'$.

Case II: $r = n$. Since $(r,k) \neq (n,2)$, $k \geq 3$.

Consider first $a = 1$. Suppose $k_1 \neq 1$. Then, since $(1,1) \in A' \in \mathcal{N}'$, we must have $A' = N'_1$, and hence $A^* = N'_b$, a contradiction to $A^* \notin \Delta_{a,b}(\mathcal{A}) = \mathcal{N}'$. So $k_1 = 1$. Thus, since $\Delta_{a,b}(\mathcal{A}) = \mathcal{N}'$, we clearly have $\mathcal{A}_0 = \{N'_i \colon i \in [k]\backslash\{1,b\}\}$, $\mathcal{A}'_1 \cup \mathcal{A}'_2 = \mathcal{S}_{\binom{[2,n]}{n-1},k}(N)$, $\mathcal{A}'_1 \cap \mathcal{A}'_2 = \{N\}$, $|\mathcal{A}'_1| + |\mathcal{A}'_2| = |\mathcal{S}_{n-1,n-1,k}([n-1] \times [1])| + 1$. By Theorem 6.1.1, it follows that for some $i \in [2]$, $\mathcal{A}'_i = \{N\}$ and $\mathcal{A}'_{3-i} = \mathcal{S}_{\binom{[2,n]}{n-1},k}(N)$. Since $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$, we thus have $\mathcal{A} = \{A \in \mathcal{S}_{n,n,k}((1,b)) \colon A \cap N \neq \emptyset\} \cup \{N'_i \colon i \in [k]\} \cong \mathcal{N}'$ or $\mathcal{A} = \mathcal{N}'$; since $\mathcal{A} \neq \Delta_{a,b}(\mathcal{A}) = \mathcal{N}'$, the former holds.

If $2 \leq a \leq n$ then, by the same argument for the corresponding sub-case $2 \leq a \leq r+1$ of Case I, $N = [2,n] \times [1]$ and $\mathcal{A} = \mathcal{S}_{n,n,k}((1,k_1))((N\backslash\{(a,1)\}) \cup \{a,b\}) \cup \{N\backslash\{(a,1)\} \cup \{(a,b),(1,i)\} \colon i \in [k]\} \cong \mathcal{N}_{n,n,k}$.

Lemma 6.3.3 *Let $a \in [n]$, $b \in [2,k]$ and $(3,k) \neq (n,2)$. Suppose $\mathcal{A}$ is an intersecting* *sub-family of $\mathcal{S}_{n,3,k}$ and $\mathcal{A} \neq \Delta_{a,b}(\mathcal{A}) \cong \mathcal{T}_{n,3,k}$. Then $\mathcal{A} \cong \mathcal{T}_{n,3,k}$.*

Proof. We may assume without loss of generality that

$$\Delta_{a,b}(\mathcal{A}) = \mathcal{T}' := \{A \in \mathcal{S}_{n,3,k} \colon |A \cap T| \geq 2\},$$

where $T := [3] \times \{k'\}$ and $k' \in [k]$. Since $\mathcal{A} \neq \Delta_{a,b}(\mathcal{A})$, there exists $A^* \in \mathcal{A} \backslash \Delta_{a,b}(\mathcal{A})$ such that $A' := \delta_{a,b}(A^*) \in \Delta_{a,b}(\mathcal{A}) \backslash \mathcal{A}$. Since $A' \in \mathcal{T}'$, $|A' \cap T| \geq 2$. Thus, since $(a,1) \in A'$, we have $(a,1) \in T$ because otherwise we get $|A^* \cap T| \geq |A' \cap T| = 2$ contradicting $A^* \notin \Delta_{a,b}(\mathcal{A}) = \mathcal{T}'$. So $a \in [3]$, $k' = 1$ and $\mathcal{T}' = \mathcal{T} := \mathcal{T}_{n,3,k}$. We may assume that $a = 1$.

Let $\mathcal{A}_1 := \mathcal{A}((1,1))$, $\mathcal{A}_1' := \mathcal{A}\langle(1,1)\rangle$, $\mathcal{A}_2 := \mathcal{A}((1,b))$, $\mathcal{A}_2' := \mathcal{A}\langle(1,b)\rangle$, $\mathcal{A}_0 := \mathcal{A} \backslash (\mathcal{A}_1 \cup \mathcal{A}_2)$. Let $Z := [2,3] \times [1]$. Since $\Delta_{1,b} = \mathcal{T}$, we clearly have $\mathcal{A}_0 = \{A \in \mathcal{S}_{n,3,k}]\{(1,1),(1,b)\}[\colon\ Z \subset A\}$, $\mathcal{A}_1' \cup \mathcal{A}_2' = \mathcal{S}_{\binom{[2,n]}{2},k}(Z)$, $\mathcal{A}_1' \cap \mathcal{A}_2' = \{Z\}$, $|\mathcal{A}_1'| + |\mathcal{A}_2'| = |\mathcal{S}_{\binom{[2,n]}{2},k}(Z)| + 1$. By Theorem 6.1.1, it follows that for some $i \in [2]$, $\mathcal{A}_i' = \{Z\}$ and $\mathcal{A}_{3-i}' = \mathcal{S}_{\binom{[2,n]}{2},k}(Z)$. Since $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$, we thus have $\mathcal{A} = \{A \in \mathcal{S}_{n,3,k} \colon A \cap (\{(1,b)\} \cup Z) \neq \emptyset\} \cong \mathcal{T}$ or $\mathcal{A} = \mathcal{T}$. Since $\mathcal{A} \neq \Delta_{a,b}(\mathcal{A}) = \mathcal{T}$, the former holds. □

Lemma 6.3.4 *Let $a \in [4]$, $b \in [2,k]$ and $k \geq 3$. Suppose $\mathcal{A}$ is an intersecting sub-family of $\mathcal{S}_{4,4,k}$ and $\mathcal{A} \neq \Delta_{a,b}(\mathcal{A}) \cong \mathcal{T}_{4,4,k}$. Then $\mathcal{A} \cong \mathcal{T}_{4,4,k}$.*

Proof. By the argument in the proof of Lemma 6.3.3, we may assume that $a = 1$ and $\Delta_{1,b}(\mathcal{A}) = \mathcal{T} := \mathcal{T}_{4,4,k}$. Taking Z, $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_1'$, $\mathcal{A}_2$ and $\mathcal{A}_2'$ to be as in the proof of Lemma 6.3.3, we now (similarly) have

$$\mathcal{A}_0 = \{A \in \mathcal{S}_{4,4,k}]\{(1,1),(1,b)\}[\colon Z \subset A\}, \tag{6.9}$$

$$\mathcal{A}_1' \cup \mathcal{A}_2' = \mathcal{S}' := \mathcal{S}_{\binom{[2,4]}{3},k}(Z), \tag{6.10}$$

$$\mathcal{A}_1' \cap \mathcal{A}_2' = \{Z \cup \{(4,k')\} \colon k' \in [k]\}. \tag{6.11}$$

For $j = 1, 2$, let ${}_j\mathcal{S}' := \{A' \in \mathcal{S}' \colon |A \cap Z| = j\}$ and ${}_j\mathcal{A}'_i := \{A'_i \in \mathcal{A}'_i \colon |A \cap Z| = j\}$, $i = 1, 2$. By (6.10) and (6.11),

$$ {}_1\mathcal{A}'_1 \cup {}_1\mathcal{A}'_2 = {}_1\mathcal{S}', \tag{6.12} $$

$$ {}_2\mathcal{A}'_1 = {}_2\mathcal{A}'_2 = {}_2\mathcal{S}'. \tag{6.13} $$

Suppose ${}_1\mathcal{A}'_1 \neq \emptyset$ and ${}_1\mathcal{A}'_2 \neq \emptyset$. For each $i \in [2]$, let $A'_i \in {}_1\mathcal{A}'_i$. Suppose that $A'_1 \cap A'_2 \cap Z = \emptyset$. Then, for some $i \in [2]$, $k_1, k_2 \in [2, k]$ and $k', k'' \in [k]$, we have $A'_i = \{(2, 1), (3, k_1), (4, k')\}$ and $A'_{3-i} = \{(2, k_2), (3, 1), (4, k'')\}$; we may assume that $i = 1$. Since $\mathcal{A}$ is intersecting, we have ${}_1\mathcal{A}'_1, {}_1\mathcal{A}'_2$ cross-intersecting, and hence $k' = k''$. We have $|[k]\backslash\{k'\}| \geq 2$ as $k \geq 3$. Let $k_3, k_4 \in [k]\backslash\{k'\}$, $k_3 \neq k_4$. By (13), $A'_3 := \{(2, 1), (3, k_1), (4, k_3)\}$ and $A'_4 := \{(2, k_2), (3, 1), (4, k_4)\}$ are in $\mathcal{A}'_1 \cup \mathcal{A}'_2$. Since $A'_2 \cap A'_3 = \emptyset$, we must have $A'_3 \in \mathcal{A}'_2$. Similarly, $A'_4 \in \mathcal{A}'_1$ as $A'_1 \cap A'_4 = \emptyset$. But this is a contradiction (to $\mathcal{A}$ intersecting) because $A'_3 \cap A'_4 = \emptyset$. So $A'_1 \cap A'_2 = \{z\}$ for some $z \in Z$, and hence, by the same argument, $A''_1 \cap A'_2 = A''_2 \cap A'_1 = \{z\}$ for any $A''_1 \in {}_1\mathcal{A}'_1$ and $A''_2 \in {}_1\mathcal{A}'_2$. This gives us ${}_1\mathcal{A}'_1 \cup {}_1\mathcal{A}'_2 \subseteq \mathcal{S}'(z)$, a contradiction to (6.12).

Therefore, ${}_1\mathcal{A}'_1 = \emptyset$ or ${}_1\mathcal{A}'_2 = \emptyset$. Thus, since $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$, it follows by (6.9), (6.12) and (6.13) that $\mathcal{A} = \{A \in \mathcal{S}_{4,4,k} \colon |A \cap (\{(1, b)\} \cup Z)| \geq 2\} \cong \mathcal{T}$ or $\mathcal{A} = \mathcal{T}$. Since $\mathcal{A} \neq \Delta_{a,b}(\mathcal{A}) = \mathcal{T}$, the former holds. □

We now come to the proof of 6.1.2, in which we use the Hilton-Milner Theorem. As in Chapter 3, we use $\mathcal{N}_{n,r}$ to denote the 'Hilton-Milner-type' family $\binom{[n]}{r}(1)([2, r+1]) \cup \{[2, r+1]\}$.

Proof of Theorem 6.1.2. The result is trivial for $r = 2$ because a 2-uniform non-centred intersecting family can only be of the form $\{\{a, b\}, \{a, c\}, \{b, c\}\}$. The case $k = 2$ and $r = n$ is also easy because for any $A := \{(a_1, k_1), ..., (a_n, k_n)\} \in \mathcal{S}_{n,n,2}$, the unique set in $\mathcal{S}_{n,n,2}$ that does not intersect A is $\{(a_1, 3 - k_1), ..., (a_n, 3 - k_n)\}$; hence an intersecting sub-family of $\mathcal{S}_{n,n,2}$ can have size at most $2^{n-1} = |\mathcal{N}_{n,n,2}|$. So we now assume that $r \geq 3$ and $(r, k) \neq (n, 2)$.

Let $\mathcal{N} := \mathcal{N}_{n,r,k}$. We will assume that

$$|\mathcal{N}| \leq |\mathcal{A}|. \tag{6.14}$$

Let $\mathcal{A}^* := \Delta_{n,k} \circ \ldots \circ \Delta_{n,2} \circ \ldots \circ \Delta_{1,k} \circ \ldots \circ \Delta_{1,2}(\mathcal{A})$. So $|\mathcal{A}^*| = |\mathcal{A}|$. By Lemma 5.3.1, $\mathcal{A}^*$ is intersecting. By (6.14) and Lemma 6.3.1, $\mathcal{A}^*$ is non-centred. By Lemmas 6.3.2 - 6.3.4, if $\mathcal{A}^* \cong \mathcal{N}$ or $\mathcal{A}^* \cong \mathcal{T}_{n,3,k}$ or $\mathcal{A}^* \cong \mathcal{T}_{4,4,k}$ then $\mathcal{A} \cong \mathcal{N}$ or $\mathcal{A} \cong \mathcal{T}_{n,3,k}$ or $\mathcal{A} \cong \mathcal{T}_{4,4,k}$. We may therefore assume that $\mathcal{A} = \mathcal{A}^*$. Taking $X := [n] \times [1]$, Corollary 5.3.2 gives us

$$A_1 \cap A_2 \cap X \neq \emptyset \text{ for any } A_1, A_2 \in \mathcal{A}. \tag{6.15}$$

Define $\mathcal{A}^{(q)} := \{A \in \mathcal{A} \colon |A \cap X| = q\}$ and $\mathcal{A}_X^{(q)} := \{A \cap X \colon A \in \mathcal{A}^{(q)}\}$; define $\mathcal{N}^{(q)}$ and $\mathcal{N}_X^{(q)}$ similarly. By (6.15), $\bigcup_{q=1}^{r} \mathcal{A}^{(q)}$ is a partition for $\mathcal{A}$. Define w_q as in the proof of Theorem 6.1.1. So

$$|\mathcal{A}| = \sum_{q=1}^{r} |\mathcal{A}^{(q)}| \leq \sum_{q=1}^{r} |\mathcal{A}_X^{(q)}| w_q, \quad |\mathcal{N}| = \sum_{q=1}^{r} |\mathcal{N}^{(q)}| = \sum_{q=1}^{r} |\mathcal{N}_X^{(q)}| w_q. \tag{6.16}$$

By considering such partitions and summations, it is easy to check that $|\mathcal{T}_{n,3,k}| = |\mathcal{N}_{n,3,k}|$ and $|\mathcal{T}_{4,4,k}| = |\mathcal{N}_{4,4,k}|$. It remains to show that equality holds in (6.14) and that $\mathcal{A} \in \{\mathcal{N}_{n,r,k}, \mathcal{T}_{n,3,k}, \mathcal{T}_{4,4,k}\}$.

Let $\mathcal{A}_X := \bigcup_{p=1}^{r} \mathcal{A}_X^{(q)}$. Since $\mathcal{A}$ is non-centred, it follows by (6.15) that

$$\mathcal{A}_X \text{ is non-centred.} \tag{6.17}$$

An immediate implication of (6.15) and (6.17) is that

$$\mathcal{A}_X^{(1)} = \emptyset = \mathcal{N}_X^{(1)}. \tag{6.18}$$

Consider $2 \leq p \leq \min\{r, n/2\}$. If $\mathcal{A}_X^{(p)}$ is non-centred then, by Theorem 1.2.2, we have $|\mathcal{A}_X^{(p)}| \leq |\mathcal{N}_{n,p}|$, and hence $|\mathcal{A}_X^{(p)}| \leq |\mathcal{N}_X^{(p)}|$; note that if $p = r$ then $|\mathcal{N}_{n,p}| = |\mathcal{N}_X^{(p)}|$, and if $p < r$ then $|\mathcal{N}_{n,p}| < |\mathcal{N}_X^{(p)}|$ unless $p = 2$, $\mathcal{A}_X^{(2)} \cong \binom{[3]\times[1]}{2}$ and either $r = 3$ or $r = 4 = n$. Now suppose $\mathcal{A}_X^{(p)}$ is centred and $x \in \bigcap_{A \in \mathcal{A}_X^{(p)}} A$. By (6.17), there exists

$B \in \mathcal{A}_X$ such that $x \notin B$. Thus, by (6.15), $\mathcal{A}_X^{(p)} \subseteq \binom{X}{p}(x)(B)$, and hence $|\mathcal{A}_X^{(p)}| \leq |\mathcal{N}_X^{(p)}|$ with equality only if $p < r$ and $\mathcal{A}_X^{(p)} \cong \mathcal{N}_X^{(p)}$.

Therefore, we have just shown that

$$|\mathcal{A}_X^{(p)}| \leq |\mathcal{N}_X^{(p)}|, \quad p = 1, ..., \min\{r, \lfloor n/2 \rfloor\}; \tag{6.19}$$

$$p \leq \min\{r, n/2\},\ p < r,\ |\mathcal{A}_X^{(p)}| = |\mathcal{N}_X^{(p)}|,\ \mathcal{A}_X^{(p)} \not\cong \mathcal{N}_X^{(p)}$$
$$\Rightarrow\ p = 2,\ \min\{r, n-1\} = 3,\ \mathcal{A}_X^{(p)} = \binom{T}{2} \text{ for some } T \in \binom{X}{3}. \tag{6.20}$$

Case I: $r \leq n/2$ (so $n \geq 6$ as $r \geq 3$). Then, by (6.14), we have equalities in (6.19). By (6.20), it follows that either $\mathcal{A}_X^{(2)} \cong \mathcal{N}_X^{(2)}$ or $r = 3$ and $\mathcal{A}_X^{(2)} \cong \binom{T}{2}$ $(T \in \binom{X}{3})$.

Suppose $\mathcal{A}_X^{(2)} \cong \mathcal{N}_X^{(2)}$. Then $\mathcal{A}_X^{(2)} = \binom{X}{2}(x)(A^*)$ for some set $A^* \in \binom{X\backslash\{x\}}{r}$. Let $\mathcal{B} := \mathcal{S}_{n,r,k}(x)(A^*) \cup \{A^*\}$. Clearly, for any $C \in \mathcal{S}_{n,r,k}\backslash\mathcal{B}$ there exists $A \in \mathcal{A}_X^{(2)}$ such that $A \cap C = \emptyset$; thus, by (6.15), $\mathcal{A} \subseteq \mathcal{B}$. Since $\mathcal{B} \cong \mathcal{N}$, it follows by (6.14) that $\mathcal{A} = \mathcal{B}$.

Now suppose $r = 3$ and $\mathcal{A}_X^{(2)} \cong \binom{T}{2}$ $(T \in \binom{X}{3})$. Let $\mathcal{T}' := \{A \in \mathcal{S}_{n,3,k} \colon |A \cap T| \geq 2\}$. Clearly, for any $C \in \mathcal{S}_{n,r,k}\backslash\mathcal{T}'$ (i.e. $|C \cap T| \leq 1$) there exists $A \in \mathcal{A}_X^{(2)}$ such that $A \cap C = \emptyset$; thus, by (6.15), $\mathcal{A} \subseteq \mathcal{T}'$. Since $|\mathcal{T}'| = |\mathcal{N}_{n,3,k}|$, it follows by (6.14) that $\mathcal{A} = \mathcal{T}'$. So $\mathcal{A} \cong \mathcal{T}_{n,3,k}$.

Case II: $r > n/2$. Suppose $n = r = 3$. Then, by (6.17), (6.18) and $\mathcal{A}_X^{(3)} \in \{\emptyset, X\}$, we clearly must have $\mathcal{A}_X^{(2)} \cong \binom{[3]\times[1]}{2}$. By the argument in Case I, $\mathcal{A} \subseteq \mathcal{T}_{3,3,k}$. Since $\mathcal{T}_{3,3,k} = \mathcal{N}_{3,3,k}$, it follows by (6.14) that $\mathcal{A} = \mathcal{N}_{3,3,k}$.

We now consider $n \geq 4$. Let $n - r \leq p \leq n/2$. Note that since we are assuming $(r,k) \neq (n,2)$,

$$w_p \geq w_{n-p} \text{ with strict inequality if } p < n/2. \tag{6.21}$$

By (6.15), for any $A \in \mathcal{A}_X^{(p)}$ and $B \in \mathcal{A}_X^{(n-p)}$, we cannot have $A = X\backslash B$; therefore

$$|\mathcal{A}_X^{(p)}| + |\mathcal{A}_X^{(n-p)}| \leq \binom{n}{n-p} = |\mathcal{N}_X^{(p)}| + |\mathcal{N}_X^{(n-p)}| \tag{6.22}$$

(note that if $p > n - r$ then $\mathcal{N}_X^{(p)} = \binom{X}{p}((1,1))$ and $\mathcal{N}_X^{(n-p)} = \binom{X}{n-p}((1,1))$, and if $p = n - r$ then $\mathcal{N}_X^{(p)} = \binom{X}{p}((1,1))\backslash\{X\backslash([2r+1] \times [1])\}$ and $\mathcal{N}_X^{(n-p)} = \binom{X}{n-p}((1,1)) \cup$

$\{[2r+1]\times[1]\}$). We have

$$\begin{aligned}|\mathcal{A}^{(p)}|+|\mathcal{A}^{(n-p)}| &= |\mathcal{A}_X^{(p)}|w_p+|\mathcal{A}_X^{(n-p)}|w_{n-p}\\ &\leq |\mathcal{N}_X^{(p)}|w_p+(\binom{n}{n-p}-|\mathcal{N}_X^{(p)}|)w_{n-p} \text{ (by (6.19), (6.21), (6.22))}\\ &= |\mathcal{N}_X^{(p)}|w_p+|\mathcal{N}_X^{(n-p)}|w_{n-p}=|\mathcal{N}^{(p)}|+|\mathcal{N}^{(n-p)}|. \quad (6.23)\end{aligned}$$

Suppose $n-r\leq 2$. Then, by (6.16), (6.19) and (6.23), $|\mathcal{A}|\leq|\mathcal{N}|$ with equality only if equality holds in (6.23). By (6.14), equality holds in (6.23) indeed. Note that we therefore have $|\mathcal{A}_X^{(2)}|=|\mathcal{N}_X^{(2)}|$ by (6.21). By (6.20), either $\mathcal{A}_X^{(2)}\cong\mathcal{N}_X^{(2)}$ or $\min\{r,n-1\}=3$ and $\mathcal{A}_X^{(2)}\cong\binom{T}{2}$ ($T\in\binom{X}{3}$). By the argument in Case I, it follows that $\mathcal{A}\cong\mathcal{N}$ or $\mathcal{A}\cong\mathcal{T}_{n,r,k}$, and the latter holds only if $r=3$ or $n=r=4$.

Finally, suppose $n-r>2$ instead. By (6.16), (6.19) (for $p=1,2,...,n-r-1$) and (6.23) (for $p=n-r,...,\lfloor n/2\rfloor$), $|\mathcal{A}|\leq|\mathcal{N}|$ with equality iff equality holds in (6.19) for $p=1,2,...,n-r-1$ and in (6.23) for $p=n-r,...,\lfloor n/2\rfloor$. By (6.14), $|\mathcal{A}|=|\mathcal{N}|$. Since we thus have equality in (6.19) for $p=2$, it follows by (6.20) that either $\mathcal{A}_X^{(2)}\cong\mathcal{N}_X^{(2)}$ or $r=3$ (note that $n-1>3$ as $n-r>2$ and $r\geq 3$) and $\mathcal{A}_X^{(2)}\cong\binom{T}{2}$ ($T\in\binom{X}{3}$). As above, this yields $\mathcal{A}\cong\mathcal{N}$ or $\mathcal{A}\cong\mathcal{T}_{n,3,k}$. □

6.4 Conjecture 5.1.1 is true for $k\geq k_0(\mathcal{F})$

Let $\mathcal{F}$ be a family. It is trivial that if $\alpha(\mathcal{F})=1$ then $\mathcal{S}_{\mathcal{F},k}$ is strictly EKR for all k. If $\alpha(\mathcal{F})=2$, $k\geq 2$ and $\mathcal{A}$ is a non-centred intersecting sub-family of $\mathcal{S}_{\mathcal{F},k}$, then $\mathcal{A}$ can only be a triangle $\{A\in\mathcal{S}_{\mathcal{F},k}\colon |A\cap\{(a_1,k_1),(a_2,k_2),(a_3,k_3)\}|=2\}$ for some distinct $a_1,a_2,a_3\in U(\mathcal{F})$, and hence the centred sub-family $\{\{(a_1,k_1),(a',k')\}\colon (a',k')\in(\{a_2\}\times[k])\cup(\{a_3\}\times[k])\}$ of $\mathcal{S}_{\mathcal{F},k}$ is larger than $\mathcal{A}$. This proves Conjecture 5.1.1 for families $\mathcal{F}$ with $\alpha(\mathcal{F})\leq 2$.

We now consider families $\mathcal{F}$ with $\alpha(\mathcal{F})\geq 3$, and we give the two proofs, mentioned in Section 6, of the fact that there exists an integer $k_0(\mathcal{F})$ such that Conjecture 5.1.1 is true if $k\geq k_0(\mathcal{F})$. We then compare the two bounds that come out of the two proofs. The following is an ingredient common to both proofs.

Lemma 6.4.1 *Let $n, k \in \mathbb{N}$, $k \geq 3$. Let $b_{n,k} \in \mathbb{N}$ such that for all $r \in [n]$, the size of a largest non-centred intersecting sub-family of $\mathcal{S}_{[r],k}$ is not greater than $b_{n,k}$. Let $\mathcal{F}$ be a family with $\alpha(\mathcal{F}) \leq n$, and let $\mathcal{A}$ be a non-centred intersecting sub-family of $\mathcal{S}_{\mathcal{F},k}$. Then $|\mathcal{A} \cap \mathcal{S}_{X,k}| \leq b_{n,k}$ for all $X \in \mathcal{F}$.*

Proof. Let $X \in \mathcal{F}$, and let $\mathcal{A}_X := \mathcal{A} \cap \mathcal{S}_{X,k}$. If $\mathcal{A}_X$ is non-centred then $|\mathcal{A}_X| \leq b_{n,k}$ is a simple consequence of having $\alpha(\mathcal{F}) \leq n$. So suppose $\mathcal{A}_X$ is centred, i.e. $|\bigcap_{A \in \mathcal{A}_X} A| \geq 1$.

Case 1: $|\bigcap_{A \in \mathcal{A}_X} A| = 1$. Let (x, y) be the unique member of $\bigcap_{A \in \mathcal{A}_X} A$. Since $\mathcal{A}$ is non-centred, there exists $A^* \in \mathcal{A}$ such that $(x, y) \notin A^*$. Let $A' := A^* \cap U(\mathcal{S}_{X,k})$, and choose $A'' \in \mathcal{S}_{X,k}$ such that $(x, y) \notin A''$ and $A' \subset A''$. Clearly, $\mathcal{A}_X \cup \{A''\}$ is a non-centred intersecting sub-family of $\mathcal{S}_{X,k}$, and hence $|\mathcal{A}_X| \leq b_{n,k} - 1$.

Case 2: $|\bigcap_{A \in \mathcal{A}_X} A| \geq 2$. Let $I := \bigcap_{A \in \mathcal{A}_X} A$, and let $(x_1, y_1), ..., (x_{|I|}, y_{|I|})$ be the distinct elements of I. Since $I \subseteq A$ for any $A \in \mathcal{A}_X$, $x_1, ..., x_{|I|}$ are distinct. If $|I| = |X|$ then I is the unique member of $\mathcal{A}_X$, so suppose $|I| < |X|$. Let $x^* \in X \backslash \{x_1, ..., x_{|I|}\}$. It is easy to see that, given that $k \geq 3$, we can choose two sets $A_1, A_2 \in \mathcal{S}_{X,k}$ such that $A_1 \cap I = \{x_1, y_1\}$, $A_2 \cap I = \{x_2, y_2\}$ and $A_1 \cap A_2 = \{(x^*, 1)\}$. So $\mathcal{A}_X \cup \{A_1, A_2\}$ is a non-centred intersecting sub-family of $\mathcal{S}_{X,k}$, and hence $|\mathcal{A}_X| \leq b_{n,k} - 2$. □

We now give the first proof, borrowing some ideas from the proof of Theorem 1.2.4 in [25].

Theorem 6.4.2 *For a family $\mathcal{F}$ with $\alpha(\mathcal{F}) \geq 3$, let $k_0(\mathcal{F}) := \binom{n}{\lfloor n/2 \rfloor}(3n-3)^n|\mathcal{F}|+n+1$ where $n := \alpha(\mathcal{F})$. Then $\mathcal{S}_{\mathcal{F},k}$ is strictly EKR for all $k \geq k_0(\mathcal{F})$.*

Proof. Let $k \geq k_0 := k_0(\mathcal{F})$, and let $\mathcal{B}$ be a non-centred intersecting sub-family of $\mathcal{S}_{[n],k}$. We first show that there exists a set I of size at most $3n-3$ such that $|B \cap I| \geq 2$ for any $B \in \mathcal{B}$. If $\mathcal{A}$ is 2-intersecting then we take I to be a set in $\mathcal{B}$. Now suppose $\mathcal{A}$ is not 2-intersecting. Then there exist $B_1, B_2 \in \mathcal{B}$ such that $|B_1 \cap B_2| = 1$. Let b be the unique element of $B_1 \cap B_2$. Since $\mathcal{B}$ is non-centred, there exists $B_3 \in \mathcal{A}$ such that $b \notin B_3$. Let $I := B_1 \cup B_2 \cup B_3$. Since $\mathcal{B}$ is intersecting, $B \cap I \neq \emptyset$ for all $B \in \mathcal{B}$. Suppose there exists $B^* \in \mathcal{B}$ such that $|B^* \cap I| = 1$. Since $B_1 \cap B_2 = \{b\}$ and $B^* \cap B_i \neq \emptyset$ for each $i \in [2]$, we must then have $B^* \cap (B_1 \cup B_2) = \{b\}$. Thus, by our supposition, $B^* \cap I = \{b\}$. But then $B^* \cap B_3 = \emptyset$, a contradiction. So $|B \cap I| \geq 2$ for all $B \in \mathcal{B}$. Now $|I| = |B_1 \cup B_2| + |B_3| - |B_3 \cap (B_1 \cup B_2)|$, and hence, by the above, $|I| = (2n-1) + n - (|B_3 \cap B_1| + |B_3 \cap B_2|) \leq 3n - 3$ as required.

Let J be the smallest set such that $I \subset [n] \times J$. So $1 \leq |J| \leq 3n-3$. For each $i \in [2,n]$, let $\mathcal{B}_i := \{B \in \mathcal{B} \colon |B \cap ([n] \times J)| = i\}$. So, by the above, $\bigcup_{i=2}^{n} \mathcal{B}_i$ is a partition for $\mathcal{B}$. Let q be the quantity $\sum_{i=2}^{n} |\{A \in \mathcal{S}_{[n],k} \colon |A \cap ([n] \times J)| = i\}|$. We therefore have

$$
\begin{aligned}
|\mathcal{B}| = \sum_{i=2}^{n} |\mathcal{B}_i| < q &= \sum_{i=2}^{n} \binom{n}{i} (|J|)^i (k - |J|)^{n-i} \\
&< \sum_{i=2}^{n} \binom{n}{i} (3n-3)^i (k-1)^{n-i} < \binom{n}{\lfloor n/2 \rfloor} (3n-3)^n \sum_{i=2}^{n} (k-1)^{n-i} \\
&= \left(\frac{k_0(\mathcal{F}) - n - 1}{|\mathcal{F}|} \right) \left(\frac{1 - (k-1)^{n-1}}{1 - (k-1)} \right) \leq \frac{(k-1)^{n-1} - 1}{|\mathcal{F}|}.
\end{aligned}
$$

Since $n := \alpha(\mathcal{F})$, it follows by Lemma 6.4.1 with $b_{n,k} = ((k-1)^{n-1} - 1)/|\mathcal{F}|$ that if $\mathcal{A}$ is a non-centred intersecting sub-family of $\mathcal{S}_{\mathcal{F},k}$ then $|\mathcal{A}| \leq (k-1)^{n-1} - 1$ and hence $|\mathcal{A}| < k^{n-1}$. This concludes the proof because, since $n := \alpha(\mathcal{F})$ implies that there exists a set $X \in \mathcal{F}$ of size n, the size of a largest star of $\mathcal{S}_{\mathcal{F},k}$ is at least as large as the size k^{n-1} of a star of $\mathcal{S}_{X,k}$. □

We now give the second proof, which is based on Theorem 6.1.2 with $r = n$.

Theorem 6.4.3 *For a family $\mathcal{F}$ with $\alpha(\mathcal{F}) \geq 3$, let $k_0(\mathcal{F}) := \left(\binom{n-1}{\lfloor (n-1)/2 \rfloor} + 1 \right) |\mathcal{F}| + 2$ where $n := \alpha(\mathcal{F})$. Then $\mathcal{S}_{\mathcal{F},k}$ is strictly EKR for all $k \geq k_0(\mathcal{F})$.*

Proof. Let $k \geq k_0 := k_0(\mathcal{F})$. Let $\mathcal{A}$ be a non-centred intersecting sub-family of $\mathcal{S}_{\mathcal{F},k}$. For any $X \in \mathcal{F}$, let $\mathcal{A}_X := \mathcal{A} \cap \mathcal{S}_{X,k}$.

