Learners' shifting perceptions of randomness.

Thesis

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Learners' Shifting Perceptions of Randomness

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Abstract

This is a phenomenographic study of learners' perceptions of randomness. Underpinning the study is a sense of randomness as a dynamic process, in which events emerge unpredictably from a generating process, into the determined past, through the curtain separating past from future. Previous research into perceptions of randomness has often used tasks in which people are asked to recognise randomness in a given sequence of outcomes. Other research has asked people to make up sequences of random outcomes. Such tasks do not carry the sense of randomness as dynamic.

Interview tasks in this study were designed to present the dynamic sense of randomness using outcomes from random generators such as dice, coins and sampling bags *in situ*. Interviewees were invited to talk about their experiences of making sense of the emerging sequence of outcomes.

The analysis of the interview transcripts addressed two related questions:

- What do learners believe about randomness?
- To what situations and circumstances do learners consider randomness to be an appropriate model?

The first question encompasses learners' expectations of a random generator and the methods, strategies and heuristics by which learners discern what is random from what is not. Drawing on local and global meanings of randomness identified by Pratt (1998), the thesis argues that learners' perspectives on randomness shift rapidly and frequently between the local and global as the learners seek to interpret the observed outcomes. Learners' interpretations of and beliefs about randomness are found to have significant impact upon the kinds of situations to which learners are able and willing to apply the model of randomness.

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PART 1:

Background to the study

Chapter 1: Introduction

What is randomness? When you think of "randomness", what else comes to mind? How would you react if a process, which you thought should be random, resulted in a run of the same outcome? How do you reason about whether a process is random or not?

These are some of the questions that are investigated in my study of how teenage learners think about randomness, what they think it is, and when they think it is applicable. In this chapter, I outline what I am interested in, why I have become interested in it, why it matters and to whom. In section 1.1, I outline the place of randomness as a topic within the school mathematics curriculum in England, and its role in the wider national agenda of creating a numerate and statistically literate democracy. My own personal perspective on the importance and difficulty of understanding the concept of randomness has played a significant role in the shaping of this study as well as providing a personal motivation to carry out this research.

1.1 My aim for my project

Across the world in the last thirty years, probability education has played an increasingly important part in the basic education of school children. The school mathematics curriculum in England contains much more content relating to statistics and probability than was the case fifty years ago. Since 1989, when the National Curriculum was introduced, the *Handling Data* strand has become compulsory for all children aged 5 to 16. Key statistical ideas within *Handling Data* in the latest version include measures of central tendency (location), measures of variation (spread) and the concept of the distribution of a set of values. Pupils are also required to develop understanding of measures of uncertainty (probability) and to use probability to make and justify decisions under uncertainty.

Behind these ideas lie the critically important concepts of 'randomness', 'variability' and 'distribution'. Within this National Curriculum, there are few mentions of the word 'random', but a central one is "Pupils should be taught to... see that random processes are unpredictable" (DfEE, 1999, pp 40, 54 and 69). This injunction appears in the programmes of study for each of Key Stages 3 and 4, and is therefore required to be part of pupils' learning experiences from the ages of 11 to 16. Related to this,

> Pupils should be taught to:
>
> compare experimental data and theoretical probabilities
>
> understand that if they repeat an experiment, they may - and usually will - get different outcomes, and that increasing sample size generally leads to better estimates of probability and population parameters.
>
> (DfEE, 1999, p71)

Randomness underpins the notion of random sampling and random variable. The idea is also fundamental to a sound understanding of probability in many applications.

Knowing when it is appropriate to question whether a process is 'random' is one of the few instances where the term 'random' is mentioned in the National Curriculum. It is important in knowing when and how to apply probabilistic reasoning to solve problems. However, there is, within the school curriculum in the UK and in many other English-speaking countries, reliance upon a few simple contexts within which to study randomness. Most common amongst these contexts are the traditional canonical settings for probability: dice, coins, urns, playing cards and spinners. Most of these contexts are used to represent situations that can be considered to have a finite set of equally likely outcomes. This can result in the misapprehension that randomness requires equally likely outcomes.

Reliance on a narrow range of contexts for randomness may also lead to limited awareness among pupils of the variety of contexts to which randomness is applicable as a model. Critical reading of the news media, and other activities that are part of being a functioning member of a modern western democracy, require citizens to have some understanding and informed experience of random variation. Within the International Statistical Institute (ISI) there has been recently increased attention given to developing statistical literacy among the population (Gal, 2003). A similar concern was part of the motivation for the launch in 2004 of Significance, a new popular journal produced in the UK by the Royal Statistical Society (RSS).

Understanding the importance of statistical variation is particularly important to citizens when they need to evaluate or interpret statistical reports in the news media. For example, newspapers often report on geographical clusters of occurrence of some disease, suggesting that some local factor may be contributing to the cause. It is important to recognise that some geographical clusters will occur by chance when cases of a disease are randomly scattered in the population. The important judgement to make is whether such a cluster is abnormally large. For example, there have been stories in the media for several years suggesting health risks arising from living near to high-voltage electricity pylons. A recent item on the BBC news reported on a study that "found children who had lived within 200m of high voltage lines at birth had a 70% higher risk of leukaemia than those 600m or more away" (BBC News website, 3 June 2005). The item went on to report a debate between experts from different organisations, some of whom called for immediate action to prevent houses and schools from being built close to such pylons, and others suggesting that here was no cause for concern and that other factors were likely to be the significant triggers for childhood leukaemia. Experience of experimenting with random processes can help learners to develop awareness of variation and of clustering.

In this study I aim to find out more about how learners think about randomness and how they discern when a situation can be thought of as random. As a consequence, I hope that it will be possible to discern some of the obstacles to appreciating the mathematical use of the term 'random'.

This study is concerned with the conceptions of randomness held by learners in secondary schools. As such, the primary context in which the concept occurs for the subjects in this study is in relation to the study of probability.

My motivation to undertake this study arises in part from my own experience of learning probability and statistics in secondary school. From an early age, I had formed a sense of randomness as a description for outcomes that were unpredictable and ever-changing. The idea of randomness referred to the anticipation and uncertainty of what the outcome would be, up to the instant in which the outcome occurred and was observed. Once I had observed the outcome, it was fixed and immutable; that was the outcome and nothing could change it now. Once the outcome had emerged from the curtain separating the future from the past, it seemed to me that it was, in an important sense, no longer random. Although it is now hard for me to be clear about how I arrived at this sense, I suspect it was derived from my experiences of playing games involving dice and playing cards.

During the beginning of my formal study of statistics, at A level, I felt a deep sense of unease about the way in which randomness was introduced into the discussion. I felt that my preconceptions about randomness were different from what I understood from the presentation of randomness in my lessons. In particular, I was introduced to many different statistical tables, all presented in a single volume: tables of various distributions (binomial, Poisson, normal, student's t) and a set of random number tables. I remember clearly feeling deeply bothered by the random number tables: they were not random in any sense that I understood.

Later, I was introduced to the ideas of hypothesis testing, and estimation using confidence intervals. I remember experiencing a sense of enlightenment when I recognised that a confidence interval was a particular outcome from a random process, and that the level of confidence expressed a probability related to this. Today I see computer simulation as an important tool in helping students to understand the idea of a confidence interval, but such tools were not available then. Looking back on these and other early experiences of learning probability and statistics, I can see that my early understanding of randomness played an important part in the way in which I made sense of these ideas. In particular, the knitting together of my understanding of randomness and probability distribution was very important.

I came to this study of randomness as a teacher of mathematics in secondary schools. When I was teaching probability and statistics, I became aware of a variety of difficulties experienced by learners, not all of them like mine. Some pupils appeared to lack experience of experimenting with random generators; I had a sense that these found the idea of a probability distribution more difficult. Later, when I carried out research, using a computer-based probability simulation to enable pupils to work on some counter-intuitive probability problems involving conditional probabilities, I observed that some pupils appeared to have difficulty understanding how the computer could possibly produce random outcomes (Wilder, 1993). Through these experiences I came to question precisely how my pupils thought about randomness.

Previous research into learners' conceptions of randomness has pointed to the fact that learners have significant difficulties, but the research does not seem to me to have listened closely to what learners believe about random phenomena, and how they think about and interpret situations in which the outcomes are inherently unpredictable. One exception to this was the work of Pratt (1998), in which learners worked with computer-based microworlds, using simulations of random generators to construct their own meanings for what they observed. Pratt's study is important, but it moved learners directly into using and interpreting computer simulations of randomness without first considering what those learners believe about random phenomena and simple random generators apart from the computer. In this study I hope to look more closely at what learners believe about simple random generators such as dice and urns, and whether they perceive randomness in their daily experience. Although computer-based simulations of randomness are increasingly widespread, I suspected that many learners might not recognise them as examples of what is "random".

This study is not a teaching experiment and no deliberate teaching intervention is considered, neither is there any attempt to study learning or to see a change in thinking.

I see my research as lying within the area of mathematics education. I have interviewed learners aged 13 to 17 to explore their conceptions of randomness. During the interviews, I was interested in how children recognised randomness, and what they thought they were recognising. I gave the learners simple tasks using dice, coins and sampling bags, and asked them to talk about their experiences of thinking about and interacting with these tasks.

The learners were inevitably discussing difficult ideas, and often without having the benefit of sophisticated language to describe their ideas. The word 'random' was problematic, not only because it was new to some learners, but also because it has acquired a currency in modern 'youth' culture that is different in subtle ways from the technical usage that statisticians want. Thinking about these issues raised questions as to whether even statisticians really know what they mean by 'random'!

1.2 The structure of the study

This study is organised in four parts.

Chapters 1 to 5 form Part 1, which provides the background to the study. Chapter 2 reviews relevant literature relating to epistemology of randomness, while Chapter 3 moves on to review literature about psychology and the pedagogy of randomness. Chapter 4 is a brief outline of the aims of the study. The methodology and the methods used in the study are discussed in Chapter 5.

Part 2, Chapters 6 to 9, covers stage 1 of the study. Chapter 6 is an account of various initial investigations, which provided valuable data and informed the development of an analysis of the stage 1 interviews. Chapter 7 contains an account of the conduct of the stage 1 interviews, including a description of the interview tasks, an account of the selection of interviewees and a short description of each interviewee, as well as a brief account of the process of analysis. An account of the main themes that emerged is given in Chapter 8, illustrated by short extracts from the interview transcripts, while Chapter 9 provides a deeper account of the idea of shifting perspectives on randomness as it emerged from the stage 1 interviews.

Part 3, Chapters 10 to 13, covers the second and final stage of the study. Chapter 10 contains an account of the development of a new interview task, designed to approach the idea of shifting perspectives from a different direction. Chapter 11 describes the conduct of the stage 2 interviews in a similar way to the description of stage 1 interviews in Chapter 7. The shifting perspectives that emerged in stage 2 interviews were similar to those in stage 1, but there was also evidence of a second kind of shift; these shifts are described in Chapter 12. In Chapter 13, evidence from stage 2 interviews is related to the material from Chapter 8 to provide a developed account of the other themes that emerged across the two stages.

Part 4 concludes the study with Chapter 14: a summary of the findings and a discussion of implications for further research and for teaching.

Chapter 2: What is Randomness?

In this chapter, I review some of the literature relating to the concept of randomness. I include consideration of some philosophical issues behind what is meant by randomness, and of some differences in meaning that occur in the various contexts in which the concept is used.

Based on my experience, it is natural to begin my review of the literature by looking first at approaches to probability. I outline different approaches to probability and the implications of these for the concept of randomness, before exploring the meaning of 'randomness' and its applications in modern thought. This leads to a discussion of random phenomena and of the relationship between different ideas of randomness and approaches to probability.

Chapter 3 contains a review of the research literature relating to ways in which naïve users think of randomness.

2.1 Probability

Hawkins and Kapadia (1984), in a review of research into understanding of probability, distinguished four approaches to probability. I propose to discuss each of these in turn and to consider what each has to say about randomness. I believe that the first three of these are different in kind from the fourth.

1. "*A priori*" probability (or theoretical probability) is calculated using an assumption of equal likelihood.

2. *Frequentist* probability (or empirical probability) is calculated from the observed relative frequencies in repeated trials.

3. *Subjective* and *intuitive* probabilities are an expression of a degree of personal belief. Such probabilities may be "embryonic precursors" to formal probability (see 4).

4. *Formal* probability (sometimes objective or normative probability) is that precisely calculated using the mathematical laws of probability. The calculation may be based on an "a priori" or a "frequentist" approach.

(Hawkins and Kapadia 1984, p 349)

2.1.1 "*a priori*" probability

The very early theoretical work on probability, for example by Cardano in the 16^{th} Century (*Liber de Ludo Aleae*, c1625), supposes a set of equiprobable outcomes (David, 1962). Similar approaches were taken later by De Moivre (*Doctrine of Chances*, 1756) and Laplace (*Philosophical Essay on Probabilities*, 1814), amongst others. This classical *a priori* interpretation of probability appears to have been the first to have been subject to calculations using the probability calculus. In that sense it may be thought of as the simplest, and it may be a natural point of entry for young children beginning to learn about probability. The approach is also relatively easy to implement within computer simulation software, and much of the probability simulation software for children that has been written about requires the application of simple ideas based on this approach to build simulations, for example, ProbSim (Konold and Miller 1992) and ChanceMaker (Pratt, 1998). However, there are many probabilistic situations in which it appears inappropriate or impossible to assume a set of equally likely outcomes. Some examples of such situations are discussed later in this chapter.

The equally likely outcomes at the heart of the *a priori* approach are sometimes identified with the notion of randomness. In this case, the idea of randomness is at the heart of the process by which the outcomes are generated. The 'randomness' of the sequence of successive observed outcomes is a necessary consequence of the idea that the generating process consists of a set of equally likely outcomes.

2.1.2 *Frequentist* probability

The idea of long run relative frequency underlies the frequentist approach to probability. This idea was developed by von Mises (1939). In a formal definition of this approach, the probability of an attribute A relative to a collective w is the limiting relative frequency of A in w (Hájek, 2001). This approach has some intuitive appeal in experimental cases where the experiment can be replicated as many times as required. For example, when a drawing pin is tossed onto a level surface it will either land on its back (point upwards) or on its side. These two outcomes cannot be assumed to be equally likely, and the best way to determine the probability of each outcome is to repeat the experiment a large number of times and to observe the long-run relative frequency. In such a case, the sequence of observed outcomes might be considered random, but in this case the focus of attention is upon the fact that the sequence is disordered.

There are however limitations to the application of this approach. One difficulty is that there is no simple way of seeing how any finite sequence of observed outcomes can be an approximation to a collective, since a collective is infinite. More fundamentally, there needs to exist a collective w in which the relative frequency of the attribute may be examined. In many real situations to which probabilities might be applied, it is not clear what this collective should be, or even whether such a collective exists. For example, in considering the question "what is the probability that it will be raining at 9am tomorrow?" such a collective consists of only a single event and the frequentist approach is inapplicable. Of course, many people might blur the definition of the appropriate collective in cases like this, and argue from a collection of somewhat similar events such as on how many days was it raining at 9am in the last month, or on how many days was it raining at 9am on days in this week of the year for the last five years. More likely however, is that people abandon the frequentist approach and construct a probability based upon the weather now and a subjective judgement of how it is likely to change in the next few hours, influenced inevitably by factors that bias the judgement towards optimism or pessimism.

These problems concerned Karl Popper and led him to develop the 'propensity' interpretation of probability (Popper, 1957, in Miller, 1986). In Popper's approach, probability is a property of a situation describing a propensity for a particular physical situation to give rise to a particular outcome. There are several different variants of the propensity approach but these all draw on Popper and the differences appear to be small (for example, Mellor, 1971). The main intention of such propensity approaches seems to be to resolve the problem of single-case probabilities within a modified frequentist approach. A fundamental difficulty with such approaches is that of explaining and justifying what is a propensity and how it can be identified, particularly in those difficult situations of single-case probabilities.

An issue in this study might be to consider whether the event that it will rain at 9am tomorrow can be considered to be a random event. Just as it is difficult to define what is meant by the probability of such an event, so it is difficult to define in what sense such an event can be considered to be random. Popper's propensity approach has little to say about this.

2.1.3 *Subjective* degrees of belief

The view of probabilities as subjective degrees of belief is quite different from previous interpretations and does not necessarily require the concept of randomness. In the most common treatment, a degree of belief is described in terms of betting behaviour: a person's degree of belief in A is p if and only if the person is prepared to pay up to p units for a bet that pays 1 if *A*, 0 if not *A* (based on Hájek 2001, p. 369). The problem here is that there is no clear objective justification to relate such a probability to 'reality'. However, since the probabilities can be treated within the probability calculus, it is possible to update them in the light of evidence by conditioning, using application of Bayes' theorem. This gives rise to the Bayesian approach to probabilities. According to Bayesian epistemology, a rational person's subjective degrees of belief that an event *A* will occur, P(*A*), should be updated on acquiring evidence *E* to the conditional probability of *A* given *E*, P(*A*|*E*). The conditional probability is obtained using Bayes' theorem and requires that the person should also be able to quantify their subjective degrees of belief that *E* will occur and that both *A* and *E* will occur. The Bayesian approach assumes that any rational person's degrees of belief will satisfy the axioms of probability, but there is some evidence that people often fail to reason according to probabilistic norms (Kahneman, Slovic and Tversky, 1981).