By Lemma 6.4.1 with $b_{n,k} = |\mathcal{N}_{n,n,k}|$ and Theorem 6.1.2 with $r = n$, we have

$|\mathcal{A}_X| \leq |\mathcal{N}_{n,n,k}|$ for all $X \in \mathcal{F}$. Therefore,

$$\begin{aligned}
|\mathcal{A}| &= \sum_{X\in\mathcal{F}} |\mathcal{A}_X| \leq \sum_{X\in\mathcal{F}} |\mathcal{N}_{|X|,|X|,k}| \leq \sum_{X\in\mathcal{F}} |\mathcal{N}_{n,n,k}| \\
&= (k^{n-1} - (k-1)^{n-1} + k - 1)|\mathcal{F}| \\
&= ((1 + (k-1))^{n-1} - (k-1)^{n-1} + k - 1)|\mathcal{F}| \\
&= \left(\sum_{i=0}^{n-1} \binom{n-1}{i} (k-1)^{n-1-i} - (k-1)^{n-1} + k - 1\right) |\mathcal{F}| \\
&= \left(\sum_{i=1}^{n-1} \binom{n-1}{i} (k-1)^{n-1-i} + k - 1\right) |\mathcal{F}| \\
&< \left(\binom{n-1}{\lfloor (n-1)/2 \rfloor} \sum_{i=1}^{n-1} (k-1)^{n-1-i} + k - 1\right) |\mathcal{F}| \\
&< \left(\binom{n-1}{\lfloor (n-1)/2 \rfloor} + 1\right) \frac{1 - (k-1)^{n-1}}{1 - (k-1)} |\mathcal{F}| \\
&< (k_0 - 2)\frac{k^{n-1}}{k-2} \leq k^{n-1}.
\end{aligned}$$

Similarly to the proof of Theorem 6.4.2, this concludes the proof. □

It is obviously clear that the value of $k_0(\mathcal{F})$ in Theorem 6.4.3 is significantly better than the one in Theorem 6.4.2. As we have shown, this improvement is the result of removing a 'non-delicate' argument borrowed from [25] and applying Theorem 6.1.2 instead.

Chapter 7

t-intersecting families of signed sets and partial permutations

7.1 Introduction

A natural question that arises from Conjecture 5.1.1 is whether a similar statement for t-intersecting families of signed sets is true. Theorem 1.4.4 tells us that for $k \geq t+1$, $\mathrm{ex}(\mathcal{S}_{[n],k}; t)$ contains trivial t-intersecting families. Together with Conjecture 5.1.1, this seems to suggest that for any family $\mathcal{F}$ and any $k \geq t+1$, $\mathrm{ex}(\mathcal{S}_{\mathcal{F},k}; t)$ contains trivial t-intersecting families. This is not true if $t > 1$. Indeed, consider $(r-t+1)(2+\frac{t-1}{2}) < n < (r-t+1)(t+1)$, and let $\mathcal{G} := \{A \in \binom{[n]}{r} \colon |A \cap [t+2]| \geq t+1\}$. By Theorem 1.2.5 (with $m = 1$), $|\mathcal{G}| > |\binom{[n]}{r}[t]|$. Let $\mathcal{B} := \{A \in \mathcal{S}_{[r],t+1} \colon |A \cap ([t+2] \times [1])| \geq t+1\}$. By Theorem 1.4.4 (with $n = r$ and $m = 0$), $|\mathcal{B}| = |\mathcal{S}_{[r],t+1}[[t] \times [1]]|$. Taking $\mathcal{A}$ to be the non-trivial t-intersecting family $\{A \in \mathcal{S}_{\binom{[n]}{r},t+1} \colon |A \cap ([t+2] \times [1])| \geq t+1\}$, we therefore get

$$|\mathcal{A}| - |\mathcal{S}_{\binom{[n]}{r},t+1}[[t] \times [1]]| = |\mathcal{G}||\mathcal{B}| - \left|\binom{[n]}{r}[t]\right||\mathcal{B}| > 0,$$

which proves what we set out to show.

However, we suggest the following conjecture.

Conjecture 7.1.1 *For any t there exists $k_0(t) \in \mathbb{N}$ such that for any family $\mathcal{F}$ with*

$\alpha(\mathcal{F}) \geq t$ *and any* $k \geq k_0(t)$*, the largest* t*-intersecting sub-families of* $\mathcal{S}_{\mathcal{F},k}$ *are trivial.*

The example that we gave above simply proves that if $t > 1$ and $k_0(t)$ exists then $k_0(t) > t + 1$. We are able to prove the following relaxation of the statement of the conjecture.

Theorem 7.1.2 *For* $t \leq r$ *there exists* $k_0(r,t) \in \mathbb{N}$ *such that for any family* $\mathcal{F}$ *with* $t \leq \alpha(\mathcal{F}) \leq r$ *and any* $k \geq k_0(r,t)$*, the largest* t*-intersecting sub-families of* $\mathcal{S}_{\mathcal{F},k}$ *are trivial.*

We in fact show that $k_0(r,t)$ can be taken to be $\binom{r}{t}\binom{r}{t+1}$.

We next pose a similar problem for t-intersecting sub-families of $\mathcal{S}^*_{\mathcal{F},k}$.

Conjecture 7.1.3 *For any* t *there exists* $k_0^*(t) \in \mathbb{N}$ *such that for any family* $\mathcal{F}$ *with* $\alpha(\mathcal{F}) \geq t$ *and any* $k \geq k_0^*(t)$*, the largest* t*-intersecting sub-families of* $\mathcal{S}^*_{\mathcal{F},k}$ *are trivial.*

This generalises Conjecture 1.5.3: take $\mathcal{F} = \{[k]\}$, $k \geq k_0^*(t)$. We are able to prove the following analogue of Theorem 7.1.2.

Theorem 7.1.4 *For* $t \leq r$ *there exists* $k_0^*(r,t) \in \mathbb{N}$ *such that for any family* $\mathcal{F}$ *with* $t \leq \alpha(\mathcal{F}) \leq r$ *and any* $k \geq k_0^*(r,t)$*, the largest* t*-intersecting sub-families of* $\mathcal{S}^*_{\mathcal{F},k}$ *are trivial.*

The value we compute for $k_0^*(r,t)$ is $\binom{r}{t}\binom{3r-2t+1}{\lfloor\frac{3r-2t+1}{2}\rfloor}\frac{r!}{(r-t-1)!} + r + 1$. Theorem 1.5.4 is an immediate consequence of this result: take $\mathcal{F} = \binom{[k]}{r}$, $k \geq k_0^*(r,t)$.

We now proceed to the proofs of the two theorems above, employing the notation in Section 2.1 as we go along.

7.2 t-intersecting families of signed sets

We shall base the proof of Theorem 7.1.2 on the compression technique. We point out that this can be avoided using an argument similar to the one for Theorem 7.1.4; however, the compression technique enables us to obtain a neater proof and a better value for $k_0(r,t)$.

For $(a,b) \in [n] \times [2,k]$, let the compression $\Delta_{a,b}$ be as defined in Section 5.4.

Lemma 7.2.1 *Let $\mathcal{F} \subseteq 2^{[n]}$, $k \geq 3$ and $(a, b) \in [n] \times [2, k]$. Suppose $\mathcal{A}$ is a non-trivial t-intersecting sub-family of $\mathcal{S}_{\mathcal{F},k}$ and $\Delta_{a,b}(\mathcal{A}) \subseteq \mathcal{S}_{\mathcal{F},k}[Z]$ for some $Z \in \binom{[n]\times[k]}{t}$. Then $|\mathcal{A}| < |\mathcal{S}_{\mathcal{F},k}[Z]|$.*

Proof. Let $Y := \{z\colon (z, l) \in Z \text{ for some } l \in [k]\}$. Given that $\Delta_{a,b}(\mathcal{A}) \subseteq \mathcal{S}_{\mathcal{F},k}[Z]$, $\mathcal{A} \subset \mathcal{S}_{\mathcal{F}[Y],k}$ and, since $\mathcal{A}$ is non-trivial, there exists $A_2 \in \mathcal{A}$ such that $|A_2 \cap Z| = t - 1$ and $Z \subseteq A_1 := \delta_{a,b}(A_2)$. So $(a, 1) \in Z$ and $Z' := Z \backslash \{(a, 1)\} \subset A$ for all $A \in \mathcal{A}$. Let $Y' := Y \backslash \{a\}$. Setting $\mathcal{F}' := \{F \backslash Y'\colon F \in \mathcal{F}[Y']\}$ and $\mathcal{A}' := \{A \backslash Z'\colon A \in \mathcal{A}[Z']\}$, we then have $\mathcal{A}' \subset \mathcal{S}_{\mathcal{F}'(a),k}$ (as $\mathcal{A} \subset \mathcal{S}_{\mathcal{F}[Y],k}$ and $Y = Y' \cup \{a\}$) and $|\mathcal{A}'| = |\mathcal{A}|$. Since $\mathcal{A}$ is a non-trivial t-intersecting family and $|Z'| = t - 1$, $\mathcal{A}'$ is a non-trivial intersecting family.

For $F' \in \mathcal{F}'(a)$, let $\mathcal{A}'_{F'} := \mathcal{A}' \cap \mathcal{S}_{F',k}$. Since $\mathcal{A}'$ is intersecting, $\mathcal{A}'_{F'}$ is intersecting. Suppose $\mathcal{A}'_{F'} \neq \emptyset$. If $\mathcal{A}'_{F'}$ is non-trivial then, Theorem 1.4.1 or Theorem 6.1.2, $|\mathcal{A}'_{F'}| < k^{|F'|-1}$. Suppose $\mathcal{A}'_{F'}$ is trivial; so $\mathcal{A}'_{F'} \subseteq \mathcal{S}_{F',k}((c, d))$ for some $(c, d) \in F' \times [k]$. Since $\mathcal{A}'$ is non-trivial, there exists $A' \in \mathcal{A}'$ such that $(c, d) \notin A'$. Thus, since $\mathcal{A}'$ is intersecting, we actually have $\mathcal{A}'_{F'} \subseteq \{A \in \mathcal{S}_{F',k}((c, d))\colon A \cap A' \neq \emptyset\}$, and hence we again get $|\mathcal{A}'_{F'}| < k^{|F'|-1}$.

We therefore have

$$|\mathcal{A}| = |\mathcal{A}'| = \sum_{F' \in \mathcal{F}'(a)} |\mathcal{A}'_{F'}| < \sum_{F' \in \mathcal{F}'(a)} k^{|F'|-1} = \sum_{F \in \mathcal{F}[Y]} k^{|F|-t},$$

and the result follows since $\sum_{F \in \mathcal{F}[Y]} k^{|F|-t} = |\mathcal{S}_{\mathcal{F},k}[Z]|$. □

Proof of Theorem 7.1.2. Let $\mathcal{F}$ be a family with $t \leq \alpha(\mathcal{F}) \leq r$. We may assume that $\mathcal{F} \subseteq 2^{[n]}$ for some $n \in \mathbb{N}$. Let $k \geq k_0(r, t) := \binom{r}{t}\binom{r}{t+1}$, and let $\mathcal{A}^*$ be a non-trivial t-intersecting sub-family of $\mathcal{S}_{\mathcal{F},k}$.

Let $\mathcal{A} := \Delta_{n,k} \circ \ldots \circ \Delta_{n,2} \circ \ldots \circ \Delta_{1,k} \circ \ldots \circ \Delta_{1,2}(\mathcal{A}^*)$. So $\mathcal{A} \subset \mathcal{S}_{\mathcal{F},k}$ and $|\mathcal{A}| = |\mathcal{A}^*|$. Also, setting $X := [n] \times [1]$, it follows by Corollary 5.3.2 that

$$|A \cap B \cap X| \geq t \text{ for any } A, B \in \mathcal{A}. \tag{7.1}$$

Suppose $\mathcal{A}$ is a trivial t-intersecting family, i.e. $\mathcal{A} \subseteq \mathcal{S}_{\mathcal{F},k}[Y]$ for some $Y \in \binom{S}{t}$,

$S \in \mathcal{S}_{\mathcal{F},k}$. By Lemma 7.2.1, we then have $|\mathcal{A}^*| < |\mathcal{S}_{\mathcal{F},k}[Y]|$, and hence we are done.

Now suppose $\mathcal{A}$ is a non-trivial t-intersecting family. If $|A' \cap X| = t$ for some $A' \in \mathcal{A}$ then, by (7.1), $A' \cap X \subseteq A$ for all $A \in \mathcal{A}$, which contradicts $\mathcal{A}$ non-trivial. So $|A \cap X| \geq t+1$ for all $A \in \mathcal{A}$, and hence we obtain a crude bound for the size of $\mathcal{A}_F := \mathcal{A} \cap \mathcal{S}_{F,k}$ $(F \in \mathcal{F})$ as follows:

$$|\mathcal{A}_F| \leq |\{A \in \mathcal{S}_{F,k} \colon |A \cap (F \times [1])| \geq t+1\}| < \binom{|F|}{t+1} k^{|F|-t-1}. \tag{7.2}$$

Let $B \in \mathcal{A}$. Since $\mathcal{A}$ is t-intersecting (by (7.1)), each $A \in \mathcal{A}$ must contain at least one of the sets in $\binom{B}{t}$, and hence $\mathcal{A} = \bigcup_{C \in \binom{B}{t}} \mathcal{A}[C]$. Choose $C^* \in \binom{B}{t}$ such that $|\mathcal{A}[C]| \leq |\mathcal{A}[C^*]|$ for all $C \in \binom{B}{t}$. We then have

$$|\mathcal{A}| = |\bigcup_{C \in \binom{B}{t}} \mathcal{A}[C]| \leq \sum_{C \in \binom{B}{t}} |\mathcal{A}[C]| \leq \binom{|B|}{t} |\mathcal{A}[C^*]| \leq \binom{r}{t} |\mathcal{A}[C^*]|. \tag{7.3}$$

Set $\mathcal{G} := \{F \in \mathcal{F} \colon \mathcal{A}[C^*] \cap \mathcal{S}_{F,k} \neq \emptyset\}$ and $\mathcal{C} := \bigcup_{G \in \mathcal{G}} \mathcal{S}_{G,k}[C^*]$. Bringing all the pieces together, we get

$$\begin{aligned}
|\mathcal{C}| &= \sum_{G \in \mathcal{G}} k^{|G|-t} \geq \sum_{G \in \mathcal{G}} k_0(r,t) k^{|G|-t-1} = \sum_{G \in \mathcal{G}} \binom{r}{t}\binom{r}{t+1} k^{|G|-t-1} \\
&> \binom{r}{t} \sum_{G \in \mathcal{G}} |\mathcal{A}_G| && \text{(by (7.2))} \\
&\geq \binom{r}{t} |\mathcal{A}[C^*]| \geq |\mathcal{A}|, && \text{(by (7.3))}
\end{aligned}$$

and hence $|\mathcal{A}^*| < |\mathcal{C}| \leq |\mathcal{S}_{\mathcal{F},k}[C^*]|$ as $|\mathcal{A}^*| = |\mathcal{A}|$. □

7.3 t-intersecting families of partial permutations

The proof of Theorem 7.1.4 is based on ideas from the preceding section and ideas used by Erdős, Ko and Rado [25] for their result concerning t-intersecting sub-families of $\binom{[n]}{r}$. Unfortunately, the compression technique fails to work for intersecting sub-families of $\mathcal{S}^*_{[n],k}$.

Let $l(n,k,t)$ be the size of a largest non-trivial t-intersecting sub-family of $\mathcal{S}^*_{[n],k}$, and let $P_j := \{(i,i)\colon i \in [j]\}$.

Lemma 7.3.1 *For $c, n, t \in \mathbb{N}$, $t \leq n$, let $k_0^*(c,n,t) := c\binom{3n-2t+1}{\lfloor\frac{3n-2t+1}{2}\rfloor}\frac{n!}{(n-t-1)!}+n+1$. Then for any $k \geq k_0^*(c,n,t)$,*

$$|\mathcal{S}^*_{[n],k}[P_t]| > c\max\{l(n,k,t), |\mathcal{S}^*_{[n],k}[P_{t+1}]|\}.$$

Proof. The result is trivial if $n = t$, so we assume that $n \geq t+1$. Let $k \geq k_0^*(c,n,t)$, and let $\mathcal{A} \subset \mathcal{S}^*_{[n],k}$ be a non-trivial t-intersecting family of size $l(n,k,t)$. Choose $A_1, A_2 \in \mathcal{A}$ such that $|A_1 \cap A_2| \leq |A \cap B|$ for all $A, B \in \mathcal{A}$.

Suppose $|A_1 \cap A_2| \geq t+1$. Let $(i^*, j^*) \in [n] \times [k]$ such that $(i^*, j^*) \in A_1 \cap A_2$. Let $j' \in [k]$ such that $(i, j') \notin A_1 \cup A_2$ for all $i \in [n]$ (note that such a j' exists since $k \geq k_0^*(c,n,t) > |A_1 \cup A_2|$). Let $A_1' := (A_1 \backslash \{(i^*, j^*)\}) \cup (i^*, j')$. By choice of j', $A_1' \in \mathcal{S}^*_{[n],k}$. Let $\mathcal{A}' := \mathcal{A} \cup A_1'$. Since $|A_1' \cap A_2| < |A_1 \cap A_2|$, it follows by choice of A_1 and A_2 that $A_1' \notin \mathcal{A}$, and hence $|\mathcal{A}'| = |\mathcal{A}| + 1$. Also by choice of A_1 and A_2, we have $|A \cap B| \geq t+1$ for all $A, B \in \mathcal{A}$, which implies that $\mathcal{A}'$ is t-intersecting. Since $\mathcal{A} \subset \mathcal{A}'$ and $\mathcal{A}$ is non-trivially t-intersecting, $|\bigcap_{A' \in \mathcal{A}'} A'| \leq |\bigcap_{A \in \mathcal{A}} A| < t$. So $\mathcal{A}'$ is a non-trivial t-intersecting sub-family of $\mathcal{S}^*_{[n],k}$ of size greater than $|\mathcal{A}|$; but this contradicts $|\mathcal{A}| = l(n,k,t)$. We therefore conclude that $|A_1 \cap A_2| = t$. Thus, since $\mathcal{A}$ is non-trivially t-intersecting, there exists $A_3 \in \mathcal{A}$ such that $A_1 \cap A_2 \not\subseteq A_3$, and hence $|A_1 \cap A_2 \cap A_3| < t$.

Let $I := A_1 \cup A_2 \cup A_3$. Suppose there exists $A^* \in \mathcal{A}$ such that $|A^* \cap I| < t+1$. Since $|A_1 \cap A_2| = t$ and $|A^* \cap A_i| \geq t$ for each $i \in [2]$, we must then have $A^* \cap (A_1 \cup A_2) = A_1 \cap A_2$. Thus, by our supposition, $A^* \cap I = A_1 \cap A_2$. But then $A^* \cap A_3 = A_1 \cap A_2 \cap A_3$, which gives the contradiction that $|A^* \cap A_3| < t$. Therefore

$$|A \cap I| \geq t+1 \text{ for all } A \in \mathcal{A}. \tag{7.4}$$

Now $|I| = |A_1 \cup A_2| + |A_3| - |A_3 \cap (A_1 \cup A_2)|$. Since $|A_1 \cup A_2| = 2n - |A_1 \cap A_2| = 2n - t$ and $|A_3 \cap (A_1 \cup A_2)| = |A_3 \cap A_1| + |(A_3 \cap A_2) \backslash A_1| \geq t + (t - |A_3 \cap A_2 \cap A_1|) \geq 2t - (t-1) =$

$t+1$, it follows that

$$|I| \leq (2n-t)+n-(t+1) = 3n-2t-1.$$

Taking J to be the smallest set such that $I \subset [n] \times J$, we then have

$$n \leq |J| \leq 3n-2t-1.$$

For each $i \in [t+1, n]$, let $\mathcal{A}_i := \{A \in \mathcal{A}\colon |A \cap ([n] \times J)| = i\}$. By (7.4), $\bigcup_{i=t+1}^{n} \mathcal{A}_i$ is a partition for $\mathcal{A}$. Let $x := \sum_{i=t+1}^{n} |\{A \in \mathcal{S}^*_{[n],k}\colon |A \cap ([n] \times J)| = i\}|$. We therefore have

$$\begin{aligned}
l(n,k,t) = |\mathcal{A}| &= \sum_{i=t+1}^{n} |\mathcal{A}_i| < x = \sum_{i=t+1}^{n} \binom{|J|}{i}\binom{n}{i} i! \binom{k-|J|}{n-i}(n-i)! \\
&< \sum_{i=t+1}^{n} \binom{3n-2t+1}{i}\binom{n}{i} i! \binom{k-n}{n-i}(n-i)! \\
&\leq \sum_{i=t+1}^{n} \binom{3n-2t+1}{i} \frac{n!}{(n-i)!}(k-n)^{(n-i)} \\
&\leq \binom{3n-2t+1}{\lfloor \frac{3n-2t+1}{2} \rfloor} \frac{n!}{(n-t-1)!} \sum_{i=t+1}^{n} (k-n)^{(n-i)} \\
&= \left(\frac{k_0^*(c,n,t)-n-1}{c}\right)\left(\frac{1-(k-n)^{n-t}}{1-(k-n)}\right) \leq \frac{(k-n)^{n-t}-1}{c} \\
&< \frac{1}{c}\left(\frac{(k-t)!}{(k-n)!}\right) = \frac{|\mathcal{S}^*_{[n],k}[P_t]|}{c}.
\end{aligned}$$

The result now follows since we also have $|\mathcal{S}^*_{[n],k}[P_{t+1}]| < x$. □

Proof of Theorem 7.1.4. Let $\mathcal{F}$ be a family with $t \leq \alpha(\mathcal{F}) \leq r$. Let $k_0^*(\binom{r}{t}, n, t)$ be as in the statement of Lemma 7.3.1 with $c = \binom{r}{t}$. Let

$$k \geq k_0^*(r,t) := \max\{k_0^*(\binom{r}{t}, n, t)\colon n \in [r]\}. \tag{7.5}$$

Note that therefore $k_0^*(r,t) = k_0^*(\binom{r}{t}, r, t)$. Let $\mathcal{A}$ be a non-trivial t-intersecting subfamily of $\mathcal{S}^*_{\mathcal{F},k}$. So $\mathcal{A} \subseteq \mathcal{S}^*_{\mathcal{F}^{(\geq t)},k}$, and we may therefore assume that $\mathcal{F} = \mathcal{F}^{(\geq t)}$.

For any $F \in \mathcal{F}$ and any family $\mathcal{B} \subseteq \mathcal{S}^*_{\mathcal{F},k}$, set $\mathcal{B}_F := \mathcal{B} \cap \mathcal{S}^*_{F,k}$. For all $F \in \mathcal{F}$, choose

$F' \in \mathcal{S}^*_{\binom{F}{t},k}$. We show that for all $F \in \mathcal{F}$,

$$\binom{r}{t}|\mathcal{A}_F| < |\mathcal{S}^*_{F,k}[F']|. \tag{7.6}$$

If $\mathcal{A}_F$ is non-trivially t-intersecting then this follows immediately from (7.5) and Lemma 7.3.1. Now suppose $\mathcal{A}_F$ is a trivial t-intersecting family. Setting $T := \bigcap_{A \in \mathcal{A}_F} A$, we then have $|T| \geq t$. If $|T| \geq t+1$ then (7.6) again follows immediately from (7.5) and Lemma 7.3.1. It remains to consider $|T| = t$. Since $\mathcal{A}$ is a non-trivial t-intersecting family, there exists $A_1 \in \mathcal{A}$ such that $T \not\subseteq A_1$ and hence $|T \cap A_1| < t$. Let $D_1 := A_1 \cap (F \times [k])$. Let F_1 be the subset of F such that $D_1 \in \mathcal{S}^*_{F_1,k}$. Let E_2 be the subset of F such that $T \in \mathcal{S}^*_{E_2,k}$. Let $F_2 := E_2 \backslash F_1$, and let T' be the set in $\mathcal{S}^*_{F_2,k}$ given by $T' := T \cap (F_2 \times [k])$. If $T' \neq \emptyset$ and $(x_1, y_1), ..., (x_{|T'|}, y_{|T'|})$ are the distinct elements of T' then take $D_2 := \{(x_i, y_i + 1 \mod k) \colon i \in [|T'|]\}$; otherwise take $D_2 := \emptyset$. Let $F_3 := F \backslash (F_1 \cup F_2)$. If $F_3 \neq \emptyset$ then take D_3 to be a set in $\mathcal{S}^*_{F_3,k}$; otherwise take $D_3 := \emptyset$. Now let $A_2 := D_1 \cup D_2 \cup D_3$. Clearly, $A_2 \in \mathcal{S}^*_{F,k}$. Therefore $\mathcal{A}_F \cup \{A_2\}$ is a non-trivial t-intersecting sub-family of $\mathcal{S}^*_{F,k}$ because $|\bigcap_{A' \in \mathcal{A}_F \cup \{A_2\}} A'| = |T \cap A_2| = |T \cap D_1| = |T \cap A_1| < t$ and for all $A \in \mathcal{A}_F$, $|A_2 \cap A| \geq |D_1 \cap A| = |A_1 \cap A| \geq t$. By (7.5) and Lemma 7.3.1, it follows that $\binom{r}{t}|\mathcal{A}_F \cup \{A_2\}| < |\mathcal{S}^*_{F,k}[F']|$, and hence (7.6).

Now, as in the proof of Theorem 7.1.2, by choosing $B \in \mathcal{A}$ and $C^* \in \binom{B}{t}$ such that $|\mathcal{A}[C]| \leq |\mathcal{A}[C^*]|$ for all $C \in \binom{B}{t}$, we get

$$|\mathcal{A}| \leq \binom{r}{t}|\mathcal{A}[C^*]|. \tag{7.7}$$

Set $\mathcal{G} := \{F \in \mathcal{F} \colon \mathcal{A}[C^*] \cap \mathcal{S}^*_{F,k} \neq \emptyset\}$ and $\mathcal{C} := \bigcup_{G \in \mathcal{G}} \mathcal{S}^*_{G,k}[C^*]$. Bringing all the pieces together, we get

$$|\mathcal{S}^*_{\mathcal{F},k}[C^*]| \geq |\mathcal{C}| = \sum_{G \in \mathcal{G}} |\mathcal{C}_G| > \sum_{G \in \mathcal{G}} \binom{r}{t}|\mathcal{A}_G| \geq \binom{r}{t}|\mathcal{A}[C^*]| \geq |\mathcal{A}|,$$

where the strict inequality and the last inequality follow by (7.6) and (7.7) respectively. Hence result. □

Chapter 8

The Erdős-Ko-Rado properties of set systems defined by double partitions

8.1 Introduction

A *double partition* P of a finite set V is a partition of V into *large sets* V_i $(0 \le i \le n)$ (the *top partition*), each partitioned into k_i *small sets* $V_{i1}, \dots, V_{ik_i}$. The family $\mathcal{V}(\mathsf{P})$ *induced by* P is the family of subsets of V that intersect each large set in at most one of its small sets. Note that $\mathcal{S}_{2^{[n]},k}$ can be re-formulated as $\mathcal{V}(\mathsf{P})$ with $k_i = k$ and $|V_{ij}| = 1$ for all $i \in [n]$ and $j \in [k]$.

Here, we are interested in the EKR and strict EKR properties of $\mathcal{V}(\mathsf{P})^{(r)}$ for different values of r, particularly for $r \le \mu(\mathcal{V}(\mathsf{P}))/2$.

Let P be a double partition. Throughout the chapter, we shall assume that for $0 \le i \le n$, the small sets V_{ij} are presented in non-increasing order of size: $|V_{i1}| \ge |V_{i2}| \ge \dots \ge |V_{ik_i}| \ge 1$. Note that therefore $\mu(\mathcal{V}(\mathsf{P})) = \sum_{i=0}^{k} |V_{ia_i}|$ and $\alpha(\mathcal{V}(\mathsf{P})) = \sum_{i=0}^{k} |V_{i1}|$. The elements of each small set V_{ij} are given some arbitrary ordering and denoted by $v_{ij1}, \dots, v_{ija_{ij}}$, where $a_{ij} = |V_{ij}|$.

The case $V = V_0, k_0 = 1$ gives $\mathcal{V}(\mathsf{P}) = 2^V$. The EKR properties for this particularly simple case are therefore given by the EKR Theorem and the Hilton-Milner Theorem (see Section 1.2).

The EKR problem for the case when the small sets are singletons has attracted much attention. Theorem 1.4.3 solves the problem completely for the case when, furthermore, the large sets are non-singleton sets of the same cardinality.

Theorem 8.1.1 (Theorem 1.4.3 rephrased) *Let* P *be a double partition of* V *into* n *large sets each of cardinality* $k \geq 2$, *where each small set is a singleton. Then, for each* $r \in [n]$,
(i) $\mathcal{V}(\mathsf{P})^{(r)}$ *is EKR, and*
(ii) strictly so unless $k = 2$ *and* $r = n \geq 3$.

Holroyd, Spencer and Talbot [40] extended Theorem 8.1.1(i) as follows.

Theorem 8.1.2 (Holroyd, Spencer, Talbot [40]) *Let* P *be a double partition of* V *into* n *large sets each of cardinality at least* 2, *where each small set is a singleton. Then* $\mathcal{V}(\mathsf{P})^{(r)}$ *is EKR for all* $r \in [n]$.

The case $r = n$ is given by Theorem 1.4.1(i), and it is easy to see that this special case implies that $\mathcal{V}(\mathsf{P})^{\alpha(\mathcal{V}(\mathsf{P}))}$ is EKR for any partition P.

For the case when all small sets are again singletons and at least one large set is also a singleton, Bey [4] and Holroyd, Spencer and Talbot [40] independently obtained the following generalisation of the EKR Theorem.

Theorem 8.1.3 (Bey [4], Holroyd, Spencer, Talbot [40]) *Let* P *be a double partition of* V *into* n *large sets, where at least one large set is a singleton and each small set is a singleton. Then* $\mathcal{V}(\mathsf{P})^{(r)}$ *is EKR if* $r \leq n/2$.

For $r > n/2$, Bey [4] determined a list of families such that $\text{ex}(\mathcal{V}(\mathsf{P})^{(r)})$ must contain a member in the list.

A slightly stronger version of Theorem 3.1.4 of Holroyd and Talbot [41] and Theorem 3.1.5 may be combined in the following statement for the case when the small sets are not necessarily singletons but there are just two large sets.

Theorem 8.1.4 *Let* P *be the double partition* $V = V_0 \cup V_1$ *with* $k_1 > 1$.
(i) If $r \leq \mu(\mathcal{V}(\mathsf{P}))/2$ *then* $\mathcal{V}(\mathsf{P})^{(r)}$ *is EKR;*
(ii) if $r < \mu(\mathcal{V}(\mathsf{P}))/2$ *then* $\mathcal{V}(\mathsf{P})^{(r)}$ *is strictly EKR;*

(iii) if $r = \mu(\mathcal{V}(\mathsf{P}))/2$ *and* $k_0 = 1$ *then* $\mathcal{V}(\mathsf{P})^{(r)}$ *fails to be strictly EKR iff* $3 \leq |V_0| \leq r$ *and* $\mu(\mathcal{V}(\mathsf{P})) = \alpha(\mathcal{V}(\mathsf{P}))$.

Note that if $k_0 = 1$ then we get Theorems 3.1.4 and 3.1.5. Only when $n = 1$, as in the result above, the problem immediately reduces to the one with $k_0 = 1$; see [41]. We recall from Chapter 3 that for this "reduced" problem, the family $\{V_0 \cup V_{11}, ..., V_0 \cup V_{1a_1}\}$ of maximal sets of $\mathcal{V}(\mathsf{P})$ is a *sunflower*.

The main contribution of the present chapter is to develop the method used in [41] to allow us to prove quite a general result concerning double partitions. Before proceeding, we note that there is a considerable difference between the case when there is a set V_i that is not further partitioned (that is, V_i is both a large and a small set, so $k_i = 1$) and the case where this is not so. This requires the following modification of our notation.

Suppose that for some non-empty $S \subseteq [n]$ and for all $i \in S$, $k_i = 1$. Then replacing the large sets V_i, $i \in S$, by the single large set $\bigcup_{i \in S} V_i$ does not alter $\mathcal{V}(\mathsf{P})$. Thus we adopt the following convention: The set V_0 is the unique large set that is trivially partitioned (i.e., also a small set), and also the only large set that is allowed to be empty. We say that P is *anchored* if $V_0 \neq \emptyset$, and *unanchored* if $V_0 = \emptyset$. A double partition that is given to be unanchored may, if convenient, be described by a top partition $V = \bigcup_{j=1}^{n} V_i$ and the empty V_0 ignored.

The *width* of a double partition P is the number of non-trivially partitioned large sets.

Our main theorem concerns anchored double partitions and is as follows.

Theorem 8.1.5 *Let* P *be an anchored double partition of width* $n > 0$. *Let* $1 \leq r \leq \mu(\mathsf{P})/2$. *Then:*
(i) $\mathcal{V}(\mathsf{P})^{(r)}$ *is EKR;*
(ii) $\mathcal{V}(\mathsf{P})^{(r)}$ *fails to be strictly EKR iff* $2r = \mu(\mathcal{V}(\mathsf{P})) = \alpha(\mathcal{V}(\mathsf{P}))$, $3 \leq |V_0| \leq r$, $n = 1$.

Clearly, this result significantly generalises Theorems 8.1.3 and 8.1.4 (recall that Theorem 8.1.4 follows immediately from the statement of Theorem 8.1.5 with $n = 1$). Unlike Theorems 8.1.1 and 8.1.2, this result in general does not hold for $\mu(\mathsf{P})/2 < r < \alpha(\mathsf{P})$;

examples can be constructed easily, especially for anchored partitions of width 1 (see [41]).

Removing the anchor condition from Theorem 8.1.5 seems to make the problem much harder. However, in the special case of an unanchored double partition of width 3 where all the V_{ij} have the same cardinality, we have the following result.

Theorem 8.1.6 *Let* P *be an unanchored double partition of width 3 such that every small set is of size c. Then* $\mathcal{V}(\mathsf{P})^{(r)}$ *is strictly EKR for all* $r \leq \mu(\mathcal{V}(\mathsf{P}))/2 = 3c/2$.

8.2 Crossing sets

Let $\mathcal{Y} := \{X_0, X_1, \dots, X_l\}$ be a family of disjoint non-empty finite sets, $Y := \bigcup_{i=0}^{l} X_i$, $x_i := |X_i|$ $(0 \leq i \leq l)$, $y := |Y|$. A subset A of Y is a *crossing set* of $\mathcal{Y}$ if $A \cap X_i \neq \emptyset$ for $i = 0, 1, ..., l$. We denote by ${}^{\times}\mathcal{Y}$ the family of crossing sets of $\mathcal{Y}$; thus, for $l+1 \leq m \leq y$, ${}^{\times}\mathcal{Y}^{(m)}$ is the family of crossing m-sets of $\mathcal{Y}$. We denote $|{}^{\times}\mathcal{Y}^{(m)}|$ by $(x_0, \dots, x_l)^{(m)}$ or, where the x_i are clear from context, by $\mathbf{y}^{(m)}$. These numbers mimic the behaviour of the binomial coefficients $\binom{y}{m}$ in some respects; in particular, they have the following property.

Lemma 8.2.1 *If* $l+1 \leq m < y/2$ *and* $m < m' \leq y - m$*, then*

$$\mathbf{y}^{(m)} \leq \mathbf{y}^{(m')}$$

with equality if and only if $m' = y - m$ *and* $l = 0$.

Proof. For each $A \in {}^{\times}\mathcal{Y}^{(m)}$ there are $\binom{y-m}{m'-m}$ sets $B \in {}^{\times}\mathcal{Y}^{(m')}$ that contain A (since every m'-subset of Y containing A is also a crossing set). Moreover, any such set B has at most $\binom{m'}{m}$ subsets that belong to ${}^{\times}\mathcal{Y}^{(m)}$. Counting in two ways the pairs (A, B) with $A \in {}^{\times}\mathcal{Y}^{(m)}, B \in {}^{\times}\mathcal{Y}^{(m')}$, we obtain

$$\mathbf{y}^{(m)} \binom{y-m}{m'-m} \leq \mathbf{y}^{(m')} \binom{m'}{m}. \tag{8.1}$$

Since $\binom{m'}{m} = \binom{m'}{m'-m}$, the inequality holds under the stated conditions and is strict when $m' < y - m$.

Now consider the case $m' = y - m$. If $l = 0$ then $\mathrm{y}^{(m)} = \binom{x_0}{m} = \binom{x_0}{m'} = \mathrm{y}^{(m')}$; so assume $l \geq 1$. We shall show that the inequality (8.1) is strict by finding some $B \in {}^{\times}\mathcal{Y}^{(m')}$ having an m-subset A such that $A \notin {}^{\times}\mathcal{Y}^{(m)}$.

There exists $X_i \in \mathcal{Y}$ such that $|X_i| \leq y/2$. Let $z := |X_i|$. Choose $B \in {}^{\times}\mathcal{Y}^{(m')}$ such that $|B \cap X_i|$ is as small as possible; that is, $|B \cap X_i| = \max\{1, m' - |Y \backslash X_i|\} = \max\{1, m' - y + z\}$. Then, since $m < y/2$, we conclude

$$|B \cap (Y \backslash X_i)| = \min\{m' - 1, y - z\} \geq \min\{m' - 1, y/2\} \geq m.$$

Therefore, there exists $A \subseteq B \cap (Y \backslash X_i)$ with $|A| = m$. Then $A \notin {}^{\times}\mathcal{Y}^{(m)}$, as required. □

Remark. We note that (8.1) still holds if we replace ${}^{\times}\mathcal{Y}^{(m)}$ by any subset $\mathcal{M}$ of ${}^{\times}\mathcal{Y}^{(m)}$ and ${}^{\times}\mathcal{Y}^{(m')}$ by $\mathcal{N} := \{B \in {}^{\times}\mathcal{Y}^{(m')} \colon A \subset B \text{ for some } A \in \mathcal{M}\}$. Thus, by Hall's Marriage Theorem [36], there is an injection $f \colon {}^{\times}\mathcal{Y}^{(m)} \to {}^{\times}\mathcal{Y}^{(m')}$ such that $A \subset f(A)$ for all $A \in {}^{\times}\mathcal{Y}^{(m)}$.

Let $l + 1 \leq r \leq y$ and $v \in X_0$. We call a family ${}^{\times}\mathcal{Y}^{(r)}(v)$ a *crossing r-star of $\mathcal{Y}$*. A family $\mathcal{F}$ of crossing sets of $\mathcal{Y}$ is said to be *strongly intersecting* if $A \cap B \cap X_0 \neq \emptyset$ for any $A, B \in \mathcal{F}$.

We now prove an 'EKR-type' theorem for strongly intersecting families of crossing sets. (The proof is actually the most technically complex part of proving Theorem 8.1.5.)

Theorem 8.2.2 *Let $\mathcal{Y} := \{X_0, \ldots, X_q\}$ be a family of disjoint non-empty sets and let $Y := \bigcup_{i=0}^{q} X_i$, $2 \leq q + 1 \leq r \leq |Y|/2$. Then:*

(i) the crossing r-stars with centres in X_0 are extremal strongly intersecting sub-families of ${}^{\times}\mathcal{Y}^{(r)}$;

(ii) these are the only extremal such families, unless $3 \leq |X_0| \leq r = |Y|/2$ and $q = 1$.

Proof. Let $\mathcal{F}$ be a strongly intersecting sub-family of ${}^{\times}\mathcal{Y}^{(r)}$. A necessary condition for it to be extremal is that it be a maximal such family, and we may therefore assume this. Let $\mathcal{G} := \{A \cap X_0 \colon A \in \mathcal{F}\}$; then by maximality, $\mathcal{F} = \{A \in {}^{\times}\mathcal{Y}^{(r)} \colon A \cap X_0 \in \mathcal{G}\}$.

Thus, for any $P \in \mathcal{G}$ with $|P| = p$ and any crossing $(r-p)$-set Q of $\{X_1, \dots, X_l\}$, we have $P \cup Q \in \mathcal{F}$ so that

$$|\{A \in \mathcal{F}\colon A \cap X_0 = P\}| = (x_1, \dots, x_q)^{(r-p)}.$$

Similarly, let ${}^{\times}\mathcal{Y}^{(r)}(v)$ be a crossing r-star with $v \in X_0$ and let $\mathcal{H} := \{A \cap X_0\colon A \in {}^{\times}\mathcal{Y}^{(r)}(v)\}$. For any $P \in \mathcal{H}$ with $|P| = p$ we obtain

$$|\{A \in \mathcal{Y}^{(r)}(v)\colon A \cap X_0 = P\}| = (x_1, \dots, x_q)^{(r-p)}.$$

We shall denote $(x_1, \dots, x_q)$ by $\mathbf{x}$.

We thus have a weighted Erdős-Ko-Rado problem to solve concerning intersecting families of subsets of X_0.

It is convenient to set $w := x_0$ and $x := y - w$. Observe that for any crossing r-set A of $\mathcal{Y}$, we have $s \le |A \cap X_0| \le t$, where $s := \max\{1, r - x\}$ and $t := \min\{r - q, w\}$. Thus, partitioning $\mathcal{G}$ and $\mathcal{H}$ by cardinality, and noting that $|\mathcal{H}^{(p)}| = \binom{w-1}{p-1}$, we need to show that

$$\sum_{p=s}^{t} |\mathcal{G}^{(p)}|\mathbf{x}^{(r-p)} \le \sum_{p=s}^{t} \binom{w-1}{p-1}\mathbf{x}^{(r-p)} \tag{8.2}$$

and that, if $\mathcal{G}$ is non-centred, then the inequality is strict unless $q = 1$ and $3 \le w \le r = |Y|/2$.

To establish (8.2), it is sufficient to show that:

1. if either $p = t = w$ or $p \le w/2$, then
   $|\mathcal{G}^{(p)}|\mathbf{x}^{(r-p)} \le \binom{w-1}{p-1}\mathbf{x}^{(r-p)}$
   (that is, $|\mathcal{G}^{(p)}| \le \binom{w-1}{p-1}$);

2. if $w/2 < p \le \min\{t-1, w\}$, then
   $|\mathcal{G}^{(p)}|\mathbf{x}^{(r-p)} + |\mathcal{G}^{(w-p)}|\mathbf{x}^{(r-(w-p))} \le \binom{w-1}{p-1}\mathbf{x}^{(r-p)} + \binom{w-1}{w-p-1}\mathbf{x}^{r-(w-p))}$.

Statement 1 follows easily since if $p = w$ then $\binom{w-1}{p-1} = 1$ and $\mathcal{G}^{(p)}$ is either empty or consists of the single set X_0, and if $p \le w/2$ then $|\mathcal{G}^{(p)}| \le \binom{w-1}{p-1}$ by the EKR Theorem.

To prove Statement 2, observe that the strong intersection condition implies that no set in $\mathcal{G}^{(p)}$ can be the complement in X_0 of a set in $\mathcal{G}^{(w-p)}$, and hence $|\mathcal{G}^{(p)}|+|\mathcal{G}^{(w-p)}| \leq \binom{w}{p}$. Thus, for such a pair $p, w-p$:

$$|\mathcal{G}^{(p)}|\mathbf{x}^{(r-p)} + |\mathcal{G}^{(w-p)}|\mathbf{x}^{(r-(w-p))} \leq \left(\binom{w}{p} - |\mathcal{G}^{(w-p)}|\right)\mathbf{x}^{(r-p)} + |\mathcal{G}^{(w-p)}|\mathbf{x}^{(r-(w-p))}.$$

Since $q \leq w-p < y/2$ and $w-p < p \leq y-(w-p)$, the conditions of Lemma 8.2.1 hold with $l = q-1$, $m = r-p$, $m' = r-(w-p)$. Since the EKR Theorem gives us $|\mathcal{G}^{(w-p)}| \leq \binom{w-1}{w-p-1} = \binom{w-1}{p}$, it follows that the maximum value of $|\mathcal{G}^{(p)}|\mathbf{x}^{(r-p)} + |\mathcal{G}^{(w-p)}|\mathbf{x}^{(r-(w-p))}$ is obtained when $|\mathcal{G}^{(w-p)}| = \binom{w-1}{p} = |\mathcal{H}^{(w-p)}|$ and $|\mathcal{G}^{(p)}| = \binom{w}{p} - \binom{w-1}{p} = \binom{w-1}{p-1} = |\mathcal{H}^{(p)}|$, and unless $\mathbf{x}^{(r-p)} = \mathbf{x}^{(r-(w-p))}$, this is the only way to achieve the maximum. This already verifies (8.2) and hence part (i) of the theorem.

To prove part (ii) of the theorem, observe that (unless $|X_0| = 1$, when the theorem is trivial) $p < w/2$ for at least one $p \in [s,t]$. Thus, unless $\mathbf{x}^{(r-(w-p))} = \mathbf{x}^{(r-p)}$, we know that $\mathcal{G}^{(p)}$ is a star, say with centre v. Then every other set of $\mathcal{G}$ must intersect each element of $\mathcal{G}^{(p)}$, and hence $\mathcal{F} = {}^{\times}\mathcal{Y}^{(r)}(v)$. So the only possibility for an extremal non-star occurs when:

(a) $\mathbf{x}^{(r-(w-p))} = \mathbf{x}^{(r-p)}$ for every $p \in [s,t]$ with $p < w/2 < w-p$;

(b) there is no $p < w/2$ with $w-p > t$.

By Lemma 8.2.1, (a) happens only if $2r - w = x$ (that is, $r = |Y|/2$) and $q = 1$. Clearly we also require $|X_0| \geq 3$ in order to obtain a non-star for $\mathcal{G}$. Finally, (b) requires $w \leq r$, and part (ii) is proved.

Finally, we note that if $q = 1$ and $3 \leq |X_0| \leq r = |Y|/2$ then we may construct a non-star family $\mathcal{A}$ of crossing r-sets such that $|\mathcal{A}| = |{}^{\times}\mathcal{Y}^{(r)}(v)|$ (where $v \in X_0$) as follows. Let $\mathcal{B} := \{A \in {}^{\times}\mathcal{Y}^{(r)}(v) \colon A \cap W = \{v\}\}$, $\mathcal{C} := \{Y \backslash A \colon A \in \mathcal{B}\}$. Then define $\mathcal{A} := ({}^{\times}\mathcal{Y}^{(r)}(v) \backslash \mathcal{B}) \cup \mathcal{C}$. □

8.3 Double partitions and compressions

We shall now develop some further notation.

Let P be a double partition. Recall that, within each large set, the small sets are ordered by size. The set V_0 and the small sets V_{i1}, $1 \le i \le n$, are said to be the *floor sets*, while the remaining small sets V_{ij}, $1 \le i \le n$, $2 \le j \le k_i$, are said to be the *upper sets*. The union of the floor sets is said to be the *floor* and is denoted by F.

We now define the *compressions* that we will be used in the main proofs.

For $i = 1, ..., n$, $j = 2, ..., k_i$, define $\delta_{i,j} \colon V \to V$ by $\delta_{i,j}(v_{ijp}) := v_{i1p}$ $(p = 1, ..., a_{ij})$, and $\delta_{i,j}(v) := v$ otherwise. Thus, each $\delta_{i,j}$ maps an upper set to the corresponding floor set and leaves all other small sets unaffected.

Let $A \in \mathcal{V}(\mathsf{P})$. We may denote $\{\delta_{i,j}(x) \colon x \in A\}$ by $\delta_{i,j}(A)$; note that $\delta_{i,j}(A) \in \mathcal{V}(\mathsf{P})$. Define the *compression operation* $\Delta_{i,j}$ on sub-families $\mathcal{A}$ of $\mathcal{V}(\mathsf{P})$ by

$$\Delta_{i,j}(\mathcal{A}) := \{\delta_{i,j}(A) \colon A \in \mathcal{A}\} \cup \{A \in \mathcal{A} \colon \delta_{i,j}(A) \in \mathcal{A}\}.$$

The following lemma outlines the fundamental properties of $\Delta_{i,j}(\mathcal{A})$.

Lemma 8.3.1 *Let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{V}(\mathsf{P})$. Then*
(i) $\Delta_{i,j}(\mathcal{A}) \subseteq \mathcal{V}(\mathsf{P})$.
(ii) $|\Delta_{i,j}(\mathcal{A})| = |\mathcal{A}|$,
(iii) $\Delta_{i,j}(\mathcal{A})$ *is intersecting,*
(iv) if V' is a union of upper sets of $\mathcal{V}(\mathsf{P})$ such that $(A \cap B)\backslash V' \neq \emptyset$ for all $A, B \in \mathcal{A}$, then $(C \cap D)\backslash(V' \cup V_{ij}) \neq \emptyset$ for all $C, D \in \Delta_{i,j}(\mathcal{A})$.

Proof. (i) and (ii) are straightforward, and (iii) follows from (iv) by setting $V' = \emptyset$. We now prove (iv).

Let $C, D \in \Delta_{i,j}(\mathcal{A})$. If $C \notin \mathcal{A}$, let $A \in \mathcal{A}$ such that $\delta_{i,j}(A) = C$. If $D \in \mathcal{A}$, let $G := \delta_{i,j}(D)$ (note that in this case $G \in \mathcal{A}$); otherwise, let $B \in \mathcal{A}$ such that $\delta_{i,j}(B) = D$.

If at least one of C, D belongs to $\mathcal{A}$, we may assume $D \in \mathcal{A}$. If also $C \in \mathcal{A}$ then $(C \cap D)\backslash(V' \cup V_{ij}) \supseteq (C \cap G)\backslash V'$ (since $G \cap V_{ij} = \emptyset$), and $C, G \in \mathcal{A}$; hence $(C \cap D)\backslash(V' \cup V_{ij}) \neq \emptyset$. If $C \notin \mathcal{A}$ then $(C \cap D)\backslash(V' \cup V_{ij}) \supseteq (A \cap G)\backslash V' \neq \emptyset$.

If $C, D \notin \mathcal{A}$ then $(A \cap B)\backslash V' \neq \emptyset$; moreover, $C \cap D = \delta_{i,j}(A \cap B)$ and hence $(C \cap D)\backslash(V' \cup V_{ij}) \neq \emptyset$. □

Lemma 8.3.2 *Let* $\mathcal{A}^* := \Delta_{1,2} \circ \ldots \circ \Delta_{1,k_1} \circ \ldots \circ \Delta_{n,2} \circ \ldots \circ \Delta_{n,k_n}(\mathcal{A})$*, where* $\mathcal{A}$ *is an intersecting subfamily of* $\mathcal{V}(\mathsf{P})$*. Then*

(i) $\mathcal{A}^* \subseteq \mathcal{V}(\mathsf{P})$.

(ii) $|\mathcal{A}^*| = |\mathcal{A}|$,

(iii) $\mathcal{A}^*$ *is an intersecting sub-family of* $\mathcal{V}(\mathsf{P})$,

(iv) $A \cap B \cap F \neq \emptyset$ *for any* $A, B \in \mathcal{A}^*$,

Proof. Each part follows by repeated application of the corresponding part of Lemma 8.3.1. □

Throughout the remainder of the chapter, we use $\mathcal{A}^*$ as in the statement of Lemma 8.3.2.

Let $\mathcal{A} \subseteq \mathcal{V}(\mathsf{P})^{(r)}$ be an intersecting family. By (i) and (ii) of Lemma 8.3.1, if $\mathcal{A}$ is non-centred and $\Delta_{i,j}(\mathcal{A})$ is a star of largest size, then $\mathcal{V}(\mathsf{P})^{(r)}$ is not strictly EKR. Thus, in order to demonstrate the strict EKR property of $\mathcal{V}(\mathsf{P})^{(r)}$ by considering families that are obtained through compression operations, we must first show that a star of largest size cannot be obtained from a compression operation on a non-centred intersecting family. Now when P is anchored, then a star with centre in V_0 certainly cannot be obtained through a compression operation Δ_{ij} on any other sub-family of $\mathcal{V}(\mathsf{P})^{(r)}$. Moreover, if $x \in V_0$, $y \notin V_0$ and $r \leq \mu(\mathcal{V}(\mathsf{P}))$ then more sets of $\mathcal{V}(\mathsf{P})^{(r)}$ contain x but not y than contain y but not x, and hence the stars with centres in V_0 are precisely those of maximum size. Thus, for an anchored double partition, a star of largest size can never result from a compression operation on a non-centred intersecting family. However, for the more general case when the double partition may be unanchored, we require the following less trivial result, the proof of which also employs Lemma 3.3.3. (In the statement and proof of this lemma, we abbreviate $\mathcal{V}(\mathsf{P})$ to $\mathcal{V}$.)

Lemma 8.3.3 *Let* P *be a double partition, let* $r \leq \mu(\mathcal{V})/2$*, and suppose that* $\mathcal{A}$ *is an intersecting sub-family of* $\mathcal{V}^{(r)}$ *such that* $\mathcal{A} \neq \Delta_{i,j}(\mathcal{A}) = \mathcal{V}^{(r)}(x) := \{A \in \mathcal{V}^{(r)} \colon x \in A\}$ *for some* $x \in V$ *and some compression* $\Delta_{i,j}$*. Then* $|V_{ij}| = |V_{i1}|$ *and* $\mathcal{A} = \mathcal{V}^{(r)}(y) := \{A \in \mathcal{V}^{(r)} \colon y \in A\}$*, where* $y \in V_{ij}$ *and* $x = \delta_{i,j}(y)$ $(\in V_{i1})$.

Proof. Let $A^* \in \mathcal{A} \backslash \Delta_{i,j}(\mathcal{A})$. So $\delta_{i,j}(A^*) \in \Delta_{i,j}(\mathcal{A})$. Since $\Delta_{i,j}(\mathcal{A}) = \mathcal{V}^{(r)}(x)$, $x \in \delta_{i,j}(A^*)$. Since $A^* \notin \Delta_{i,j}(\mathcal{A}) = \mathcal{V}^{(r)}(x)$, $x \notin A^*$. So $x \in \delta_{i,j}(A^*) \backslash A^*$. So $x = \delta_{i,j}(y)$ for some $y \in A^* \cap V_{ij}$, $j > 1$, and $x \in V_{i1}$.

Let M be any maximal set of $\mathcal{V}$ that contains $A^* \cup V_{ij}$, and let $\mathcal{A}_M := \{A \in \mathcal{A} \cap \mathcal{V}^{(r)}(y) \colon A \subset M\}$. Let $N := M \backslash \{y\}$ and $\mathcal{A}'_M := \{A \backslash \{y\} \colon A \in \mathcal{A}_M\} \subseteq \binom{N}{r'}$, where $r' = r - 1 \leq \mu(\mathcal{V})/2 - 1 \leq |M|/2 - 1 = (|M| - 1)/2 - 1/2 < |N|/2$. Suppose $A' \in \mathcal{A}'_M$ and $B' \notin \mathcal{A}'_M$ for some $B' \in \binom{N \backslash A'}{r'}$. Then $A'' := A' \cup \{y\} \in \mathcal{A}_M$, $B'' := B' \cup \{y\} \notin \mathcal{A}$, and $\delta_{i,j}(B'') \notin \mathcal{A}$ since $\delta_{i,j}(B'') \cap A'' = \emptyset$. So $\delta_{i,j}(B'') \in \mathcal{V}^{(r)}(x) \backslash \Delta_{i,j}(\mathcal{A})$, a contradiction. Therefore, if $A' \in \mathcal{A}'_M$ then $B' \notin \mathcal{A}'_M$ for all $B' \in \binom{N \backslash A'}{r'}$. Also, $A^* \backslash \{y\} \in \mathcal{A}'_M$. By Lemma 3.3.3, $\mathcal{A}'_M = \binom{N}{r'}$. Hence $\mathcal{A}_M = \{A \in \mathcal{V}^{(r)}(y) \colon A \subset M\}$.

Since $2r \leq \mu(\mathcal{V})$, for any $A \in \mathcal{V}^{(r)} \backslash \mathcal{V}^{(r)}(y)$ there exists $B \in \mathcal{A}_M$ such that $A \cap B = \emptyset$. So $\mathcal{A} \subseteq \mathcal{V}^{(r)}(y)$. Since $|\mathcal{V}^{(r)}(x)| \geq |\mathcal{V}^{(r)}(y)|$ (as $|V_{i1}| \geq |V_{ij}|$) and $|\mathcal{A}| = |\Delta_{i,j}(\mathcal{A})| = |\mathcal{V}^{(r)}(x)|$, it follows that $|\mathcal{A}| = |\mathcal{V}^{(r)}(y)| = |\mathcal{V}^{(r)}(x)|$, and hence $|V_{ij}| = |V_{i1}|$. □

8.4 Proof of Theorem 8.1.5

Let P be anchored. In the proof that follows, we abbreviate $\mathcal{V}(\mathsf{P})$ to $\mathcal{V}$.

If $r = 1$ then there is nothing to prove, so we may assume $r \geq 2$ and thus $\mu(\mathcal{V}) \geq 4$. Moreover, $|V| \geq 5$ since V_1 is non-trivially partitioned. Since a non-centred family of 2-sets must be of size 3, it immediately follows that $\mathcal{V}$ is strictly 2-EKR. We therefore assume $3 \leq r \leq \mu(\mathcal{V})/2$.

Now let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{V}^{(r)}$ such that

$$|\mathcal{A}'| \leq |\mathcal{A}| \text{ for any intersecting family } \mathcal{A}' \subset \mathcal{V}^{(r)}. \tag{8.3}$$

By Lemmas 8.3.2 and 8.3.3, we may assume that $\mathcal{A} = \mathcal{A}^*$, and hence (by Lemma 8.3.2(iv)) that

$$A \cap B \cap F \neq \emptyset \text{ for any } A, B \in \mathcal{A}. \tag{8.4}$$

Let x be a fixed element of V_0, and let $\mathcal{J}$ be the star of $\mathcal{V}^{(r)}$ with centre x.

We now develop a notation for partitioning sub-families of $\mathcal{F}^{(r)}$ in accordance with their intersections with the upper sets.

Let $U := \{(i,j): 1 \leq i \leq n, 2 \leq j \leq k_i\}$; that is, U is the set of subscript pairs associated with the upper sets of P. By (8.4), each set of $\mathcal{A}$ intersects at least one floor set and thus at most $r-1$ upper sets. This is true also of $\mathcal{J}$. Thus, let

$$\mathcal{U} := \{S \subseteq U: |S| < r, (i_1, j_1) \neq (i_2, j_2) \in S \text{ implies } i_1 \neq i_2\}$$

(note that $\emptyset \in \mathcal{U}$). Then a family $\mathcal{B}$ of elements of $\mathcal{V}^{(r)}$ each intersecting at least one floor set is partitioned as follows: $\mathcal{B} = \bigcup_{S \in \mathcal{U}} \mathcal{B}_S$ where $\mathcal{B}_S$ is the sub-family of $\mathcal{B}$ whose sets intersect all the sets V_{ij}, $(i,j) \in S$, and no other upper sets. For $S \in \mathcal{U}$, let F_S denote the union of those floor sets that are not 'under' any of the upper sets of S: $F_S = F \backslash \bigcup_{(i,j) \in S} V_{i1}$. Then, for $S \neq \emptyset$, a sub-family $\mathcal{B}_S$ is a family of crossing r-sets in which F_S takes the role of X_0 and the upper sets take the role of the X_i for $i \geq 1$ (see Section 8.2); moreover, for $\mathcal{B}_S = \mathcal{A}_S$, we have $\mathcal{B}_S$ strongly intersecting by (8.4).

Therefore, by Theorem 8.2.2(i), $|\mathcal{A}_S| \leq |\mathcal{J}_S|$ for each $S \in \mathcal{U} \backslash \{\emptyset\}$. By the EKR Theorem, we also have $|\mathcal{A}_\emptyset| \leq |\mathcal{J}_\emptyset|$. Thus $|\mathcal{A}| \leq |\mathcal{J}|$ (which proves (i)). By (8.3), $|\mathcal{A}| = |\mathcal{J}|$, and hence $|\mathcal{A}_S| = |\mathcal{J}_S|$ for each $S \in \mathcal{U}$.

For any $S \in \mathcal{U}$, if we can show that $\mathcal{A}_S = (\mathcal{V}^{(r)}(v))_S$ for some $v \in F$, then it follows that $\mathcal{A} \subseteq \mathcal{V}^{(r)}(v)$, since for all $A \in \mathcal{V}^{(r)}]v[$ there exists $B \in (\mathcal{V}^{(r)}(v))_S$ such that $A \cap B = \emptyset$, as every maximal set is of size $\geq 2r$. We have already noted (in Section 8.3) that $|\mathcal{V}^{(r)}(v)|$ is maximised only if $v \in V_0$; hence, if we show that $\mathcal{A}_S = (\mathcal{V}^{(r)}(v))_S$ for some $v \in F$, then, by (8.3), $\mathcal{A} = \mathcal{V}^{(r)}(v)$ where $v \in V_0$.

If $r < \mu(\mathcal{V})/2$ or $r = \mu(\mathcal{V})/2 < \alpha(\mathcal{V})/2 = |F|/2$ then by taking $S = \emptyset$ and applying the Hilton-Milner Theorem we indeed obtain $\mathcal{A}_S = (\mathcal{V}^{(r)}(v))_S$ for some $v \in F$ (since $|\mathcal{A}_S| = |\mathcal{J}_S|$).

If $r = \mu(\mathcal{V})/2 = \alpha(\mathcal{V})/2$ and $n > 1$ then we choose S such that $|S| \geq 2$. By Theorem 8.2.2, $\mathcal{A}_S = (\mathcal{V}^{(r)}(v))_S$ for some $v \in F$.

It remains to consider the case $n = 1$. Recall that we are assuming $r \geq 3$. If $|W| < 3$ or $|W| > r$ then we take $S = \{(1,j)\}$, $j > 1$, and again apply Theorem 8.2.2.

If $3 \leq |W| \leq r$ then the non-centred intersecting family $(\mathcal{J}\backslash\{A \in \mathcal{J}: A \cap V_0 = \{x\}\}) \cup \{A \in \mathcal{V}^{(r)}: A \cap V_0 = V_0\backslash\{x\}\} \subset \mathcal{V}^{(r)}$ has size equal to $|\mathcal{J}|$. Thus the strict EKR property fails only in the cases stated in the theorem. □

8.5 Proof of Theorem 8.1.6

Recall that P is unanchored with $n = 3$, $a_{ij} = c$ $(j = 1, ..., k_i,\ i = 1, 2, 3)$ and $V_0 = \emptyset$. For simplicity, we assume that $k_1 \leq k_2 \leq k_3$. As in Section 8.3, $V_{11} \cup V_{21} \cup V_{13}$ is the floor, denoted by F, and as in Section 8.4, we abbreviate $\mathcal{V}(\mathsf{P})$ to $\mathcal{V}$.

Suppose $2r \leq \mu(\mathsf{P})$ $(= \alpha(\mathsf{P}))$ and $\mathcal{A}$ is an intersecting sub-family of $\mathcal{V}^{(r)}$ that is not a star. By Lemma 8.3.3, $\mathcal{A}^*$ is not a star either. Thus, using Lemma 8.3.2, we may assume that $\mathcal{A} = \mathcal{A}^*$ and that $A \cap B \cap F \neq \emptyset$ for any $A, B \in \mathcal{A}$.

Let $D_i := \{0, ..., k_i\}$, $i = 1, 2, 3$. For any $(d_1, d_2, d_3) \in D_1 \times D_2 \times D_3$, let $\mathcal{A}_{d_1,d_2,d_3}$ be the sub-family of sets $A \in \mathcal{A}$ such that $A \cap V_{id_i} \neq \emptyset$ for all i such that $d_i \neq 0$, and $A \cap V_{ij} = \emptyset$ otherwise. So the families $\mathcal{A}_{d_1,d_2,d_3}$ partition $\mathcal{A}$. Let $\mathcal{J}$ be the star of $\mathcal{V}^{(r)}$ with centre v_{111} and partition it similarly. Note that $\mathcal{J}$ is a star of largest size.

By Lemma 8.3.2(iv), for any (d_1, d_2, d_3), $(d'_1, d'_2, d'_3) \in D_1 \times D_2 \times D_3$ such that $\mathcal{A}_{d_1,d_2,d_3} \neq \emptyset$ and $\mathcal{A}_{d'_1,d'_2,d'_3} \neq \emptyset$, we must have $d_i = d'_i = 1$ for some $i \in [3]$. We now consider two cases.

Case 1: $\{i \in [3]: d_i = 1\} = \{i'\}$ for some $\mathcal{A}_{d_1,d_2,d_3} \neq \emptyset$. Then $d_{i'} = 1$ for any $\mathcal{A}_{d_1,d_2,d_3} \neq \emptyset$. Thus, let Q be the double partition obtained from P by deleting the small sets $V_{12}, \ldots, V_{1k_1}$; then $\mathcal{A}$ is a subfamily of $\mathcal{G} := \mathcal{V}(\mathsf{Q})$. Now Q is an anchored partition of width 2, and hence, by Theorem 8.1.5, $\mathcal{G}^{(r)}$ is strictly EKR. So $|\mathcal{A}| < |\mathcal{V}^{(r)}(v_{i'11})| \leq |\mathcal{J}|$.

Case 2: $|\{i \in [3]: d_i = 1\}| > 1$ whenever $\mathcal{A}_{d_1,d_2,d_3} \neq \emptyset$. So the non-empty classes can only be $\mathcal{A}_{1,1,1}$, $\mathcal{A}_{d_1,1,1}$, $\mathcal{A}_{1,d_2,1}$, and $\mathcal{A}_{1,1,d_3}$, $d_i \in D_i$ $(i = 1, 2, 3)$.

Let $\mathcal{A}_0 := \mathcal{A}_{1,1,1} \cup \mathcal{A}_{0,1,1} \cup \mathcal{A}_{1,0,1} \cup \mathcal{A}_{1,1,0}$ and, similarly, $\mathcal{J}_0 := \mathcal{J}_{1,1,1} \cup \mathcal{J}_{0,1,1} \cup \mathcal{J}_{1,0,1} \cup \mathcal{J}_{1,1,0}$. (These are the sub-families of $\mathcal{A}$ and $\mathcal{J}$ that consist of r-subsets of F). By Theorem 1.2.1, $|\mathcal{A}_0| \leq |\mathcal{J}_0|$.