2.1.4 *Formal* Probability

The fourth approach to probability identified by Hawkins and Kapadia deals with a different set of issues and is not clearly identifiable as an approach in the same sense as the previous three. It refers to a mathematical model (the probability calculus), whereas the previous three have been more about how a model can be related to contexts in the material world.

The idea of "Formal" probability here seems to refer to the agreed "normative" probability calculus, based on the commonly accepted axioms for probability. It is not necessary to restrict this to objective (*a priori* or *frequentist*) probabilities as the axioms and the rules that follow from them can be applied equally well to subjective degrees of belief. Indeed, this is exactly what is done in standard Bayesian statistics, where subjective probabilities may be updated in the light of experience (which may be empirical observation) using Bayes Theorem.

2.1.5 Summary

To a greater or lesser extent, behind each of these approaches to probability lies some conception of randomness or uncertainty, or chance. In the classical *a priori* approach, randomness can be identified with the equally likely outcomes that lie at the heart of the process. This is in contrast to the frequentist approach, in which the observer's experience of randomness is really focused on the disorder within the emerging sequence of observed outcomes. When the situation consists of a single case of an unpredictable outcome, randomness may be considered to be unpredictability. Unpredictability might also be considered a simile for randomness in cases for which subjective probability judgements are used.

In order to take this discussion further, it is necessary to turn now to explore how the term 'random' is understood.

2.2 Dictionary definitions of Random

For lay people the term 'random' may be rather less clearly defined than it is for statisticians. One simple device by which the meaning of the term for different people can be investigated is to examine the definitions of the word in a current dictionary. In the following dictionary definitions, taken from two widely-used single volume dictionaries, each set of definitions begins with a general definition, which attempts to encompass the meaning of the term in lay use. This is followed in each case by some more specialist uses in statistics and other fields.

Oxford dictionary[1]

Random: made or done without method or conscious choice

[1] These are from the Concise Oxford Dictionary. Of course there are many more obscure definitions and meanings in the full OED. I am only concerned here with illustrating the meanings that a lay person might meet.

Random: with equal chances for each item

Random (of masonry) with stones of irregular size and shape

Collins dictionary

Random: lacking any definite plan or prearranged order; haphazard

Random: (statistics: of a variable) having several possible experimental values any one of which is uncertain and depends on chance

Random: (statistics: of a sampling process) carried out so that each member of the population has an equal probability of being selected

Random: noun, a printing term for lines of type under a headline (also called a bank)

There is some similarity between the first meanings identified in each of these definitions, but there are also significant differences, which may indicate differing ways of thinking of randomness in particular contexts. The first definition seems to suggest a focus on the *process* of selection. The second seems to describe more the *outcomes* of the selection process. This distinction between process and outcomes turns out to be significant.

The idea that 'randomness' may be thought of in two distinct ways is proposed in the following sentence taken from a paper by Falk and Konold.

> One major source of confusion is the fact that randomness involves two distinct ideas: *process* and *pattern.*
> (Zabell, 1992, in Falk and Konold, 1997, p 306)

This distinction is potentially very significant for understanding the different ways in which people think about randomness. The idea of randomness as process may indicate the unpredictability of any individual event in the process. This may be seen to be expressed in the two dictionary definitions discussed above in the phrases "without conscious choice" and "lacking any definite plan". However, the randomness of a process may usually only be judged by consideration of the outcomes as lacking pattern by which prediction might be aided. Falk and Konold (1997), in discussing this distinction, cite Spencer-Brown (1957), who writes of *primary randomness* and *secondary randomness* in discussing similar ideas. Primary randomness describes the randomness of the causal process, while secondary randomness is that which is discerned in the sequence of outcomes. The importance of this distinction between *process* and *outcome* lies in the fact that they may not appear to be exactly equivalent: a finite sequence may be highly patterned and so not appear random, and yet still have been produced by a random process.

Each of these definitions defines randomness in terms of what it is not, rather than stating what it is. Herein lies another of the fundamental difficulties with the concept: that it is difficult to define or to describe directly. It may be that many people's concept of randomness is based on an exclusion of other causes, and has nothing beyond this. There may be a parallel here with the mathematical concept of infinity, which might be thought of as not finite, and therefore as "going on for ever", "without end".

A number of related terms can be identified in a thesaurus; their meanings are closely related to, and possibly even overlapping with, randomness: for example, haphazard, chance, fortuitous, serendipitous, aleatory, arbitrary, casual, stray, occasional, indefinite, indiscriminate, non-specific, unspecific, unspecified, unordered, unorganised, undirected, unplanned, accidental, uncalculated, unsystematic, adventitious, incidental, hit-or-miss, irregular, erratic.[2]

Some of these terms are clearly no longer current, and might even be considered obsolete, but others may be used by lay people in conversation to refer to features of randomness without directly using the word 'random'. For naïve users of the concept, the word 'random' and its derivatives may be less familiar than more everyday terms such as chance, haphazard, unplanned or accidental. However, the difficulty is that such terms are inherently less precise and are open to misinterpretation.

2.3 A structured examination of the idea of randomness

The question "What is Randomness?" can be considered a philosophical one; it certainly touches on philosophical issues. Dembski (2000), writing in The Concise Routledge Encyclopaedia of Philosophy, identifies five different approaches to randomness used by philosophers.

- Randomness as the output of a chance process
- Randomness via mixing
- Randomness as pattern-breaking
- Randomness as a measure of computational complexity
- Randomness as mimicking chance (sometimes referred to as pseudo-randomness)

I have taken these five approaches as an organising framework for the following discussion in which I identify some of the ways in which randomness might be seen by children in secondary schools, and by their parents.

2.3.1 The output of a chance process

This definition focuses upon the process generating the output and links the idea of randomness with that of chance. However, the term 'chance process' is also similar to 'randomness' in that it is difficult to define precisely. In ancient times, chance processes were often used as a basis for making decisions. Societies in ancient Greece, Jerusalem, Mesopotamia, and in China, all demonstrated a belief that a purely chance mechanism, which eliminated the possibility of human manipulation, would "give the gods a clear channel through which to express their divine will" (Bennett, 1998, p28). The idea appears

[2] These have all come from The Oxford Thesaurus, 1991 edition.

to be that a chance process has no earthly cause and can therefore be viewed as a pure expression of divine will. Such a view is still widespread today: an event that might be considered a chance outcome will be interpreted as something that was 'meant to be', that is due to fate or destiny. Terms such as 'good fortune' and luck also come to mind in this context. Thus, an element of our present concept of randomness comes from the early historical idea that something without known cause is caused by 'chance'. Again, randomness is seen in terms of what it is not.

It is interesting to reflect on whether people today think of 'random' events as having a 'cause'. For children in school, who conduct experiments with random outcomes in order to learn about probability, there might be some difficulty in accepting the teacher's agenda if they hold the idea that such outcomes are governed by divine will or even by luck, fate or destiny. To understand the implications of the mathematics of probability may require a different view of the events.

This seems to me to be the most general and unrestricted definition, but also perhaps one that says least about what the output will be like. The idea applies equally to the outcomes from rolling a die, or tossing a coin, or waiting time at Oxford Circus for the next bus to Warren Street, provided that you accept these are chance processes and are unpredictable. Unfortunately, each of these particular processes could also be viewed as deterministic, if only we knew enough information about, for example, the motion of the die. There is a potentially circular argument here, in that the reference to a chance process begs all the questions that we are asking of the term 'random'.

An important question here is whether randomness, in the sense of a purely chance process, can ever be shown to exist. If one approaches empirical study with the assumption that all observable phenomena may be accounted for deterministically, then any apparent chance in what is observed can be thought of as observations for which the deterministic explanation has yet to be found. If one has all knowledge, in particular, knowledge of the exact values of all the parameters, any system is deterministic, even if chaotic. The problem is that such complete and perfect knowledge is not possible.

Falk and Konold (1990) write of two distinct ways of viewing probability: the *epistemic* and the *ontic*. According to the epistemic view, uncertainty is introduced into science to account for the scientist's ignorance of the many variables that affect the data, and because of the error in measurement, as in the previous discussion. The randomness in evolutionary biology can be thought of as arising from the deterministic interaction of multiple variables in a highly complex process; the complexity of the process and the variety of different variables are beyond human capacity for analysis and so the outcomes are viewed as irreducibly random. Even people who believe that nature is ultimately deterministic may use probability in this way. In contrast, ontic probabilism sees chance as an irreducible part of natural phenomena, represented to some degree for example in theories of quantum physics. In most accounts of quantum physics, the physical universe is fundamentally random; quantum events exhibit random outcomes that can never be accounted for by hidden variables (Eagle, 2005). There is still some debate about whether ontic probability is an acceptable view. For example, Ayer (1980) explores the use of the chance concept, and distinguishes ways in which people speak of events as 'happening by chance', but at the end he remains unconvinced about the existence of ontic probability.

Even if one rejects the idea of ontic probabilism, epistemic probabilism can open the possibility of using randomness (and the idea of a chance process) as a powerful model for some aspects of situations where events are unpredictable. For example, the outcomes from rolling a die or waiting time at Oxford Circus for the next bus to Warren Street can each be modelled usefully as outcomes from a chance process.

2.3.2 Randomness via mixing

There is a certain intuitive appeal in the idea that, when things are mixed, the result is 'random'. Piaget and Inhelder (1975) used the idea of the random mixture in their study of children's ideas about chance published in 1951 (English translation, 1975). For Piaget, "chance is an essential characteristic of an irreversible mix, while at the other extreme is mechanical causality, characterised by its intrinsic reversibility" (p xvii). He supposed that the idea of chance must therefore be dependent upon the prior understanding of 'reversible operations', since he claims that chance phenomena are those which are not reducible to reversible operations.

One of the common early experiences of randomness for many children is that of selecting objects from a mixture. Situations such as these become recognised in 'urn problems' - one of the icons for randomness - picking a counter (or a chip or some other object) from a mixture of counters (or chips or other objects) in an urn (or a bag or a pot). Selecting an item from a random mixture of items is a commonly experienced context for early probability problems in primary school and in early secondary school.

The idea is important in the study of fluids in physics as it is fundamental to the theory of statistical mechanics. Consider for example a large jar of water containing a small sachet of blue dye that is about to be punctured. At the instant that the sachet is punctured all the blue dye is concentrated inside the sachet in one corner of the jar. Over time, forces act on the water to move the fluid around in the jar so that eventually the particles of blue dye become thoroughly mixed throughout the water in the jar and the fluid achieves a 'uniform' mixture. As someone said to me: "It's like putting hot water into a bath that has grown cold; it takes time to diffuse its way around and raise the overall temperature."

2.3.3 Pattern-breaking

Within this account of randomness, the focus is squarely upon the sequence of outcomes, the output of the random process. I expect that most people are looking for continuous pattern breaking as evidence of randomness, to judge whether a process can be considered 'random'. In this sense, I see this view of randomness as being closest to an informal view that might be used by people outside formal mathematics or statistics. The idea is intuitively simple: if I can discern a repeating pattern in the emerging sequence generated by a process, and this pattern is sustained, then the process is not random.

Yet there is a catch within this reasoning. Consider for example a process that repeatedly generates outcomes of 0 or 1 with equal probabilities. Every finite sequence of 0-1 outcomes, whether patterned or not, could be generated by this infinite process at some time. So, although any pattern that appears in a sequence of outcomes from this process will end eventually, finite sequences with apparent patterns will always continue to emerge. This means that searching for pattern in the emerging sequence of outcomes is not necessarily an effective means of arriving at a judgement of the randomness of the underlying process.

2.3.4 A measure of computational complexity

Chaitin (see 1975 and 1988) defined the complexity of a character string in terms of its minimal description. Any sequence comprising a simple pattern, such as 10101010101010101010101010101010, can be described concisely. Descriptions such as "1010…" or "start with 10 and then continue to alternate all the way" may suggest how this might be done. More complex sequences cannot be so compressed.

The definition and measurement of this idea of complexity has led more recently to a new conceptualisation of randomness (Falk and Konold 1997).

> The algorithmic randomness of a binary-digit sequence is the bit length of the shortest computer program that can reproduce the sequence.
>
> (Falk and Konold, 1997, p306)

One aspect of the intuitive appeal of this definition lies in the fact that an incompressible sequence must be without pattern. The idea of randomness as complexity may be considered equivalent to that of randomness as pattern breaking, in the sense that it may be difficult or impossible to distinguish empirically between behaviours arising from the one or the other. Bar-Hillel and Wagenaar (1991) draw on evidence from a variety of previous studies to suggest that the pattern breaking view of randomness is common, but Falk and Konold (1997) use much of the same evidence to justify a claim that people may be judging randomness as complexity.

2.3.5 Pseudo-randomness

Pseudo-randomness is used in electronic random number generators. The 'pseudo-random' outcome is generated by some algorithm, designed to produce a sequence of outcomes that look like a random sequence, in the sense that outcomes are distributed as they would be by a random process. The sequence of outcomes has no pattern and is essentially unpredictable without detailed knowledge of the algorithm and its inputs. However, the sequence is completely determined by the starting point: if the iterative function that generates the sequence and the initial seed value are known, then the sequence generated by this pseudo-random process is entirely predictable. In this sense, such a sequence of numbers is not random at all, any more than would be the list in a printed table of 'random' numbers, as used by statisticians in the past.

For all practical purposes, since the algorithm and its seed value are not known, the sequence can be treated as the outcome of a 'random' process, and is indistinguishable from the output of any other 'random' process.

2.4 Philosophical accounts of Randomness

Recent accounts of randomness by modern philosophers have re-examined the concept in attempts to clarify it, and to seek to extend the rigour of the idea to contexts in which it seems natural to apply it. In this section I shall refer to just two writers whose ideas have brought me some enlightenment as I have considered the ways in which pupils in secondary school speak of and think about randomness. The first offers an interpretation of randomness that fits with my sense that randomness provides a model for thinking about situations that are uncertain or unpredictable. The second extends this by defining randomness as unpredictability, and in so doing seems to me to extend the applicability of the model.

2.4.1 A model for incomplete knowledge

Kyburg (1974) offers a helpful interpretation of randomness consisting of four parts.

> the object that is supposed to be a random member of a class;
>
> the set of which the object is a random member (population or collective);
>
> the property with respect to which the object is a random member of the given class;
>
> the knowledge of the person giving the judgement of randomness.
>
> (Kyburg 1974)

This interpretation is interesting for two important reasons. First it draws attention to the importance of considering the class of objects from which the random outcomes are drawn. This encourages us to be clear about the meaning of the term in a particular context. Second, Kyburg draws attention to the important role of the state of knowledge of the person making the judgement that the situation is 'random'. As he says: "[R]andomness...

is going to be a concept which is relative to our body of knowledge, which will somehow reflect what we know and what we don't know." (Kyburg, 1974, p217)

Thus, whether or not the object is considered a random member of the class depends on the state of the judge's knowledge of the object. Such an interpretation is much more generally applicable to frequentist situations and to situations where there is some partial information affecting the judgement of randomness.

This interpretation of randomness fits comfortably with the idea of randomness as a model for situations of uncertainty. It allows the use of pseudo-random generators without difficulty, since the user has not the knowledge of the underlying algorithm to enable her to predict the outcomes. For example, the sequence of integers in the decimal expansion of pi may appear to be the outcome of a random process to a person who did not know their source, even though they are a determined sequence.

I find Kyburg's interpretation also reminiscent of the frequentist definition of probability in that it includes explicit reference to the underlying class from which the random object is drawn. This is interesting in that adopting Kyburg's view of randomness would draw me towards a frequentist approach to probability. The work of Gigerenzer (1993) in countering the heuristics and biases programme of Kahneman and Tversky (1972) has also suggested strongly that naïve users are able to reason more readily and more accurately when probabilities are represented in terms of frequencies. The possible significance of frequencies in assisting people to reason about situations of uncertainty will be discussed further in Chapter 3.

My own view is that randomness is a powerful model for thinking about many situations where the outcomes are unpredictable or uncertain. I do not at present want to be concerned with the debate about ontic probabilism, whether randomness is in fact an essential aspect of reality. For me it is enough that the idea can be applied to thinking about situations that I experience.