Now, for $d_2 > 1$, $\mathcal{A}_{1,d_2,1}$ is a family of crossing r-sets for $\mathcal{Y} := \{V_{11} \cup V_{31}, V_{d_2}\}$, obeying the conditions of Theorem 8.2.2. Thus, for all $d_2 \in [2, k_2]$, we have $|\mathcal{A}_{1,d_2,1}| \leq |^{\times}\mathcal{Y}^{(r)}(v_{111})| = |\mathcal{J}_{1,d_2,1} \cup \mathcal{J}_{1,d_2,0}|$. Similarly, if $d_3 > 1$ then $|\mathcal{A}_{1,1,d_3}| \leq |\mathcal{J}_{1,1,d_3} \cup \mathcal{J}_{1,0,d_3}|$. In particular, we note that if $k_3 > k_1$ then $|\mathcal{A}_{1,1,d_3}| \leq |\mathcal{J}_{1,1,d_3} \cup \mathcal{J}_{1,0,d_3}|$ for $k_1 + 1 \leq d_3 \leq k_3$.

The remaining sub-families $\mathcal{A}_{d_1,d_2,d_3}$ that need to be compared with sub-families of $\mathcal{J}$ are $\{\mathcal{A}_{1,1,d}, \mathcal{A}_{d,1,1} : 2 \leq d \leq k_1\}$. Our strategy is to show that $|\mathcal{A}_{1,1,d}| + |\mathcal{A}_{d,1,1}| < |\mathcal{J}_{1,0,d}| + |\mathcal{J}_{1,1,d}| + |\mathcal{J}_{1,2,d}|$, $d = 2, ..., k_1$, from which the result clearly follows, since we shall have made comparisons linking all the sub-families of $\mathcal{A}$ with sub-families of $\mathcal{J}$, and at least one of these comparisons involves a strict inequality.

Let us fix $d \in [2, k_1]$ and define $\mathcal{A}' := \mathcal{A}_{1,1,d} \cup \mathcal{A}_{d,1,1}$. We now define two bijections, $\delta_1 : V_{31} \to V_{11}$ and $\delta_2 : V_{1d} \to V_{3d}$, as follows.

$$\delta_1(v_{31p}) = v_{11p} \quad (p = 1, \ldots, c);$$

$$\delta_2(v_{1dp}) = v_{3dp} \quad (p = 1, \ldots, c).$$

For any $X_1 \subseteq V_{31}$, $X_2 \subseteq V_{1d}$, we may denote $\{\delta_1(x) : x \in X_1\}$ and $\{\delta_2(x) : x \in X_2\}$ by $\delta_1(X_1)$ and $\delta_2(X_2)$ respectively. Now define an injective mapping $\delta : \mathcal{A}_{d,1,1} \to \binom{V_{11} \cup V_{21} \cup V_{3d}}{r}$ by

$$\delta(A) = \delta_1(A \cap V_{31}) \cup (A \cap V_{21}) \cup \delta_2(A \cap V_{1d}) \quad (A \in \mathcal{A}_{d,1,1}).$$

Define the compression Δ on $\mathcal{A}'$ by

$$\Delta(\mathcal{A}') = \mathcal{A}_{1,1,d} \cup \{\delta(A) : A \in \mathcal{A}_{d,1,1}\} \cup \{A \in \mathcal{A}_{d,1,1} : \delta(A) \in \mathcal{A}_{1,1,d}\}.$$

Now let $\mathcal{B} := \Delta(\mathcal{A}')$. Thus, $\mathcal{B} = \mathcal{B}_{1,1,d} \cup \mathcal{B}_{d,1,1}$ where $\mathcal{B}_{1,1,d} = \mathcal{A}_{1,1,d} \cup (\mathcal{B} \backslash \mathcal{A})$ and $\mathcal{B}_{d,1,1} = \mathcal{B} \backslash \mathcal{B}_{1,1,d}$.

Claim 8.5.1 *(i)* $|\mathcal{B}| = |\mathcal{A}|$.
(ii) $A \cap B \cap (V_{11} \cup V_{21}) \neq \emptyset$ *for any* $A, B \in \mathcal{B}$.

Proof. (i) is straightforward.

We now define $f : \mathcal{A}' \to \mathcal{B}$ by $f(A) = \delta(A)$ if $A \in \mathcal{A}_{d,1,1}$ and $\delta(A) \notin \mathcal{A}_{1,1,d}$, and

$f(A) = A$ otherwise. So f is a bijection. We prove (ii) by showing that

$$f(A) \cap f(B) \cap (V_{11} \cup V_{21}) \neq \emptyset \text{ for any } A, B \in \mathcal{A}'. \tag{8.5}$$

We recall that, by Lemma 8.3.2(iv), $A \cap B \cap F \neq \emptyset$ for any $A, B \in \mathcal{A}$. If $A, B \in \mathcal{A}_{1,1,d}$ then (8.5) is immediate. If $A \in \mathcal{A}_{1,1,d}$ and $B \in \mathcal{A}_{d,1,1}$ then $f(A) \cap f(B) \cap V_{21} = A \cap B \cap V_{21} \neq \emptyset$, and hence (8.5). Suppose $A, B \in \mathcal{A}_{d,1,1}$. Since $A \cap B \cap (V_{21} \cup V_{31}) \neq \emptyset$, if $\delta(A), \delta(B) \notin \mathcal{A}_{1,1,d}$ then (8.5) is straightforward. Suppose $\delta(A) \in \mathcal{A}_{1,1,d}$ and $\delta(B) \notin \mathcal{A}_{1,1,d}$. Since $\delta(A) \cap B \cap V_{21} \neq \emptyset$, we have $A \cap \delta(B) \cap V_{21} \neq \emptyset$, and hence (8.5). Finally, suppose $\delta(A), \delta(B) \in \mathcal{A}_{1,1,d}$. So $A \cap B \cap V_{21} \neq \emptyset$ because $A \cap \delta(B) \cap V_{21} \neq \emptyset$; hence (8.5). □

By Theorem 8.2.2,

$$|\mathcal{B}_{1,1,d}| \leq |\mathcal{J}_{1,0,d}| + |\mathcal{J}_{1,1,d}|. \tag{8.6}$$

By Claim 8.5.1(ii), we have $A \cap B \cap V_{21} \neq \emptyset$ for all $A, B \in \mathcal{B}_{d,1,1}$. By Theorem 8.2.2, $|\mathcal{B}_{d,1,1}| \leq |\mathcal{J}_{1,2,d}|$. If $|\mathcal{B}_{d,1,1}| < |\mathcal{J}_{1,2,d}|$ then we are done.

Suppose $|\mathcal{B}_{d,1,1}| = |\mathcal{J}_{1,2,d}|$. By Theorem 8.2.2(ii), there exists $v' \in V_{21}$ such that $\mathcal{B}_{d,1,1} = \mathcal{K}_{d,1,1}$ where $\mathcal{K} := \mathcal{V}^{(r)}(v')$. Let $\mathcal{C} := \{A \in \mathcal{B}_{d,1,1} \colon A \cap V_{21} = v'\}$. $\mathcal{C} \neq \emptyset$ since $2r \leq \mu(G) = 3c$. Let $C \in \mathcal{C}$. If there exists $A \in \mathcal{B}_{1,1,d}$ such that $v' \notin A$ then $A \cap C = \emptyset$, a contradiction. So $\mathcal{B}_{1,1,d} \subseteq \mathcal{K}_{1,1,d}$, and hence $|\mathcal{B}_{1,1,d}| \leq |\mathcal{K}_{1,1,d}| = |\mathcal{J}_{1,1,d}|$. Since $2r \leq 3c$, we have $|\mathcal{J}_{1,0,d}| > 0$, and hence a strict inequality in (8.6). It follows that $|\mathcal{A}| < |\mathcal{J}|$. □

Chapter 9

An extension of the Erdős-Ko-Rado Theorem and multiple cross-intersecting families

9.1 Introduction

For the purpose of this chapter only, we define families $\mathcal{A}^*$ and $\mathcal{A}'$ on a family $\mathcal{A} \subseteq 2^{[n]}$ as follows:

$$\mathcal{A}^* := \{A \in \mathcal{A}\colon A \cap B \neq \emptyset \text{ for all } B \in \mathcal{A}\},$$
$$\mathcal{A}' := \mathcal{A} \backslash \mathcal{A}^* = \{A \in \mathcal{A}\colon A \cap B = \emptyset \text{ for some } B \in \mathcal{A}\}.$$

Here we first show that the elegant cycle method by which Katona [42] obtained a beautiful short proof of the EKR Theorem extends to a proof of the significant extension of the EKR Theorem that was revealed in Section 1.1 and that is stated formally as Theorem 9.2.2 below. We then demonstrate the usefulness and importance of Theorem 9.2.2 by showing that it yields a slight extension of Theorem 1.6.3 almost immediately.

9.2 Results and proofs

If σ is a *cyclic ordering* of the elements of a set X, and the elements of a subset A of X are consecutive in σ, then we say that A *meets* σ.

Lemma 9.2.1 *Let $r \leq n/2$. Let σ be a cyclic ordering of $[n]$, and let $\mathcal{B} \subseteq \mathcal{C} := \{B \in \binom{[n]}{r} : B$ meets $\sigma\}$. Then*

$$|\mathcal{B}^*| + \frac{r}{n}|\mathcal{B}'| \leq r,$$

and if $r < n/2$ then equality holds iff either $|\mathcal{B}^*| = r$ and $\mathcal{B}' = \emptyset$ or $\mathcal{B}^* = \emptyset$ and $\mathcal{B}' = \mathcal{C}$.

Proof. Clearly there are n r-subsets of $[n]$ that meet σ, i.e. $|\mathcal{C}| = n$. So the result is straightforward if $\mathcal{B}^* = \emptyset$. Suppose $\mathcal{B}^* \neq \emptyset$. Let $B^* \in \mathcal{B}^*$, and let $x_1, ..., x_r$ be the consecutive points in σ such that $B^* = \{x_1, ..., x_r\}$. For $i \in [r]$, let C_i be the r-set in $\mathcal{C}$ beginning with x_i in σ, and let C'_i be the r-set in $\mathcal{C}$ ending with x_i in σ. Let $\mathcal{D} := \{C_1, ..., C_r\} \cup \{C'_1, ..., C'_r\}$. Note that $B^* = C_1 = C'_r$ and hence $\mathcal{D} = \{B^*\} \cup \{C_2, ..., C_r\} \cup \{C'_1, ..., C'_{r-1}\}$. By the definitions of $\mathcal{B}^*$ and $\mathcal{B}'$, we have $\mathcal{B}^* \cup \mathcal{B}' \subseteq \mathcal{D}$ (because $B^* \in \mathcal{B}^*$) and, since $r \leq n/2$, $C'_{j-1} \notin \mathcal{B}^* \cup \mathcal{B}'$ for any $j \in [2, r]$ such that $C_j \in \mathcal{B}^*$. It follows that there are at least $|\mathcal{B}^*| - 1$ sets in $\mathcal{D} \backslash (\mathcal{B}^* \cup \mathcal{B}')$, and hence $|\mathcal{B}'| \leq |\mathcal{D}| - |\mathcal{B}^*| - (|\mathcal{B}^*| - 1) = 2r - 2|\mathcal{B}^*|$. So

$$|\mathcal{B}^*| + \frac{r}{n}|\mathcal{B}'| \leq |\mathcal{B}^*| + \frac{1}{2}|\mathcal{B}'| \leq |\mathcal{B}^*| + \frac{1}{2}(2r - 2|\mathcal{B}^*|) = r,$$

and it is immediate from this expression that if $\frac{r}{n} < \frac{1}{2}$ then equality holds throughout iff $|\mathcal{B}^*| = r$ and $\mathcal{B}' = \emptyset$. Hence result. □

Theorem 9.2.2 *Let $r \leq n/2$ and $\mathcal{A} \subseteq \binom{[n]}{r}$. Then*

$$|\mathcal{A}^*| + \frac{r}{n}|\mathcal{A}'| \leq \binom{n-1}{r-1},$$

and if $r < n/2$ then equality holds iff either $|\mathcal{A}^*| = \binom{n-1}{r-1}$ and $\mathcal{A}' = \emptyset$ or $\mathcal{A}^* = \emptyset$ and $\mathcal{A}' = \binom{[n]}{r}$.

Proof. If $\mathcal{A}' = \emptyset$ then the result is trivial, so we consider $\mathcal{A}' \neq \emptyset$. Let $\mathcal{E} := \binom{[n]}{r}$. For a cyclic ordering σ of $[n]$, a family $\mathcal{F} \subseteq \mathcal{E}$ and a set $E \in \mathcal{E}$, let $\mathcal{F}_\sigma := \{F \in \mathcal{F}: F$ meets $\sigma\}$

and

$$\Phi(\sigma, E) := \begin{cases} 1 & \text{if } E \text{ meets } \sigma; \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$(\mathcal{A}^*)_\sigma \cup (\mathcal{A}')_\sigma = (\mathcal{A}_\sigma)^* \cup (\mathcal{A}_\sigma)' \quad \text{and} \quad (\mathcal{A}^*)_\sigma \subseteq (\mathcal{A}_\sigma)^*. \tag{9.1}$$

Let N be the set of all $(n-1)!$ cyclic orderings of $[n]$. Note that any r-subset of $[n]$ meets $r!(n-r)!$ cyclic orderings of $[n]$. We therefore have

$$\begin{aligned} r!(n-r)!\left(|\mathcal{A}^*| + \frac{r}{n}|\mathcal{A}'|\right) &= \sum_{A^* \in \mathcal{A}^*} r!(n-r)! + \frac{r}{n} \sum_{A' \in \mathcal{A}'} r!(n-r)! \\ &= \sum_{A^* \in \mathcal{A}^*} \sum_{\sigma \in N} \Phi(\sigma, A^*) + \frac{r}{n} \sum_{A' \in \mathcal{A}'} \sum_{\sigma \in N} \Phi(\sigma, A') \\ &= \sum_{\sigma \in N} \left(\sum_{A^* \in \mathcal{A}^*} \Phi(\sigma, A^*) + \frac{r}{n} \sum_{A' \in \mathcal{A}'} \Phi(\sigma, A') \right) \\ &= \sum_{\sigma \in N} \left(|(\mathcal{A}^*)_\sigma| + \frac{r}{n} |(\mathcal{A}')_\sigma| \right) \\ &\leq \sum_{\sigma \in N} \left(|(\mathcal{A}_\sigma)^*| + \frac{r}{n} |(\mathcal{A}_\sigma)'| \right) \quad \text{(by (9.1))} \qquad (9.2) \\ &\leq \sum_{\sigma \in N} r \quad \text{(by Lemma 9.2.1)} \qquad (9.3) \\ &= r(n-1)!, \end{aligned}$$

which yields the inequality in the theorem.

Now suppose $r < n/2$ and we have equality in the theorem. So we have equality in (9.2) and (9.3). The former equality and (9.1) clearly give us

$$(\mathcal{A}^*)_\sigma = (\mathcal{A}_\sigma)^* \quad \text{and} \quad (\mathcal{A}')_\sigma = (\mathcal{A}_\sigma)'. \tag{9.4}$$

The equality in (9.3) and Lemma 9.2.1 give us that for any $\sigma \in N$, if $(\mathcal{A}_\sigma)' \neq \emptyset$ then $(\mathcal{A}_\sigma)' = \mathcal{E}_\sigma$ (and $(\mathcal{A}_\sigma)^* = \emptyset$). Thus, by (9.4),

$$\text{for any } \sigma \in N, \text{ if } (\mathcal{A}')_\sigma \neq \emptyset \text{ then } (\mathcal{A}')_\sigma = \mathcal{E}_\sigma. \tag{9.5}$$

Let A be any set in $\mathcal{A}'$; recall that we are considering $\mathcal{A}' \neq \emptyset$. Let B be any set in $\binom{[n]\backslash A}{r}$. We can choose $\sigma_{A,B} \in N$ such that both A and B meet $\sigma_{A,B}$. Since $A \in (\mathcal{A}')_{\sigma_{A,B}}$ and $B \in \mathcal{E}_{\sigma_{A,B}}$, we have $B \in (\mathcal{A}')_{\sigma_{A,B}}$ by (9.5). So $B \in \mathcal{A}'$. Therefore $\mathcal{A}' = \mathcal{E}$ by Lemma 3.3.3. Hence result. □

For convenience, we state our slightly extended version of Theorem 1.6.3 in full. Note that the slight improvement is given by parts (ii) and (iii) below.

Theorem 9.2.3 (Extension of Theorem 1.6.3) *If $r \leq n/2$, $k \geq 2$, and $\mathcal{A}_1$, ..., $\mathcal{A}_k$ are cross-intersecting sub-families of $\binom{[n]}{r}$ then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \begin{cases} \binom{n}{r} & \text{if } k \leq \frac{n}{r}; \\ k\binom{n-1}{r-1} & \text{if } k \geq \frac{n}{r}. \end{cases}$$

Suppose equality holds and $\mathcal{A}_1 \neq \emptyset$:

(i) if $k < n/r$ then $\mathcal{A}_1 = \binom{[n]}{r}$ and $\mathcal{A}_i = \emptyset$ for $i = 2, ..., k$;

(ii) if $k > n/r$ then $\mathcal{A}_1 = ... = \mathcal{A}_k$ and $|\mathcal{A}_1| = \binom{n-1}{r-1}$;

(iii) if $k = n/r > 2$ then $\mathcal{A}_1, ..., \mathcal{A}_k$ are as in (i) or (ii).

Proof. Let $\mathcal{A} := \bigcup_{i=1}^{k} \mathcal{A}_i$. Clearly $\mathcal{A}^* = \bigcup_{i=1}^{k} \mathcal{A}_i^*$ and $\mathcal{A}' = \bigcup_{i=1}^{k} \mathcal{A}_i'$. Suppose $\mathcal{A}_i' \cap \mathcal{A}_j' \neq \emptyset$, $i \neq j$. Let $A \in \mathcal{A}_i' \cap \mathcal{A}_j'$. Then there exists $A_i \in \mathcal{A}_i'$ such that $A \cap A_i = \emptyset$, which is a contradiction because $A \in \mathcal{A}_j$. So $\mathcal{A}_i' \cap \mathcal{A}_j' = \emptyset$ for $i \neq j$, and hence $|\mathcal{A}'| = \sum_{i=1}^{k} |\mathcal{A}_i'|$. Applying Theorem 9.2.2, we therefore get

$$\sum_{i=1}^{k} |\mathcal{A}_i| = \sum_{i=1}^{k} |\mathcal{A}_i'| + \sum_{i=1}^{k} |\mathcal{A}_i^*| \leq |\mathcal{A}'| + k|\mathcal{A}^*| \leq \binom{n}{r} + (k - \frac{n}{r})|\mathcal{A}^*|. \tag{9.6}$$

Suppose $k < \frac{n}{r}$. Then $\sum_{i=1}^{k} |\mathcal{A}_i| \leq \binom{n}{r}$, and equality holds iff $\mathcal{A}^* = \emptyset$ and $\mathcal{A} = \mathcal{A}' = \binom{[n]}{r}$. If $A \in \mathcal{A}_1$ and $B \in \binom{[n]\backslash A}{r}\backslash \mathcal{A}_1$ then $B \notin \mathcal{A}_i$, $i = 2, ..., k$, and hence $B \in \binom{[n]}{r}\backslash \mathcal{A}$.

Thus, if $\mathcal{A} = \binom{[n]}{r}$ then the conditions of Lemma 3.3.3 hold for $\mathcal{A}_1$ (recall that $\mathcal{A}_1 \neq \emptyset$), and therefore $\mathcal{A}_1 = \mathcal{A} = \binom{[n]}{r}$. Hence (i).

Next, suppose $k > \frac{n}{r}$. Then, by (9.6) and Theorem 9.2.2,

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \binom{n}{r} + (k - \frac{n}{r})\binom{n-1}{r-1} = k\binom{n-1}{r-1},$$

and equality holds iff $\mathcal{A}_1^* = \ldots = \mathcal{A}_k^* = \mathcal{A}^*$ and $|\mathcal{A}^*| = \binom{n-1}{r-1} = |\mathcal{A}|$. Hence (ii).

Finally, suppose $k = \frac{n}{r}$. Then, by (9.6), $\sum_{i=1}^{k} |\mathcal{A}_i| \leq |\mathcal{A}'| + \frac{n}{r}|\mathcal{A}^*| \leq \binom{n}{r}$. Suppose $k > 2$. Thus, since $k = \frac{n}{r}$, $r < \frac{n}{2}$. Therefore, if $\mathcal{A}^* = \emptyset$ then $\mathcal{A}$ is as in the case $k < \frac{n}{r}$, and, since $|\mathcal{A}'| + \frac{n}{r}|\mathcal{A}^*| \leq \binom{n}{r}$ implies $|\mathcal{A}^*| + \frac{r}{n}|\mathcal{A}'| \leq \binom{n-1}{r-1}$, it is immediate from Theorem 9.2.2 that if $\mathcal{A}^* \neq \emptyset$ then $\mathcal{A}^*$ is as in the case $k > \frac{n}{r}$. Hence (iii). □

Chapter 10

Erdős-Ko-Rado with monotonic non-decreasing separations

10.1 Introduction

For a *monotonic non-decreasing (mnd) sequence* of non-negative integers $\{d_i\}_{i\in\mathbb{N}}$ (i.e. $0 \leq d_1 \leq d_2 \leq ...$) and a set $X \subset \mathbb{N}$, we define

$$\mathcal{P}(\{d_i\}_{i\in\mathbb{N}}) := \{\{a_1, ..., a_r\} \subset \mathbb{N} \colon r \in \mathbb{N}, a_{i+1} > a_i + d_{a_i}, i = 1, ..., r-1\},$$
$$\mathcal{P}_X(\{d_i\}_{i\in\mathbb{N}}) := \mathcal{P}(\{d_i\}_{i\in\mathbb{N}}) \cap 2^X.$$

If $X = [n]$ then we also write $\mathcal{P}_n(\{d_i\}_{i\in\mathbb{N}})$. For convenience and neatness of notation, we assume that $\{d_i\}_{i\in\mathbb{N}}$ is some fixed mnd sequence, and we drop the argument '$(\{d_i\}_{i\in\mathbb{N}})$' from any of the notation for the families defined above unless we consider a different sequence.

In this chapter, we are concerned with the extremal intersecting sub-families of $\mathcal{P}_n^{(r)}$. Due to some fundamental differences, we will treat the case $d_1 > 0$ separately from the complementary case $d_1 = 0$. One difference has to do with the extremal structures. Another difference is that, as we will show, the EKR property holds for all r if $d_1 > 0$, whereas it is not guaranteed to hold for $d_1 = 0$ and $\alpha(\mathcal{P}_n)/2 < r < \alpha(\mathcal{P}_n)/2$. A simple example for the latter case is that if $d_i = 0$ for all $i \in \mathbb{N}$ and $n/2 < r < n$ then $\mathcal{P}_n^{(r)}$ is non-centred, intersecting, and of course larger than the star of $\mathcal{P}_n^{(r)}$ with centre 1;

other examples with $0 = d_1 < d_{n-1}$ can be easily constructed.

For the case $d_1 > 0$, we determine every single extremal structure and exactly when it arises (i.e. for which sequences $\{d_i\}_{i \in \mathbb{N}}$ it is extremal); the proof is self-contained.

For the case $d_1 = 0$, we determine precisely the cases when $\mathcal{P}_n^{(r)}$ has the strict and non-strict EKR property for $r \leq \alpha(\mathcal{P}_n)/2$. The proof is based on the EKR Theorem, the Hilton-Milner Theorem and Theorem 9.2.3. Although we do not give an extensive list of all the extremal structures for the non-strict EKR case, we give a characterisation in terms of necessary and sufficient conditions that their sets must satisfy.

The answer to the EKR problem for the case when $d_i = d$ for all $i \in [n-1]$ is known. If $d = 0$ then $\mathcal{P}_n^{(r)} = \binom{[n]}{r}$, and hence we know precisely what are the extremal intersecting sub-families of $\mathcal{P}_n^{(r)}$; see Section 1.2.

Theorem 10.1.1 (Erdős, Ko, Rado [25], Hilton, Milner [38]) *Suppose $d_i = 0$ for all $i \in [n-1]$. Let $\mathcal{A}$ be an extremal intersecting sub-family of $\mathcal{P}_n^{(r)}$.*
(i) If $r < n/2$ (which is equal to $\alpha(\mathcal{P}_n)/2$) then $\mathcal{A}$ is a star of $\mathcal{P}_n^{(r)}$.
(ii) If $r = n/2$ then for any $A \in \mathcal{P}_n^{(r)} = \binom{[n]}{r}$, exactly one of A and $[n]\backslash A$ is in $\mathcal{A}$.
(iii) If $r > n/2$ then $\mathcal{A} = \{\mathcal{P}_n^{(r)}\}$.

Holroyd, Spencer and Talbot proved the EKR property for $d > 0$, but they left the problem of determining the whole set of extremal structures open.

Theorem 10.1.2 (Holroyd, Spencer, Talbot [40]) *If $d_i = d > 0$ for all $i \in [n-d-1]$ then the star of $\mathcal{P}_n^{(r)}$ with centre 1 is an extremal intersecting sub-family of $\mathcal{P}_n^{(r)}$.*

To be able to state our main results, we need to develop some further notation and definitions. We point out to the reader that, for various purposes (such as statements, proofs, explanations) in this chapter, the notation in Section 2.1 will also be used, and heavily so for the proofs of the main results.

10.2 Further notation, definitions and main results

For a finite set $A \subset \mathbb{N}$, let

$$l(A) := \min\{a \in A\}, \quad u(A) := \max\{a \in A\}.$$

For $i, r \in \mathbb{N}$, define $P_{i,r} := \{p_1, ..., p_r\} \in \mathcal{P}$ by $p_1 := i$ and $p_{j+1} := p_j + d_{p_j} + 1$, $j = 1, ..., r-1$ (if $r > 1$). We need to define $P_{i,0} := \emptyset$.

For $r \leq \alpha(\mathcal{P}_n)$, let

$$k_{n,r} := \max\{i \colon u(P_{i,r}) \leq n\}.$$

For $i \geq 2$, let

$$E_i := \{a \in [i-1] \colon a + d_a \geq i\}, \quad e_i := |E_i|.$$

Clearly, since $\{d_i\}_{i \in \mathbb{N}}$ is mnd,

$$E_i = [j, i-1] \text{ for some } j \in [i-1].$$

For any $z \in \mathbb{Z} := \{0\} \cup \mathbb{N} \cup \{-n \colon n \in \mathbb{N}\}$, let $s_z \colon \mathcal{P} \to 2^{\mathbb{N}}$ be defined by

$$s_z(A) = \{a + z \colon a \in A\}.$$

We will often use the fact that

$$A \in \mathcal{P},\ l(A) \geq 2,\ x \in [l(A) - 1] \ \Rightarrow\ s_{-x}(A) \in \mathcal{P},$$

which is again a consequence of $\{d_i\}_{i \in \mathbb{N}}$ being mnd.

We say that $\mathcal{P}_{[x,y]}$ is *symmetric* if $\mathcal{P}_{[x,y]} = \mathcal{P}_{[x,y]}(\{d_i^* = d\}_{i \in \mathbb{N}})$ for some $d \in \mathbb{N} \cup \{0\}$, otherwise we say that $\mathcal{P}_{[x,y]}$ is *asymmetric*. Note that if $\alpha(\mathcal{P}_{[x,y]}) > 1$ then $\mathcal{P}_{[x,y]}$ is symmetric iff $e_y = d_x$.

Suppose $d_1 = d_3 = 1$, $y \in P_{3,r} = s_1(P_{2,r})$, $r \geq 2$, and for

$$m := \begin{cases} \max\{a \in [y] \colon d_a = 1\} & \text{if } \mathcal{P}_y \text{ is asymmetric;} \\ y & \text{if } \mathcal{P}_y \text{ is symmetric,} \end{cases}$$

$m = 2t + 1$ for some $t \in \mathbb{N}$. Then we say that $\mathcal{P}_y^{(r)}$ is *type I*, and we say that $\mathcal{A} \subset \mathcal{P}_y^{(r)}$ is *special* iff $\mathcal{A} = \{A_1, ..., A_q\} \cup (\mathcal{P}_y^{(r)}(1) \backslash \{B_1, ..., B_q\})$ for some $q \in [t]$, where

$$A_1 := P_{3,r} = P_{3,t} \cup P_{m+2,r-t}, \quad B_t := P_{1,t} \cup P_{m+1,r-t},$$

and for all $i \in [t-1]$ (if $t > 1$),

$$A_{i+1} := \{2j : j \in [i]\} \cup \{2j+1 : j \in [i+1, t]\} \cup P_{m+2,r-t}, \quad B_i := s_{-1}(A_{i+1}).$$

Clearly, a special family as above is $\mathcal{P}_y^{(r)}(y)$ iff $q = t$ and $\mathcal{P}_y$ is symmetric. Also note that

$$\begin{aligned} &\mathcal{A}\ (\subset \mathcal{P}_y^{(r)}) \text{ special; } \mathcal{P}_y \text{ asymmetric or } \mathcal{A} \neq \mathcal{P}_y^{(r)}(y) \\ &\quad \Rightarrow \mathcal{P}_y^{(r)}(1)(y) \cup \{P_{1,r}, P_{3,r}\} \subseteq \mathcal{A}. \end{aligned} \tag{10.1}$$

That a special family is intersecting is not difficult to check; however, for the sake of completeness, this is proved in Section 10.4 (Lemma 10.4.4).

If $\mathcal{P}_y$ is asymmetric, $y \in P_{k,r} = s_{k-1}(P_{1,r})$, $k := k_{y,r}$, and $k \leq d_1 + 1$ then we say that $\mathcal{P}_y^{(r)}$ is *type II*. Note that $P_{k,r} = s_{k-1,r}(P_{1,r})$ implies $P_{i,r} = s_1(P_{i-1,r})$, $i = 2, ..., k$ (if $k > 1$). An example of a type II family is $\mathcal{P}_{10}^{(3)}(\{d_i^*\}_{i \in \mathbb{N}})$ with $d_1^* = d_2^* = d_3^* = 2$ and $d_4^* = d_5^* = d_6^* = 3$.

This brings us to our first and main result.

Theorem 10.2.1 *Suppose $d_1 > 0$ and $2 \leq r \leq \alpha(\mathcal{P}_n)$.*
(i) If $\mathcal{P}_n^{(r)}$ is type I then $\mathrm{ex}(\mathcal{P}_n^{(r)}) = \{\mathcal{P}_n^{(r)}(1)\} \cup \{\mathcal{A} \subset \mathcal{P}_n^{(r)} : \mathcal{A}$ *special* $\}$.
(ii) If $\mathcal{P}_n^{(r)}$ is type II, or $\mathcal{P}_n$ is symmetric but $\mathcal{P}_n^{(r)}$ is not type I, then $\mathrm{ex}(\mathcal{P}_n^{(r)}) = \{\mathcal{P}_n^{(r)}(1), \mathcal{P}_n^{(r)}(n)\}$.
(iii) In any other case, $\mathrm{ex}(\mathcal{P}_n^{(r)}) = \{\mathcal{P}_n^{(r)}(1)\}$.

Clearly, this immediately yields the strict and non-strict EKR property for $d_1 > 0$.

Corollary 10.2.2 *If $d_1 > 0$ and $r \leq \alpha(\mathcal{P}_n)$ then $\mathcal{P}_n^{(r)}$ is EKR, and strictly so unless $\mathcal{P}_n^{(r)}$ is type I.*

The following is our result for the complementary case $d_1 = 0$.

Theorem 10.2.3 *Suppose $d_1 = 0 < d_{n-1}$ and $r \leq \alpha(\mathcal{P}_n)/2$. Let $m := \min\{i \in [n] : d_i \neq 0\}$. Let $\mathcal{A} \subset \mathcal{P}_n^{(r)}$.*
(i) If $n \in P_{1,2r}$ and $m = 2r-1$ then $\mathcal{A} \in \mathrm{ex}(\mathcal{P}_n^{(r)})$ iff

(a) $\mathcal{A}]\{2r-1, 2r, ..., n\}[= \binom{[2r-2]}{r} \backslash \{[2r-2]\backslash A \colon A \in \mathcal{A}\langle 2r-1\rangle\langle n\rangle\}$, $\mathcal{A}\langle 2r-1\rangle\langle n\rangle$ *intersecting,*

(b) $\mathcal{A}\langle i\rangle \cap \binom{[2r-2]}{r-1} = \mathcal{A}\langle n\rangle \cap \binom{[2r-2]}{r-1} \in \mathrm{ex}(\binom{[2r-2]}{r-1})$, $i = 2r-1, ..., n-1$, *and*

(c) $\mathcal{A}\langle n\rangle \cap \binom{[2r-2]}{r-1}$ *and* $\mathcal{A}\langle 2r-1\rangle\langle n\rangle$ *are cross-intersecting.*

(ii) If $n \in P_{1,2r}$ *and* $r+2 \leq m \leq 2r-2$ *then* $\mathcal{A} \in \mathrm{ex}(\mathcal{P}_n^{(r)})$ *iff for some* $j \in [m-1]$ *and* $\mathcal{H}_0 \subseteq \binom{[m-1]\backslash\{j\}}{r}$, $\mathcal{A} = \mathcal{H}_0 \cup (\mathcal{P}_n^{(r)}(j) \backslash \{P_{1,2r}\backslash A \colon A \in \mathcal{H}_0\})$.