2.4.2 Unpredictability

The concept of randomness is used in such a variety of ways that there are suggestions that it may be impossible to adequately account for all of the various applications through a single united concept. "Indeed, it seems highly doubtful that there is anything like a unique notion of randomness there to be explicated" (Howson and Urbach, 1993, p324). However, in a more recent paper Eagle (2005) attempts to achieve such unification when he proposes that "randomness is to be understood as a special case of the epistemic concept of the unpredictability of a process" (p1). The National Curriculum for Mathematics in England requires that "pupils should be taught to see that random processes are unpredictable". Clearly the meaning of unpredictability in Eagle's paper is much more rigorous and precise than that intended in the National Curriculum, but there is a resonance of ideas here which is worth pursuing a little.

Eagle begins his paper with an analysis of various scientific uses of the concept from which he identifies four competing demands that he suggests any rigorous analysis of randomness as a single concept would need to satisfy. Eagle goes on to consider standard approaches to defining randomness – the frequentist approach of Von Mises, and the complexity approach of Kolmogorov and Chaitin – and he shows that each fails to meet some of the four demands of randomness that he has identified. He develops a rigorous approach to predictability and unpredictability and uses this to build an analysis of randomness as unpredictability. Finally he argues that his approach provides features that meet the intuitive demands of the concept.

The four demands that Eagle identifies as required for an intuitive use of the concept are interesting for this study to the extent that they resonate with pupils' experiences of randomness. They are outlined below.

> *Statistical Testing*: The concept needs to encompass random sampling and randomized experiments. This requires that random sequences may be generated as required, and it should be possible to judge whether a given sequence is random.
>
> *Finite Randomness*: The concept needs to be applicable to finite phenomena and to single events.
>
> *Explanation and Confirmation*: The suggestion that a system is random should indicate an explanation for why the system shows particular behaviour. Also, such a suggestion should be confirmable by empirical observation.
>
> *Determinism*: Randomness should be compatible with determinism, so that the idea can be used where appropriate to describe the behaviour of, for example, chaotic systems.
>
> (Drawn from Eagle, 2005, pp7-8)

In the following paragraphs I shall briefly review each of these demands in order to consider the relevance of speaking about randomness as unpredictability for children meeting the concept of randomness in secondary school.

Statistical Testing

The idea of random sampling is an important new idea introduced in Key Stage 3 of the English National Curriculum for Mathematics. Pupils are expected to select random samples and to be able to justify this. They therefore need to be able to work with pseudo-random numbers generated by computer, and possibly also with random number tables. This requires at least an informal idea of the characteristics that might be expected of a random sequence and some ideas about how a random sequence can be recognised. While the treatment of these ideas for secondary school pupils is necessarily informal, their discussion still raises some of the difficult issues underpinning the concept of randomness.

To see randomness as unpredictability makes the recognition of a random process intuitive and emphasises the idea that whether an individual considers a process to be random must depend upon that individual's state of knowledge about the process. Pseudo-random numbers are self-evidently unpredictable for the pupil using the computer software. However, it might be instructive to pupils to work with a simple random number generator in a graphic calculator, where the seed can be controlled and so a predetermined sequence of pseudo-random numbers can be generated.

Finite randomness

The mathematics National Curriculum in England requires that "pupils should be taught to see that random processes are unpredictable". This inevitably means that pupils will consider finite processes and single events. It is often natural to speak of a single event as random, even though the standard frequentist explanation of probability, which has often been presented in secondary textbooks, uses a view of randomness which is incompatible with such usage. However, if an event is clearly unpredictable, then randomness may intuitively be an appropriate model to consider.

Explanation and Confirmation

The problem of how to recognise, test or confirm whether a process is random is a particularly difficult one for pupils with the standard views of randomness. Typically pupils might use the pattern breaking idea, but as discussed earlier, this can lead to difficulties. However, if the pupil is simply looking to see if the outcomes are unpredictable, some of these issues can be avoided.

Determinism

Some pupils may consider that a process that is essentially deterministic cannot be seen as random, and vice versa. Such a belief would be difficult to reconcile with approaches whereby some phenomena are seen as both deterministic and random.

2.5 Randomness and Distribution

I have discussed earlier the idea that in both mathematical and everyday contexts 'randomness' is often thought of in terms of that which it is not: particularly that it is without pattern and without order. Yet, in order to develop ideas of probability and distribution, the pattern and order that emerge from the long run behaviour of random phenomena are critically important. This apparent contradiction between 'randomness' as without pattern, and 'distribution' as a pattern that emerges from random behaviour in the long run, is interesting.

The statistician David Moore, writing with intention of influencing teaching of chance and data in schools (1990) has given a definition of randomness that makes clear the relationship between randomness, distribution and probability.

> Phenomena having uncertain individual outcomes but a regular pattern in many repetitions are called *random*. "Random" is not a synonym for "haphazard" but a description of a kind of order different from the deterministic one that is popularly associated with science and mathematics. Probability is the branch of mathematics that describes randomness.
>
> (Moore, 1990, p98)

The regular pattern that becomes increasingly discernible in many repetitions is the distribution. When a random process generates outcomes, the manner in which the observed outcomes are distributed amongst all the possible outcomes will produce, in the long run, an emergent pattern. The pattern becomes clearer as the process continues to generate more outcomes. This pattern may be described as the distribution of the outcomes.

Wilensky writes of emergent phenomena as a new computational field "in which global patterns emerge from local interactions" (Wilensky, 1997, p180). He places probability distributions as "canonical cases of emergent phenomena", stating that "they are stable global structures that emerge from the interactions of multiple distributed agents" (ibid). In standard statistics classes, students do not experience this emergence of order from disorder, and they learn about distributions only through such global characteristics as mean and variance. Wilensky suggests that the "connection between the micro- level of the phenomenon and the macro- level" is critically important for the learner to experience, and its absence from the learner's experience is an example of the disconnection of learning that causes what he terms "epistemological anxiety". He uses computer-based random simulations as environments in which learners can experience and experiment with the emergence of distribution as long-run pattern.

Thus there is a sense in which randomness as locally unpredictable, and distribution as long run pattern can be seen as two sides of the same coin. Then, probability in the frequentist sense can be associated with the long run emergent pattern.

However, as was noted earlier, "a priori" probabilities can also be defined classically from an assumption of equally likely outcomes based on symmetries in a random generator. In this case, the assumption that all outcomes in the base set of outcomes are equally likely is justified only to the extent that the emergent long-run distribution approaches the classically derived probability distribution.

In the case of a probability assigned to a single event, based upon subjective degrees of belief, there is no identifiable distribution. Consider for example the likelihood of there being a thunderstorm during my holiday week in the seaside town where I am due to go next summer. While it is intuitively natural to view the occurrence of a thunderstorm during the specified week as a random event, it is not possible to see an emergent distribution because there is no conceivable set of repetitions: the specific week will only occur once.

2.6 A Dynamic Concept: a personal view

I see "randomness" as a dynamic concept: a concept involving unpredictable change. For me, the word "random" describes something that cannot be 'pinned down' or 'printed off' in a permanent record without losing something of the essence of what it is to be "random". I trace this way of thinking back, in my own experience of learning, to when I was first introduced to random number tables during my A level studies. I felt very strongly that these tables were, in a fundamental sense, the antithesis of what I understood by the word "random". They were entirely predictable. The sequence of numbers on the page was always the same whenever I opened the book to use the tables, and I therefore felt very strongly the need to invent some more 'random' (unpredictable) way of selecting numbers from the page. I was trying to make the numbers selected more truly unpredictable, more dynamic, and in this sense more random.

Years later, I felt a similar sense of unease about the pseudo-random numbers generated on my calculator when I first realised that there was a deterministic algorithm that generated them. For me at that time, the calculator's pseudo-random numbers would be predictable if I only knew the formula, and so they could not be random. More recently I have observed that some of my students appeared to experience some unease about pseudo-random number generators when they first used them in my courses, being concerned that the numbers generated by them could not be trusted to behave as though they were random.

In time I came to recognise that my own state of knowledge about the underlying algorithm generating pseudo-random numbers was such that I could not predict the outcomes at all. I find it interesting that I have never been able similarly to resolve my sense of unease about random number tables. While the calculator's random number generator seems to embody a dynamic sense of randomness that cannot be pinned down, and is always changing, printed tables of random numbers are static, permanent and unchanging.

It would seem that there are at least two levels of appreciating randomness. The first is to recognise the use of the term random to describe something where the outcomes are unpredictable in an absolute sense. The second level, which may be later in developing, is where the outcomes would be predictable if one had good enough knowledge, but since one does not, the outcomes are essentially unpredictable and can be seen as random.

2.7 Process and outcome

A key issue to be considered in this study is whether 'randomness' is seen by people as being a property of the generating process or of the sequence of outcomes generated. I have argued above that I see randomness as a model to describe a process. In this way it can be seen as providing explanation for the outcomes observed. The sequence of observed outcomes might be described as 'random' if it were considered to have arisen from a process that was seen as being a random process, that is a process that is reasonably considered to be modelled by 'randomness'. This is a key aspect of the notion of randomness that I wish to convey to my students when I am teaching. One question for me in this study is whether learners see randomness as a process in this way.

An important distinction for me to make is between the random process and the random pattern in a sequence of outcomes that is generated by a random process (Zabell, 1992).

Previous research into perceptions of randomness has not always been clear about this distinction (Nickerson, 2002), but some writers have discussed the issue explicitly (Falk and Konold, 1997; Wagenaar, 1991). The position that randomness is a property of a process rather than of the outcomes was adopted by Wagenaar (1991), and related to this is the view that any outcome from a random process is considered a random outcome (Pollatsek and Konold, 1991). The problem here is that it is only by observing the outcomes from a process that one can judge whether the process is random.

A process is deemed random if it generates a sequence of unpredictable outcomes. However, a judgement or perception that a process is random will often be based on examination of a sequence of outcomes, which if the process is random will be expected to be without pattern. Unfortunately, while the argument would be true if infinite sequences were examined, in practice only finite sequences are ever considered, and short finite sequences produced by a random process are likely to contain apparent patterns. A real sequence of 20 outcomes from tossing a coin – such as HTTTTHTHTHHTHHHHHHHH which arose in one of my interviews and provoked much consternation in the interviewee – when viewed as a record of an experiment in the past, feels very different from the sense of unfolding surprise and anticipation when I experience it one outcome at a time in a real experiment.

2.8 Concluding remarks

The following themes that were identified in this chapter will be revisited in later chapters.

1. Definitions of randomness seem to be in two kinds:
   - Randomness is defined in the negative
   - Equally likely outcomes EQUIVALENT to randomness
2. Furthermore, the ways people view randomness can be classified in different ways:

- Caused / Not caused:

  Some people see a chance process as involving (or even being caused by) luck or divine will. Is Chance Caused or not caused?

- Local / Global:

  Randomness and distribution are two sides of the same phenomenon. Randomness can be viewed at the local level or at the global level.

- Process and outcome:

  It is possible to consider that randomness is in the process or the product.

3. The conflict between randomness and determinism as ideas that appear to be opposites has been shown in two distinct ways in the discussion in this chapter.

   - Randomness can be applied as a model for incomplete knowledge. This means that randomness may be considered as a model for a situation that is essentially deterministic, albeit complex. There is a potential contradiction here in the conjunction of two apparently opposite ideas: determinism and randomness.
   - A related conflict of determinism with randomness may lie within the idea, within the frequentist approach to probability, of a determined distribution emerging from the disorder of successive random outcomes.

4. Some other ideas and questions were identified in this chapter and will be referred to in later chapters.

   - Randomness can be seen as pattern breaking.
   - Randomness can be seen as equivalent to unpredictability.
   - Can people see a single event as random? If so, what might they mean by this?

5. Finally, I have outlined my personal view of randomness as a dynamic concept that describes a generating process and the outcomes as they emerge from that process.

Chapter 3: Randomness: Teaching and Learning Issues

In this chapter, I begin with a discussion of the history of probability and randomness and some possible implications for learning. I consider some research findings about teaching and learning randomness and related ideas of probability. There are necessarily some areas of overlap between issues discussed in Chapter 2 and Chapter 3. In particular, some literature considered in this chapter was referenced in Chapter 2, where it exemplified or related to some approaches to conceiving of randomness. When such cross-references occur, I have referred back to Chapter 2.

3.1 Luck, agency and cause

In medieval Europe, according to Hacking (1975), there were considered to be two distinct classes of knowledge. Knowledge acquired through demonstration was known as *scientia*, while knowledge, opinion or belief that required the testimony of authority or God-given signs was referred to as *opinio*. The word 'probable' described opinions that had received approval from some authority. A 'probable' belief or opinion was therefore 'approved' by an appropriate authority, but it could never completely change from being *opinio* to *scientia*, no matter how much approval it received.

During the sixteenth and seventeenth centuries, the concept of evidence came to incorporate the Stoic idea of *signa*: signs that are recognised by experience to be often associated with or suggestive of a phenomenon (Hacking, 1975). This provided a bridge between opinion and demonstrable knowledge. As Hobbes wrote in 1640: "...signs are but conjectural; and according as they have often or seldom failed, so their assurance is more or less; but never full and evident..." (Hobbes, 1650). The word probability came to describe the degree to which a belief had moved, considering the evidence of signs, from complete ignorance to certain knowledge. In other words, it conveys the extent to which *opinio* is also *scientia*. A special case of this was the tendency for experiments with coins or dice to produce stable relative frequencies in the long run.

The emergence of the idea of probability in seventeenth-century Europe was followed by its rapid spread as it was applied to various areas of science, accompanied by a change in the concept of evidence (Hacking 1975). Following "the emergence of probability" described by Hacking (1975) there was a dramatic change in the world view expressed by the leading thinkers of the day. This remarkable shift has been aptly described as "The Probabilistic Revolution" (see Krüger, Daston and Heidelberger, 1987; Krüger, Gigerenzer and Morgan, 1987). This conceptual revolution between the seventeenth and nineteenth centuries represents the change from a purely deterministic world view, framed in terms of universal laws and constrained by the religious sensibilities of the day, to a view in which probability and randomness were central and indispensable.

The manner in which the function of the word 'probability' has changed with the emergence of the new concept may provide insight into some of the difficulties experienced by today's students as they grapple with notions of probability and randomness. Words such as 'chance' and 'luck', as they are used in daily life, may convey ideas that owe more to a pre-seventeenth-century view of the world than the modern view. The word 'luck', for example, might be used to express a belief that the world is not random or probabilistic, and that events are determined by the intervention of 'fate', or 'fortune', or even by a capricious god.

One way of thinking about students' talk about chance events may be to see the students as sometimes reflecting pre-seventeenth-century views of evidence, rather than a modern probabilistic view. The conceptual revolution that took place during the seventeenth century with the emergence of probability was so sudden and so significant as to represent a discontinuity. It seems likely that the strongly held pre-seventeenth-century worldview might persist in popular culture in much the same way that Aristotelian thinking about motion has been found to persist (for example, diSessa, 1982; Burns, 1988).

It may be that in discussion of situations involving uncertainty, between student and student, or between student and teacher, students will appear to switch between the "public discourse" involving the language of chance, and the "private musings" in which they still think in deterministic patterns. Wagenaar (1988) has observed that "in the minds of many people, luck and chance often seem to act as real causes". The suggestion is that gamblers see chance and luck as separate causal mechanisms; for example, when faced by a long run of reds at roulette, they may see chance as having broken down, and luck playing a more important role. Luck is thought to influence a bettor's choice through future outcomes and explains important consequences and accomplishment, whereas chance lies behind surprise, fun and coincidence. Wagenaar (1988) suggested that a belief in luck could be a basic fallacy of gamblers. Hawkins, Joliffe and Glickman (1992) point out that the existence of beliefs, such as the gamblers' belief in luck, may produce classroom difficulties similar to those reported in the science education literature and cited in Garfield and Ahlgren (1988) in which belief systems conflict with school science. Such beliefs are often found to be particularly robust in the face of challenge.

More significantly, the work of both Hacking (1975) and Wagenaar (1988) indicate that there may be serious disagreement between statisticians and naïve learners about the meanings of words used by statisticians. This possibility of a failure of communication between teacher and student, or between researcher and subject, (or even between statistician and teacher) has serious implications for anyone involved in statistical education or in researching notions of chance and randomness.

Kath Truran has reported that children aged between 7 and 12, in interviews, frequently mentioned luck when discussing the behaviour of simple random generators, such as dice, spinners and urns (Truran, 1998). The children also sometimes expressed animistic beliefs in a controlling force, or in a strategy or skill by which they could control outcomes.

Few researchers have explicitly investigated the role of cultural views of probability and randomness in the development of probabilistic understanding. A study by Phillips and Wright (1977) suggested that students from eastern cultures in Asia showed less inclination to apply probabilistic thinking than those from Western Europe. A more recent study by Amir and Williams (1999) focused explicitly on the relationship between culture as represented in beliefs, language and experience, and the development of informal and formal probabilistic knowledge. Findings suggested that a significant proportion of children, aged 11 to 12 years, showed some superstitious belief which could be related to their informal probabilistic thinking. There were also significant differences in beliefs and experiences relating to chance and probability between pupils in English and Asian ethnic groups; these differences appeared to relate to different religious beliefs and cultural practices within the two groups (Amir and Williams, 1999).

3.2 Developmental studies

Beginning with the work of Piaget on the development of concepts of number, logic, shape and chance in young children, there have been various attempts to study and describe the development of children's understanding. In this section I consider the findings from three developmental studies of chance and randomness. Each of these studies says something about the relationship between randomness and distribution.

3.2.1 Piaget and Inhelder's Raindrops Task

The earliest research into the development of children's understanding of random phenomena assumed a frequentist and objective definition of probability (Piaget and Inhelder, 1975). According to Piaget and Inhelder, a child cannot distinguish between chance and necessity before the age of six or seven. According to their model, the concept of chance cannot be recognised until the idea of a reversible operation has been understood. A child who has not understood reversibility cannot tolerate disorder and seeks a reason for everything; they therefore cannot comprehend the idea of an irreversible mixture or random mixture. The ability to estimate probabilities only begins to appear after this, and requires two other abilities: to count the possible outcomes, which Piaget considers involves the logical process of counting permutations; and to compare ratios (Piaget and Inhelder, 1975).

Piaget and Inhelder suggested that children's experiences with random mixtures would lead to the development of ideas of distribution and the law of large numbers (1975). One of their tasks focused on this aspect of development regarding a uniform distribution; it was the 'raindrops task' in which children were asked to consider raindrops landing on squares of pavement. The researchers interpreted the ways in which children 'distributed' raindrops among the squares as an indication of their understanding of randomness and of distribution. The children's responses were classified broadly in three age groups, 6 to 9 years, 9 to 12 years, and over 12, representing stages of development. The youngest children tended to place raindrops in a few adjacent squares or evenly across the squares, even with a very small number of raindrops. Responses from children aged 9 to 12 were characterised by "an understanding of a progressive regularity" (Piaget and Inhelder, 1975, p52) but they showed little understanding of how the differences between numbers of raindrops in adjacent squares might vary as the number of drops increased. The oldest children attributed the progressively uniform distribution to the decreasing proportional differences rather than absolute differences. Piaget and Inhelder concluded from this task that only the children in the oldest age group had an understanding of the law of large numbers that variability decreases in proportional terms, while the middle age group had only "qualitative intuitions of proportionality" (Piaget and Inhelder, 1975, p54).

3.2.2 Green's development of the Raindrops task

Piaget's stages were investigated in a study of about 3000 secondary school students by Green (1983). Green developed a Probability Concepts Test for his study, in which he used pencil and paper versions of many of the practical experimental tasks used by Piaget, as well as drawing on tasks used by other researchers such as Kahneman and Tversky (see section 3.5 below). Green analysed the results using statistical techniques to demonstrate a hierarchy of 'difficulty' in the questions. He claimed to have shown the existence of the Piagetian stages in the development of the concept of chance.

A few questions in Green's study were specifically intended to explore children's understanding of random phenomena, and one of these was a version of Piaget and Inhelder's raindrop task. This question, which was expressed in terms of snowflakes, is reproduced in an abbreviated form in Figure 3.1 (taken from Green, 1989, p31).

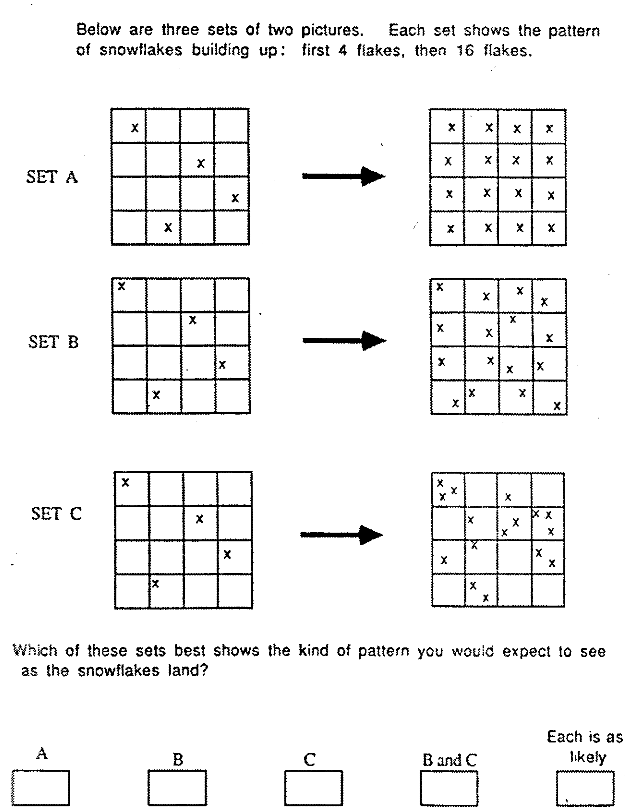

Figure 3.1: The Snowflakes Task (Green, 1989, p31)

The results from this particular question showed a remarkable decline with increasing age in the percentage of pupils selecting option C, the random option, from 26% at age 11/12 to 18% at age 15/16 (Green, 1983). This prompted Green to refine the question for use in a test for younger pupils aged 7 to 11 (Green, 1989). The refined question again produced surprising results. Although the proportion of children choosing a regular symmetrical pattern declined with age, there was not a corresponding increase with age in the proportion of children choosing either the random or the semi-random patterns.

Green speculated that the context of falling raindrops or snowflakes might be interfering with children's responses to these problems, in that the lasting impression of falling raindrops is of an even wetness everywhere. In reality, people might pay little attention to the distribution of the first few drops.

Green wrote a variation on this problem in which counters were independently placed on a 4 by 4 grid of squares in a randomly chosen square. This context was developed specifically to make clear the idea of a discrete uniform distribution. Groups of pupils first participated in an experiment in which they placed six counters randomly on the grid and saw that duplication was likely to occur. Then they were given the task in which they were asked to select the most random responses. The responses of pupils showed that random grids like set C in Figure 3.1 were chosen by much higher proportions of pupils than in the earlier snowflakes problem, and the decline with increasing age was reversed. Green's experiment demonstrated clearly that the pupils' responses are highly dependent upon the context in which the problem is set and the circumstances surrounding the test. He went on to comment on his findings:

> The increase in facility with age on some items and the lack of any such increase for other apparently very similar items shows the very subtle intellectual demands which the concept of randomness makes. What to the teacher may appear trivial and obvious may, in fact, be deep and subtle and so should not be hastily dismissed or glossed over. In studying random events, aspects of local and

> global frequency distribution and visual appearance arise together. These need to be separately considered and focused upon. To use the blanket term 'random' without exploring its different manifestations and interpretations creates rather than avoids difficulties and impoverishes the intellectual life of our students.
>
> (Green, 1989, pp38-39)

Green's reports on his work on the raindrops problems (1983, 1988, and 1989) demonstrate how difficult are the ideas of randomness and distribution for pupils to learn and for researchers to explore. He has shown that Piaget's original account (Piaget and Inhelder, 1975) may be inadequate and he has pointed in particular to the need for careful consideration of the ideas embedded within use of the term 'random'. Green's suggestion that closer attention should be given to the local and global aspects of distribution is important and will be considered further in a later chapter of this study.

3.2.3 Metz on randomness and distribution

Metz (1998) reported a large study in which she examined the probabilistic reasoning of 5- and 8-year old children and undergraduates, with the intention of distinguishing aspects of the concept of randomness that develop with age. She used various classical probability tasks, which involved subjects predicting outcomes, experimenting and reflecting on what they observed. Tasks used random generators, such as spinners and urns, and each task involved a single uncertain event in a repeatable context in which increasingly regular distributional patterns emerge with increasing numbers of repetitions. The data consisted of videotapes of participants engaged in the tasks; these were analysed by two researchers who independently coded all the tapes using an agreed coding scheme.

The study considered a construct of randomness involving "an integration of the uncertainty and unpredictability of a given event with patterns manifested over many repetitions of the event" (Metz, 1998, p349). A key finding was that many children and adults failed to integrate the uncertainty with the long-run patterns in one of two ways. One form of failure was to over-interpret the determinism of the distributional pattern, and this led people to over-state the information given by a distribution and to assume the possibility of deterministic outcomes from their model of the random generator. The other failure was to recognise uncertainty separately from the distribution, leading to a tendency to under-estimate the information given by a distribution and to treat situations as less predictable than they might have been.

3.3 Intuitions of chance – Fischbein

Learning in western education systems is characterised by a search for increasing certainty and dispelling of doubt. Western school curricula tend to be dominated by an emphasis on a deterministic world view, which Fischbein (1975) suggested leads students to over-ride and neglect probabilistic intuitions they developed in their early years. Students learn to look for one right answer, and this is difficult to reconcile with understanding that learning probability will not enable one to predict the immediate outcome of a random process with certainty.

Fischbein argued that teaching of probability should build on early probabilistic intuitions developed in young children. He proposed that intuition and reasoning each provide the individual with information about the same reality, and that they are reconcilable to each other. Since intuition is intimately linked with action, and is more readily available than explicit reasoning, it may represent an intermediate stage between mental operation and outward action. He suggested that young children develop a range of adaptive mechanisms through everyday experience, and that among these is the ability intuitively to estimate proportions and to make predictions (Fischbein, 1975). For example, he drew attention to the findings of probability matching experiments, such as those reported by Estes (1964). In the simplest experiments of this kind, the subject was asked to predict which of two events A or B will occur and, having made a prediction, was shown which of the two events actually occurred and was asked to make another prediction. The experiment continued in this way, and the subjects predictions were confirmed or falsified by the true outcome. "Probability Matching" refers to the finding that, over a large number of trials, the proportion of predictions of event A tends to match the proportion of occurrences of event A. In particular, Fischbein noted that this phenomenon was shown by children as young as three years. He interpreted this as evidence that young children have a primary intuition of relative frequency.

Other more recent studies, such as Green's study (1989) discussed in section 3.2, have echoed the finding that children appear to become more deterministic in their thinking as they get older. This has prompted some to wonder whether it is the school curriculum teaching children to think deterministically and to suppress their early intuitions about probabilistic thinking.

3.4 Sources of uncertainty

Kahneman and Tversky (1982) contrasted two ways in which ideas of randomness and probability are applied. The distinction arises from the source of the uncertainty. When the uncertainty arises from an aspect of the world *external* to the individual, the task can be interpreted as finding "the probability" of an event, and the interpretation is considered to be objective. On the other hand, if the uncertainty is seen as arising from the state of the individual's knowledge or relative ignorance about the event, then the individual has the task of finding "My probability", leading to an *internal* probability. It is worth noting that, within the Bayesian school of probability, all uncertainty is regarded as arising from some degree of ignorance, and is therefore internal.

The distinction between external and internal sources of uncertainty is further refined by Kahneman and Tversky (1982). Assessments of external uncertainty may need to be made either in singular mode, by consideration of the propensities of the possible outcomes, or in distributional or frequentist mode, by considering all similar situations observed in the past. *Distributional mode* refers to situations which can be considered as replicable, in which the distribution emerges from many repetitions; such situations can be analysed in frequentist terms. In contrast, there are situations in which probabilities can be assigned to single events: such probabilities are in the *singular mode*. For example, an assessment of whether a drawing pin is more likely to land point up or point down could be made in singular mode by a reasoned argument about the physical characteristics of the drawing pin, such as the position of its centre of gravity; or it could be made in frequentist mode by recalling the past incidents of dropped drawing pins, and judging the relative frequency of pins landing point up.

Similarly, assessments of internal uncertainty are made, in two different modes: a reasoned consideration of external evidence and arguments; and an introspective assessment of confidence. For example, the question 'How sure are you whether Rome is north of New York?' could be assessed by reason and argument, but the judgement of whether a person's name has been remembered correctly or a word has been spelled correctly will often be based on whether it looks, sounds or feels right.

Kahneman and Tversky (1982) conclude by considering some of the consequences of their analysis. Particular situations of uncertainty are often assessed in more than one mode, and it is quite possible that the person making the assessment will not be explicit about the mode in which they are working, and may not be aware of it. Some people may switch between modes as they consider a situation. The laws of probability, such as the probabilities of two complementary events summing to one, do not seem to apply equally to all variants of uncertainty. Indeed, in natural language, the very word "probability" does not seem to apply to internal judgements of uncertainty as much as to external assessments. Finally they point out that external assessment in the frequentist or distributional mode forms a link between subjective probabilities and frequentist probabilities.

This analysis has important consequences for teachers and researchers. A researcher investigating peoples' reasoning about uncertainty may need to consider whether the manner in which she has posed a problem requires the subject to work with one particular variant of uncertainty. If the subject is, in fact, reasoning in an unexpected mode, then he may not understand the language of the problem in the sense intended by the researcher, and so may not interpret the problem in the manner expected. It is also possible that either the subject or the researcher will not be expecting their assessment of the uncertainty to be consistent with the normal mathematical laws of probability. This may lead to difficulties for the researcher in interpreting the subject's response. Similar difficulties will exist for teachers introducing probability and related concepts.

3.5 Heuristics and Biases

The work of Kahneman, Tversky, and colleagues, which aimed to investigate ways in which people make judgements under uncertainty, has grown into what is known as the heuristics and biases programme (Kahneman, Slovic and Tversky, 1982). Papers arising from this research have identified several heuristics that are reported to be widely used in reasoning under uncertainty. Of these, the *representativeness* heuristic and the *availability* heuristic appear to be considered particularly often in studies of children's probabilistic reasoning (for example: Amir and Williams, 1999; Fischbein and Gazit, 1984) and are considered most relevant within the context of the present study.

3.5.1 Representativeness

Kahneman and Tversky (1972) proposed a representativeness heuristic which they claimed is widely used by people to judge the probability of events, to predict outcomes and to judge what to expect from a random process. Outcomes which most closely represent the essential characteristics of the process that generated them are judged more likely than those which do not, and people take such outcomes to be what they expect to see. In many circumstances this heuristic leads to appropriate judgements, but Kahneman and Tversky have demonstrated through a variety of experiments that under certain conditions people's use of representativeness can lead to bias. For example, the following question was put to 92 subjects.

> All the families of six children in a city were surveyed. In 72 families the *exact order* of births of boys and girls was GBGBBG. What is your estimate of the number of families surveyed in which the *exact order* of births was BGBBBB?
>
> (Kahneman and Tversky, 1972, p34)

Most subjects judged the ordered sequence GBGBBG of births of boys and girls in a family of six to be more likely than the ordered sequence BGBBBB. The same subjects viewed the birth sequence BBBGGG as less likely than GBGBBG. The second choice here is very like the idea of randomness as 'pattern-breaking', which was discussed in Chapter 2. Kahneman and Tversky suggest that their subjects judged the sequence GBGBBG to be more likely in these circumstances since it is more similar to the proportion of boys and girls in the population as a whole and it appears more random.

Alternative explanations for these results are possible. It may be that, in making a judgement between GBGBBG and BGBBBB, many subjects were in fact judging between the unordered outcomes "three boys, three girls" and "five boys, one girl" (Konold, 1990). On the other hand, the representativeness account of this situation contains two hidden assumptions: that the subjects saw the goal of the question as being to arrive at a judgement about probability; and that the concept of probability held by the subjects was 'similar' to that held by the researchers. The second of these assumptions is particularly open to question, as illustrated by the outcome approach (Konold, 1989; see section 3.7).

Kahneman and Tversky have also identified a belief in a "law of small numbers" (Tversky and Kahneman, 1971), under which even small samples are expected to be representative. The combination of these two accounts yields "the simplest and most intuitive account of subjective randomness, that of Local Representativeness" (Bar-Hillel and Wagenaar, 1991, p 444). Kahneman and Tversky propose that local representativeness provides an explanation for the widely reported gambler's fallacy according to which, when faced with a run of red on a roulette wheel, most people will wrongly predict that black is more likely to come up next "presumably because the occurrence of black will result in a more representative sequence than the occurrence of an additional red" (Tversky and Kahneman, 1974).

Local representativeness and the idea of randomness as pattern-breaking together provide an explanation for persistent features of people's perception of randomness as observed by many researchers (Bar-Hillel and Wagenaar, 1991, Falk and Konold, 1997).

3.5.2 Availability

People using what Kahneman and Tversky have termed the availability heuristic typically base their judgements under uncertainty upon specific cases which are most readily brought to mind. Two examples will serve to illustrate this heuristic. The first involves the judgement of word frequency.

> Suppose you sample a word from the English language. Is it more likely that the word starts with a K, or that K is the third letter?
>
> (Tversky and Kahneman, 1973, p166)

Using the availability heuristic, people answer such a question by assessing how easily they are able to think of instances of the two categories. Because it is easier to recall words beginning with the letter K, this category is judged by most people to be the more frequent, even though in a typical English text words with K in the third position are twice as frequent as words beginning with K. This is an example of a *bias due to the effectiveness of the search set* used in the availability heuristic (Tversky and Kahneman, 1974).

The second example concerns *bias of imaginability* (Tversky and Kahneman, 1974), arising from the fact that in certain circumstances some instances are more easily imagined than others.