(iii) If $n \notin P_{1,2r}$ *or* $m \leq r+1$ *then* $\mathcal{A} \in \mathrm{ex}(\mathcal{P}_n^{(r)})$ *iff* $\mathcal{A} = \mathcal{P}_n^{(r)}(j)$ *for some* $j \in [m-1]$.

This result and Theorem 10.1.1 give a characterisation of the extremal structures for the case $d_1 = 0$. It is easy to see that they yield the following.

Corollary 10.2.4 *If* $d_1 = 0$ *and* $r \leq \alpha(\mathcal{P}_n)/2$ *then* $\mathcal{P}_n^{(r)}$ *is EKR, and strictly so unless* $n \in P_{1,2r}$ *and* $\max\{i \in [2r-1] \colon d_i = 0\} \geq r+1$.

10.3 The key fact and the compression operation

An interesting key fact is that the 'forward' mnd separations separations d_i induce 'backward' mnd separations e_i with the following additional property.

Proposition 10.3.1 *For* $i \geq 2$, $e_i \leq e_{i+1} \leq e_i + 1$.

Proof. If $E_i = [i-1]$ or $E_i = \emptyset$ then $e_{i+1} \leq e_i + 1$ trivially. Suppose $E_i \neq [i-1]$ and $E_i \neq \emptyset$. Then $E_i = [j, i-1]$ for some $j \in [2, i-1]$. So $(j-1) + d_{j-1} < i$, and hence $E_{i+1} \subseteq E_i \cup \{i\}$. Therefore $e_{i+1} \leq e_i + 1$.

If $E_i = \emptyset$ then $e_i \leq e_{i+1}$ trivially. Suppose $E_i \neq \emptyset$. Then $E_i = [j, i-1]$ for some $j \in [i-1]$. Since $d_{j+1} \geq d_j$, we thus have $(j+1) + d_{j+1} \geq j + d_j + 1 \geq i+1$. So $[j+1, i] \subseteq E_{i+1}$, and hence $|E_i| \leq |E_{i+1}|$. Therefore $e_i \leq e_{i+1}$. □

Using the above result, we can now prove the compression lemma for our problem. For $p, q \in \mathbb{N}$, let $\Delta_{p,q} \colon 2^{\mathcal{P}} \to 2^{\mathcal{P}}$ be as defined in Section 2.2.

Lemma 10.3.2 *Let* $\mathcal{A}^*$ *be an intersecting sub-family of* $\mathcal{P}$. *Let* $p, q \in \mathbb{N}$ *such that* $d_p > 0$ *and* $d_q > 0$. *Let* $\mathcal{A} := \Delta_{p,q}(\mathcal{A}^*)$.

(i) If $p = q - 1$ then $\mathcal{A}\langle q\rangle$ and $\mathcal{A}]q[$ are intersecting.

(ii) If $p = q - 1$ and $e_p < e_q$ then $\mathcal{A}\langle q\rangle \cup \mathcal{A}]q[$ is intersecting.

(iii) If $p = q + 1$ and $d_p = d_q$ then $\mathcal{A}\langle q\rangle$ and $\mathcal{A}]q[$ are intersecting.

Proof. By Proposition 2.2.1(i), $\mathcal{A}]q[$ is intersecting.

Note that if $p = q - 1$ or $p = q + 1$ then, since $d_p > 0$ and $d_q > 0$, $\mathcal{P}[\{p, q\}] = \emptyset$.

Suppose $p = q - 1$. Let $w := \max\{1, q - e_q - 1\}$, and let $P \in \mathcal{P}]w[(q)$. Then $P \cap [w, p] = \emptyset$, $w = \max\{1, p - e_q\}$. If $p - e_q > 1$ then $1 < w \leq p - e_p$ since $e_p \leq e_q$ by Proposition 10.3.1. Since $P \cap [w, p] = \emptyset$, we thus have $P \cap [\max\{1, p - e_p\}, p] = \emptyset$, implying $(P\backslash\{q\}) \cup \{p\} \in \mathcal{P}$. So $\mathcal{P}]w[$ is (p, q)-compressed. By Proposition 2.2.1(iv), (i) follows.

Suppose $p = q - 1$ and $e_p < e_q$. Let $P \in \mathcal{P}(q)$. Then $P \cap [\max\{1, q - e_q\}, p] = \emptyset$. Since $q - e_q = p + 1 - e_q \leq p - e_p$, we thus have $P \cap [\max\{1, p - e_p\}, p] = \emptyset$, implying $(P\backslash\{q\}) \cup \{p\} \in \mathcal{P}$. So $\mathcal{P}$ is (p, q)-compressed. By Proposition 2.2.1(iii), (ii) follows.

Suppose $p = q + 1$ and $d_p = d_q$. Let $w := q + d_q + 1$, and let $P \in \mathcal{P}]w[(q)$. Then $P \cap [p, p + d_p] = P \cap [p, w] = P \cap [p, w - 1] = P \cap [q + 1, q + d_q] = \emptyset$, and hence $(P\backslash\{q\}) \cup \{p\} \in \mathcal{P}$. So $\mathcal{P}]w[$ is (p, q)-compressed. By Proposition 2.2.1(iv), (iii) follows. □

10.4 The case $d_1 > 0$

Throughout this section, we assume that $d_1 > 0$ and $\alpha(\mathcal{P}_n) \geq 2$. We set

$$n' := n - e_n - 1.$$

Note that $n' \geq 1$ since $\alpha(\mathcal{P}_n) \geq 2$. So $n' + d_{n'} < n$, and hence

$$d_{n'} \leq e_n.$$

Lemma 10.4.1 $k_{n,r} = k_{n',r-1}$.

Proof. Let $k := k_{n,r}$ and $k' := k_{n',r-1}$. So $u(P_{k,r}) \leq n < u(P_{k+1,r})$ and $u(P_{k',r-1}) \leq n' < u(P_{k'+1,r-1})$. Thus, since $u(P_{k',r-1}) + d_{u(P_{k',r-1})} + 1 \leq n' + d_{n'} + 1 \leq n' + e_n + 1 = n$, we have $u(P_{k',r}) \leq n$, and hence $k' \leq k$. Now,

$$u(P_{k,r-1}) = u(P_{k,r}) - e_{u(P_{k,r})} - 1 \leq u(P_{k,r}) - (e_n - (n - u(P_{k,r}))) - 1 = n',$$

where the first inequality follows by $n - u(P_{k,r})$ applications of Lemma 10.3.1. So $k \leq k'$. Since $k' \leq k$, the result follows. □

Lemma 10.4.2 *Suppose $1 \leq q \leq \alpha(\mathcal{P}_{y-1})$ and either $\mathcal{P}_y$ is symmetric or $\mathcal{P}_y^{(q)}$ is type II. Then $s_1(A) \in \mathcal{P}_y^{(q)}$ for any $A \in \mathcal{P}_{y-1}^{(q)}$.*

Proof. If $q = 1$ or $\mathcal{P}_y$ is symmetric then the result is straightforward. So consider $q \geq 2$ and $\mathcal{P}_y^{(q)}$ type II. Setting $k := k_{y,q}$, we then have $y \in P_{k,q} = s_{k-1}(P_{1,q})$ and $k \leq d_1 + 1$. For each $i \in [d_1 + 1]$, let $p_{i,1} < ... < p_{i,q}$ such that $P_{i,q} = \{p_{i,1}, ..., p_{i,q}\}$. By definition of $P_{i,q}$, $p_{i,j} = p_{i,j-1} + d_{p_{i,j-1}} + 1$ for each $j = [2, q]$. Since $P_{k,q} = s_{k-1}(P_{1,q})$, $p_{k,j} = p_{1,j} + k - 1$ for each $j \in [q]$. Thus, for each $j \in [2, q]$, $p_{k,j-1} + d_{p_{k,j-1}} + 1 = p_{k,j} = (p_{1,j-1} + d_{p_{1,j-1}} + 1) + k - 1$, and hence $d_{p_{k,j-1}} = d_{p_{1,j-1}} + p_{1,j-1} + k - 1 - p_{k,j-1} = d_{p_{1,j-1}}$. Therefore, for each $j \in [q-1]$, $d_{p_{k,j}} = d_{p_{1,j}}$, and hence, for each $i \in [k]$, $d_{p_{i,j}} = d_{p_{1,j}}$ (as $d_{p_{1,j}} \leq d_{p_{i,j}} \leq d_{p_{k,j}} = d_{p_{1,j}}$).

Now let $A \in \mathcal{P}_{y-1}^{(q)}$, and let $a_1 < ... < a_q \leq y - 1$ such that $A = \{a_1, ..., a_q\}$. Let $h \in [q]$, and let $A_h := \{a_{q-h+1}, ..., a_q\}$; so $|A_h| = h$. Since $y \in P_{k,q}$ and $k := k_{y,q}$, we have $P_{k_{y,h},h} = \{p_{k,q-h+1}, ..., p_{k,q}\}$ and $p_{k,q} = y$. Since $a_q \leq y - 1 = p_{k,q} - 1$ and $\{d_i\}_{i \in \mathbb{N}}$ is mnd, it follows that $a_{q-h+1} \leq p_{k,q-h+1} - 1$. So $a_j \leq p_{k,j} - 1$ for all $j \in [q]$. It is straightforward that we also have $p_{1,j} \leq a_j$ for all $j \in [q]$. So $p_{1,j} \leq a_j \leq p_{k,j} - 1$ for all $j \in [q]$. Since we established that $d_{p_{i,j}} = d_{p_{1,j}}$ for any $i \in [k]$ and $j \in [q-1]$, the result follows. □

Lemma 10.4.3 *Suppose $\mathcal{P}_n$ is asymmetric, $\mathcal{P}_n\langle n\rangle$ $(= \mathcal{P}_{n'})$ is symmetric and either $\mathcal{P}_n\langle 1\rangle$ $(= \mathcal{P}_{[d_1+2,n]})$ is symmetric or $d_2 > d_1$. Then $\alpha(\mathcal{P}_n) \leq 3$.*

Proof. Since $\mathcal{P}_n$ is asymmetric, we have $d_1 < e_n$, and hence $d_1 = ... = d_p < d_{p+1}$ for some $p \in [n']$. Since $\mathcal{P}_{n'}$ is symmetric, it follows that $(p + 1) + d_{p+1} \geq n'$. Let $p_1 < p_2 < p_3 < p_4$ such that $P_{1,4} = \{p_1, p_2, p_3, p_4\}$. So $p_1 = 1$, $p_2 = d_1 + 2$, $p_3 = p_2 + d_{p_2} + 1$, $p_4 = p_3 + d_{p_3} + 1$.

Suppose $p \leq d_1 + 1$. Then $p + 1 \leq p_2$, and hence $p_3 \geq (p+1) + d_{p+1} + 1 \geq n' + 1$, $p_4 \geq (n'+1) + d_{n'+1} + 1 \geq n + 1$. So $u(P_{1,4}) > n$, and hence $\alpha(\mathcal{P}_n) \leq 3$.

Now suppose $p \geq d_1 + 2$. So $d_2 = d_1$ as $d_1 \leq d_2 \leq d_p$ and $d_1 = d_p$. Thus, by the conditions of the lemma, $\mathcal{P}_{[d_1+2,n]}$ is symmetric. Since $d_1 + 2 \leq p$ and $d_1 = ... = d_p$, $d_{d_1+2} = d_1$. So $d_{d_1+2} < e_n$, but this contradicts $\mathcal{P}_{[d_1+2,n]}$ symmetric. □

Lemma 10.4.4 *Let $\mathcal{A} \subset \mathcal{P}_y^{(r)}$ be a special family as defined in Section 10.2. Then $\mathcal{A}$ is intersecting.*

Proof. We are required to show that for any $q \in [t]$, the sets that do not intersect A_q are members of $\{B_1, ..., B_q\}$. Recall that $d_i = 1$ for all $i \in [m]$ ($m = 2t+1$).

Consider first $q = 1$. So $A_q = P_{3,r}$. Let $B \in \mathcal{P}_y^{(r)}(1)$ such that $B \cap A_q = \emptyset$, and let $B' := B \backslash \{1\}$. Since $B \cap P_{3,r} = \emptyset$ and $d_1 = 1$, $l(B') \geq 4$. So $B'' := B' \cup \{2\} \in \mathcal{P}_{[2,y]}^{(r)}$ as $d_2 = 1$. Now, given that $y \in P_{3,r} = s_1(P_{2,r})$, $P_{2,r}$ is the unique set in $\mathcal{P}_{[2,y-1]}^{(r)}$, and hence, since $B \cap P_{3,r} = \emptyset$ implies $n \notin B''$, we have $B'' = P_{2,r}$. So $B = (P_{2,r} \backslash \{2\}) \cup \{1\} = B_1$, and hence $\mathcal{A}$ is intersecting.

Now consider $q > 1$. So $A_q = \{2j \colon j \in [q-1]\} \cup (\{2j+1 \colon j \in [q,t] \cup P_{m+2,r-t}) = P_{2,q-1} \cup P_{2q+1,r-q+1}$. Now $P_{2q+1,r-q+1} = P_{3,r} \backslash P_{3,q-1}$. Since $y \in P_{3,r} = s_1(P_{2,r})$, we have $y \in P_{2q+1,r-q+1} = s_1(P_{2q,r-q+1})$, and hence $C := P_{2q,r-q+1}$ is the unique set in $\mathcal{P}_{[2q,y-1]}^{(r-q+1)}$. Note that $C \cap A_q = \emptyset$. Let D be a set in $\mathcal{P}_{[2q-1,y]}^{(r-q+1)} \backslash \{C\}$ such that $D \cap A_q = \emptyset$. Then $y \notin D$ (since $y \in A_q$) and $2q-1 \in D$ (otherwise $D \in \mathcal{P}_{[2q,y-1]}^{(r-q+1)}$, which leads to the contradiction that $D = C$). Now $d_{2q} = 1$ and, since $2q+1 \in A_q$, $2q+1 \notin D$. So $E := (D \backslash \{2q-1\}) \cup \{2q\} \in \mathcal{P}_{[2q,y-1]}^{(r-q+1)}$, and hence $E = C$. So $D = (C \backslash \{2q\}) \cup \{2q-1\}$. Since $P_{2,q-1} \subset A_q$, $P_{1,q-1}$ is the unique set in $\mathcal{P}_{2q-2}^{(q-1)}$ that does not intersect A_q. Therefore $F_1 := P_{1,q-1} \cup C$ and $F_2 := P_{1,q-1} \cup D$ are the only sets in $\mathcal{P}_y^{(r)}$ that do not intersect A_q. It is clear from the above that $F_1 = B_{q-1}$ and $F_2 = B_q$. Hence result. □

Lemma 10.4.5 *If $\mathcal{P}_y$ is symmetric or $\mathcal{P}_y^{(r)}$ is a type II family then $|\mathcal{P}_y^{(r)}(y)| = |\mathcal{P}_y^{(r)}(1)|$.*

Proof. If $r = 1$ then the result is trivial, so we assume $r > 1$ and prove the result by induction on r. If $\mathcal{P}_y$ is symmetric then the result follows immediately by symmetry, so suppose $\mathcal{P}_y^{(r)}$ is a type II family. Clearly, $y \geq u(P_{1,r})$. If $y = u(P_{1,r})$ then $\mathcal{P}_y^{(r)} = \{P_{1,r}\} = \mathcal{P}_y^{(r)}(1) = \mathcal{P}_y^{(r)}(y)$. We now assume that $y > u(P_{1,r})$ and proceed by induction on y. Since $\mathcal{P}_y^{(r)}$ is type II, we have $y \in P_{k_{y,r}} = s_{k_{y,r}-1}(P_{1,r})$ and $k_{y,r} \leq d_1 + 1$; note that this implies $y \in P_{k_{y,r}} \backslash \{k_{y,r}\} = s_{k_{y,r}-1}(P_{1,r} \backslash \{1\})$ and $d_1 + 2 = l(P_{1,r} \backslash \{1\}) \leq l(P_{k_{y,r}} \backslash \{k_{y,r}\}) \leq (d_1 + 1) + d_{d_1+1} + 1 \leq d_{d_1+2} + (d_1 + 2)$. Since $\mathcal{P}_y\langle 1 \rangle = \mathcal{P}_{[d_1+2,y]}$, it follows that either $\mathcal{P}_y\langle 1 \rangle$ is symmetric or $\mathcal{P}_y\langle 1 \rangle^{(r-1)}$ is isomorphic to a type II family in the obvious way. Also, it is fairly straightforward that either $\mathcal{P}_y]1[$ ($= \mathcal{P}_{[2,y]}$) is symmetric or $\mathcal{P}_y]1[^{(r)}$ is isomorphic to a type II family in the obvious way. Therefore, by the inductive hypotheses, we get $|\mathcal{P}_y\langle 1 \rangle^{(r-1)}(y)| = |\mathcal{P}_y\langle 1 \rangle^{(r-1)}(d_1 + 2)|$ and $|\mathcal{P}_y]1[^{(r)}(y)| = |\mathcal{P}_y]1[^{(r)}(2)|$. So $|\mathcal{P}_y^{(r)}(y)| = |\mathcal{P}_y\langle 1 \rangle^{(r-1)}(d_1 + 2)| + |\mathcal{P}_y]1[^{(r)}(2)| = |\mathcal{P}_y^{(r)}(1)(d_1 + 2)| + |\mathcal{P}_y^{(r)}(1)]\{2, 3, ..., d_2 + 2\}[|$, and hence the result follows if $d_2 = d_1$. Since $u(P_{1,r}) < y \in P_{k_{y,r}}$, $k_{y,r} > 1$. Thus, as we showed in the proof of Lemma 10.4.2, $d_2 = d_1$ indeed. □

We now come to the proof of Theorem 10.2.1. Recall from Section 10.2 that $s_{-x}(A) \in \mathcal{P}$ if $A \in \mathcal{P}$, $l(A) \geq 2$ and $x \in [l(A) - 1]$; this tool will be used often in the proof.

Proof of Theorem 10.2.1. Let $\mathcal{J} := \mathcal{P}_n^{(r)}(1)$. If $\mathcal{P}_n^{(r)}$ is type I and $\mathcal{A}^* \subset \mathcal{P}_n^{(r)}$ is special then trivially $|\mathcal{A}^*| = |\mathcal{J}|$, and Lemma 10.4.4 tells us that $\mathcal{A}^*$ is intersecting. Lemma 10.4.5 tells us that $|\mathcal{P}_n^{(r)}(n)| = |\mathcal{J}|$ if either $\mathcal{P}_n$ is symmetric or $\mathcal{P}_n^{(r)}$ is type II. Thus, taking

$$\mathcal{A}^* \in \text{ex}(\mathcal{P}_n^{(r)}), \tag{10.2}$$

the result follows if we show that $|\mathcal{A}^*| = |\mathcal{J}|$ and that if $\mathcal{A}^* \neq \mathcal{J}$ then one of the following holds:

- $\mathcal{P}_n^{(r)}$ is type I and $\mathcal{A}^*$ is special;
- $\mathcal{A}^* = \mathcal{P}_n^{(r)}(n)$ and either $\mathcal{P}_n$ is symmetric or $\mathcal{P}_n^{(r)}$ is type II.

Given that $r \leq \alpha(\mathcal{P}_n)$, we have $\mathcal{P}_n^{(r)} \neq \emptyset$ and hence $\mathcal{A}^* \neq \emptyset$.

Suppose $r = 2$ and $\mathcal{A}^*$ is centred. Then $\mathcal{A}^* = \mathcal{P}_n^{(2)}(i)$ for some $i \in [n]$. If $e_i < d_1$ then, since $\{d_i\}_{i\in\mathbb{N}}$ is mnd, we clearly must have $i \leq d_1$, in which case $n > i + d_i$ as $\mathcal{A}^* \neq \emptyset$. So

$$|\mathcal{A}^*| = i - 1 - e_i + \max\{0, n - (i + d_i)\}$$
$$= \begin{cases} i - 1 - e_i & \text{if } e_i \geq d_1,\ n \leq i + d_i; \\ n - i - d_i & \text{if } e_i < d_1,\ n > i + d_i; \\ n - 1 - d_i - e_i & \text{if } e_i \geq d_1,\ n > i + d_i, \end{cases}$$

and hence $|\mathcal{A}^*| \leq n - 1 - d_1 = |\mathcal{J}|$ with equality iff either $i = 1$ or $i = n$ and $e_n = d_1$. Thus, by (10.2), either $\mathcal{A}^* = \mathcal{J}$ or $\mathcal{A}^* = \mathcal{P}_n^{(2)}(n)$ and $\mathcal{P}_n$ is symmetric.

Next, suppose $r = 2$ and $\mathcal{A}^*$ is non-centred. Then $\mathcal{A}^*$ can only be of the form $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}\}$ $(a_1 \neq a_2 \neq a_3 \neq a_1)$, which implies $\{a_1, a_2, a_3\} \in \mathcal{P}_n$. If $a_3 > a_2 + 2$ then $|\mathcal{P}_n^{(2)}(a_1)| \geq |\{\{a_1, a_h\} \colon h \in [a_2, a_3]\}| \geq 4 > |\mathcal{A}^*|$, which contradicts (10.2). So $a_3 = a_2 + 2$, and hence $d_{a_2} \leq 1$. Since $1 \leq d_1 \leq d_{a_2}$, $d_{a_2} = d_1 = 1$. So $|\mathcal{J}| = n - 2$, and hence, since $|\mathcal{A}^*| = 3$, $n \leq 5$ by (10.2). Also, $n \geq a_3 \geq a_2 + 2 \geq (a_1 + 2) + 2 \geq 5$. So $n = 5$, and hence $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, $d_1 = d_3 = 1$. Together with the above, this clearly settles the result for $r = 2$.

We now consider $r \geq 3$. Since $n \geq u(P_{1,\alpha(\mathcal{P}_n)})$ and $r \leq \alpha(\mathcal{P}_n)$, $n \geq u(P_{1,r})$. If $n = u(P_{1,r})$ then the result is trivial since we get $\mathcal{A}^* = \mathcal{P}_n^{(r)} = \{P_{1,r}\}$, so we assume that $n > u(P_{1,r})$ and proceed by induction on n.

Let $\mathcal{A} := \Delta_{n-1,n}(\mathcal{A}^*)$. Since $\mathcal{A}(n) \subseteq \mathcal{A}^*(n)$, we have

$$\Delta_{n-1,n}(\mathcal{A}(n)) \subseteq \mathcal{A}^*(n); \tag{10.3}$$

and since $\mathcal{A}^*$ is intersecting, the following holds:

$$A \in \mathcal{A}]n[,\ A \cap B = \emptyset \text{ for some } B \in \mathcal{A}\langle n\rangle \ \Rightarrow\ n - 1 \in A,\ \delta_{n,n-1}(A) \in \mathcal{A}^*. \tag{10.4}$$

Note that $\mathcal{P}_n\langle n\rangle = \mathcal{P}_{n'}$. Since we are considering $3 \leq r \leq \alpha(\mathcal{P}_n)$ and $n > u(P_{1,r})$, we clearly have $2 \leq r - 1 \leq \alpha(\mathcal{P}_{n'})$ and $3 \leq r \leq \alpha(\mathcal{P}_{n-1})$. So $\mathcal{A}\langle n\rangle \subset \mathcal{P}_{n'}^{(r-1)} \neq \emptyset$, $\mathcal{J}\langle n\rangle =$

$\mathcal{P}_{n'}^{(r-1)}(1) \neq \emptyset$, $\mathcal{A}]n[\subset \mathcal{P}_{n-1}^{(r)} \neq \emptyset$, $\mathcal{J}]n[= \mathcal{P}_{n-1}^{(r)}(1) \neq \emptyset$. Now, by Lemma 10.3.2(i), $\mathcal{A}\langle n\rangle$ and $\mathcal{A}]n[$ are intersecting. So the inductive hypothesis yields $|\mathcal{A}\langle n\rangle| \leq |\mathcal{J}\langle n\rangle|$ and $|\mathcal{A}]n[| \leq |\mathcal{J}]n[|$, and hence $|\mathcal{A}| \leq |\mathcal{J}|$. Since $|\mathcal{A}| = |\mathcal{A}^*|$ and $\mathcal{A}^* \in \mathrm{ex}(\mathcal{P}_n^{(r)})$, we obtain $|\mathcal{A}| = |\mathcal{J}|$ and

$$\mathcal{J} \in \mathrm{ex}(\mathcal{P}_n^{(r)}). \tag{10.5}$$

So $|\mathcal{A}\langle n\rangle| = |\mathcal{J}\langle n\rangle|$, $|\mathcal{A}]n[| = |\mathcal{J}]n[|$, and hence, since the inductive hypothesis gives us $\mathcal{J}\langle n\rangle \in \mathrm{ex}(\mathcal{P}_{n'}^{(r-1)})$ and $\mathcal{J}]n[\in \mathrm{ex}(\mathcal{P}_{n-1}^{(r)})$, we have

$$\mathcal{A}\langle n\rangle \in \mathrm{ex}(\mathcal{P}_{n'}^{(r-1)}), \tag{10.6}$$

$$\mathcal{A}]n[\in \mathrm{ex}(\mathcal{P}_{n-1}^{(r)}). \tag{10.7}$$

Thus, by the inductive hypothesis again, the following must hold:

$$\mathcal{A}\langle n\rangle = \mathcal{J}\langle n\rangle \text{ or } \mathcal{A}\langle n\rangle = \mathcal{P}_{n'}^{(r-1)}(n') \text{ or } \mathcal{A}\langle n\rangle \text{ is special;} \tag{10.8}$$

$$\mathcal{A}]n[= \mathcal{J}]n[\text{ or } \mathcal{A}]n[= \mathcal{P}_{n-1}^{(r)}(n-1) \text{ or } \mathcal{A}]n[\text{ is special.} \tag{10.9}$$

Suppose $\mathcal{A}\langle n\rangle = \mathcal{J}\langle n\rangle$. Then $\mathcal{J}(n-1) \subseteq \Delta_{n-1,n}(\mathcal{A}(n))$, and hence $\mathcal{J}(n-1) \subset \mathcal{A}^*$ by (10.3). Suppose $\mathcal{A}^*]1[(n) \neq \emptyset$. Let $A \in \mathcal{A}^*]1[(n)$ and $B := (s_{-1}(A\backslash l(A))) \cup \{1\}$. Then $B \in \mathcal{J}(n-1)$, and hence $B \in \mathcal{A}^*$. But $A \cap B = \emptyset$, contradicting $\mathcal{A}^*$ intersecting. Next, suppose $\mathcal{A}^*]1[]n[\neq \emptyset$. Let $C \in \mathcal{A}^*]1[]n[$ and $D := (s_{-1}(A\backslash(l(A) \cup u(A))) \cup \{1\}$. So $D \in \mathcal{A}\langle n\rangle$, and hence $E := D \cup \{n\} \in \mathcal{A}^*$. But $C \cap E = \emptyset$, contradicting $\mathcal{A}^*$ intersecting. So $\mathcal{A}^*]1[]n[= \emptyset$. Since we also established $\mathcal{A}^*]1[(n) = \emptyset$, $\mathcal{A}^*]1[= \emptyset$. So $\mathcal{A}^* \subseteq \mathcal{J}$. By (10.2), $\mathcal{A}^* = \mathcal{J}$.

We now consider $\mathcal{A}\langle n\rangle \neq \mathcal{J}\langle n\rangle$. Thus, by (10.8), $\mathcal{A}\langle n\rangle = \mathcal{P}_{n'}^{(r-1)}(n')$ or $\mathcal{A}\langle n\rangle$ is special. We also keep in mind that $\mathcal{A}]n[$ is as in (10.9).

Suppose $k_{n',r-1} = 1$. If $\mathcal{A}\langle n\rangle$ is special then $k_{n',r-1} = 3$, so $\mathcal{A}\langle n\rangle = \mathcal{P}_{n'}^{(r-1)}(n')$. By (10.6) and the inductive hypothesis, either $\mathcal{P}_{n'}$ is symmetric or $\mathcal{P}_{n'}^{(r-1)}$ is type II. So $u(P_{k_{n',r-1},r-1}) = n'$. Together with $k_{n',r-1} = 1$, this gives us $\mathcal{A}\langle n\rangle = \{P_{1,r-1}\} = \mathcal{J}\langle n\rangle$, a contradiction. So

$$k_{n',r-1} \geq 2. \tag{10.10}$$

Thus, by Lemma 10.4.1,

$$k_{n,r} \geq 2. \tag{10.11}$$

We will consider the case $\mathcal{P}_n$ symmetric separately from the case $\mathcal{P}_n$ asymmetric.

Case 1: $\mathcal{P}_n$ symmetric. So $|\mathcal{P}_n^{(r)}(n)| = |\mathcal{J}|$, and hence $\mathcal{P}_n^{(r)}(n) \in \mathrm{ex}(\mathcal{P}_n^{(r)})$ by (10.5).

Now, in this case, we clearly have $n \in P_{k_{n,r},r}$. By (10.11), $k_{n,r} \geq 2$. The case $k_{n,r} = 2$ is trivial since then $\mathcal{P}_n^{(r)} = \mathcal{P}_n^{(r)}(1)(n) \cup \{P_{1,r}, P_{2,r}\}$ and either $\mathcal{A}^* = \mathcal{P}_n^{(r)} \backslash \{P_{2,r}\} = \mathcal{J}$ or $\mathcal{A}^* = \mathcal{P}_n^{(r)} \backslash \{P_{1,r}\} = \mathcal{P}_n^{(r)}(n)$.

Consider next $k_{n,r} = 3$ and $d_1 = 1$. Since $\mathcal{P}_n$ is symmetric, $n = 2r + 1$. Note that this is the unique case when $\mathcal{P}_n$ is symmetric and $\mathcal{P}_n^{(r)}$ is type I. Let $A_1 := P_{3,r}$, $A_{r+1} := P_{2,r}$ and $A_{i+1} := \{2j \colon j \in [i]\} \cup \{2j+1 \colon j \in [i+1, r]\}$, $i = 1, ..., r-1$. Let $B_{r+1} := \{1\} \cup P_{5,r-1}$ and $B_i := s_{-1}(A_{i+1})$, $i = 1, ..., r$. For each $i \in [r]$, let $\mathcal{S}_i$ be the special family $\{A_1, ..., A_i\} \cup (\mathcal{J} \backslash \{B_1, ..., B_i\})$. Since $|\mathcal{S}_1| = ... = |\mathcal{S}_r| = |\mathcal{J}|$, it follows by (10.5) that $\mathcal{S}_1, ..., \mathcal{S}_r \in \mathrm{ex}(\mathcal{P}_n^{(r)})$. For each $i \in [r+1]$, $|\mathcal{A}^* \cap \{A_i, B_i\}| \leq 1$ as $A_i \cap B_i = \emptyset$. Since $|\mathcal{A}^*| = |\mathcal{J}|$ (by (10.2), (10.5)) and $\mathcal{P}_n^{(r)} \backslash \mathcal{J} = \{A_1, ..., A_{r+1}\}$, we actually have $|\mathcal{A}^* \cap \{A_i, B_i\}| = 1$ for all $i \in [r+1]$. Suppose $\mathcal{A}^* \neq \mathcal{J}$. Then $A_q \in \mathcal{A}^*$ for some $q \in [r+1]$; assume that q is the largest such integer. Suppose $q > 1$ and there exists $p \in [2, q]$ such that $A_p \in \mathcal{A}^*$ and $A_{p-1} \notin \mathcal{A}^*$; then, since $B_{p-1} \cap A_p = \emptyset$, we get the contradiction that $|\mathcal{A}^* \cap \{A_{p-1}, B_{p-1}\}| = 0$. So $A_p \in \mathcal{A}^*$ for all $p \in [q]$. Since $A_1 \cap A_{r+1} = \emptyset$, $q \leq r$. Therefore $\mathcal{A}^*$ is the special family $\mathcal{S}_q$.