> Consider a group of 10 people who form committees of k members, $2 \leq k \leq 10$. How many different committees of k members can be formed?
>
> (Tversky and Kahneman, 1974, p12)

A person who attempts this question without computation might try mentally to construct committees of various sizes and then evaluate how many exist of each size according to how easily they can be imagined. A simple scheme for constructing such committees is to partition the set of 10 members. While it is easy to construct five disjoint committees, it is impossible to find even two disjoint committees of 8 members. Since smaller committees are more easily imagined than large ones, they appear to be more frequent.

The availability heuristic is affected by several factors: the recency of incidents; the distinctiveness of incidents; the frequency with which incidents are brought to mind; the depth with which the memory of incidents is processed. The recency factor can be used to explain why motorway drivers are seen to drive at a slower speed after passing the scene of an accident that they did before. The risk of an accident is highlighted in their minds while the accident is recent and the risk appears to be higher. Distinctive risk factors can appear to be greater than those which are less distinctive: for example, the risk of shark attack in Florida is much less than the risk of heart attack, but a report of a shark attack is very distinctive and more easily recalled than report of someone having suffered a heart attack. The frequency of reporting of rail crashes in the UK in the last few years has been very high compared with the reporting of road accidents and this appears to have been accompanied by a public perception that rail travel is more dangerous than road travel, although the statistics suggest otherwise.

3.5.3 Criticism of heuristics and biases

Gigerenzer (1993) has proposed counter-arguments to the heuristics suggested by Kahneman and Tversky. He reports experiments in which carefully modified versions of the tasks used by Kahneman and Tversky have shown no evidence of biases when presented to subjects. The modifications focus on re-expressing the tasks in terms of absolute frequencies instead of probabilities, ratios or relative frequencies.

As an example consider the example of what Tversky and Kahneman (1983) have called the conjunction fallacy, illustrated by the example of Linda, the bank teller.

> Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.
>
> Which is more probable?
>
> (a) Linda is a bank teller.

(b) Linda is a bank teller and active in the feminist movement.

(Tversky and Kahneman, 1983)

In Tversky and Kahneman's research, about 90% of respondents answered that the second option (b) was more probable, a judgement that Tversky and Kahneman claimed was clearly non-normative since it violates the conjunction rule: for any two events A and B, $P(A \cap B) \leq P(A)$ and $P(A \cap B) \leq P(B)$. The explanation for this widespread fallacy, according to Tversky and Kahneman, is that it is another example of people's reliance upon the representativeness heuristic.

Gigerenzer's analysis points out two problems with this example. First, he notes that this is a situation in which the probability would not be accepted by statisticians of the frequentist school (Gigerenzer, 1997), since it is really a probability in 'singular mode'. Gigerenzer shares "the worries of those statisticians, philosophers, and psychologists who caution that the laws of probability do not apply to all kinds of statements about singular events, but apply only in well-defined circumstances" (Gigerenzer, 1996, p593). He suggests that there are good reasons for proposing that a respondent to the Linda example might quite reasonably interpret the question in such a way that the conjunction rule would be inappropriate. Secondly, he notes that Kahneman and Tversky's view of what constitutes sound reasoning ignores the content and the context. He suggests that such a view is not acceptable, and that it is necessary to identify what the word 'probable' means in order to identify what is sound reasoning in the context of this problem. He notes that in a colloquial sense the word can mean 'plausible' or 'having the appearance of truth', and he suggests that such meanings might have little relation to the mathematical interpretation of probability, implying that the conjunction rule is not relevant.

In view of these concerns, Gigerenzer reformulated the Linda problem in terms of frequencies. He replaced the question, "Which is more probable?" by a judgement involving frequencies.

> There are 100 women like Linda.
>
> How many of them are
>
> (a) bank tellers,
>
> (b) bank tellers and active in the feminist movement?
>
> (Gigerenzer, 1997)

This reformulation appears to make explicit that the problem is about mathematical probability. This version of the problem was used in a series of experiments, in which violations of the conjunction rule fell from almost 90% with the original problem to about 20% with the frequency judgements.

Gigerenzer and his colleagues have constructed a theoretical framework to predict the circumstances under which particular frequentist modifications to the probes used by Kahneman and Tversky will reduce or even nullify the biases observed using the original probes. This work is significant, but it works by narrowing the field of application of probabilistic thinking to situations in which a frequentist approach to probability can be used.

3.6 Equiprobability bias

A significant finding with potential implications for understanding of randomness is the equiprobability bias reported by Lecoutre (1992). In a series of experiments, subjects showed a tendency to assume that the set of possible outcomes in a chance setting were by nature equally likely. The finding persisted even when the subjects had some background in probability theory. Furthermore, when the chance element was masked in an 'equivalent' test item requiring the same numerical calculations, the bias towards equiprobability was significantly less.

The standard problem involved three chips in a jar: two red and one white. When two chips are randomly selected from the jar, the result is either R1: "a red and a white chip are obtained" or R2: "two red chips are obtained". The question posed was:

> Do you think the chance of obtaining each of these results is equal? Or is there more chance of obtaining one of them, and if so, which, R1 or R2?
>
> (Lecoutre, 1992, p557)

In almost all their experiments, Lecoutre and colleagues found that the proportion of respondents who stated that R1 and R2 were equally likely was at least 50%.

The "house experiment" used a mathematically equivalent problem in which the chance element was masked. The experiment involved three square cards: two cards showed a triangle (in place of the two red chips), and the third showed a square (in place of the white chip). The three cards are shown in Figure 3.2 below.

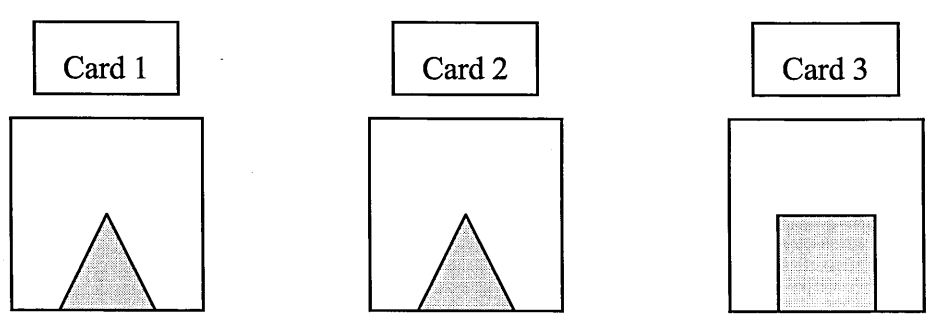

Figure 3.2: Lecoutre's three cards.

The subjects were explicitly shown that it was possible to construct a rhombus if cards 1 and 2 were drawn, or a house if cards 1 and 3 or cards 2 and 3 were drawn. Through this demonstration the chance aspect of the problem was masked. Respondents were then given the following question and were asked to select one of the four given responses.

I put the three cards in this box and I am going to draw two cards. With the two cards drawn I will be able to construct either a house or a rhombus as you have just seen.

Do you think there is:

- An equal chance of obtaining a house and a rhombus?
- More chance of obtaining a house than a rhombus?
- More chance of obtaining a rhombus than a house?
- If it is impossible for you to give an answer, why?

(Lecoutre, 1992, p562)

In response to the house question, 75% of respondents gave the correct response and only 23% gave the equiprobable response. This was a significant reduction in the proportion giving the equiprobable response, and would indicate, as Lecoutre argued, that the equiprobability bias is closely linked to people's understanding of chance. Even when the correct cognitive models are available, they are sometimes not spontaneously associated with a situation involving chance.

Other studies have reported that the equiprobability bias remains common across students of different ages and persists in both simple and compound events (see Watson, 2005; Batanero and Sanchez, 2005). However, Amir and Williams (1999), in a smaller study that looked at beliefs and experience of a narrow age range across different cultural backgrounds, found relatively little evidence of consistent use of the "equiprobability bias heuristic": only 19% of 11-12 year old pupils showed such a bias in at least one-third of their responses on a questionnaire. Amir and Williams suggest that the equiprobability bias may be linked with "a culture that regards chance in most cases as 'equiprobable', for examples in games using equiprobable dice…" (Amir and Williams, 1999, p101).

3.7 Outcome Approach

Another heuristic apparently used by people, adults as well as children, to interpret frequentist probabilities is the *outcome approach,* identified by Konold (1989). Konold interviewed college students about situations of uncertainty, and found evidence that many students hold conceptions of probability that are 'non-probabilistic' (Konold, 1989, and 1990). He argued that these people reasoned according to the outcome approach, showing a tendency to interpret a statement about the probability of an event as a prediction that the event will occur. For example, the statement that the probability of rain tomorrow is 70% is understood as a prediction that it will rain tomorrow. The prediction is interpreted as being only 70% confident, but it is nonetheless a definite prediction. A student who uses the outcome approach may interpret the fact that it rained on 70 days out of 100 as showing that the prediction was wrong 30 times out of 100; the prediction was therefore not very good. A probability of close to 50% is interpreted as an inability to make any prediction and probabilities are interpreted according to their proximity to the 'key' values of 0%, 50% and 100%. Indeed, it is only when the probability is close to 50% that the outcome-oriented individual will consider the event to be 'random'.

The objective of a person using the outcome approach appears to be that of predicting whether event A will occur in a single instance, rather than answering the question "How often will event A occur in the long run?" Of course, when considering the weather forecast, this is a perfectly natural interpretation, but it ignores the probabilistic nature of the prediction which is implied in the statement "the probability of rain tomorrow is 70%". The natural inclination of a person inclined to use the outcome approach seems to be a search for causality. Falk and Konold (1990) see this difference in objective as lying behind the outcome approach, and as characterising the difference between formal and informal views of probability. An informal view of probability is common in the weather forecast, for example, and may be widely found in colloquial uses of the language of probability. The person who adopts the outcome approach, or who considers reasoning with uncertainty as being about predictions in the short run, is under-rating the role of chance. They may have learned "to think and converse in everyday language about 'chance', 'probability', 'luck', 'randomness', and have developed a rich vocabulary with which to communicate degrees of belief" but their underlying perspective is at odds with formal theory. Their focus is on the short run, whereas the success of formal probability theory is only apparent in the long run.

Steinbring (1989) observed a similar suppression of the role of chance which arose when students were invited to evaluate a probability theory against observed outcomes. He described a classroom dialogue in which students discuss the justification, correctness and interpretation of a proposed theoretical model for an urn drawing experiment. The students considered how to judge the difference between the theory and the observed data. During the discussion chance was presented by the students (and later also by the teacher) as the "only valid pattern of justification" (p331). Faced with the suggestion that in a particular single instance the least likely event might occur, the student's response was "that's chance", with no attempt to consider the difference between the short term and long term perspectives. Steinbring suggests that this degeneration of chance "into a substitute for justification" obscures the difference between theoretical expectation and empirical facts in probability (Steinbring, 1989, p331).

3.8 Meanings of randomness

In a study of the meanings that children attributed to random phenomena, Pratt observed closely the ways in which children aged 10 and 11 years articulated their ideas and beliefs as they worked in a carefully designed computer-based domain. He identified two separate classes of meanings articulated by the children: local meanings and global meanings.

Local meanings related to the behaviour of the 'stochastic' process in the short term. They were "local in the sense that such meanings focus on trial by trial variation" (Pratt, 1998, p142). The local meanings identified by Pratt can be seen as characteristics that defined randomness for the children in his study.

- *Unpredictability*: If the next outcome is not predictable, a child might regard the experiment as random,
- *Irregularity*: If there is evidently no patterned sequence in prior results, a child might refer to the experiment as random,
- *Unsteerability*: If the child is unable to exert physical control over the outcome of the phenomenon, the experiment might be seen as random, and
- *Fairness*: If there seems to be a rough symmetry in the experiment, a child may think of the experiment as random.

(Pratt, 2005, p175)

A fifth characteristic, which related specifically to the fact that the work was done within a computer-based domain using the pseudo-random generator, was the idea of the computer-in-control. Some children appeared to be initially reluctant to see the process as truly random because it had been programmed.

Sometimes the local meanings that the children articulated were contradictory, in which case children very often responded by readily modifying the characteristics considered necessary for randomness. Some children did not notice, or suppressed and ignored, the contradictions within the meanings that they articulated. For example, when a child experienced a simulation of a spinner as unfair and yet also as unpredictable, the unfairness was ignored and the spinner was considered random because it was unpredictable.

On the other hand, *global meanings* evolved as children began to see the significance of an increase in the number of trials and recognized what emerged in the long term behaviour. In their fully evolved form, global meanings were characterized by properties that broadly corresponded to the three of the main local characteristics of randomness, and all focused on a view of randomness arising from aggregation.

> Global meanings are characterised by the following properties:
>
> - The proportion of outcomes for each possibility is predictable *(probability),*
> - The proportion of prior results for each possibility in the possibility space will stabilise as an increasing number of results is considered *(large numbers),*
> - The observer is able to exert control over these proportions through manipulation of the possibility space *(distribution).*
>
> (Pratt, 1998, pp 142-3)

The transition from local meanings to global meanings was not straightforward, and children did not move smoothly from articulating a local meaning to the corresponding global meaning. However, children appeared to construct meanings for aspects of randomness through their interaction with the tools provided within the computer microworld in a process that was lengthy and complex. Often the global meanings that emerged were expressed in terms of causal relationships.

> Whereas local meanings remain at the level of that which can not be explained deterministically, the ultimate irony is that the central meanings constructed for long term behaviour, despite being later developments, turn out after all to have features associated with deterministic behaviour...
>
> (Pratt, 1998, p225)

Thus the global meanings appeared to exist alongside the local meanings and, during the activity with the computer microworld, the children's focus moved from local to global.

Pratt does not report any movement from global to local, but it seems implicit in the task that he set and the probes that he used, that no such switch was either anticipated or looked for.

The movement of the subject's attention between local meanings and global interpretations is interesting and deserves further attention. Pratt has shown that learners can hold multiple interpretations of what they observe, and that some of these may appear to an outsider to be contradictory. He has also shown that learners appear to be adept at shifting their attention between meanings and are able readily to suppress inconsistencies. At the global level, he has noted that the meanings for randomness become apparently self-contradictory in that they appear to be deterministic in nature. Because Pratt's method made extensive use of tools in a computer-based microworld, learners were led relatively quickly from the uncertainty and messiness of viewing outcomes trial by trial to viewing representations of larger numbers of trials that had been aggregated to produce an image that pointed towards the idea of distribution.

There are important questions to be asked about what people experience as they move from looking at short sequences of outcomes to looking at increasingly long sequences, before the data has been aggregated. What leads people to shift their attention from the local meanings to looking at the possibility of some global meaning? When does this shift occur? Does attention shift in the other direction, from global interpretation to the local, trial-by-trial variation?

3.9 Pattern seeking

It has been argued by Dehaene (1998) that a sense of number is innate in the human mind. Dehaene claims that the brain centre responsible for this sense of number can be identified and that the capacities of this 'Number Sense' can tell us something about how to teach mathematics. Furthermore, some recent research has suggested that there is a particular centre in the brain, in the pre-frontal cortex, which constantly monitors for patterns in sequences of events (Meredith, 2002), a kind of compulsive pattern-perception. This finding comes from experiments in which subjects watched random sequences of images - a circle or a square - flash onto a screen. During the experiments, researchers used a high-resolution functional Magnetic Resonance Imaging machine to monitor the subjects' brain activity. Subjects were asked to press a button with the left hand when they saw a circle and with the right hand when they saw a square.

Because the sequence of circles and squares displayed was random, there were sometimes short patterns in the sequence of outcomes, either runs of the same image, or repeated alternations of circle square. When such a pattern occurred, the subjects showed brain reaction at the moment that the pattern was broken. When a long pattern was broken, the subjects tended to show an increased reaction time. Unsurprisingly, a longer sequence of an alternating pattern (circle square) was required before a response was detected, than for a run of the same image (circle circle). The researchers suggested that this unconscious searching for patterns might lie behind superstitious behaviour, where people believe that they have discerned a pattern in a sequence of random outcomes, where in reality no long-term pattern exists (Meredith, 2002). It seems likely that, if such compulsive pattern-seeking is widespread, then it might provide an alternative explanation for some of the heuristics and biases identified by Kahneman and Tversky (Kahneman, Slovic and Tversky, 1982). For example, the representativeness heuristic could be seen as seeking to match observed outcomes to a template representing the essential characteristics of the generating process.

3.10 Generation or Recognition

The stimulus tasks or activities that have most commonly been used in previous research to investigate people's ideas about randomness may be classified into two categories (Falk et al., 1997):

> *Generation* tasks, in which subjects are required to make up random sequences or sets of outcomes to simulate a series of outcomes from 'tossing a coin' or the final positions of a number of 'snowflakes landing randomly on a square grid';
>
> *Recognition* (perception or judgement) tasks, in which subjects are asked to select what they think is the 'most random' of several sets of results that might have been produced by a 'random' process, or to decide whether a given outcome was produced by a random process.

Falk and Konold suggest (1997) that recognition tasks "may be more appropriate for revealing subjective concepts of randomness" because "a person could perceive randomness 'accurately' and still be unable to reproduce it" (p302). Indeed there is clear suggestion from many studies (Nickerson, 2002; Shaughnessy, 1992; Garfield and Ahlgren, 1988) that people are generally not good at generating random sequences. This usually means that when people try to generate what they consider to be 'random sequences', the sequences generated differ in some systematic manner from sequences generated by an independent random generator. Typically, people trying to simulate a random process tend to produce more short runs, fewer long runs and more alternations between outcomes, than would be expected from a random process (Bar-Hillel and Wagenaar, 1991; Falk and Konold, 1997). Other studies have used recognition tasks to explore what sequences people consider to be maximally random, and these again show that people tend to identify randomness with those sequences having an excess of alternations between outcomes (Bar-Hillel and Wagenaar, 1991; Falk and Konold, 1997).

However, the fact that people can perceive randomness 'accurately', or not, does not reveal what they are paying attention to. It is not possible, using such tasks, to examine the role of the idea of 'process' within the subject's conception of randomness. The recognition tasks described above focus attention only on a given sequence of outcomes; they do not require the person to consider the process by which these outcomes were generated.

Because I see randomness as, in an important sense, 'dynamic', I wanted to use probes that would present subjects with a 'dynamic randomness'. Recognition tasks do not have this dynamic sense: the sequences presented to the subjects are static and fixed in the same way that a table of random numbers was 'fixed' for me. Generation tasks go some way towards the dynamic view that I want to present, since the subject may need to be aware of the essential unpredictability as they 'generate' the sequence. A subject, who has generated their own sequence, may have access to their assumptions about the particular process that they had in mind. However, I expected other tasks, involving real-time interaction with the outcomes from a random process, would go further. My initial intention was to include generation tasks in my study, accompanied by further tasks to form the basis for a discussion of what people believe about randomness in the situation presented.

3.11 Critiques of previous studies

Epstein Kainan (2000) provides a detailed and extensive review of research relating to people's perceptions and judgements of randomness. In this, she argues that many of the experiments reported in the literature, which find that people are unable to behave randomly or to recognise randomness, are open to criticism.

She discusses a variety of 'instructional biases' that might be brought about by the manner in which the researchers frame their instructions to subjects. As Falk notes:

> Randomness is not only vaguely defined; it has also received an ambivalent treatment over decades by statisticians, psychologists, and other scientists.
>
> (Falk, 1991, p215)

Instructional biases may arise from an explicit but inadequate or misleading definition. For example, some researchers have explicitly directed participants in generation tasks to produce jumbled sequences, without order (see Ayton et al, 1989). In this case the notion of a jumbled sequence, without order, is taken as representing randomness, but this is misleading since any random generator would produce discernible patterns if the sequence were sufficiently long. In fact, almost any instruction could mislead. For example, a subject who is asked to generate an arrangement of a set of cards that might be produced if the set were well-shuffled (Falk, 1981), might attempt to produce an arrangement in which the cards do not appear in any order that might be helpful in playing the game. If the instruction uses the term 'random mixture' the subject might bring their experience of creating a mixture in a cooking context where the goal is to produce a homogenous arrangement of the elements in small pieces, as in a salad mixture. Such an assumption might result in a subject tending to avoid runs of any individual outcome in generating a random sequence (Falk, 1999).

It seems unfair to require subjects to evaluate their attempts to produce sequences according to one set of criteria, when they are being judged in the research according to another. As Ayton *et al.* remarked about the potential for instruction bias:

> (This) potential instructional bias seems to us symptomatic of a deeper tacit uneasiness among investigators of the lay concept of randomness. Running somewhat hauntingly through the published reports on the psychology of randomness is the disconcerting notion that it may not be reasonable to judge the competence of experimental subjects when, strictly speaking, the task they are set requires them to do what cannot, logically, be justified.
>
> (Ayton, Hunt and Wright, 1989, p225)

Much of Epstein Kanain's discussion of methodological issues in studies of randomness relates to the question of whether an individual's behaviour can ever be random. She cites work by Rapoport and Budescu (1992 and 1997) which used a procedure that avoided any instructions concerning randomness. The studies used tasks in which the simple and well-defined objectives would be optimally achieved through a random response. Using this paradigm, Budescu and Rapoport (1994) report that, when people play two-person competitive games, in which they need to conceal their choices in order to maximise their gain, the binary sequences that they generate appear random under standard tests. However, while this might demonstrate that people can, under certain circumstances, produce behaviour which appears to be random, it does not say much about what people believe randomness to be, nor about what they perceive to be random, and why they think the way they do.

3.12 Researching Perceptions of Randomness

Previous research into peoples' understanding of 'randomness' has often focussed on the perceived randomness of a sequence of 'outcomes'. For example, many studies have required subjects to judge whether a sequence of outcomes may be described as 'random' without inviting them to pay attention to the process by which these outcomes were generated (Falk and Konold, 1997; Batanero and Serrano, 1999)

I suggest that, in research into perceptions of randomness, the lack of attention given to the dynamic sense of random is related to the lack of attention paid to the randomness of the generating process. I therefore decided that the stimulus tasks used in the present study should focus the subjects' attention on the process. However, paying attention to the process may not in itself be enough. I have also set out to encourage and enable my subjects to be aware of the surprise and anticipation that can often be experienced when one interacts with a random process at each successive outcome. Inevitably, when someone is judging whether a process that generates a sequence of discrete outcomes is random, they will pay some attention to a sequence of past outcomes from the process as well as possibly considering the nature of the generating process itself. The object of the consideration may remain static and unchanging, even if it includes the process itself, unless it includes an attempt to anticipate the future outcomes of the process. It is in anticipating the next outcomes and in the accompanying experience of surprise that the dynamic unpredictability of what is random becomes a part of what is experienced.

Previous research into people's perception of randomness has often appeared to conclude that people have 'wrong' conceptions of randomness or that human reasoning is in some sense lacking or failing. My aim in this study is not to draw such conclusions but rather to describe what the learners in my interviews actually do and how they reason. I try not to evaluate the responses of the interviewees but rather to describe them, so as to appreciate the range of perceptions and ways of thinking.

3.13 Pseudo-random numbers

An important issue is whether it would be appropriate to use computer simulation in this study. Some writers have expressed reservations about the use of computer generated pseudo-random numbers with learners. Green (1990) reported on a small study of a suite of programs for BBC microcomputers used in classrooms. He noted difficulties verifying the claims for such software.

> ...The misconceptions which are common in the field of probability (and about computers) must give cause for doubt as to whether the pupils get from computer simulations what the teachers or software writers assume. There seems to be a built-in assumption that the basis of the simulation is understood and accepted and the role of the computer (and the random number generator) is appreciated.
>
> (Green, D., 1990, p59)

Hawkins et al (1991) and Shaughnessy (1992) suggest that experience of physical versions of the experiments represented by the software may be an essential pre-requisite. As Shaughnessy puts it:

> It is as though students must proceed through the same steps as the computer in order to believe the results. It is important for us to continue to develop connections between concrete simulations and computer simulations in our teaching, and to investigate the effects of transition between the two in our research.
>
> (Shaughnessy, 1992, p485)

The random number generator in a computer is often explained to students by analogy to physical generators such as dice or spinners. The difficulty with this approach is that physical generators are not usually considered primarily as generators of random sequences, but rather their physical properties of symmetry and homogeneity are used to justify the assumption that there is a pool of equally likely outcomes. On the other hand, a random generator on a computer has no direct link with physical symmetry or other physical properties by which to justify a probabilistic model. Biehler (1991) suggests the similarity between a random generator and a pool of equally likely outcomes lies in the structure of the data generated. To judge the quality of a random generator requires detailed probabilistic analysis of the generated data.

> The difficulties with pseudo-random number generators in computer simulations really are an indication of inherent features of probability.
>
> (Biehler, 1991, page 191)

Biehler proposes that computer generators be presented to students alongside physical generators, and used to give insight into the structure of data generated by physical generators. Such an approach might help students understand that random numbers can be imitated by a mathematical algorithm, and might move towards overcoming the mystique surrounding computer generators. But this still leaves the profound philosophical conflict that the computer generates a 'random' sequence deterministically.

In view of these cautions about learners' abilities to trust pseudo-random numbers, I decided that I would not use computer simulations in the early stages of this study.

3.14 Modelling and Learning

In Chapter 1, I discussed my view that the concept of randomness involves the act of modelling some phenomena. In the sense that the act of modelling involves creating and/or working with a representation of the phenomena for some purpose, the motivation behind the act of modelling is clearly significant. The motivation of a learner choosing to use the ideas of randomness when thinking about a phenomenon is another aspect of the concept of randomness that is not considered in either recognition or generation tasks. Indeed, I suggest that the way that generation tasks and recognition tasks have been set to subjects in many studies has not recognised that randomness is a model and that the application of randomness to a situation is an act of modelling.

The extent to which a learner chooses to apply randomness as a model when thinking about a particular situation may be a significant variable. Indeed it is possible that in some circumstances a learner may choose not to apply the recognised randomness model at all, but some variation of it. Konold's identification of people using what he has termed the 'outcome approach' (Konold, 1989) would seem to suggest that some people tend to apply something other than the randomness model in some circumstances. An individual using the outcome approach is concerned to predict the outcome of a single trial, and does so by relying on a causal model which ignores frequency information, even when such information contradicts the causal model. However, there are circumstances in which it may be a rational decision to work with a deterministic model instead of a stochastic one, so one may not necessarily conclude that an individual who appears to predict a single outcome is using the outcome approach. Pratt (1998) uses the idea of crossing the road safely to make this point. If I choose a place to cross the road which is visible, and I exercise caution in choosing when to cross, then I expect to cross safely. In this sense, I minimise the risk but I choose to think deterministically: because I have minimised the risk I *will* cross safely. Few people would use a stochastic model for crossing the road, in which they would explicitly recognise that they have a specific chance of crossing safely and therefore a chance of *not* doing so.

It follows that any attempt to use tasks as a stimulus for subjects to discuss their ideas of randomness needs to pay attention to the purposes which might motivate the subject in a given context when they choose whether to apply a stochastic model or not.

Thinking with models is a common activity in many spheres of human experience. Learners' perceptions of randomness and of the behaviour of random processes must be recognised as distinct from both the mathematical models and the real world situations that they represent. Randomness is a model that is sometimes useful to describe certain kinds of phenomena. Many attempts to define mathematical modelling can be found in the literature, but one definition that I have found helpful in thinking about this study is the following:

> Modelling is thinking about one thing in terms of simpler artificial things; mathematics is thinking about these simple artificial things, including inventing new ones.
>
> (Ogborn, 1994, p13)

If modelling is described in terms of thinking, then what is thinking? At one level, thinking can involve building explanations of observed phenomena, solving problems, or making reasoned predictions. When children speak of "working things out", this is an expression of the substance of thinking. At a different level, thinking is the mental manipulation of imagined entities, in the style of mental models.

I think of mental models as general tools for thinking: mental tools for thinking about abstract concepts in terms of objects or events in the real world, using logical reasoning to extend from the real world experience of objects or events to build mental models (Johnson-Laird, 1983). Normative models of random phenomena and probability may be less precise and less explicit than in an area such as physical systems; thus mental models used by subjects to think about situations involving uncertainty, chance or randomness may be difficult to articulate.

There is an important sense in which a goal for teaching about randomness is that the idea of randomness should become a model for thinking about situations in which it is not possible to know precisely what will happen in any single instance, as discussed in Chapter 2. However such a development requires the learner to bring together thinking and experiences from various different arenas of experience, particularly from daily life and from the classroom.

From everyday life, the learner has many experiences of uncertainty, such as waiting for a bus, or reading newspaper reports of earthquakes or a tsunami. The individual learner will have tried to make some sense of each of these experiences, and may have attempted to generalise across such experiences. In attempting to make sense of uncertainty and unpredictability, they may draw upon informal lessons from home, family and possibly from a community. For example, children whose family belongs to a religious community may have been taught religious beliefs relating to a particular faith community, which may affect how they consider situations involving uncertainty, especially where such a situation involves some personal stake.

From formal education, the learner experiences lessons about the formal mathematical models of randomness and probability, and may experience applying formal models to replicable experiments. The planned and organised intention behind the provision of experiences in a formal educational setting seems initially to be quite different from the manner in which learners experience in daily life. However, the processes of sense-making – of organising, interpreting and building meaning from the formal experiences – cannot be separated from the meanings that learners have previously constructed in informal settings. This argument fits the essential principles of constructivism:

> Knowledge is actively constructed by the cognizing subject, not passively received from the environment.
>
> Coming to know is an adaptive process that organises one's experiential world; it does not discover an independent, pre-existing world outside the mind of the knower.
>
> (Kilpatrick, 1987, p7)

The processes by which mental models are acquired and become internalised as tools for thought are not well understood. I want to outline an account which builds on a Vygotskian perspective and is drawn in part from Bliss (1994, pp27-30).

Johnson-Laird (1983) views a mental model as a tool for thought, as a working representation of a phenomenon held in the mind. He sees the individual as searching for examples and counter-examples against which to judge and match their developing mental model. Bliss suggests that two important processes need to be examined. First, there is the process of internalisation, whereby action and experience become organised, abstracted and internalised in the mind as tools for thought. Secondly, she considers the expression of ideas and thinking for communication, reflection and inspection; this process of externalisation she suggests can help an individual to refine, clarify and develop their thinking.

According to Piaget, the development of mental processes arises when the individual is engaged in unconscious mental activity to abstract general schemes from experiences and actions. This Piagetian internalisation is within the individual and the mental schemes abstracted are unconscious. This essentially private process seems to me to be inadequate to describe the process by which individuals learn.

From a Vygotskian perspective, the process of internalisation is concerned with relations between social processes and the individual's activities, and is mediated through language. Schemes are built from experiences of external social activities in which individuals share experiences and share their understanding of these experiences using language. Examples of such activities relating to randomness might be an informal family game of 'snakes and ladder' or a discussion at the bus-stop about whether or when the late-running bus will arrive. Within a classroom, the activity might be an activity using dice to simulate traffic flow at a set of traffic lights. The expression, interpretation and exchange of ideas and models enables development of a language within the mind of the individual for thinking about, reflecting upon, internalising and owning versions of these ideas and models, as they become internal mental processes. However, the process of building a shared understanding through conversation, listening to and reading from others, and trying express, may sometimes leave mismatches between the internalised mental models held by different individuals, and may leave mismatches between an individual's mental model and the equivalent idea in formal mathematics.

The process of externalising ideas and models is the relationship between thought and language. Vygotsky discussed this relationship in ways that express the subtle and yet dynamic nature of the process.

> The relation of thought to word is not a thing but a process, a continual movement back and forth from thought to word and from word to thought… Every thought tends to connect something with something else, to establish a relationship between things. Every thought moves, grows and develops,

> fulfils a function, solves a problem. This flow of thought occurs as an inner movement through a series of planes.
>
> (Vygotsky, 1934, edited and translated 1962, p125, from Bliss, 1994, p29-30)

This process is capable of infinite variety and may stop at any point. Many thoughts are never externalised and many cognitive activities are tacit. Indeed, as Bliss points out, there is a distinction to be made "between mental models acquired without explicit instruction and those deriving from teaching" (p29). Bliss suggests that the second are likely to be more readily available for conscious thought and reflection, while the first will often be less easily externalised. Vygotsky suggests that externalisation of thought enables reflection, which in turn aids the development of understanding. Bliss notes that, while actions and speech are transient, externalisation in other media, such as writing, drawing and painting, is more permanent and can enable more extended reflection.

My intention in this study is not to explore the processes of internalisation and externalisation, but to explore what has been internalised by learners about randomness. Thus I am concerned with devising activities which will encourage and enable learners to externalise their ideas to me. I now need to discuss how what is internalised in the mind of an individual can be related to external stimuli.

The diagram in Figure 3.3 below is developed from a similar figure that I devised in an earlier study (Wilder, 1993). It represents schematically some of the different ways in which one thing represents another. Boxes in the top half of the figure represent things outside the mind of the learner, while those in the bottom half are representations held in the learner's mind. The box in the top left might represent an activity in which the learner experiments with a cubical die. The box in the bottom left would then represent the set of expectations held by the learner about how the die will behave when it is rolled, the outcomes that will be observed. This box might also contain the learner's ideas about how to interact with a die, and what to do if it does not behave as expected. The expectations and ideas that constitute the learner's mental model will have been acquired from previous experiences of playing games involving dice, many of which are likely to have occurred in social settings. In such settings, the learner may have observed behaviours such as blowing on the die 'for luck' before rolling it, or they may have experienced the frustration of trying to 'get a six' to start the game!