Now suppose that either $d_1 = 1$ and $k_{n,r} \geq 4$ or $d_1 > 1$ and $k_{n,r} \geq 3$. Then, by Lemma 10.4.1, $\mathcal{P}_{n'}^{(r-1)}$ is not type I, and hence $\mathcal{A}\langle n \rangle$ is not special. So $\mathcal{A}\langle n \rangle = \mathcal{P}_{n'}^{(r-1)}(n')$, and hence $A_1 := P_{k_{n',r-1},r-1} \cup \{n\} \in \mathcal{A}^*$, $A_2 := (A_1 \backslash l(A_1)) \cup \{l(A_1) - 1\} \in \mathcal{A}^*$ (we have $l(A_2) \geq 2$ because, since $l(A_1) = k_{n',r-1}$ and $k_{n,r} \geq 3$, $l(A_1) \geq 3$ by Lemma 10.4.1). Let $\mathcal{A}' := \Delta_{2,1}(\mathcal{A}^*)$. By Lemma 10.3.2(iii), $\mathcal{A}'\langle 1 \rangle$ and $\mathcal{A}']1[$ are intersecting. By an argument similar to the one for $\mathcal{A}$ above, $\mathcal{A}'\langle 1 \rangle$ and $\mathcal{A}']1[$ must obey conditions similar to (10.8) and (10.9); in particular, $\mathcal{A}'\langle n\rangle$ must be $\mathcal{P}^{(r-1)}_{[d_1+2,n]}(d_1+2)$ or $\mathcal{P}^{(r-1)}_{[d_1+2,n]}(n)$ (note that, since $\mathcal{P}_n$ is symmetric, $\mathcal{A}'\langle n\rangle$ cannot be isomorphic to a special family just like $\mathcal{A}\langle n\rangle$ cannot be special). Suppose $\mathcal{A}'\langle 1\rangle = \mathcal{P}^{(r-1)}_{[d_1+2,n]}(d_1+2)$. Taking $A_3 := s_{-1}(A_1)$, we then have $A_4 := (A_3\backslash\{l(A_3), l(A_3\backslash l(A_3))\}) \cup \{1, d_1+2\} \in \mathcal{A}^*$. If $l(A_1) = d_1+2$ then $A_2 \cap A_4 = \emptyset$, otherwise $A_1 \cap A_4 = \emptyset$; a contradiction. So $\mathcal{A}'\langle 1\rangle = \mathcal{P}^{(r-1)}_{[d_1+2,n]}(n)$. Since $\mathcal{P}_n$ is symmetric, we can use an argument similar to the one we applied for the case $\mathcal{A}\langle n\rangle = \mathcal{J}\langle n\rangle$ to obtain $\mathcal{A}^* = \mathcal{P}^{(r)}_n(n)$.

Case 2: $\mathcal{P}_n$ asymmetric. Note that therefore $e_n > 1$. As we showed above, the following are the cases that must be investigated.

Sub-case 2.1: $\mathcal{A}\langle n\rangle = \mathcal{P}^{(r-1)}_{n'}(n')$. Thus, by (10.6) and the inductive hypothesis, either $\mathcal{P}_{n'}$ is symmetric or $\mathcal{P}^{(r-1)}_{n'}$ is type II. So

$$n' \in P_{k_{n',r-1},r-1} = s_{k_{n',r-1}-1}(P_{1,r-1}), \tag{10.12}$$

where $k_{n',r-1} \geq 2$ by (10.10).

Suppose $\mathcal{A}]n[$ is special but not $\mathcal{P}^{(r)}_{n-1}(n-1)$. By definition, we have $k_{n-1,r} = 3$, $d_1 = 1$, $u(P_{3,r}) = n-1$, and hence $u(P_{1,r+1}) = u(\{1\} \cup P_{3,r}) = n-1$. So $u(P_{1,r}) = (n-1) - e_{n-1} - 1 \leq n - e_n - 1 = n'$, where the inequality follows by Proposition 10.3.1. Since $k_{n',r-1} = k_{n,r} \geq k_{n-1,r} = 3 > d_1 + 1$ (where the first equality is given by Lemma 10.4.1), $\mathcal{P}^{(r-1)}_{n'}$ is not type II. So $\mathcal{P}_{n'}$ is symmetric with $e_{n'} = d_1 = 1$. Suppose $u(P_{1,r}) < n'$. Since $\mathcal{P}_{n'}$ is symmetric, we then get $P_{2,r} = s_1(P_{1,r})$ and $u(P_{2,r}) \leq n'$. So $A_1 := P_{2,r-2} \cup \{n'\} \in \mathcal{A}\langle n\rangle$. By (10.1), $P_{1,r} \in \mathcal{A}]n[$. Since $A_1 \cap P_{1,r} = \emptyset$, (10.4) gives us $n-1 \in P_{1,r}$, which contradicts $u(P_{3,r}) = n-1$. So $u(P_{1,r}) = n'$. Since $P_{3,r-1} = P_{1,r}\backslash\{1\}$ and $\mathcal{A}\langle n\rangle = \mathcal{P}^{(r-1)}_{n'}(n')$, we therefore have $P_{3,r-1} \in \mathcal{A}\langle n\rangle$, and hence $A_2 := P_{3,r-1} \cup \{n\} \in \mathcal{A}^*$. Since $P_{3,r} = \delta_{n-1,n}(A_2)$, we obtain $P_{3,r} \in \mathcal{A}^*$ by (10.3). Now, since $\mathcal{A}]n[$ is special, $P_{3,r} = s_1(P_{2,r})$ and, by (10.1), $A_3 := \{1, n-1\} \cup (P_{2,r-1}\backslash\{2\}) \in \mathcal{A}]n[$. So $A_2 \cap A_3 = \emptyset$, and hence $A_4 := \delta_{n,n-1}(A_3) \in \mathcal{A}^*$ by (10.4). But then $P_{3,r} \cap A_4 = \emptyset$, a contradiction. We therefore conclude that $\mathcal{A}]n[= \mathcal{J}]n[$ or $\mathcal{A}]n[= \mathcal{P}^{(r)}_{n-1}(n-1)$.

Sub-sub-case 2.1.1: $\mathcal{A}]n[= \mathcal{J}]n[$. Let $A_1 := (P_{k_{n',r-1}-1,r-1}\backslash\{n'-1\}) \cup \{n'\}$. Note that $n'-1 \in P_{k_{n',r-1}-1,r-1}$ by (10.12). Since $\mathcal{A}\langle n\rangle = \mathcal{P}^{(r-1)}_{n'}(n')$, we thus have $A_1 \in \mathcal{A}\langle n\rangle$.

Suppose $k_{n',r-1} > 2$. Then $A_2 := (P_{k_{n',r-1}-2,r-1} \backslash \{k_{n',r-1} - 2\}) \cup \{1, n-2\} \in \mathcal{A}]n[$. By (10.12), we have $P_{k_{n',r-1}-1,r-1} = s_1(P_{k_{n',r-1}-2,r-1})$, and hence $A_1 \cap A_2 = \emptyset$. But then (10.4) gives us $n - 1 \in A_2$, a contradiction. So $k_{n',r-1} \leq 2$. By (10.10), $k_{n',r-1} = 2$, and hence $u(P_{2,r-1}) = n'$ by (10.12). Thus, $A_3 := P_{2,r-1} \in \mathcal{A}\langle n \rangle$ and, by (10.12), $n' - 1 = u(P_{1,r-1})$. Suppose $d_{n'-1} < e_n$. Then $(n'-1) + d_{n'-1} + 1 \leq n' + e_n - 1 = n - 2$, and hence $A_4 := P_{1,r-1} \cup \{n-2\} \in \mathcal{P}$ (as $n'-1 = u(P_{1,r-1})$). So $A_4 \in \mathcal{A}]n[$. Since (10.12) implies $A_3 \cap A_4 = \emptyset$, (10.4) gives us $n - 1 \in A_4$, a contradiction. So $d_{n'-1} = d_{n'} = e_n$. Thus, since $u(P_{1,r-1}) = n' - 1$ and $P_{2,r-1} = s_1(P_{1,r-1})$ (by (10.12)), we have $P_{1,r} = P_{1,r-1} \cup \{(n'-1) + e_n + 1\} = P_{1,r-1} \cup \{n-1\}$, $P_{2,r} = P_{2,r-1} \cup \{n' + e_n + 1\} = P_{2,r-1} \cup \{n\}$, and hence $P_{2,r} = s_1(P_{1,r-1} \cup \{n-1\}) = s_1(P_{1,r})$. So $\mathcal{P}_n^{(r)}$ is type II. Now $u(P_{1,r}) = n-1$ implies $\mathcal{A}]n[= \{P_{1,r}\}$. Since $P_{2,r-1} \in \mathcal{P}_{n'}^{(r-1)}(n') = \mathcal{A}\langle n \rangle \subseteq \mathcal{A}^*\langle n \rangle$ and $P_{1,r} \cap P_{2,r-1} = \emptyset$, it follows by (10.4) that $\mathcal{A}^*]n[= \emptyset$ and $\mathcal{A}^*]n'[(n) = \{(P_{1,r} \backslash \{n-1\}) \cup \{n\}\}$. So $\mathcal{A}^*]n'[(n) = \mathcal{P}_n^{(r)}]n'[(n)$ as $u(P_{1,r} \backslash \{n-1\}) = u(P_{1,r-1}) = n' - 1$. Since $\mathcal{A}(n')(n-1) = \emptyset$, we have $\mathcal{A}^*(n')(n) = \mathcal{A}(n')(n)$, and hence $\mathcal{A}^*(n')(n) = \mathcal{P}_n^{(r)}(n')(n)$. Therefore $\mathcal{A}^* = \mathcal{P}_n^{(r)}(n)$.

Sub-sub-case 2.1.2: $\mathcal{A}]n[= \mathcal{P}_{n-1}^{(r)}(n-1)$. Suppose $d_{n'} < e_n$. Then $A_1 := P_{k_{n',r-1},r-1} \cup \{n-1\} \in \mathcal{A}]n[$. Recall that $\mathcal{A}\langle n \rangle = \mathcal{P}_{n'}^{(r-1)}(n')$. Thus, by (10.12), $A_2 := P_{k_{n',r-1},r-1} \cup \{n\} \in \mathcal{A}(n)$, and hence $A_2 \in \mathcal{A}^*$. Since $A_1 = \delta_{n-1,n}(A_2)$, $A_1 \in \mathcal{A}^*$ by (10.3). By (10.10), $k_{n',r-1} - 1 \geq 1$; so let $A_3 := P_{k_{n',r-1}-1,r-1} \cup \{n-1\}$. We have $A_3 \in \mathcal{A}]n[$, and $A_2 \cap A_3 = \emptyset$ since $A_2 = s_1(A_3)$ by (10.12). Hence $A_4 := \delta_{n,n-1}(A_3) \in \mathcal{A}^*$ by (10.4). But $A_1 \cap A_4 = A_2 \cap A_3 = \emptyset$, a contradiction. So $d_{n'} = e_n$, which implies $e_{n-1} = e_n$.

Let $A \in \mathcal{A}]n[$. Since $n - 1 \in A$ and $(n-1) - e_{n-1} - 1 = n - e_n - 2 \leq n' - 1$, we have $n' \notin A$ and $B := A \backslash \{n-1\} \in \mathcal{P}_{n'-1}^{(r-1)}$. Since $\mathcal{P}_{n'}$ is symmetric or $\mathcal{P}_{n'}^{(r-1)}$ is type II, Lemma 10.4.2 gives us $s_1(B) \in \mathcal{P}_{n'}^{(r-1)}$. So $C := (s_1(B) \backslash u(s_1(B))) \cup \{n', n\} \in \mathcal{P}_n^{(r)}(n')(n)$. Since $\mathcal{P}_n^{(r)}(n')(n) = \mathcal{A}(n) \subseteq \mathcal{A}^*(n)$, $C \in \mathcal{A}^*$. Since $A \cap C = \emptyset$, it follows by (10.4) that $A \notin \mathcal{A}^*$ and $\delta_{n,n-1}(A) \in \mathcal{A}^*]n'[(n)$. We have therefore shown that $\mathcal{A}]n[(n') = \emptyset$, $\mathcal{A}^*]n[= \emptyset$ and $\mathcal{A}^*]n'[(n) = \Delta_{n,n-1}(\mathcal{A}]n[) = \Delta_{n,n-1}(\mathcal{P}_{n-1}^{(r)}(n-1)) = \mathcal{P}_n^{(r)}]n'[(n)$. Since $\mathcal{A}]n[(n') = \emptyset$ implies $\mathcal{A}^*(n')(n) = \mathcal{A}(n')(n)$, we have $\mathcal{A}^*(n')(n) = \mathcal{P}_n^{(r)}(n')(n)$. So $\mathcal{A}^* = \mathcal{P}_n^{(r)}(n)$.

We now need to show that $\mathcal{P}_n^{(r)}$ is type II. Since $e_{n-1} = e_n$ and $\mathcal{P}_n$ is asymmetric, $\mathcal{P}_{n-1}$ is asymmetric. Since $\mathcal{A}]n[= \mathcal{P}_{n-1}^{(r)}(n-1) \in \mathrm{ex}(\mathcal{P}_{n-1}^{(r)})$, it follows by the inductive hypothesis that $\mathcal{P}_{n-1}^{(r)}$ is type II, and hence $k_{n-1,r} \leq d_1+1$. By the inductive hypothesis for $\mathcal{A}\langle n\rangle = \mathcal{P}_{n'}^{(r-1)}(n') \in \mathrm{ex}(\mathcal{P}_{n'}^{(r-1)})$, either $\mathcal{P}_{n'}^{(r-1)}$ is type II or $\mathcal{P}_{n'}$ is symmetric. If $\mathcal{P}_{n'}^{(r-1)}$ is type II then, by definition, $k_{n',r-1} \leq d_1+1$. We now show that this inequality also holds if $\mathcal{P}_{n'}$ is symmetric.

So suppose we instead have $\mathcal{P}_{n'}$ symmetric and $k_{n',r-1} > d_1+1$. Then $\{1,n\} \cup P_{k_{n',r-1},r-1} \in \mathcal{P}_n^{(r+1)}$, and hence $r < \alpha(\mathcal{P}_n)$. If $\mathcal{P}_n\langle 1\rangle$ is symmetric or $d_2 > d_1$ then Lemma 10.4.3 gives us $\alpha(\mathcal{P}_n) \leq 3$, and hence $r \leq 2$; but we are now considering $r \geq 3$. So $\mathcal{P}_n\langle 1\rangle$ is asymmetric and $d_2 = d_1$. Let $\mathcal{A}' := \Delta_{2,1}(\mathcal{A}^*)$. By Lemma 10.3.2(iii), $\mathcal{A}'\langle 1\rangle$ and $\mathcal{A}']1[$ are intersecting. The inductive hypothesis gives us $|\mathcal{A}'\langle 1\rangle| \leq |\mathcal{P}_n\langle 1\rangle^{(r-1)}(d_1+2)| = |\mathcal{P}_n^{(r)}(1)(d_1+2)|$ and $|\mathcal{A}']1[| \leq |\mathcal{P}_n]1[^{(r)}(2)| = |\mathcal{P}_n^{(r)}(1)]\{2,3,...,d_2+2\}[| \leq |\mathcal{P}_n^{(r)}(1)]\{2,3,...,d_1+2\}[|$, and therefore $|\mathcal{A}'| \leq |\mathcal{P}_n^{(r)}(1)(d_1+2)| + |\mathcal{P}_n^{(r)}(1)]\{2,3,...,d_1+2\}[| = |\mathcal{P}_n^{(r)}(1)|$. Since $|\mathcal{A}'| = |\mathcal{A}|$ and $\mathcal{A} \in \mathrm{ex}(\mathcal{P}_n^{(r)})$, it follows that $|\mathcal{A}'\langle 1\rangle| = |\mathcal{P}_n\langle 1\rangle^{(r-1)}(d_1+2)|$ and $|\mathcal{A}']1[| = |\mathcal{P}_n]1[^{(r)}(2)|$. Since the inductive hypothesis gives us $\mathcal{P}_n\langle 1\rangle^{(r-1)}(d_1+2) \in \mathrm{ex}(\mathcal{P}_n\langle 1\rangle^{(r-1)})$, we therefore have $\mathcal{A}'\langle 1\rangle \in \mathrm{ex}(\mathcal{P}_n\langle 1\rangle^{(r-1)})$. Thus, by the inductive hypothesis, one of the following holds: (a) $\mathcal{A}'\langle 1\rangle = \mathcal{P}_n\langle 1\rangle(d_1+2)^{(r-1)}$, (b) $\mathcal{A}'\langle 1\rangle = \mathcal{P}_n\langle 1\rangle(n)^{(r-1)}$, (c) $\mathcal{A}'\langle 1\rangle$ is isomorphic to a special family. Suppose (a) holds. Then $P_{d_1+2,r-1} \in \mathcal{A}'\langle 1\rangle$, and hence $P_{1,r} \in \mathcal{A}'(1)$. So $P_{1,r} \in \mathcal{A}^*$ as $\mathcal{A}'(1) \subset \mathcal{A}^*$; but this clearly contradicts $\mathcal{A}^* = \mathcal{P}_n^{(r)}(n)$ and (10.11). Suppose (c) holds. Since $\mathcal{P}_n\langle 1\rangle$ is asymmetric, (10.1) then gives us $P_{d_1+2,r-1} \in \mathcal{A}'\langle 1\rangle$; but, as we have just shown, this leads to a contradiction. Hence (b) holds. Since $\mathcal{P}\langle 1\rangle$ is asymmetric, it follows by the inductive hypothesis that $\mathcal{P}_n\langle 1\rangle^{(r-1)}$ is isomorphic to a type II family, and hence, by definition of a type II family, we must have $r-1 = \alpha(\mathcal{P}_n\langle 1\rangle)$; but this clearly contradicts $r < \alpha(\mathcal{P}_n)$.

Therefore, as we claimed earlier, $k_{n',r-1} \leq d_1+1$. So $k_{n,r} \leq d_1+1$ by Lemma 10.4.1. Now, since $\mathcal{P}_{n-1}^{(r)}$ is type II, $n-1 \in P_{k_{n-1,r},r} = s_{k_{n-1,r}-1}(P_{1,r})$. Since $e_{n-1} = e_n$, it follows that $n'-1 = (n-1)-e_{n-1}-1 \in P_{k_{n-1,r},r-1} = s_{k_{n-1,r}-1}(P_{1,r-1})$. Since either $\mathcal{P}_{n'}^{(r-1)}$ is type II or $\mathcal{P}_{n'}$ is symmetric, $n' \in P_{k_{n',r-1},r-1} = s_{k_{n',r-1}-1}(P_{1,r-1})$. Therefore $P_{k_{n',r-1},r-1} = s_1(P_{k_{n-1,r},r-1})$ and $k_{n-1,r} = k_{n',r-1}-1$. Since $d_{n'} = e_n$, we have $n'+d_{n'}+1 = n$, and hence $P_{k_{n,r},r} = P_{k_{n',r-1},r-1} \cup \{n\}$ as $n' = u(P_{k_{n',r-1},r-1})$. Bringing together the relations we have just established, we get

$$P_{k_{n,r},r} = s_1(P_{k_{n-1,r},r-1}) \cup \{n\} = s_1(P_{k_{n-1,r},r-1} \cup \{n-1\}) = s_1(P_{k_{n-1,r},r})$$
$$= s_1(s_{k_{n-1,r}-1}(P_{1,r})) = s_{k_{n-1,r}}(P_{1,r}) = s_{k_{n',r-1}-1}(P_{1,r}).$$

Together with Lemma 10.4.1, this gives us $P_{k_{n,r},r} = s_{k_{n,r}-1}(P_{1,r})$. Since we also established $k_{n,r} \leq d_1 + 1$ and $n \in P_{k_{n,r},r}$, $\mathcal{P}_n^{(r)}$ is type II.

Sub-case 2.2: $\mathcal{A}\langle n \rangle$ special, $\mathcal{A}\langle n \rangle \neq \mathcal{P}_{n'}^{(r-1)}(n')$. By definition and (10.1), $n' \in P_{3,r-1} = s_1(P_{2,r-1})$ and $P_{1,r-1}, P_{3,r-1} \in \mathcal{A}\langle n \rangle$. Taking $Q_1 := P_{1,r-1} \cup \{n\}$ and $Q_3 := P_{3,r-1} \cup \{n\}$, we then have $Q_1, Q_3 \in \mathcal{A}^*(n)$ (as $\mathcal{A}\langle n \rangle \subseteq \mathcal{A}^*\langle n \rangle$).

Suppose $\mathcal{A}]n[= \mathcal{P}_{n-1}^{(r)}(n-1)$. So $A_1 := s_{-1}(Q_3) = P_{2,r-1} \cup \{n-1\} \in \mathcal{A}]n[$ and $A_2 := P_{1,r-1} \cup \{n-1\} \in \mathcal{A}]n[$. Since $A_2 = \delta_{n-1,n}(Q_1)$, it follows by (10.3) that $A_2 \in \mathcal{A}^*$. Since $A_1 \cap Q_3 = \emptyset$, we have $A_3 := P_{2,r-1} \cup \{n\} \in \mathcal{A}^*$ by (10.4). Now, by (10.7) and the inductive hypothesis, we should have $\mathcal{P}_{n-1}^{(r)}$ type II or $\mathcal{P}_{n-1}$ symmetric, and hence $P_{2,r} = s_1(P_{1,r})$; but then $A_2 \cap A_3 = \emptyset$, a contradiction. So $\mathcal{A}]n[\neq \mathcal{P}_{n-1}^{(r)}(n-1)$.

Next, suppose $\mathcal{A}]n[$ is special. Then $A_1 := s_1(P_{2,r}) = P_{3,r} = P_{3,r-1} \cup \{n-1\} \in \mathcal{A}]n[$ and, by (10.1), $A_2 := \{1, n-1\} \cup (P_{2,r-1} \backslash \{2\}) \in \mathcal{A}]n[$. Since $A_1 = \delta_{n-1,n}(Q_3)$, (10.3) gives us $A_1 \in \mathcal{A}^*$. Since $A_2 \cap Q_3 = A_2 \cap \delta_{n,n-1}(A_1) = \emptyset$, (10.4) gives us $A_3 := \delta_{n,n-1}(A_2) \in \mathcal{A}^*$. But $A_1 \cap A_3 = \emptyset$, a contradiction.

Therefore $\mathcal{A}]n[= \mathcal{J}]n[$. Suppose $d_{n'-1} < e_n$. Then $(n'-1) + d_{n'-1} + 1 \leq n' + e_n - 1 = n - 2$. Now $n' - 1 \in P_{2,r-1}$ as $n' \in P_{3,r-1} = s_1(P_{2,r-1})$. Taking $A_1 := \{1, n-2\} \cup (P_{2,r-1} \backslash \{2\})$, we thus get $A_1 \in \mathcal{A}]n[\cap \mathcal{A}^*$. However, since $Q_3 \ni n' \leq n - 2 - d_{n'-1} \leq n - 3$ implies $n - 2 \notin Q_3$, we also get $A_1 \cap Q_3 = \emptyset$, a contradiction. So $d_{n'-1} = e_n$, and hence $d_{n'-1} = d_{n'} = e_n$ (as $d_{n'} \leq e_n$). Thus, since $u(P_{2,r-1}) = n' - 1$ and $u(P_{3,r-1}) = n'$, $u(P_{2,r}) = (n'-1) + d_{n'-1} + 1 = n' + e_n = n - 1$ and similarly $u(P_{3,r}) = n$. So $P_{3,r} = P_{3,r-1} \cup \{n\} = s_1(P_{2,r-1} \cup \{n-1\}) = s_1(P_{2,r})$. Given that $\mathcal{A}\langle n \rangle$ is special, $d_1 = d_3 = 1$. Since $\mathcal{P}_n$ is asymmetric, we therefore have $e_n > 1$, and hence $m := \max\{a \in [n] \colon d_a = 1\} \leq n'$. Thus, since $\mathcal{A}\langle n \rangle$ is special, $m = 2t + 1$ for some $t \in [n']$. So $\mathcal{P}_n^{(r)}$ is type I.

It remains to show that $\mathcal{A}^*$ is special. Let us take A_i, B_i, $i = 1, ..., t$, to be as in the definition of a special family with $y = n$ in Section 10.2. Since $n \in P_{3,r} = P_{3,t} \cup P_{m+2,r-t} = s_1(P_{2,r}) = s_1(P_{2,t} \cup P_{m+1,r-t})$, we have $n \in P_{m+2,r-t} = s_1(P_{m+1,r-t})$, and hence $n \in A_i$ and $n-1 \in P_{m+1,r-t} \subset B_i$ for all $i \in [t]$. For each $i \in [t]$, let $A'_i := A_i \backslash \{n\}$, $B'_i := B_i \backslash \{n-1\}$, $B''_i := B'_i \cup \{n\}$. If $r = t+1$ then $P_{m+2,r-t} = P_{m+2,1} = \{m+2\}$, and hence $n = m+2$, which clearly contradicts $m \leq n'$ and $d_{n'} = e_n > 1$ (which we established above). So $r \geq t+2$, and hence $P_{m+2,r-t-1} \neq \emptyset$, $P_{m+1,r-t-1} \neq \emptyset$. Clearly, for all $i \in [t]$, $A'_i = (A_i \backslash P_{m+2,r-t}) \cup P_{m+2,r-t-1}$ and $B'_i = (B_i \backslash P_{m+1,r-t}) \cup P_{m+1,r-t-1}$ (recall that $P_{m+1,r-t} \subset B_i$). Therefore, since $\mathcal{A}\langle n \rangle$ is special, $\mathcal{A}\langle n \rangle = \{A'_1, ..., A'_q\} \cup (\mathcal{P}_{n'}^{(r-1)}(1) \backslash \{B'_1, ..., B'_q\})$ for some $q \in [t]$. So $\mathcal{A}_1^* := \mathcal{A}(n) = \{A_1, ..., A_q\} \cup (\mathcal{P}_n^{(r)}(1)(n) \backslash \{B''_1, ..., B''_q\})$. Since $\mathcal{A}(n) \subseteq \mathcal{A}^*(n)$, $\mathcal{A}_1^* \subseteq \mathcal{A}^*$. Now, we also have $\mathcal{A}]n[= \mathcal{J}]n[= \mathcal{P}_{n-1}^{(r)}(1)$. So $\mathcal{A}_2^* := \mathcal{P}_{n-1}^{(r)}(1)]n-1[= \mathcal{A}]n[]n-1[\subset \mathcal{A}^*$. Finally, consider $A \in \mathcal{A}]n[(n-1) = \mathcal{P}_{n-1}^{(r)}(1)(n-1)$. If $A = B_i$ for some $i \in [q]$ then, since $A_i \cap B_i = \emptyset$ and $A_i \in \mathcal{A}_1^*$, we must have $A \notin \mathcal{A}^*$ and $(A \backslash \{n-1\}) \cup \{n\} \in \mathcal{A}^*$. If $A \notin \{B_1, ..., B_q\}$ then $(A \backslash \{n-1\}) \cup \{n\} \in \mathcal{A}_1^*$, and hence $A \in \mathcal{A}^*$ by (10.3). Setting $\mathcal{A}_3^* := \mathcal{A}^* \backslash (\mathcal{A}_1^* \cup \mathcal{A}_2^*)$, we therefore have $\mathcal{A}_3^* = (\mathcal{P}_{n-1}^{(r)}(1)(n-1) \backslash \{B_1, ..., B_q\}) \cup \{B''_1, ..., B''_q\}$. So $\mathcal{A}^* = \mathcal{A}_1^* \cup \mathcal{A}_2^* \cup \mathcal{A}_3^* = \{A_1, ..., A_q\} \cup (\mathcal{P}_n^{(r)}(1) \backslash \{B_1, ..., B_q\})$. So $\mathcal{A}^*$ is special. □

10.5 The case $d_1 = 0$

We start with a lemma concerning sets in hereditary families.

Lemma 10.5.1 *Let $\mathcal{F}$ be hereditary. If there exist $F_1, F_2 \in \mathcal{F}$ such that $F_1 \cap F_2 = \emptyset$ and $|F_1| = |F_2| = \alpha(\mathcal{F})$ then for any $F \in \mathcal{F}$ there exists $F' \in \mathcal{F}$ such that $F \cap F' = \emptyset$ and $|F| + |F'| > \alpha(\mathcal{F})$.*

Proof. Let $F \in \mathcal{F}$. If $F \subseteq F_1$ then the result follows immediately by taking $F' \in \binom{F_2}{\alpha(\mathcal{F})+1-|F|}$. If $F \nsubseteq F_1$ then $|F_1 \backslash F| \geq |F_1| - (|F|-1) = \alpha(F) + 1 - |F|$, and hence the result follows by taking $F' \in \binom{F_1 \backslash F}{\alpha(\mathcal{F})+1-|F|}$. □

The converse of this result is not true; to see this, take $\mathcal{F}$ to be $2^{[n+1]} \backslash \{[n+1]\}$.

Also, in a sense, the conditions on F_1 and F_2 cannot be improved; by considering $\mathcal{F} = 2^{F_1} \cup 2^{F_2}$ and $F = \{f\}$, it is easy to see that we can neither allow F_1 and F_2 to have non-empty intersection nor allow F_1 or F_2 to have size less than $\alpha(\mathcal{F})$.

Lemma 10.5.2 *If $d_1 > 0$, $\alpha(\mathcal{P}_n) \geq 3$ and $n \in P_{1,\alpha(\mathcal{P}_n)}$ then for any $A \in \mathcal{P}_n]2[$ there exists $A' \in \mathcal{P}_n(2)$ such that $A \cap A' = \emptyset$ and $|A| + |A'| \geq \alpha(\mathcal{P}_n)$.*

Proof. Let $1 = p_1 < p_2 < ... < p_{\alpha(\mathcal{P}_n)}$ such that $P_{1,\alpha(\mathcal{P}_n)} = \{p_1, ..., p_{\alpha(\mathcal{P}_n)}\}$. We have $\mathcal{P}_n\langle 2\rangle \subset 2^{[a,n]}$, where $a := 2 + d_2 + 1$. Let $a =: q_1 < ... < q_{\alpha(\mathcal{P}_n)-2}$ such that $P_{a,\alpha(\mathcal{P}_n)-2} = \{q_1, ..., q_{\alpha(\mathcal{P}_n)-2}\}$. So

$$p_2 = 1 + d_1 + 1 < 2 + d_2 + 1 = q_1 < p_2 + d_{p_2} + 1 = p_3, \tag{10.13}$$

and if $\alpha(\mathcal{P}_n) \geq 4$ then, proceeding inductively, we also get

$$p_i = p_{i-1} + d_{p_{i-1}} + 1 < q_{i-2} + d_{q_{i-2}} + 1 = q_{i-1} < p_i + d_{p_i} + 1 = p_{i+1}, \tag{10.14}$$

$i = 3, ..., \alpha(\mathcal{P}_n) - 1$. Let $F_1 := P_{p_3,\alpha(\mathcal{P}_n)-2} = P \backslash \{p_1, p_2\}$, $F_2 := P_{a,\alpha(\mathcal{P}_n)-2}$. By (10.13) and (10.14), $F_1, F_2 \in \mathcal{P}_n\langle 2\rangle$ and $F_1 \cap F_2 = \emptyset$. Since $|F_1| = \alpha(\mathcal{P}_n) - 2$, $\alpha(\mathcal{P}_n\langle 2\rangle) \geq \alpha(\mathcal{P}_n) - 2$. By definition of a, $P_{a,\alpha(\mathcal{P}_n\langle 2\rangle)} \in \mathcal{P}_n\langle 2\rangle$ (for the same reason that $P_{1,\alpha(\mathcal{P}_n)} \in \mathcal{P}_n$, being that $\{d_i\}_{i\in\mathbb{N}}$ is mnd). So $u(P_{a,\alpha(\mathcal{P}_n\langle 2\rangle)}) \leq n$. Now $n = p_{\alpha(\mathcal{P}_n)}$ as we are given that $n \in P_{1,\alpha(\mathcal{P}_n)}$.

Suppose $\alpha(\mathcal{P}_n\langle 2\rangle) > \alpha(\mathcal{P}_n) - 2$. Then $q_{\alpha(\mathcal{P}_n)-2} \in P_{a,\alpha(\mathcal{P}_n\langle 2\rangle)} \backslash \{u(P_{a,\alpha(\mathcal{P}_n\langle 2\rangle)})\}$. Together with (10.13) and (10.14), this gives us $u(P_{a,\alpha(\mathcal{P}_n\langle 2\rangle)}) \geq q_{\alpha(\mathcal{P}_n)-2} + d_{q_{\alpha(\mathcal{P}_n)-2}} + 1 > p_{\alpha(\mathcal{P}_n)}$, contradicting $u(P_{a,\alpha(\mathcal{P}_n\langle 2\rangle)}) \leq n = p_{\alpha(\mathcal{P}_n)}$. So $\alpha(\mathcal{P}_n\langle 2\rangle) = \alpha(\mathcal{P}_n) - 2 = |F_1| = |F_2|$.

Let $A \in \mathcal{P}_n]2[$. Suppose $A \in \mathcal{P}_n\langle 2\rangle$. By Lemma 10.5.1, there exists $A'' \in \mathcal{P}_n\langle 2\rangle$ such that $A \cap A'' = \emptyset$ and $|A| + |A''| \geq \alpha(\mathcal{P}_n\langle 2\rangle) + 1 = \alpha(\mathcal{P}_n) - 1$. Hence $A' := A'' \cup \{2\} \in \mathcal{P}_n(2)$, $A \cap A' = \emptyset$ and $|A| + |A'| \geq \alpha(\mathcal{P}_n)$. Now suppose $A \notin \mathcal{P}_n\langle 2\rangle$. We have $A^* := A \cap [a, n] \in \mathcal{P}_n\langle 2\rangle \cup \{\emptyset\}$. If $A^* \neq \emptyset$ then we apply the argument for A above to get $|A^*| + |A'| \geq \alpha(\mathcal{P}_n)$ for some $A' \in \mathcal{P}_n(2)$ such that $A^* \cap A' = \emptyset$, which clearly yields the result. Suppose $A^* = \emptyset$. Let $A' := F_1 \cup \{2\}$. So $A \cap A' = \emptyset$ and

$|A| + |A'| \geq 1 + (\alpha(\mathcal{P}_n) - 1) = \alpha(\mathcal{P}_n).$ □

Proof of Theorem 10.2.3. We start with (i), for which we have $d_{2r-2} = 0$ and $d_{2r-1} = n - 2r$. We first consider $\mathcal{A} \in \mathrm{ex}(\mathcal{P}_n^{(r)})$ and prove the necessary condition. Let $\mathcal{B} := \mathcal{P}_n^{(r)}(1)$. Let $\mathcal{A}_0 = \mathcal{A} \cap \binom{[2r-2]}{r}$, $\mathcal{A}_2 := \mathcal{A}(2r-1)(n)$ and $\mathcal{A}_{1,i} := \mathcal{A}(i)\backslash\mathcal{A}_2$, $i = 2r-1, ..., n$. Define $\mathcal{B}_0$, $\mathcal{B}_2$, and $\mathcal{B}_{1,i}$ similarly. Note that since $(2r-1) + d_{2r-1} + 1 = n$ (and $d_i \geq d_{2r-1}$ for all $i \geq 2r$), if $A \in \mathcal{A}$ and $|A \cap [2r-1, n]| > 1$ then $A \cap [2r-1, n] = \{2r-1, n\}$. So $\mathcal{A}_0 \cup \mathcal{A}_2 \cup \bigcup_{i=2r-1}^{n} \mathcal{A}_{1,i}$ is a partition for $\mathcal{A}$. Let $\mathcal{A}_2' := \mathcal{A}\langle 2r-1\rangle\langle n\rangle \subseteq \binom{[2r-2]}{r-2}$ and $\mathcal{A}_{1,i}' := \mathcal{A}\langle i\rangle \cap \binom{[2r-2]}{r-1} = \mathcal{A}_{1,i}\langle i\rangle$, $i = 2r-1, ..., n$. Define $\mathcal{B}_2'$ and $\mathcal{B}_{1,i}'$ $(i = 2r-1, ..., n)$ similarly. So

$$|\mathcal{A}| = |\mathcal{A}_0| + |\mathcal{A}_2'| + \sum_{i=2r-1}^{n} |\mathcal{A}_{1,i}'|, \quad |\mathcal{B}| = |\mathcal{B}_0| + |\mathcal{B}_2'| + \sum_{i=2r-1}^{n} |\mathcal{B}_{1,i}'| \tag{10.15}$$

Clearly, $\mathcal{A}_0$ and $\mathcal{A}_2'$ must be cross-intersecting. So

$$|\{A, [2r-2]\backslash A\} \cap (\mathcal{A}_0 \cup \mathcal{A}_2')| \leq 1 \text{ for all } A \in \binom{[2r-2]}{r-2} \cup \binom{[2r-2]}{r}, \tag{10.16}$$

and hence

$$|\mathcal{A}_0| + |\mathcal{A}_2'| \leq \binom{2r-2}{r} = |\mathcal{B}_0| + |\mathcal{B}_2'|. \tag{10.17}$$

Let us now consider $\mathcal{A}_{1,i}'$, $i = 2r-1, ..., n$. These families must also be cross-intersecting. Thus, by Theorem 9.2.3, we have

$$\sum_{i=2r-1}^{n} |\mathcal{A}_{1,i}'| \leq (n - 2r + 2)\binom{2r-3}{r-2} = \sum_{i=2r-1}^{n} |\mathcal{B}_{1,i}'|. \tag{10.18}$$

By (10.15), (10.17) and (10.18), we have $|\mathcal{A}| \leq |\mathcal{B}|$. Thus, since $\mathcal{A} \in \mathrm{ex}(\mathcal{P}_n^{(r)})$, we actually have $|\mathcal{A}| = |\mathcal{B}|$. It follows that the inequalities in (10.17) and (10.18) are actually equalities. By Theorem 9.2.3 and the EKR Theorem, an equality in (10.18) yields $\mathcal{A}_{1,2r-1}' = ... = \mathcal{A}_{1,n}' \in \mathrm{ex}(\binom{[2r-2]}{r-1})$; hence (b).

Since $d_{n-1} > 0$ and $2r \leq \alpha(\mathcal{P}_n)$, $n \geq 2r+1$. So the sets of $\mathcal{A}_{1,2r}$ do not intersect with those of $\mathcal{A}_2$ on $[2r-1, n]$, and hence $\mathcal{A}'_{1,2r}$ and $\mathcal{A}'_2$ are cross-intersecting. By the equalities in (b), (c) follows.

Since we established equality in (10.17), we also have equality in (10.16), which implies that $\mathcal{A}_0 = \binom{[2r-2]}{r} \backslash \{[2r-2]\backslash A \colon A \in \mathcal{A}'_2\}$. Thus, to obtain (a), it remains to show that $\mathcal{A}'_2$ is intersecting. Suppose there exist $A_1, A_2 \in \mathcal{A}'_2$ such that $A_1 \cap A_2 = \emptyset$. So $[2r-2]\backslash(A_1 \cup A_2) = \{x, y\}$ for some distinct $x, y \in [2r-2]$. Let $A_3 := A_1 \cup \{x\}$ and $A_4 := A_2 \cup \{y\}$. So $A_3 \cap A_2 = \emptyset$ and $A_4 \cap A_1 = \emptyset$. Since $\mathcal{A}'_{1,2r}$ and $\mathcal{A}'_2$ are cross-intersecting (see above), we therefore get $A_3, A_4 \notin \mathcal{A}'_{1,2r}$. Since $A_4 = [2r-2]\backslash A_3$, this implies that $\mathcal{A}'_{1,2r} \notin \mathrm{ex}(\binom{[2r-2]}{r-1})$ (see Theorem 10.1.1(ii)), a contradiction to (b). So $\mathcal{A}'_2$ is intersecting. Hence (a).

We now prove the sufficiency condition in (i). So let $\mathcal{A}$ be a sub-family of $\mathcal{P}_n^{(r)}$ that obeys (a), (b) and (c). Define $\mathcal{A}_0$, $\mathcal{A}_2$ and $\mathcal{A}_{1,i}$, $i = 2r-1, ..., n$, as above. As we showed above, $\mathcal{A}_0 \cup \mathcal{A}_2 \cup \bigcup_{i=2r-1}^{n} \mathcal{A}_{1,i}$ is a partition for $\mathcal{A}$. By definition, $\mathcal{A}_0$, $\mathcal{A}_2$ and $\mathcal{A}_{1,i}$, $i = 2r-1, ..., n$, are intersecting. By (a), $\mathcal{A}_0 \cup \mathcal{A}_2$ is intersecting. By (b) and (c), $\mathcal{A}_2 \cup \bigcup_{i=2r-1}^{n} \mathcal{A}_{1,i}$ is intersecting. If $A \in \bigcup_{i=2r-1}^{n} \mathcal{A}_{1,i}$ then $A \cap [2r-2] = r-1$ and hence A intersects each set in $\binom{[2r-2]}{r}$; so $\mathcal{A}_0$ and $\bigcup_{i=2r-1}^{n} \mathcal{A}_{1,i}$ are cross-intersecting. Therefore $\mathcal{A}$ is intersecting. Now, it is immediate from (a), (b) and (c) that the bounds in (10.17) and (10.18) are attained. So $\mathcal{A} \in \mathrm{ex}(\mathcal{P}_n^{(r)})$.

We now prove (ii) and (iii) by induction on n. We first consider $\mathcal{A}^* \in \mathrm{ex}(\mathcal{P}_n^{(r)})$ and prove the necessary conditions for $\mathcal{A}^*$. Unlike we did in the Proof of Theorem 10.2.1, we do not use $\Delta_{n-1,n}$ because if $\alpha(\mathcal{P}_n)/2 = r > \alpha(\mathcal{P}_{n-1})/2$ (which is possible) then we cannot apply the inductive hypothesis. Instead, we work with $\mathcal{A} := \Delta_{m,m+1}(\mathcal{A}^*)$. By Lemma 10.3.2(ii), $\mathcal{A}]m+1[\cup \mathcal{A}\langle m+1\rangle$ is intersecting. We have $\mathcal{A}]m+1[\subset \mathcal{P}_n^{(r)}]m+1[= \mathcal{P}_n]m+1[^{(r)}$ and $\mathcal{A}\langle m+1\rangle \subset \mathcal{P}_n^{(r)}\langle m+1\rangle = \mathcal{P}_n\langle m+1\rangle^{(r'')}$, where $r'' = r-1$. Since $m, m+d_m+1 \in P_{1,\alpha(\mathcal{P}_n)} \in \mathcal{P}_n$, we have $\alpha(\mathcal{P}_n) = \alpha(\mathcal{P}_n]m+1[)$ and

$$r'' \leq (\alpha(\mathcal{P}_n) - 2)/2 = \alpha(\mathcal{P}_n\langle m\rangle\langle m+d_m+1\rangle)/2 \leq \alpha(\mathcal{P}_n\langle m+1\rangle)/2. \qquad (10.19)$$

Observe that $\mathcal{P}_n]m+1[$ and $\mathcal{P}_n\langle m+1\rangle$ are isomorphic to $\mathcal{P}_{n'}(\{d'_i\}_{i\in\mathbb{N}})$ and $\mathcal{P}_{n''}(\{d''_i\}_{i\in\mathbb{N}})$ respectively, where $n' = n-1$, $n'' = \max\{m-1, n-d_{m+1}-2\}$, and mnd sequences $\{d'_i\}_{i\in\mathbb{N}}$ and $\{d''_i\}_{i\in\mathbb{N}}$ are given by

$$d'_i := \begin{cases} d_i = 0 & \text{if } i \in [m-1]; \\ d_m - 1 & \text{if } i = m; \\ d_{i+1} & \text{if } i \in \mathbb{N}\backslash[m], \end{cases} \quad \text{and } d''_i := \begin{cases} d_i = 0 & \text{if } i \in [m-1]; \\ d_{i+d_{m+1}+2} & \text{if } i \in \mathbb{N}\backslash[m-1]. \end{cases}$$

Therefore, we can apply the inductive hypothesis or Theorem 10.1.1 to each of $\mathcal{A}]m+1[$ and $\mathcal{A}\langle m+1\rangle$ to get

$$|\mathcal{A}]m+1[| \leq |\mathcal{P}_n^{(r)}]m+1[(1)|, \quad |\mathcal{A}\langle m+1\rangle| \leq |\mathcal{P}_n^{(r)}\langle m+1\rangle(1)|, \tag{10.20}$$

and hence $|\mathcal{A}| \leq |\mathcal{P}_n^{(r)}(1)|$. Since $|\mathcal{A}| = |\mathcal{A}^*|$ and $\mathcal{A}^* \in \operatorname{ex}(\mathcal{P}_n^{(r)})$, $\mathcal{A} \in \operatorname{ex}(\mathcal{P}_n^{(r)})$. So we actually have equalities in (10.20), and hence

$$\mathcal{A}]m+1[\in \operatorname{ex}(\mathcal{P}_n^{(r)}]m+1[), \quad \mathcal{A}\langle m+1\rangle \in \operatorname{ex}(\mathcal{P}_n^{(r)}\langle m+1\rangle). \tag{10.21}$$

ClaClaim 10.5.3 *Suppose $a \in [m]$ and $\mathcal{A}]m+1[= \mathcal{P}_n^{(r)}]m+1[(a)$. Then $a \in [m-1]$ and $\mathcal{A} = \mathcal{A} = \mathcal{P}_n^{(r)}(a)$.*

PrProof. Suppose $\mathcal{A}]m+1[= \mathcal{P}_n^{(r)}]m+1[(a)$, $a \in [m]$. Then, since $a \in P_{1,\alpha(\mathcal{P}_n)} \in \mathcal{P}_n$ and $r \leq \alpha(\mathcal{P}_n)/2$, for any $A \in \mathcal{P}_n^{(r)}\langle m+1\rangle]a[$ there exists $A' \in \mathcal{A}]m+1[$ such that $A \cap A' = \emptyset$. Since $\mathcal{A}]m+1[\cup\mathcal{A}\langle m+1\rangle$ is intersecting, it follows that $\mathcal{A}\langle m+1\rangle \subseteq \mathcal{P}_n^{(r)}\langle m+1\rangle(a)$. So $\mathcal{A} = \mathcal{P}_n^{(r)}(a)$ as $\mathcal{A} \in \operatorname{ex}(\mathcal{P}_n^{(r)})$. If $a = m$ then $\mathcal{A}\langle m+1\rangle = \emptyset$, and hence $|\mathcal{A}| = |\mathcal{P}_n^{(r)}]m+1[(a)| \leq |\mathcal{P}_n^{(r)}]m+1[(1)| < |\mathcal{P}_n^{(r)}(1)|$, contradicting $\mathcal{A} \in \operatorname{ex}(\mathcal{P}_n^{(r)})$. □

Claim 10.5.4 *Suppose $n \in P_{1,2r}$ and $m \leq 2r-2$. Let $j \in [m-1]$. Let $A \in \mathcal{P}_n^{(r)}]j[]m+1[$ such that $A \cap [m,n] \neq \emptyset$. Then there exists $A' \in \mathcal{P}_n^{(r)}(j)(m+1)$ such that $A \cap A' = \emptyset$.*

Proof. Let $\mathcal{Q} := \mathcal{P}_n \cap 2^{[m,n]}$. So $\mathcal{Q}$ is isomorphic to $\mathcal{P}_{n-(m-1)}(\{d_{i+m-1}\}_{i\in\mathbb{N}})$. Clearly, $n \in P_{1,2r}$ implies $n \in P_{m,\alpha(\mathcal{Q})}$ and $\alpha(\mathcal{P}_n) = 2r$. Since $m \leq 2r-2$ and $\alpha(\mathcal{Q}) = \alpha(\mathcal{P}_n) - (m-1) = 2r-(m-1)$, $\alpha(\mathcal{Q}) \geq 2r-(2r-3) = 3$. Let $B := A \cap [m,n] \in \mathcal{Q}]m+1[$. By Lemma 10.5.2, there exists $B' \in \mathcal{Q}(m+1)$ such that $B \cap B' = \emptyset$ and

$|B| + |B'| = \alpha(\mathcal{Q})$. Let $A' := B' \cup ([m-1]\backslash A)$. So $|A'| = |B'| + |[m-1]\backslash A| = (\alpha(\mathcal{Q}) - |B|) + ((m-1) - (r - |B|)) = \alpha(\mathcal{Q}) + m - 1 - r = r$. Since $j \notin A$, $j \in A'$. The truth of the claim is now clear. □

Note that $P_{1,2r} \in \mathcal{P}_n$ since $2r \leq \alpha(\mathcal{P}_n)$.

Consider first $n \notin P_{1,2r}$. Since $m \in P_{1,2r}$, $m+1 \notin P_{1,2r}$. So $P_{1,2r} \in \mathcal{P}_n]m+1[]n[$. By (10.21) and the inductive hypothesis, it follows that $\mathcal{A}]m+1[= \mathcal{P}_n^{(r)}]m+1[(j)$ for some $j \in [m]$ (note that if $d_m = 1$ then $d'_m = 0$ and $d'_{m+1} > 0$). By Claim 10.5.3, $\mathcal{A} = \mathcal{P}_n^{(r)}(j)$ and $j \in [m-1]$.

Now consider $n \in P_{1,2r}$ and $m \leq 2r-2$. Let $m =: p_1 < p_2 < ... < p_{\alpha(\mathcal{Q})} := n$ such that $P_{m,\alpha(\mathcal{Q})} = \{p_1, ..., p_{\alpha(\mathcal{Q})}\}$, where $\mathcal{Q}$ is as in the Proof of Claim 10.5.4 and hence $\alpha(\mathcal{Q}) \geq 3$; note that $P_{1,2r} = [m-1] \cup P_{m,\alpha(\mathcal{Q})}$. Let $m'' := (m+1) + d_{m+1} + 1$. Let $m'' =: q_1 < ... < q_{\alpha(\mathcal{Q})-2}$ such that $P_{m'',\alpha(\mathcal{Q})-2} = \{q_1, ..., q_{\alpha(\mathcal{Q})-2}\}$. Similarly to (10.13) and (10.14), we have

$$p_2 = m + d_m + 1 < (m+1) + d_{m+1} + 1 = q_1 < p_2 + d_{p_2} + 1 = p_3, \tag{10.22}$$

and if $\alpha(\mathcal{Q}) \geq 4$ then

$$p_i = p_{i-1} + d_{p_{i-1}} + 1 < q_{i-2} + d_{q_{i-2}} + 1 = q_{i-1} < p_2 + d_{p_2} + 1 = p_{i+1}, \tag{10.23}$$

$i = 3, ..., \alpha(\mathcal{Q}) - 1$. Let $P''_{1,2r''} := \{p''_1, ..., p''_{2r''}\} \in \mathcal{P}(\{d''_i\}_{i\in\mathbb{N}})$, where $p''_1 := 1$ and $p''_{l+1} := p''_l + d''_{p''_l} + 1$, $l = 1, ..., 2r''-1$. Clearly, $p''_j = j$, $j = 1, ..., m-1$, and $p''_l = q_{l-m+1} - d_{m+1} - 2$, $l = m, ..., 2r''$. Note that $2r'' = 2r - 2 = (m-1) + \alpha(\mathcal{Q}) - 2$ (as $P_{1,2r} = [m-1] \cup P_{m,\alpha(\mathcal{Q})}$). Now, by (10.22) and (10.23), $q_{\alpha(\mathcal{Q})-2} < p_{\alpha(\mathcal{Q})}$. So we have $p''_{2r''} = p''_{m+\alpha(\mathcal{Q})-3} = q_{\alpha(\mathcal{Q})-2} - d_{m+1} - 2 < n - d_{m+1} - 2 = n''$. By the inductive hypothesis, it follows that $\mathcal{A}\langle m+1\rangle = \mathcal{P}_n^{(r)}\langle m+1\rangle(j)$ for some $j \in [m-1]$. Therefore

$$\mathcal{A}(m+1) = \mathcal{P}_n^{(r)}(m+1)(j). \tag{10.24}$$

Let $\mathcal{H}_0 \cup \mathcal{H}_1 \cup \mathcal{H}_2$ be the partition of $\mathcal{A}]m+1[$ given by $\mathcal{H}_0 := \mathcal{A}]m+1[\cap\binom{[m-1]}{r}$, $\mathcal{H}_1 := \{A \in \mathcal{A}]m+1[: P_{m,\alpha(\mathcal{Q})} \subseteq A\}$, $\mathcal{H}_2 := \mathcal{A}]m+1[\backslash(\mathcal{H}_0 \cup \mathcal{H}_1)$. Define a partition

$\mathcal{I}_0 \cup \mathcal{I}_1 \cup \mathcal{I}_2$ of $\mathcal{P}_n^{(r)}]m+1[(j)$ similarly. Since $\mathcal{A}$ is intersecting, it follows by (10.24) and Claim 10.5.4 that

$$\mathcal{H}_1 \subseteq \mathcal{I}_1, \quad \mathcal{H}_2 \subseteq \mathcal{I}_2. \tag{10.25}$$

Suppose $m \leq r+1$. If $m < r+1$ then $\mathcal{H}_0 = \emptyset$, and if $m = r+1$ then $\mathcal{H}_0 = \{[m-1]\} \in \mathcal{P}_n^{(r)}]m+1[(j)$. Together with (10.24) and (10.25), this gives us $\mathcal{A} \subseteq \mathcal{P}_n^{(r)}(j)$. Since $\mathcal{A} \in \mathrm{ex}(\mathcal{P}_n^{(r)})$, $\mathcal{A} = \mathcal{P}_n^{(r)}(j)$.

Now suppose $m \geq r+2$. If $A \in \mathcal{H}_0 \backslash \mathcal{I}_0$ then $P_{1,2r} \backslash A \in \mathcal{I}_1 \backslash \mathcal{H}_1$; hence $|\mathcal{H}_0| + |\mathcal{H}_1| \leq |\mathcal{I}_0| + |\mathcal{I}_1|$ as $\mathcal{H}_1 \subseteq \mathcal{I}_1$ (by (10.25)). By (10.20), (10.21) and (10.25), it follows that $\mathcal{H}_2 = \mathcal{I}_2$ and $|\mathcal{H}_0| + |\mathcal{H}_1| = |\mathcal{I}_0| + |\mathcal{I}_1|$. We now prove that $\mathcal{A} = (\mathcal{P}_n^{(r)}(j) \backslash \{P_{1,2r} \backslash A \colon A \in \mathcal{H}_0\}) \cup \mathcal{H}_0$ by showing that for any $A \in \binom{[m-1]}{r}$, $P_{1,2r} \backslash A$ is the unique set in $\mathcal{P}_n^{(r)}$ that does not intersect A. Indeed, let $A' \in \mathcal{P}_n^{(r)}$ such that $A \cap A' = \emptyset$. Then $A' = A_1' \cup A_2'$ for some $A_1' \subseteq [m-1] \backslash A$ and $A_2' \in \mathcal{Q}$. Suppose $A_1' \neq [m-1] \backslash A$ or $|A_2'| < \alpha(\mathcal{Q})$ $(= 2r - m + 1)$; then $|A'| < (m-1-r) + (2r-m+1) = r$, a contradiction. So $A_1' = [m-1] \backslash A$ and $|A_2'| = \alpha(\mathcal{Q})$. Clearly, since $n \in P_{m,\alpha(\mathcal{Q})} = P_{1,2r} \backslash [m-1]$, $P_{m,\alpha(\mathcal{Q})}$ is the only set in $\mathcal{Q}$ of size $\alpha(\mathcal{Q})$. So $A_2' = P_{m,\alpha(\mathcal{Q})}$, and hence $A' = P_{1,2r} \backslash A$.

We conclude the proof of the necessary conditions in (ii) and (iii) by showing that $\mathcal{A}^* = \mathcal{A}$. Suppose $\mathcal{A}^* \neq \mathcal{A}$ instead. Then there exists $A^* \in \mathcal{A}^* \backslash \mathcal{A}$ such that $A := \delta_{m,m+1}(A^*) \in \mathcal{A} \backslash \mathcal{A}^*$. Now we have shown that for some $j \in [m-1]$ and $\mathcal{H}_0 \subseteq \binom{[m-1] \backslash \{j\}}{r}$, $\mathcal{A} = (\mathcal{P}_n^{(r)}(j) \backslash \{P_{1,2r} \backslash A \colon A \in \mathcal{H}_0\}) \cup \mathcal{H}_0$ (where $\mathcal{H}_0 = \emptyset$ if $n \notin P_{1,2r}$ or $m \leq r+1$). Thus, since $m \in A$, $A \in \mathcal{P}_n^{(r)}(j)(m)$. Therefore $A^* \in \mathcal{P}_n^{(r)}(j)(m+1) \backslash \mathcal{A}$, but this is a contradiction because, since $m+1 \notin P_{1,2r}$, $\mathcal{P}_n^{(r)}(j)(m+1) \subset \mathcal{A}$.

It remains to prove the sufficiency conditions in (ii) and (iii). We have shown that for any intersecting family $\mathcal{A} \subset \mathcal{P}_n^{(r)}$, $|\mathcal{A}| \leq |\mathcal{P}_n^{(r)}(1)|$. This already proves the sufficiency condition in (iii) because for any $j \in [2, m-1]$, $\mathcal{P}_n^{(r)}(j)$ is isomorphic to $\mathcal{P}_n^{(r)}(1)$. Therefore the sufficiency condition in (ii) follows from the already established fact that if $n \in P_{1,2r}$, $r+2 \leq m \leq 2r-2$ and $A \in \binom{[m-1]}{r}$ then $P_{1,r} \backslash A$ is the unique set in $\mathcal{P}_n^{(r)}$ that does not intersect A. $\square$

Chapter 11

Graphs with the Erdős-Ko-Rado property

11.1 A graph-theoretical formulation and result

A *graph* G is a pair $(V(G), E(G))$ such that $E(G) \subseteq \binom{V(G)}{2}$. $V(G)$ and $E(G)$ are called the *vertex set* and the *edge set* of G respectively. If $v, w \in V(G)$ and $\{v, w\} \in E(G)$ then v and w are said to be *adjacent* and edge vw is said to be *incident* to v and w. If G has no edges incident to a vertex x then x is said to be a *singleton.*

In the following, we represent an edge $\{v, w\}$ of a graph by the abbreviation vw.

A set $I \subseteq V(G)$ is said to be an *independent set* if the vertices in I are pair-wise non-adjacent. We denote the family of all independent sets of vertices of G by $\mathcal{I}_G$. We shall abbreviate $\alpha(\mathcal{I}_G)$ and $\mu(\mathcal{I}_G)$ to $\alpha(G)$ and $\mu(G)$ respectively; so $\alpha(G)$ denotes the *independence number* $\max\{|I| : I \in \mathcal{I}_G\}$ and $\mu(G)$ denotes the *minimum* cardinality of a *maximal* independent set of vertices of G.

A graph G is said to be *connected* if for any $\{v, w\} \in \binom{V(G)}{2} \backslash E(G)$ there exist $v_1, ..., v_p \in V(G)$ such that $vv_1, v_pw \in E(G)$ and if $p > 1$ then $v_iv_{i+1} \in E(G)$ for $i = 1, ..., p-1$. If G is a disjoint union of connected graphs $G_1, ..., G_q$ then G_j $(j \in [q])$ is said to be a *component* of G.

It is interesting that many EKR-type results can be expressed in terms of the EKR or strict EKR property of $\mathcal{I}_G^{(r)}$ for some graph G and $r \in X \subseteq [\alpha(G)]$. Before coming to the crux of this chapter, we give a brief review of such results, recalling certain well-known classes of graphs and also defining new ones as we go along.

Let E_n denote the *empty graph* on n vertices, i.e. the graph consisting of n singletons. By the EKR Theorem and the Hilton-Milner Theorem, we have the following.

Theorem 11.1.1 *Let $r \leq n/2$. Then $\mathcal{I}_{E_n}^{(r)}$ is EKR, and strictly so if $r < n/2$.*

Theorem 1.5.2 for permutations and partial permutations can also be phrased in a graph-theoretical form as follows.

Theorem 11.1.2 *Let G be the graph defined by $V(G) := [n] \times [n]$ and $E(G) := \{\{(i,j), (i',j')\} \in \binom{V(G)}{2} : i = i'$ or $j = j'\}$. Then $\mathcal{I}_G^{(r)}$ is strictly EKR for all $r \in [n]$.*

Note that the case $r = n$ is actually Theorem 1.5.1.

Suppose G is a graph whose vertex set has a partition $V(G) = V_1 \cup ... \cup V_p$ into *partite sets* such that any two vertices are adjacent iff they belong to distinct partite sets. Such a graph is said to be a *complete multipartite graph* of *order p*. If $|V_1| = ... = |V_p| = 1$ then G is called a *complete graph*, and it is denoted by K_p.

Theorem 1.4.3 can be rephrased as follows.

Theorem 11.1.3 *Let $r \leq n$ and $k \geq 2$. Let G be a disjoint union of n copies of K_k. Then $\mathcal{I}_G^{(r)}$ is EKR, and strictly so unless $r = n$ and $k = 2$.*

Similarly, Theorem 8.1.2 can be rephrased as follows.

Theorem 11.1.4 *If G is a disjoint union of complete graphs each of order at least 2 then $\mathcal{I}_G^{(r)}$ is EKR for all $r \leq n$.*

In [41], parts (i) and (ii) of Theorem 8.1.4 were actually phrased in the graph-theoretical form and hence along the following lines.

Theorem 11.1.5 *Let G be a disjoint union of two complete multipartite graphs. Let $r \leq \mu(G)/2$. Then $\mathcal{I}_G^{(r)}$ is EKR, and strictly so if $r < \mu(G)/2$.*

Similarly, Theorem 8.1.5 can be rephrased as follows.

Theorem 11.1.6 *Let G be a disjoint union of k complete multipartite graphs and a non-empty set V_0 of singletons. Let $1 \leq r \leq \mu(G)/2$. Then:*
(i) $\mathcal{I}_G^{(r)}$ is EKR;
(ii) $\mathcal{I}_G^{(r)}$ fails to be strictly EKR iff $2r = \mu(G) = \alpha(G)$, $3 \leq |V_0| \leq r$, $k = 1$.

We now introduce the first of two definitions that are crucial for achieving the target of this chapter, which is revealed towards the end of this section.

Definition 11.1.7 *For a monotonic non-decreasing (mnd) sequence $\{d_i\}_{i\in\mathbb{N}}$ of non-negative integers, let $M := M(\{d_i\}_{i\in\mathbb{N}})$ be a graph such that $V(M) = \mathbb{N}$ and for $a, b \in V(M)$ with $a < b$, $ab \in E(M)$ iff $b \leq a + d_a$. Let $M_n := M_n(\{d_i\}_{i\in\mathbb{N}})$ be the sub-graph induced from M by the subset $[n]$ of $V(M)$. We refer to M_n as an mnd graph.*

Suppose $M_n = M_n(\{d_i = d\}_{i\in\mathbb{N}})$, $d \in \mathbb{N}$, and G is a copy of M_n. Then G is called a *d'th power of a path*, and if $d = 1$ then G is also simply called a *path.*

Let $\mathcal{P}_n(\{d_i\}_{i\in\mathbb{N}})$ be as defined in Chapter 10, and let $\mathcal{P}_n := \mathcal{P}_n(\{d_i\}_{i\in\mathbb{N}})$. Note that $\mathcal{P}_n^{(r)} = \mathcal{I}_{M_n}^{(r)}$. Thus, by Theorems 10.2.1 and 10.2.3, we have the following.

Theorem 11.1.8 *(i) If $d_1 > 0$ and $r \leq \alpha(\mathcal{I}_{M_n})$ then $\mathcal{I}_{M_n}^{(r)}$ is EKR, and strictly so unless $\mathcal{P}_n^{(r)}$ is type I (see Chapter 10 for definition).*
(ii) If $d_1 = 0$ and $r \leq \alpha(\mathcal{I}_{M_n})/2$ then $\mathcal{I}_{M_n}^{(r)}$ is EKR, and strictly so if $r < \alpha(\mathcal{I}_{M_n})/2$.

Note that Theorem 10.1.2 gives the "non-strict part" in (i) above for the special case when M_n is a power of a path; see [40].

We now come to our second important definition. First of all, a *directed graph* (or *digraph*) D is a pair $(V(D), E(D))$ such that $E(D) \subseteq V(D) \times V(D)$.

Definition 11.1.9 *(i) For $n > 2$, $1 \leq k < n-1$, $0 \leq q < n$, let ${}_qD_n^{k,k+1}$ be the digraph $(\{v_i \colon i \in [n]\}, E)$ such that if $1 \leq i \leq q$ then $(v_i, v_{i+s\,(\mathrm{mod}\,n)}) \in E$ iff $s \in [k+1]$, and if $q+1 \leq i \leq n$ then $(v_i, v_{i+s\,(\mathrm{mod}\,n)}) \in E$ iff $s \in [k]$.*
(ii) Let ${}_qC_n^{k,k+1}$ be the graph such that $vw \in E({}_qC_n^{k,k+1})$ iff $(v,w) \in E({}_qD_n^{k,k+1})$ or $(w,v) \in E({}_qD_n^{k,k+1})$. ${}_0C_n^{k,k+1}$ is called a k'th power of a cycle (or simply a cycle if $k = 1$) and also denoted by C_n^k. If $q > 0$ then we call ${}_qC_n^{k,k+1}$ a modified k'th power of a cycle.

A nice EKR-type result of Talbot [58] for *separated sets* can be stated as follows.

Theorem 11.1.10 (Talbot [58]) *Let $r \leq \alpha(C_n^k)$. Then $\mathcal{I}_{C_n^k}^{(r)}$ is EKR, and strictly so unless $k = 1$ and $n = 2r + 2$.*

The clique number $\mathrm{cl}(G)$ of a graph G is the size of a largest complete sub-graph of G. Hilton and Spencer proved the following.

Theorem 11.1.11 (Hilton and Spencer [39]) *Let G be a disjoint union of graphs $P, C_1, ..., C_n$ such that $\mathrm{cl}(P) \leq \min\{\mathrm{cl}(C_i) \colon i \in [n]\}$, where P is a power of a path and C_i ($i \in [n]$) is a power of a cycle. Then $\mathcal{I}_G^{(r)}$ is EKR for all $r \leq \alpha(G)$.*

As we explain later, the work in this chapter is inspired by the following result.

Theorem 11.1.12 (Holroyd, Spencer, Talbot [40]) *Let G be a disjoint union of n components consisting of complete graphs, paths, cycles, and at least one singleton. Then $\mathcal{I}_G^{(r)}$ is EKR for all $r \leq n/2$.*

Note that, unlike all the preceding theorems, this result does not live up to Conjecture 1.3.4 because (for any graph G) $\mu(G)$ is at least as large as the number of components of G, and there is no bound as to how much larger it can be.