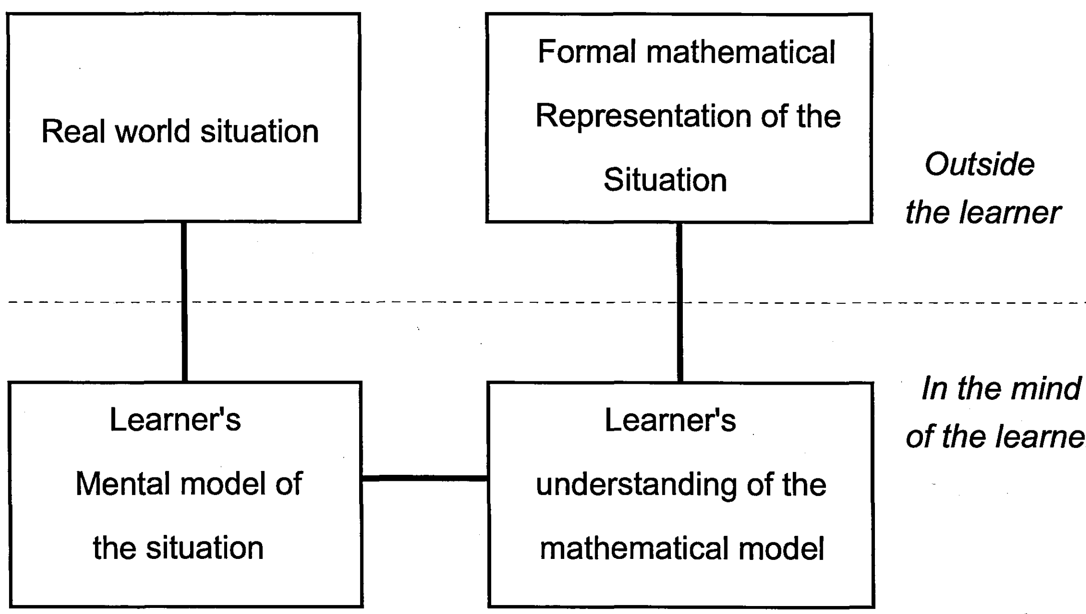

Figure 3.3: Representations and models

Boxes on the right hand side of Figure 3.3 represent mathematical ideas, and are likely to have been the subject of formal educational experiences. I anticipated that, in this study, all the learners would have had some formal introduction to probability. Thus, when presented with the task of experimenting with a cubical die, the learner might call to mind their understanding of a probabilistic model for the outcomes from rolling a fair die. This would be in the box in the bottom right, and it might or might not be consistent with a formal mathematical model for a fair die.

The heavier vertical and horizontal lines joining the boxes represent some of the interfaces across which these various representations interact. Suppose, for example, that the cubical die in the real world situation is discovered by experiment to be biased. There might be initially an inconsistency, a mismatch, with the learner's mental model of the situation. The learner might be expected to adapt their mental model, which would create an inconsistency with the probabilistic model that was called to mind earlier. In attempting to modify their understanding of a suitable mathematical model, it is possible that the learner could be faced with creating a mathematical model that is outside their previous experience of formal mathematical models, in which case there might arise a vacancy in the box in the top right.

The focus of attention in this study is on the box in the bottom left and its interfaces with the top left and the bottom right. I shall not be much concerned in this study with formal mathematical models outside the mind of the learner, but only with the learner's interpretation and understanding of the mathematics: that is, with what the learner believes that they know.

3.15 Concluding remarks

A variety of themes have been discussed in this chapter. In this closing section I summarise those that will be important in later chapters. I have related some of these themes to the bullet points in section 2.8 at the end of Chapter 2.

Reflection upon the historical development of probability and statistics alongside evidence from psychological studies of people's beliefs about luck and chance may indicate that children still hold beliefs in spurious controlling forces or agencies when they come to study probability in school. Indeed some studies (such as Phillips and Wright, 1977, and Amir and Williams, 1999) have indicated that such influences are more prevalent amongst pupils from some cultural and ethnic groups. This relates to point 2a) in Chapter 2, that randomness is seen as either caused or not caused.

The work of Piaget (1951, translated 1975), Green (1989) and Metz (1998) indicate that randomness, distribution and the law of large numbers are a related set of ideas in the development of the concept of randomness. Green points to a failure to consider local uncertainty separately from global aspects of randomness, and Metz suggests that integration of these two aspects is a critical development in understanding of randomness. This issue may relate to the potential conflict between the ideas of randomness and determinism (Chapter 2, point 3b). From a different perspective, Pratt's work has also indicated the critical nature of the transition from working only with local meanings of randomness to holding global meanings of deterministic distribution alongside local uncertainty (Pratt, 1998). This relates to point 2b) in Chapter 2 that randomness can be seen either locally or globally.

Of the widely reported heuristics used by people of all ages in making judgements under uncertainty, some are likely to have particular significance in recognising and interpreting situations in which randomness might be applied. I list below those heuristics discussed in this chapter, which I expect to find young people using in their strategies to discern random from non-random behaviour.

- Representativeness (Kahneman and Tversky, 1972) is widely reported amongst children and adults.
- Availability (Tversky and Kahneman, 1973) is reported to be used in making judgements about risk and may be related to people's beliefs about how hard it is to roll a six with a fair die.
- Equiprobability bias (Lecoutre, 1992) may affect people reasoning about random generators. I wonder whether the equiprobability bias is related to the belief held by some people that randomness is equivalent to having equally likely outcomes (Chapter 2, point 1b).
- Outcome approach (Konold, 1989)

- Pattern seeking as a strategy is likely to be closely related to the idea that randomness is pattern-breaking (Chapter 2, point 4a).

The distinction between distributional and singular uncertainties (Kahneman and Tversky, 1982) relates to the question, identified at the end of Chapter 2, of whether people see single events as being random. Similarly, the distinction between internal and external sources of uncertainty may be significant. While randomness might be expected to be readily applied to situation of external uncertainty, this might not be the case for internal sources of uncertainty.

A variety of methodological issues has been raised in this chapter, relating to methods used in previous studies reported in the research literature. In discussion of the distinction between recognition tasks and generation tasks, I do not see recognition tasks as embodying my idea of randomness as dynamic and I therefore have a preference for generation tasks. However, when people perform generation tasks, I see the need to interview them about their experience of generating outcomes to explore the focus of their attention.

Computer generated pseudo-random numbers are likely to be significant in the teaching and learning about randomness and probability. However, I consider that they would introduce an added level of complexity that is best excluded for the purposes of this study.

Finally I have drawn upon ideas about mental models and a Vygostkian approach to learning to outline a model of learning and understanding about randomness. This model has enabled me to identify my focus for this study as the learner's informal mental models for randomness.

Chapter 4: Aims of this Study

The fundamental purpose of this study is to explore learners' perceptions of randomness. There are two distinct, but related, sets of questions that I sought to address under this heading.

The first set of questions relates to what learners believe about randomness.

- What do learners believe randomness to be?
- What do learners expect from a random generator?
- How do learners recognise what they believe to be random and discern the random from what they consider to be non-random?

The second set of questions relates to the idea that randomness is an important model to be applied to describe a process.

- To what situations and circumstances do learners consider randomness to be an appropriate model?
- What is the range and variety of situations and circumstances for which learners consider that randomness is an appropriate model?
- How are learners' responses to this second set of questions related to their responses to the first set?

In this study, I was not concerned with trying to develop any kind of teaching approach, and I did not seek to effect any change in learners' understanding. Rather, I aimed to identify and describe the range and variety of what learners understand about randomness.

A fundamental question for me in beginning this study was to identify what randomness is. In Chapter 2, I have identified various ways in which randomness is conceived by statisticians and by philosophers. This in itself has presented some major challenges for me in undertaking this study as randomness is defined in several different ways. However, my starting point has been my own personal perspective on randomness as a dynamic concept, which I described in section 2.6. This view of randomness is the one that I intend to present in my research.

Two important considerations relating to my personal view of randomness lie behind my approach to the study. The first is that randomness is a model for a process, not only a description of a sequence of outcomes. Secondly, randomness is dynamic.

I wanted to try to engage learners in discussion of randomness as process, as well as considering how and whether to interpret observed outcomes as from a random process. As noted in Chapter 3, many previous studies have considered people's interpretation of finite sequences of outcomes, often quite short sequences, and have not paid sufficient attention to the subject's ideas about the generating process.

My view of randomness as a dynamic concept has also led me to make criticisms, in Chapter 3, of the methods adopted in much previous research into perceptions of randomness. I have set out to develop short tasks or probes to encourage learners to consider what they expect from the situation presented to them, and to decide whether they believe the situation to be random. An important feature that I wanted to build into these tasks was that they should be dynamic in the sense that I have described, and that they encourage the individual to attend not only to the outcomes as they appear, but also to the generating process.

I also wanted to consider whether it is possible to develop a modification of the standard generation task, in which learners not only attempt to emulate a random generator by producing random outcomes, but also reflect on what they were thinking while so doing. Although it is possible that a generation task leads the individual to focus almost exclusively upon the generated outcomes, I wonder whether by encouraging people to reflect upon and to talk about the experience of generating random outcomes, I might be able to access some ideas about the generating process that they are trying to emulate.

As discussed in Chapters 2 and 3, I decided not to introduce computer simulations of randomness in this study. My concern was with what people believe about randomness; to introduce pseudo-randomness into this discussion seems to me to be a diversion from my main purpose. However, I do not mean to suggest that computers have no role in children learning to experience and understand randomness. Indeed, I see one outcome of my study as being to contribute to the more effective design of software in which children can experience and experiment with random phenomena.

My aim was to explore learner's existing informal notions of randomness and their interfaces with both the real world and with the learner's own formal mathematical understanding of randomness. I wanted learners to express their mental models of randomness – that is, to express those working internalised representations of randomness that they used to solve problems and to interact with real situations. (See Figure 3.3)

Chapter 5: Methodology and Methods

In this chapter, I outline the considerations that have shaped the development of the methods used in my study. In Chapter 3, I discussed my own view of the concept of randomness as being dynamic. I contrasted this view with the ideas implicit in methods used in other studies of people's perceptions of randomness that have been reported in the literature and I considered how my own view of randomness relates to some critiques of previous studies. In this chapter, I draw upon the discussion in Chapter 3 to show how consideration of these critiques and of my own perspective on randomness, and my reading about phenomenography, have shaped the development of the tasks used in this study and led me to use in-depth interviews with a few subjects rather than short probes with a larger sample. The present study is set within a phenomenographic framework.

5.1 Methodology

From an early stage, the development of this study was influenced by my reading about phenomenography as a methodological framework for describing what I was aiming to achieve in my research. As the study evolved, the design moved away from being a purely phenomenographic study, but the influence remains and needs to be acknowledged.

Attempting to probe people's awareness of randomness causes considerable methodological difficulties because of the essentially negative nature of randomness (not determined, not predictable, without pattern). Its meaning as a word could be probed using a Wittgensteinian approach by looking, for example, at the various ways the word is used in the media and in interviews, but I wanted to try to probe beneath the surface of use to try to uncover what makes it such a difficult concept to grasp and to use appropriately. Because I was interested in exploring the variety of experiences and uses of the term amongst students, I turned to phenomenography, which is intended to identify the range of ways of experiencing a phenomenon.

5.1.1 Phenomenography

Phenomenography is related to, but distinct from, phenomenology. Phenomenology originated in the writing of the German philosopher, Husserl, in 1900, and later developed to a philosophical movement that included scholars such as Heidegger, Sartre, Merleau-Ponty, Gadamer and Schűtz. Broadly it represents an approach to the study of phenomena through experience: the researcher's account of her experience of the phenomenon provides the data. Phenomenological research has been influential in sociology and psychology, amongst other areas.

Phenomenography is a much more recent development and arose originally in Gothenburg, Sweden, in the 1970's, in research into conceptions of what it means to learn, and to teach (Marton, 1981; Säljö, 1982). It has since been applied to studying different ways of understanding the content of learning (Linder, 1989; Renström, Andersson and Marton, 1990), and to studying different conceptions of the world.

Marton (1994) suggests that, in phenomenography, the data to be analysed relates to many people's experience of the phenomenon being studied. Phenomenographic methods provide a systematic approach to the collection and analysis of this data. Within phenomenology, the researcher attempts to provide a 'thick description' of the phenomenon based upon her own experiences (direct observation of or 'living in' the phenomenon). In contrast, the phenomenographic researcher uses data from many different subjects and seeks to produce broad categories to describe differences between the ways people experience, conceptualise, understand, perceive and apprehend the phenomenon (Marton, 1994). The aim of phenomenographic research is to identify a limited number of qualitatively different ways of experiencing the phenomenon, and to characterise these in terms of relationships between them, perhaps forming hierarchies in relation to some criteria. This aim closely matches what I hoped to achieve in relation to learners' understanding and thinking about 'randomness'.

Data collection in a phenomenographic study usually involves a large number of individual interviews conducted as a dialogue, in which the interviewee is encouraged to reflect on aspects of the phenomenon, which may yet be unconsidered in the mind of the interviewee. In Marton's words: "The more we can make things which are unthematised and implicit into objects of reflection, and hence thematised and explicit, the more fully do we explore awareness" (1994). Or again, "As phenomenography is empirical research, the researcher is not studying his or her own awareness and reflection, but that of their subjects" (1994). Marton describes the interview as a dialogue, enabling the 'thematisation' of previously implicit aspects of the subject's experience. The data therefore represents an account of these various aspects jointly constituted by interviewer and interviewee.

In a phenomenographic approach, analysis of the data is usually conducted iteratively. The first stage is to focus on "similarities and differences between the ways in which the phenomenon appears to the participants", beginning with an attempt to identify what is immediately relevant in the interview as an expression of experience of the phenomenon. This leads on to identifying "distinct ways of understanding (or experiencing) the phenomenon", looking for similarities and differences between the 'relevant' quotations from the interview and grouping them under these headings. Having identified the groups, the researcher can look for relationships between the groups to see how the "distinct ways of understanding (or experiencing) the phenomenon" are related. The ordered or partially ordered set of categories of description is referred to as the outcome space and constitutes the outcome of the study.

There are undoubtedly difficulties within the philosophical foundations of phenomenography, some of which might be represented by the difficulty of such fundamental questions as "What is a way of experiencing something?" (This question appears on Hasselgren's website (Hasselgren, webref) and represents recent concerns of phenomenographic researchers). However, it is not my intention to engage with these here as the approach has been shown to be effective in previous studies (Marton, 1994; Linder, 1989; Renström, Andersson and Marton, 1990; Runesson, 1999). Rather, I intend to use the pragmatic intentions of phenomenographic research to represent in general terms what I hope to achieve in relation to learners' experience and understanding of 'randomness'. I use interviews to probe learners' experience and understanding of what it means to say that something is 'random', based around the learners' immediate experience of addressing tasks which involve the idea of randomness in some way. For example, this entails examining when a subject chooses to describe something as 'random' and when as 'not random', and how a subject describes and reasons about random situations.

5.1.2 Why interviews?

Much previous research into perceptions of randomness has focused on how subjects respond to written problems with multiple choice responses (see for example, Green, 1983). Subjects' written responses are then examined and interpreted. Other research has interviewed subjects about problem situations and used subjects' responses to identify heuristics that appear to underpin 'judgements under uncertainty' (Kahneman and Tversky, 1972; Kahneman, Slovic and Tversky, 1982).

Pratt (1998) took a very different approach. In his study, he created a computer microworld for exploring randomness, and interviewed pupils as they worked in this microworld. The idea of using computer software to provide an environment in which learners make explicit their ideas about randomness is an important one, to which I may need to return after the present study. However, my concern in this study has been to discover how learners interact with and think about situations that they would usually expect to be random. In this study, I considered that to have worked with pseudo-random generators would have been an unnecessary additional complication, as discussed in section 3.13.

In a recent experimental study of school students' perceptions of randomness, Batanero and Serrano (1999) adopted a quantitative approach. Data was collected from 277 students and a series of significance tests was conducted to demonstrate where students' ability to recognise randomness shows significant changes from age 14 to age 17. In follow-up interviews, the researchers asked the students to explain how they could tell whether the given set of outcomes was 'random' or not. The children's explanations were used to infer how the children were reasoning about randomness. Because the children's explanations of how they recognised randomness were based on their interpretation of given sequences of outcomes (recognition tasks), I was not convinced that the inferences drawn in this study were valid. The recognition of randomness in fixed sequences of outcomes does not seem to me to hold the 'dynamic' sense of random that I believe is important. This study attempted to do something similar to what I wanted to do, but the study had not explored the students' reasoning in depth.

In my study, my intention was to probe subjects' understanding of randomness to a greater extent than had been done previously. Semi-structured interviews were organised around a sequence of 'stimulus' tasks or activities. The tasks provided a context in which this greater depth of insight could emerge. Questions asked in the interviews sought to probe why subjects hold the conceptions that they do, and whether there are limitations to the circumstances in which subjects will apply these conceptions.