The idea of the graph-theoretical formulation we have been discussing emerged in [41], in which Holroyd and Talbot in fact initiated the study of the general EKR problem for independent sets of graphs and made Conjecture 1.3.4. By proving Theorem 11.1.5, they provided an example of a graph G such that G obeys the conjecture and, as we demonstrate in a stronger fashion below, $\mathcal{I}_G^{(r)}$ may not be EKR if $\mu(G)/2 < r < \alpha(G)$ (it is easy to see that for such a graph G, $\mathcal{I}_G^{(r)}$ is EKR for $r = \alpha(G)$). They gave various other examples of graphs H and values $r > \mu(H)/2$ for which $\mathcal{I}_H^{(r)}$ is *not* EKR, and one particularly interesting example of this kind has $r = \alpha(H)$. The idea behind Conjecture 1.3.4 is that if I is any maximal independent set of a graph G with $\mu(G) \geq 2r$, then, since $|I| \geq \mu(G)$, it holds by the EKR Theorem that $\binom{I}{r}$ is EKR, and strictly so if $\mu(G) > 2r$.

We now show that there are graphs G such that $\mu(G) < \alpha(G)$ and $\mathcal{I}_G^{(r)}$ is not EKR for all $\mu(G)/2 < r < \alpha(G)$. Indeed, let G be the graph consisting of a 3-set V_0 of singletons and a complete bipartite graph with partite sets V_1 and V_2 of size 5 and 4 respectively. So $7 = \mu(G) < \alpha(G) = 8$. For $r \in [\alpha(G)]$, let $\mathcal{J}_r$ be a star of $\mathcal{I}_G^{(r)}$ with centre $x \in V_0$, and let $\mathcal{A}_r := (\mathcal{J}_r \backslash \{A \in \mathcal{J}_r \colon A \cap V_0 = \{x\}\}) \cup \{A \in \mathcal{I}_G^{(r)} \colon A \cap V_0 = V_0 \backslash \{x\}\}$. Clearly $\mathcal{J}_r$ is a star of $\mathcal{I}_G^{(r)}$ of largest size. For all $\mu(G)/2 < r < \alpha(G)$, we have $|\mathcal{A}_r| > |\mathcal{J}_r|$. This proves what we set out to show.

Conjecture 1.3.4 seems very hard to prove or disprove. However, restricting the problem to some classes of graphs with singletons makes it tractable. Theorem 11.1.1 and the example that we gave above demonstrate the fact that when an arbitrary number of singletons are allowed in a graph G, $\mathcal{I}_G^{(r)}$ may not be EKR for $r > \mu(G)/2$.

We now come to the objective of this chapter, which is to provide an improvement of the techniques in [40] that enables us to confirm the conjecture for the class of graphs in Theorem 11.1.12 and even larger classes. The key idea that leads us to this improvement is to consider a suitable larger class of graphs, namely to allow copies of mnd graphs and modified powers of cycles in the disjoint union specified in Theorem 11.1.12. Since the proof goes by induction, we will need to perform certain deletions on the original graph. When a deletion is performed on a power of a cycle, which is the most difficult component to treat, we obtain a modified power of a cycle (mpc) or a power of a path, and if a deletion is performed on an mpc then we obtain an mnd graph or another mpc. So the idea is that every time a deletion is performed, the resulting graph is in the admissible class. Although not necessary for our main aim, we show that our method allows us to include *trees* (connected graphs that contain no cycles as sub-graphs) as components; the scope is to illustrate the fact that the method we employ works for many classes of graphs.

Theorem 11.1.13 *Conjecture 1.3.4 is true if G is a disjoint union of complete multipartite graphs, copies of mnd graphs, powers of cycles, modified powers of cycles, trees, and at least one singleton.*

As from the next section, we employ the notation in Section 2.1.

11.2 The compression operation for independent sets of graphs

For $v \in V(G)$, let $N_G(v)$ be the set of *neighbours* of v in G, i.e.

$$N_G(v) := \{w \in V(G)\backslash\{v\} : vw \in E(G)\}.$$

As in [40], we use $G - v$ to denote the graph obtained from G by deleting $v \in V(G)$ (and hence edges incident to v), and $G \downarrow v$ to denote the graph obtained by deleting also all vertices in $N_G(v)$. Note that

$$\mathcal{I}_G\langle v\rangle = \mathcal{I}_{G\downarrow v}, \quad \mathcal{I}_G]v[= \mathcal{I}_{G-v}.$$

For $u, v \in V(G)$, let $\Delta_{u,v} : \mathcal{I}_G \to \mathcal{I}_G$ be defined as in Section 2.2.

Lemma 11.2.1 *Let $uv \in E(G)$. Let $\mathcal{A}^* \subset \mathcal{I}_G^{(r)}$ be an intersecting family, and let $\mathcal{A} := \Delta_{u,v}(\mathcal{A}^*)$.*
(i) If $N_G(u) \subseteq N_G(v) \cup \{v\}$ then $\mathcal{A}\langle v\rangle \cup \mathcal{A}]v[$ is intersecting.
(ii) If $|N_G(u)\backslash(N_G(v) \cup \{v\})| \leq 1$ then $\mathcal{A}\langle v\rangle$ and $\mathcal{A}]v[$ are intersecting.

Proof: Since $uv \in E(G)$, $\mathcal{I}_G[\{u, v\}] = \emptyset$.

Suppose $N_G(u) \subseteq N_G(v) \cup \{v\}$. Then, for any independent set $I \in \mathcal{I}_G]u[(v)$, we have $N_G(u) \cap (I\backslash\{v\}) \subseteq N_G(v) \cap (I\backslash\{v\}) = \emptyset$, and hence $(I\backslash\{v\}) \cup \{u\} \in \mathcal{I}_G$. So $\mathcal{I}_G$ is (u, v)-compressed, and hence (i) follows by Proposition 2.2.1(ii).

Now suppose $|N_G(u)\backslash(N_G(v) \cup \{v\})| \leq 1$. Then $N_G(u)\backslash(N_G(v) \cup \{v\}) = \{w\}$ for some $w \in V(G)\backslash\{v\}$. So $N_{G-w}(u)\backslash(N_{G-w}(v) \cup \{v\}) = \emptyset$, and hence, similarly to the above, $(I\backslash\{v\}) \cup \{u\} \in \mathcal{I}_{G-w}$ for any independent set $I \in \mathcal{I}_{G-w}]u[(v)$. Thus, since $\mathcal{I}_G]w[= \mathcal{I}_{G-w}$, $\mathcal{I}_G]w[$ is (u, v)-compressed. So (ii) follows by Proposition 2.2.1(i),(iv).

11.3 Graph deletion lemmas

The following property of graphs will have a very important role in our improvement of Theorem 11.1.12.

Lemma 11.3.1 *Let G be a graph, and let $v \in V(G)$. Then*

$$\min\{\mu(G \downarrow v), \mu(G - v)\} \geq \mu(G) - 1.$$

Proof: The inequality $\mu(G \downarrow v) \geq \mu(G) - 1$ follows by Lemma 4.4.1(i) since $\mathcal{I}_G\langle v\rangle = \mathcal{I}_{G\downarrow v}$. The inequality $\mu(G-v) \geq \mu(G)-1$ follows by Lemma 4.4.1(ii) since $\mathcal{I}_G]v[= \mathcal{I}_{G-v}$ and $\mathcal{I}_G$ is a hereditary family. □

Corollary 11.3.2 *Let $r \leq \mu(G)/2$, and let $v, w \in V(G)$. Then:*
(i) $r - 1 < \mu(G \downarrow v)/2$;
(ii) $r - 1 \leq \mu(G - v \downarrow w)/2$.

Proof. By $r \leq \mu(G)/2$ and Proposition 11.3.1, we have
(i) $r - 1 < (\mu(G) - 1)/2 \leq \mu(G \downarrow v)/2$ and
(ii) $r - 1 \leq (\mu(G) - 2)/2 \leq (\mu(G - v) - 1)/2 \leq \mu(G - v \downarrow w)/2$. □

One of the various properties of non-singleton trees (i.e. trees containing at least two vertices and hence at least one edge) is that they contain vertices which have only one neighbour. To see this, consider picking any vertex in a graph G and then traversing vertices without visiting any vertex twice until no new vertex can be visited; if G is a non-singleton tree then, since a tree contains no cycles, this procedure stops when a vertex with one neighbour has been visited.

Lemma 11.3.3 *Let T be a tree with $|V(T)| \geq 2$, and let $w \in V(T)$ such that $N_T(w)$ consists only of one vertex v. Then*

$$\mu(T - v) \geq \mu(T).$$

Proof. Let Z be a maximal independent set of $T - v$. Since w is a singleton of $T - v$, we must have $w \in Z$. So Z is also a maximal independent set of T because $vw \in E(T)$. Hence result. □

Lemma 11.3.4 *Let $M_n(\{d_i\}_{i\in\mathbb{N}})$ be as in Definition 11.1.7, and let $M_n := M_n(\{d_i\}_{i\in\mathbb{N}})$. If $d_1 > 0$ then*

$$\mu(M_n - 2) \geq \mu(M_n).$$

Proof: Let Z be a maximal independent set of $M_n - 2$. Then $1 \in Z$ or $1z \in E(M_n - 2)$ for some $z \in Z$. Suppose $1 \in Z$. Since $d_1 > 0$, we have $12 \in E(M_n)$, and hence Z is a maximal independent set of M_n. Now suppose $1z \in E(M_n - 2)$ for some $z \in Z$. Then, by definition of M_n, $z \leq 1 + d_1 < 2 + d_2$, and hence $2z \in E(M_n)$. Thus, Z is again a maximal independent set of M_n. Therefore $\mu(M_n - 2) \geq \mu(M_n)$. □

Lemma 11.3.5 *Let ${}_qC_n^{k,k+1}$ be as in Definition 11.1.9. If $q > 0$ then*

$$\mu({}_qC_n^{k,k+1} - v_{k+2}) \geq \mu({}_qC_n^{k,k+1}).$$

Proof. Let $C := {}_qC_n^{k,k+1}$ and $V := V(C)$. If $N_C(v_1) = V\backslash\{v_1\}$ then trivially $\mu(C - v_{k+2}) = \mu({}_qC_n^{k,k+1}) = 1$. So suppose $N_C(v_1) \neq V\backslash\{v_1\}$. Let Z be a maximal independent set of $C - v_{k+2}$, and let $s := \min\{i\colon v_i \in Z\}$, $t := \max\{i\colon v_i \in Z\}$. If $s \leq k+1$ then $v_s v_{k+2} \in E(C)$, and hence Z is also maximal in C. Suppose $s \geq k+3$. Suppose also that $v_{k+2}v_s \notin E(C)$. Then $v_{k+1}v_s \notin E(C - v_{k+2})$ and, since $q < n$ (by definition of C) and $s \leq t \leq n$, $v_t v_{k+1} \notin E(C - v_{k+2})$. So $Z \cup \{v_{k+1}\} \in \mathcal{I}_{C-v_{k+2}}$, but this contradicts the maximality of Z. So $v_{k+2}v_s \in E(C)$, and hence Z is also maximal in C. Therefore $\mu(C - v_{k+2}) \geq \mu(C)$. □

Lemma 11.3.6 *Let C_n^k be as in Definition 11.1.9. If $n \geq 2k+2$ then*

$$\mu(C_n^k - v_{k+1} - v_{2k+2}) \geq \mu(C_n^k).$$

Proof. Let Z be a maximal independent set of $C_n^k - v_{k+1} - v_{2k+2}$. If Z contains $z \in \{v_{k+2}, ..., v_{2k+1}\}$ then $zv_{k+1}, zv_{2k+2} \in E(C_n^k)$, and hence Z is also maximal in C_n^k. Now consider $Z \cap \{v_{k+2}, ..., v_{2k+1}\} = \emptyset$. Thus, if $zv_{k+1}, zv_{2k+2} \notin E(C_n^k)$ for all $z \in Z$ then $Z \cup \{v\}$ is an independent set of $C - v_{k+1} - v_{2k+2}$ for all $v \in \{v_{k+2}, ..., v_{2k+1}\}$, but this is a contradiction. We therefore have $zw \in E(C_n^k)$ for some $z \in Z$ and $w \in \{v_{k+1}, v_{2k+1}\}$. Suppose $w = v_{k+1}$ and $Z \cup \{v_{2k+2}\}$ is an independent set of C_n^k.

Then $zv_{2k+1} \notin E(C_n^k - v_{k+1} - v_{2k+2})$, and hence $Z \cup \{v_{2k+1}\}$ is an independent set of $C_n^k - v_{k+1} - v_{2k+2}$, a contradiction. By symmetry, we can neither have both $w = v_{2k+2}$ and $Z \cup \{v_{k+1}\}$ an independent set of C_n^k. Therefore there exist $z_1, z_2 \in Z$ such that $z_1v_{k+1}, z_2v_{2k+2} \in E(C_n^k)$, and hence Z is maximal in C_n^k. So $\mu(C_n^k - v_{k+1} - v_{2k+2}) \geq \mu(C_n^k)$. □

11.4 Proof of result

We shall now use the lower bounds obtained for $\mu({}_qC_n^{k,k+1} - v_{k+2})$, $\mu(C_n^k - v_{k+1} - v_{2k+2})$, and $\mu(P_n^{k\uparrow} - v_2)$ in terms of $\mu({}_qC_n^{k,k+1})$, $\mu(C_n^k)$, and $\mu(P_n^{k\uparrow})$ respectively to prove Theorem 11.1.13.

Lemma 11.4.1 *Let G be a graph containing an edge vw and a singleton x. Suppose $2 \leq r \leq \mu(G)$. Then $|\mathcal{I}_G^{(r)}(v)| < |\mathcal{I}_G^{(r)}(x)|$.*

Proof. Since x is a singleton, $A\backslash\{y\} \cup \{x\} \in \mathcal{I}_G^{(r)}$ for any $A \in \mathcal{I}_G^{(r)}]x[$ and $y \in A$. Setting $\mathcal{J} := \{A\backslash\{v\} \cup \{x\} \colon A \in \mathcal{I}_G^{(r)}(v)]x[\}$, it follows that $\mathcal{J} \subseteq \mathcal{I}_G^{(r)}(x)]v[$. Given that $vw \in E(G)$, we have $\mathcal{I}_G(v)(w) = \emptyset$, and hence actually $\mathcal{J} \subseteq \mathcal{I}_G^{(r)}(x)]v[\backslash\mathcal{I}_G^{(r)}(x)(w)$; also, $\mathcal{I}_G^{(r)}(x)(w) \subseteq \mathcal{I}_G^{(r)}(x)]v[$, and hence $|\mathcal{J}| \leq |\mathcal{I}_G^{(r)}(x)]v[| - |\mathcal{I}_G^{(r)}(x)(w)|$. We therefore have

$$\begin{aligned}
|\mathcal{I}_G^{(r)}(v)| &= |\mathcal{I}_G^{(r)}(v)(x)| + |\mathcal{I}_G^{(r)}(v)]x[| = |\mathcal{I}_G^{(r)}(v)(x)| + |\mathcal{J}| \\
&\leq |\mathcal{I}_G^{(r)}(x)(v)| + |\mathcal{I}_G^{(r)}(x)]v[| - |\mathcal{I}_G^{(r)}(x)(w)| \\
&= |\mathcal{I}_G^{(r)}(x)| - |\mathcal{I}_G^{(r)}(x)(w)|.
\end{aligned}$$

Now, since $\{x, w\} \in \mathcal{I}_G^{(2)}$ and $2 \leq r \leq \mu(G)$, there exists $I \in \mathcal{I}_G^{(r)}$ such that $\{x, w\} \subset I$, i.e. $\mathcal{I}_G^{(r)}(x)(w) \neq \emptyset$. Hence result. □

Lemma 11.4.2 *Let G be a graph, and let $r \leq \mu(G)/2$. Let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{I}_G^{(r)}$ such that $\mathcal{A}\langle v\rangle = \mathcal{I}_{G\downarrow v}^{(r-1)}(y) \neq \emptyset$ for some $y \in V(G \downarrow v)$. Then $\mathcal{A} \subseteq \mathcal{I}_G^{(r)}(y)$.*

Proof. Suppose there exists $A \in \mathcal{A}]v[$ such that $y \notin A$. We are given that $\mathcal{I}_{G\downarrow v}^{(r-1)}(y) \neq \emptyset$, and so $\mathcal{I}_G(v)(y) \neq \emptyset$. Therefore there exists a maximal independent set Y of G

such that $v, y \in Y$. Given that $2r \leq \mu(G)$, we have $2r \leq |Y|$. Since $y, v \in Y\backslash A$, it follows that $\binom{Y\backslash A}{r}[\{y,v\}] \neq \emptyset$. Let $A' \in \binom{Y\backslash A}{r}[\{y,v\}]$. So $A'\backslash\{v\} \in \mathcal{I}_{G\downarrow v}^{(r-1)}(y)$, and hence $A' \in \mathcal{A}(v)$. But $A \cap A' = \emptyset$, which contradicts $\mathcal{A}$ intersecting. Hence result. $\square$

Proof of Theorem 11.1.13. By induction on $|E(G)|$. If $|E(G)| = 0$ then the result is given by Theorem 11.1.1, so we assume that $|E(G)| > 0$. This means that G contains a non-singleton component. If G consists solely of complete multipartite graphs and singletons then the result is given by Theorem 11.1.6. We now consider the case when G contains a component G_1 that is neither a singleton nor a complete mulitpartite graph.

Let G_2 be the graph obtained by removing G_1 from G. Note that

$$\mu(G) = \mu(G_1) + \mu(G_2). \tag{11.1}$$

By our definition of component, G_1 is connected, and hence G_1 contains no singletons. Thus, since G contains at least one singleton, G_2 contains some singleton x.

Let $r \leq \mu(G)/2$, and let $\mathcal{A}^* \in \mathrm{ex}(\mathcal{I}_G^{(r)})$. Let $\mathcal{J} := \mathcal{I}_G^{(r)}(x)$. So $|\mathcal{J}| \leq |\mathcal{A}^*|$. By Lemma 11.4.1, $\mathcal{J}$ is a largest star of $\mathcal{I}_G^{(r)}$, and for any $v \in V(G_1)$, $\mathcal{J}\langle v\rangle$ and $\mathcal{J}]v[$ are largest stars of $\mathcal{I}_{G\downarrow v}^{(r-1)}$ of $\mathcal{I}_{G-v}^{(r)}$ respectively.

Now G_1 is a tree or a copy of an mnd graph or a modified power of a cycle or a power of a cycle. We consider each of these four possibilities separately and in the order we have listed them. We will actually show that in each of the first three cases, $\mathcal{I}_G^{(r)}$ is in fact strictly EKR even if $r = \mu(G)/2$.

Case I: G_1 is a tree T, $|V(T)| \geq 2$. So there exists $u \in V(G_1)$ such that $N_{G_1}(u)$ consists solely of one vertex v (see the preceding section). Let $\mathcal{A} := \Delta_{u,v}(\mathcal{A}^*)$. Since $N_G(u) = N_{G_1}(u) = \{v\}$, it follows by Lemma 11.2.1(i) that $\mathcal{A}\langle v\rangle \cup \mathcal{A}]v[$ is intersecting.

Since G_1 contains no cycles, $G_1 - v$ and $G_1 \downarrow v$ contain no cycles, and hence $G_1 - v$ and $G_1 \downarrow v$ are disjoint unions of trees and singletons. So $G - v$ and $G \downarrow v$ belong to the class of graphs specified in the theorem.

By Corollary 11.3.2(i), $r-1 < \mu(G \downarrow v)/2$. By Lemma 11.3.3, $\mu(G_1 - v) \geq \mu(G_1)$; so $\mu(G-v) = \mu(G_1 - v) + \mu(G_2) \geq \mu(G_1) + \mu(G_2) = \mu(G) \geq 2r$.

Therefore, since $\mathcal{A}\langle v\rangle \subset \mathcal{I}_{G\downarrow v}^{(r-1)}$ and $\mathcal{A}]v[\subset \mathcal{I}_{G-v}^{(r)}$, the inductive hypothesis gives us $|\mathcal{A}\langle v\rangle| \leq |\mathcal{J}\langle v\rangle|$ and $|\mathcal{A}]v[| \leq |\mathcal{J}]v[|$. So $|\mathcal{A}| \leq |\mathcal{J}|$. Since $|\mathcal{A}^*| = |\mathcal{A}|$ and $\mathcal{A}^* \in \mathrm{ex}(\mathcal{I}_G^{(r)})$, $|\mathcal{A}\langle v\rangle| = |\mathcal{J}\langle v\rangle|$ and $|\mathcal{A}]v[| = |\mathcal{J}]v[|$. Since $r-1 < \mu(G \downarrow v)/2$, it follows by the inductive hypothesis that $\mathcal{A}\langle v\rangle = \mathcal{I}_{G\downarrow v}(y)$ for some $y \in V(G \downarrow v)$. Thus, by Lemma 11.4.2, $\mathcal{A} \subseteq \mathcal{I}_G^{(r)}(y)$. If y is not a singleton of G then Lemma 11.4.1 gives us $|\mathcal{I}_G^{(r)}(y)| < |\mathcal{J}|$, but this leads to the contradiction that $|\mathcal{A}^*| < |\mathcal{J}|$. So y is a singleton of G, and hence $\mathcal{A}^* \subseteq \mathcal{I}_G^{(r)}(y)$ (as $\mathcal{A} \subseteq \mathcal{I}_G^{(r)}(y)$). Therefore $\mathcal{I}_G^{(r)}$ is strictly EKR.

Case II: G_1 is a copy of an mnd graph $M_n := M_n(\{d_i\}_{i\in\mathbb{N}})$. We may assume that $G_1 = M_n$. Since G_1 contains no singletons, $n \geq 2$ and $d_1 \geq 1$. Let $v := 2$ and $u := 1$, and let $\mathcal{A} := \Delta_{u,v}(\mathcal{A}^*)$. By definition of M_n and $d_1 \geq 1$, $N_{G_1}(u) \subset N_{G_1}(v) \cup \{v\}$. Since $N_G(u) = N_{G_1}(v)$, it follows by Lemma 11.2.1(i) that $\mathcal{A}\langle v\rangle \cup \mathcal{A}]v[$ is intersecting.

Clearly, $G_1 - v$ is a copy of $M_{n-1}(\{d_i'\}_{i\in\mathbb{N}})$, where $d_1' = d_1 - 1$ and $d_i' = d_{i+1}$ for all $i \geq 2$. Also, if $n \leq 2 + d_2$ then $G_1 \downarrow v = (\emptyset, \emptyset)$, and if $n > 2 + d_2$ then $G_1 \downarrow v$ is a copy of $M_{n-2-d_2}(\{d_i''\}_{i\in\mathbb{N}})$, where $d_i'' = d_{i+2+d_2}$ for all $i \geq 1$. So $G - v$ and $G \downarrow v$ belong to the class of graphs specified in the theorem.

The rest follows as in the preceding case, except that we get $\mu(G_1 - v) \geq \mu(G_1)$ by Lemma 11.3.4.

Case III: G_1 is a modified k'th power of a cycle ${}_qC_n^{k,k+1}$. So $q > 0$. Let v_i, $i = 1, ..., n$, be as in Definition 11.1.9. Let $u := v_{k+1}$ and $v := v_{k+2}$, and let $\mathcal{A} := \Delta_{u,v}(\mathcal{A}^*)$. Since $q < n$ (by definition), $(v_s, u) \notin E({}_qD_n^{k,k+1})$ for all $s \geq k+1$. Since $q > 0$, $(v_s, v) \in E({}_qD_n^{k,k+1})$ for all $s \leq k+1$. Therefore $N_G(u) = N_{G_1}(u) \subseteq N_{G_1}(v) \cup \{v\}$, and hence, by Lemma 11.2.1(i), $\mathcal{A}\langle v\rangle \cup \mathcal{A}]v[$ is intersecting.

Clearly, if $q < k+1$ then $G_1 - v$ is a copy of ${}_{n-(k+1-q)}C_{n-1}^{k-1,k}$, and if $q \geq k+1$ then $G_1 - v$ is a copy of ${}_{q-k-1}C_{n-1}^{k,k+1}$. If $N_{G_1}(v) \cup \{v\} = V(G_1)$ then $G_1 \downarrow v = (\emptyset, \emptyset)$. Suppose $N_{G_1}(v) \cup \{v\} \neq V(G_1)$. Then $V(G_1 \downarrow v) = \{v_m, ..., v_n\}$, where

$$m = \begin{cases} 2k+3 & \text{if } q < k+2; \\ 2k+4 & \text{if } q \geq k+2. \end{cases}$$

Let $n' := n - m + 1$. By considering the bijection $\beta\colon V(G_1 \downarrow v) \to [n']$ defined by $\beta(v_i) = n - i + 1$ ($i \in [m, n]$), it is easy to see that $G_1 \downarrow v$ is a copy of $M_{n'}(\{d_j\}_{j\in\mathbb{N}})$, where

$$d_j = \begin{cases} k & \text{if } j \leq n - (q + k + 1); \\ k+1 & \text{if } j > n - (q + k + 1). \end{cases}$$

So $G - v$ and $G \downarrow v$ belong to the class of graphs specified in the theorem.

The rest follows as in Case I, except that we get $\mu(G_1 - v) \geq \mu(G_1)$ by Lemma 11.3.5.

Case IV: G_1 is a power of a cycle C_n^k. Let v_i, $i = 1, ..., n$, be as in Definition 11.1.9, and let $u := v_k$ and $v := v_{k+1}$. If $n < 2k+2$ then $N_{G_1}(v) \cup \{v\} = N_{G_1}(v) \cup \{v\} = V(G_1)$, $G_1 \downarrow v = (\emptyset, \emptyset)$, $G_1 - v$ is a copy of ${}_{n-k-1}C_{n-1}^{k-1,k}$, $\mu(G_1 - v) = \mu(G_1) = 1$, and hence, by the same line of argument for each of the preceding cases, we conclude that $\mathcal{I}_G^{(r)}$ is strictly EKR. Now suppose $n \geq 2k + 2$. Let $\mathcal{A} := \Delta_{u,v}(\mathcal{A}^*)$. Since $N(u) \backslash (N(v) \cup \{v\}) = \{v_n\}$, it follows by Lemma 11.2.1(ii) that $\mathcal{A}\langle v\rangle$ and $\mathcal{A}]v[$ are intersecting.

Clearly, $G_1 \downarrow v$ is a power of a path. As in Case I, it follows that $|\mathcal{A}\langle v\rangle| \leq |\mathcal{J}(x)|$.

Now $G_1 - v$ is a path (if $k = 1$) or a copy of ${}_{n-k-1}C_{n-1}^{k-1,k}$ (if $k > 1$); however, we are not guaranteed that $\mu(G_1 - v) = \mu(G_1)$ (this is the case if, for example, $G_1 = C_4^1$). Let $\mathcal{B}^* := \mathcal{A}]v[$. Let $u' := v_{2k+1}$ and $v' := v_{2k+2}$, and let $\mathcal{B} := \Delta_{u',v'}(\mathcal{B}^*)$. Clearly, $N_{G-v}(u') = N_{G_1-v}(u') \subset N_{G_1}(v') \cup \{v'\}$. Thus, by Lemma 11.2.1(ii), $\mathcal{B}\langle v'\rangle \cup \mathcal{B}]v'[$ is intersecting.

If $k = 1$ then $G_1 - v - v'$ is a disjoint union of a path and a singleton, and if $k > 1$ then $G_1 - v - v'$ is a copy of ${}_{n-2k-2}C_{n-2}^{k-1,k}$. It is easy to see that $G_1 - v \downarrow v'$ is a power of a path. So $G - v - v'$ and $G - v \downarrow v'$ belong to the class of graphs specified in the theorem.

By Corollary 11.3.2(ii), $r - 1 \leq \mu(G - v \downarrow v')/2$. By Lemma 11.3.3, $\mu(G_1 - v - v') \geq \mu(G_1)$; so $\mu(G - v - v') = \mu(G_1 - v - v') + \mu(G_2) \geq \mu(G_1) + \mu(G_2) = \mu(G) \geq 2r$.

Therefore, since $\mathcal{B}\langle v'\rangle \subset \mathcal{I}_{G-v\downarrow v'}^{(r-1)}$ and $\mathcal{B}]v'[\subset \mathcal{I}_{G-v-v'}^{(r)}$, the inductive hypothesis gives us $|\mathcal{B}\langle v'\rangle| \leq |\mathcal{J}]v[\langle v'\rangle|$ and $|\mathcal{B}]v'[| \leq |\mathcal{J}]v[]v'[|$. So $|\mathcal{B}^*| = |\mathcal{B}| \leq |\mathcal{J}]v[|$. Since $|\mathcal{A}| = |\mathcal{A}\langle v\rangle| + |\mathcal{B}^*| \leq |\mathcal{J}\langle v\rangle| + |\mathcal{J}]v[|$, we have $|\mathcal{A}| \leq |\mathcal{J}|$, and hence $\mathcal{I}_G^{(r)}$ is EKR.

Now suppose $r < \mu(G)/2$. Since $|\mathcal{A}^*| = |\mathcal{A}|$ and $|\mathcal{A}^*| \in \mathrm{ex}(\mathcal{I}_G^{(r)})$, we must have $|\mathcal{A}\langle v\rangle| = |\mathcal{J}\langle v\rangle|$ and $|\mathcal{B}^*| = |\mathcal{J}]v[|$. By Corollary 11.3.2(i), we have $r-1 < \mu(G \downarrow v)/2$, and hence, by the inductive hypothesis, $\mathcal{A}\langle v\rangle = \mathcal{I}_{G\downarrow v}(y_1)$ for some $y_1 \in V(G \downarrow v) \subset V(G)\backslash\{u,v\}$. Since $|\mathcal{B}^*| = |\mathcal{J}]v[|$, we have $|\mathcal{B}\langle v'\rangle| = |\mathcal{J}]v[\langle v'\rangle|$ and $|\mathcal{B}]v'[| = |\mathcal{J}]v[]v'[|$. Since $r < \mu(G)/2$, $r-1 < (\mu(G)-2)/2 \leq \mu(G-v\downarrow v')/2$ by Lemma 11.3.1. Thus, by the inductive hypothesis, $\mathcal{B}\langle v'\rangle = \mathcal{I}_{G-v\downarrow v}^{(r-1)}(y_2)$ for some $y_2 \in V(G-v\downarrow v)$. By Lemma 11.4.2, $\mathcal{B} \subseteq \mathcal{I}_{G-v}^{(r)}(y_2)$. If y_2 is not a singleton of $G-v$ then Lemma 11.4.1 gives us $|\mathcal{I}_{G-v}^{(r)}(y_2)| < |\mathcal{J}]v[|$, but this leads to the contradiction that $|\mathcal{B}^*| < |\mathcal{J}]v[|$. So y_2 is a singleton of $G-v$, and hence, since $G_1 - v$ contains no singletons, $y_2 \in V(G)\backslash V(G_1) \subset V(G)\backslash\{u,v\}$. Note that, by definition of $\mathcal{B}$, $\mathcal{B}(v') \subseteq \mathcal{B}^*$. Thus, since $\mathcal{B}\langle v'\rangle = \mathcal{I}_{G-v\downarrow v}^{(r-1)}(y_2)$, $\mathcal{I}_{G-v}^{(r)}(y_2)(v') \subseteq \mathcal{A}]v[$. Suppose $y_1 \neq y_2$. Let $A_1 \in \mathcal{I}_{G-v}^{(r)}(y_2)(v')]\{u,y_1\}[$. So $A_1 \in \mathcal{A}]v[$, $\{u,v\} \cap A_1 = \emptyset$, and hence $A_1 \in \mathcal{A}^*$. Let Y be a maximal independent set of G containing y_1 and v. Since $2r \leq \mu(G) \leq |Y|$ and $\{y_1, v\} \cap A_1 = \emptyset$, $\binom{Y\backslash A_1}{r}[\{y_1,v\}] \neq \emptyset$. Let $A_2 \in \binom{Y\backslash A_1}{r}[\{y_1,v\}]$. Since $\mathcal{A}\langle v\rangle = \mathcal{I}_{G\downarrow v}^{(r-1)}(y_1)$, $A_2 \in \mathcal{A}(v)$. Now, by definition of $\mathcal{A}$, $\mathcal{A}(v) \subseteq \mathcal{A}^*$. Hence $A_2 \in \mathcal{A}^*$. But $A_1 \cap A_2 = \emptyset$, which contradicts $\mathcal{A}^*$ intersecting. So $y_1 = y_2$. Since $y_2 \notin \{u,v\}$ and $\mathcal{B} \subseteq \mathcal{I}_{G-v}^{(r)}(y_2)$, we clearly have $\mathcal{B}^* \subseteq \mathcal{I}_{G-v}^{(r)}(y_2)$. So we have $\mathcal{A}^* = \mathcal{A}(v) \cup \mathcal{B}^* \subseteq \mathcal{I}_G^{(r)}(y_2)$. This proves that $\mathcal{I}_G^{(r)}$ is strictly EKR. □

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