The style of interview that I envisaged was rather like the clinical interviews used by Piaget. In a critical analysis of the clinical interview procedure in mathematics education research, Ginsburg (1981) suggested that clinical interviews can legitimately be used for three different research purposes: to discover cognitive activities; to identify cognitive activities; and to evaluate levels of competence. My purpose in this study has been to discover the variety of mental structures, processes and thought patterns that learners use as they interact with instances of randomness, and to explore what the learner is aware of and to what aspects of the experience of randomness they give attention. I have not been seeking to evaluate competence, and before I started this study I did not have a detailed map of the variety of thought processes that I might find. My purpose therefore fits closely the first of Ginsburg's three purposes for clinical interviews.

However, it is important to recognise that an interview is a conversation between two people: the interviewer and the interviewee. The data arising from the interview cannot be seen as independent of the interviewer-researcher, especially when, as in this case, the interview is semi-structured to allow freedom to follow the ideas that emerge. The product of an interview is, in the words of Kvale "knowledge… created inter the points of view of the interviewer and the interviewee" (Kvale, 1996, p124).

A critical element in the success of these interviews was the choice of probes to be the focus for discussion about the idea of 'randomness'. I assumed initially that this stimulus should take the form of a task or activity. However, such tasks can be varied in several ways. The development of the tasks used is discussed in section 5.2.

5.1.3 Design and conduct of interviews

The purpose of a qualitative research interview is to obtain a qualitative description of the interviewee's lived experience, and of the meanings that the interviewee interprets from their experiences (Kvale, 1996). In a semi-structured interview, one of the aims is to allow the interviewee the freedom to talk about what is important to them within the parameters of the focus of the interview, and to express their views. No formal script is used, although an agenda – a structure or framework – for the conversation has been established beforehand. The existence of the framework helps to organise the interview to ensure that important aspects of the focus are not omitted. The framework also provides an organisation for the process of analysis. Kvale has suggested that a semi-structured interview:

> ...has a sequence of themes to be covered, as well as suggested questions. Yet at the same time there is an openness to changes of sequence and forms of questions in order to follow up the answers given and the stories told by the subjects.
>
> (Kvale, 1996, p.124)

An interview with a single interviewee is relatively easy to transcribe from audiotape. In interviews with pairs or small groups, it can be difficult to identify from the tape precisely who was speaking at any instant. Also people interrupt one another; the transcription can become difficult to record accurately. In a one-to-one interview, the opinions and views expressed stem only from the exchange between interviewer and interviewee, which can help make the process of analysis more manageable.

Questions, probes and prompts need to be expressed in a clear, straightforward way, in order not to confuse the interviewee. It is important to avoid using words or phrases that are unlikely to be understood by the respondents. The interviewer needs to pay close attention to the interviewee's response to any question in order to be aware of times when the interviewee has not understood, or might have misunderstood. Where possible, questions should be kept short; lengthy and complex questions are difficult to understand.

> Dynamically, the questions should promote a positive interaction; keep the flow of the conversation going and motivate the subjects to talk about their experiences and feelings. The questions should be easy to understand, short and devoid of academic language.
>
> (Kvale, 1996, p.130)

Care is needed in establishing the mood and direction of the interview in the first few minutes. The interviewer needs briefly to introduce the purpose of the interview, explain the use of the tape-recorder, and invite the interviewee to ask any questions. The first few minutes on an interview are very important in establishing a relationship between interviewee and interviewer. The interviewer needs to listen attentively, and show interest in what the interviewee says. It is also important that the interviewer should be at ease and totally clear about what is to be discussed. As far as possible, the early stages of the interview should be kept simple, to allow the interviewee and interviewer to get used to each other. As the interview progresses, the nature of the questions and the issues addressed may become more demanding and searching.

At the end of the interview, the interviewer again allows the interviewee to ask any questions or to add any further comments. At this stage, it is possible that the interviewee may be experiencing some tension of anxiety, particularly if the interview has been searching or demanding, so it is important to allow them time to unwind a little and to emerge from the experience (Kvale, 1996).

During an interview, the interviewer listens closely to what is said by the interviewee. Interventions from the interviewer are sometimes necessary to direct the conversation in productive directions or to encourage the interviewee to reflect more deeply and expand upon an idea expressed. However, it is important that the interviewer should limit their interventions since the intention is to hear the experiences and reflections of the interviewee. Theman noted particularly the need to take nothing for granted in the interviewee's experience and understanding of the focus of the study.

> We can never take for granted that we know what our subjects know about a certain phenomenon, even though we are able to talk about it in very similar general terms.
>
> (Theman, 1979, p3)

Related to this, Theman shows how the interviewer's interventions can lead to several problems. First, he noted, with reference to an interview he conducted, how easily an interviewer can unintentionally anticipate the interviewee's utterances and hence fail to record the evidence that a key idea originated from the interviewee. He commented: "When the subject... almost says the thing I want to hear, it works like a trigger, it makes me immediately take the lead" (Theman, 1979, p5). The interviewer's intervention means that, in the recorded dialogue, the interviewee's conception is inseparable from that of the interviewer.

A similar difficulty can arise from the fact that one of the interviewer's aims within the interview is to bring the interviewee to work on the edge of their existing understanding, which means discussing knowledge and understanding that is either new or to some extent unreflected. During such discussion, the interviewee is likely to experience significant insecurity which might result in a dependency on the interviewer. If the interviewee is left out of their depth, then this kind of exchange can pose a threat to the relationship of trust between interviewer and interviewee, but if the interviewer provides support too readily then the interviewee may come to expect support whenever they experience some difficulty, which would undermine the dynamic of the interview process. Theman gave several examples of how delicate the balance between challenge and trust can be (Theman, 1979).

In a phenomenographic interview, in which the focus is to explore the varieties of ways of understanding a topic, one specific difficulty is that the interviewer will need to seek to explore the boundaries of the subject's understanding, and perhaps also watch the subject in the process of building new understanding. Theman (1979) drew attention to the consequence that, when the interviewee declares they have reached the limits of their understanding, it is necessary to check this claim in various ways. For example, it may be that the interviewee simply did not understand the last question, and that when the question is rephrased they can proceed further. Or it may simply be that the subject has had enough and wishes to stop the interview process. In either case, further questioning may be perceived as unwelcome. Theman suggested that the interviewer needs to establish a clear agreement with the interviewee from the start of the interview that further questioning is the object of the research. By establishing a more equal and open exchange between interviewer and interviewee, in which the interviewee is encouraged to make use of the interviewer as a resource in their search for understanding, Theman hoped that the interviewer would be able to probe more deeply the interviewee's conception.

Theman's approach illustrates the dilemma faced in conducting interviews of this kind. The interviewer needs to keep in mind the danger that suggestions made to the interviewee, and other interventions made during the interview, will potentially distort the interviewee's expression of their understanding. And yet, a purely objective style of questioning will do little to build the relationship of trust between interviewer and interviewee necessary to encourage the interviewee to express their ideas openly and honestly.

Having considered the issues discussed in this chapter, I adopted an approach of in-depth interviews, along the lines of clinical interviews (Ginsburg, 1981) in which my follow-up questions depended on the responses I got from subjects.

The purpose of my enquiry was not only to identify the variety of ways in which the people interviewed thought about randomness, but also to shed light on why randomness is hard to grasp intellectually. The analysis of the data is not statistical, but is designed to probe beneath the surface of meaning and conceptual development. Thus it was important to learn from each interview to inform the next. Issues that emerged in conversation in earlier interviews sometimes became a focus for more extensive discussion in later interviews. Although the structure of the interviews remained broadly the same, there were differences of emphasis, arising both from a desire to probe particular issues, and from the serendipitous nature of the tasks that were the focus of discussion in the interviews.

5.1.4 Concept maps

A particular kind of probe that was considered was the use of concept maps. A concept map offers a method of displaying visually the ideas, words, methods and problems related to a central theme. In the case of the present enquiry, the central theme would be 'randomness'. Related ideas might include random generators of various kinds, probability, chance or disorder. The diagram produced consists of a set of nodes, each representing an idea, word, theme or problem, and each connected to related ideas by lines or arrows. Where appropriate, an arrow could be used to indicate the direction of a relationship between two nodes. The idea behind using concept mapping as a research probe is that the map produced by a subject has been found to give insight into the subject's understanding of a mathematical topic, such as function (McGowan, 1998; Williams, 1998). My hope was that it might be used similarly to consider students' understanding of 'randomness', and could be used later as the focus for discussion about the ideas and links included in the map, and what had been left out.

Various formats are possible for a concept map. Perhaps the most simple is a 'spider diagram' with related ideas and themes radiating out from the central theme, and these in turn connecting to further ideas radiating out from them. Other formats might be a 'hierarchy', using a diagram looking more like a tree diagram, or a system akin to a flowchart. Drawing on McGowan's work on understanding of function (McGowan, 1998) the most appropriate format for investigating a learner's conception of randomness was a spider diagram.

Concept mapping has been used in other research studies in mathematics education (Williams, 1998; McGowan, 1998). The method has also been used in assessment of children's understanding of scientific concepts (Comber and Johnson, 1995; Roth, 1994). McGowan (1998), in a study of students' conceptions of 'function' in mathematics, identified a number of constraints on the interpretation of concept maps produced.

- The amount of time the student spends constructing the map.
- How much information and how many connections between elements the student is able to record within the constraints of 2 dimensions and limited space.
- The student's ability to categorise, organise and reflect upon her own perceptions and actions.

McGowan (1998) described how, during the course of her study, she revised the instructions she gave for creating a concept map. She instructed subjects to conduct an initial brainstorming of ideas, and recommended subjects to put these onto small post-its, which could then be arranged and rearranged as additional ideas were added. The final map was not to be drawn until the student felt she had arrived at an arrangement that reflected the organisation and connections appropriately. This approach has the benefit of keeping the emerging concept map 'fluid' so that the subject can continue to interact with it and refine it until they feel happy with the outcome.

As well as the difficulty for the student of reflecting upon, and then organising and expressing, her own ideas about 'function' within the constraints of a concept map, there is the issue that many students are not familiar with concept mapping, and so need to be carefully inducted into the method. Williams (1998), in a report of her use of concept maps in a study of conceptual knowledge of function, emphasised that she had instructed all her volunteers at some length about what she meant by a concept map. She showed participants examples of other peoples' concept maps of different concepts before asking them to produce their own concept maps of 'function'. Williams' account indicates that there might be an additional constraint on the use of concept maps to add to the list offered by McGowan above:

- The extent of the student's prior experience in producing concept maps and the depth of her understanding of the idea of a concept map.

The use of concept mapping with students requires careful explanation to students of what is required, and the students need time to formulate their ideas as well as expressing and organising them onto paper in the form of a concept map. I have given an account in Chapter 6 of some further issues that arose when I used this method with students.

5.2 Methods

I begin this section with a brief outline of the structure of the whole study.

5.2.1 Outline of the study

The study has passed through several stages.

In the preliminary stage, I made an initial exploration of how people use the words random and randomness, and what ideas people have about the situations in which the ideas of randomness are applicable. In this stage, the methods were informal, taking advantage of opportunities to discuss ideas with groups of people who were gathered together for other purposes. I used sentence completion tasks and I tried concept mapping. I also tried early versions of some of the tasks that were developed later.

The first stage consisted of a series of nine clinical interviews around a collection of 'dynamic' tasks which are described below in section 5.2.3. Interviewees were school students aged from 13 to 17. Interviews were recorded on audiotape and, shortly after each interview, the tape was transcribed.

In the preliminary analysis of the first stage interviews, a detailed commentary was added into the interview transcript. This included a detailed record of the outcomes generated in the 'dynamic' tasks and a detailed description of my impressions of how the interviewee responded to what happened.

When all the interviews had been completed and the transcripts and commentaries prepared, the transcripts and the interview commentaries were analysed for themes that occurred across interviews. This first analysis was difficult and lengthy; at first it produced a very long list of themes, from which emerged four major questions relating to randomness, which I considered to provide motivation for the behaviours and utterances of the interviewees. The questions were;

- What is randomness?
- How is it expressed?
- How is it recognised?
- Is it caused?

In the review towards the end of the analysis of the first stage interviews, the big idea of 'shifting perspectives' was recognised and described for the first time. In order to investigate this idea further, and to see this idea from a different direction, a new task was developed. This was the 'counters' task.

The second stage consisted of a further nine clinical interviews with a new sample of students, aged between 13 and 17. In each interview in the second round, the tasks from the first round were preceded by the counters task. However, a key focus of the interview questions in this second stage was to explore further the ways in which the interviewee's way of seeing the tasks changed. By watching the behaviour of the interviewee and listening closely to what they said, I tried to see when the interviewee's attention shifted in various ways. Again, the interviews were recorded on audiotape and a detailed transcription was made and a commentary was added.

In the analysis of the transcriptions and commentaries of the second stage interviews, I initially searched for and identified instances of the themes identified in the analysis of the first stage. As a consequence of examining the new data, the themes were reconsidered and refined.

5.2.2 Permission

All participants were asked to give their permission for their responses to be used in the reported study. Permission was also obtained from parents. In order to respect the anonymity of subjects, names used in the report are pseudonyms.

5.2.3 Development of a process task

I discussed in Chapter 3 the varieties of task that have been used in previous studies to investigate people's understanding of randomness. I presented there an analysis of the advantages and disadvantages of *generation* tasks and *recognition* tasks. These considerations led me to reject recognition tasks as the main stimulus tasks for my interviews, and to favour generation tasks. However, in discussion with colleagues, it was suggested that generation tasks might not really be directing the subjects' attention to the randomness of the generating process, as I had hoped. My experience with a subject in an early interview reinforced this doubt. It seemed possible that subjects were actually generating a set of outcomes with the properties that they believed were characteristic of a random sequence of outcomes. There are at least two distinct reasons why this might be so. First, the subject may have little awareness of the process aspect of randomness, and so be finding alternative approaches to the task. If this were true, then possibly any stimulus task would fail to direct attention and awareness towards the process. Alternatively, generation tasks may simply not be the best tasks to direct attention towards process.

In an attempt to focus more directly upon randomness as process, I developed a new task for use in interviews. In this task, the interviewee is given a random generator, such as a die, asked to examine it and use it to generate results, which are recorded by the interviewer as they are generated. The interviewee is then invited to consider the outcomes so far, and to talk about what they think will happen next. Follow-up questions explore how the interviewee has arrived at any prediction, how confident they are in what they say, and how they feel about the generator so far. In order to provoke further the interviewee's awareness of the process, the first stage interviews explored the use of unfamiliar dice as generators. Most interviews began with the use of a severely biased die. The intention was that interviewee's expectations of how a die should behave might be challenged rapidly, opening the possibility for discussion about the interviewee's preconceptions. After this a second, unfamiliar die was presented - this time a spherical die - which from the interviewee's perspective may or may not be fair. A third die was a standard cubical die, but with a deep crack dividing the whole of one face and parts of two others. I hoped that the possibility of bias in each of the second and third die would provoke in the subject a conscious awareness of how a random process using a die would be expected to behave, and that this would help the interviewee to externalise this awareness during the interview. This task and the other tasks used in the stage 1 interviews are discussed in greater detail in Chapter 7.

From the phenomenographic perspective, there is a danger in using activities such as these as the core for the interviews: the interview could become focused on the interviewee's experience, understanding and awareness of the activities rather than the concept of 'randomness' itself. That is to say, the activity could acquire greater significance in the interviewee's experience during the interview than the underlying concept of randomness, towards which the activities are intended to direct the interviewee's attention. In the early stages of my research, I could do no more than be aware of this possibility. I consider this danger further in my discussion of the lessons to be learned from the first stage of interviews in Chapter 8, and I consider a response to the issue in the design of the second iteration in Chapter 10. The conduct of the stage 2 interviews is discussed in Chapter 11.

5.2.4 Interventions during the tasks

My approach as interviewer and researcher in this study has been to intervene during the interviews in whatever way seemed sensible in the moment, and to describe my interventions as fully as possible. My descriptions have included the utterances and behaviours from both interviewee and myself that I perceived as leading up to my intervention. By providing the 'thick' description, I have aimed to preserve the authenticity and the trustworthiness of my account, even though I acknowledge that complete objectivity is impossible.

The generators used in my process tasks were familiar objects: dice and coins. There was therefore a danger that interviewees might consider the tasks trivial and fail to engage with the nature of randomness. I hoped that I would generate some element of surprise in presenting the biased die first, and that this would draw the interviewee into reflecting more deeply on the outcomes and the generating processes. I also proposed to ask the interviewee to pause in the process of generating outcomes when I observed sequences of outcomes that might be provocative. Examples might be a run of three successive sixes in rolling a die, or a run of four or five successive heads when tossing a coin, or later in the sequence of outcomes, an under-representation of a possible outcome from the sequence of outcomes observed. At such moments, I would invite the interviewee to comment on what they saw and to consider what they thought might occur on the next outcome.

An important factor in perceptions of randomness is 'culture'. Amir and Williams (1999) operationalise 'culture' as *beliefs, language,* and *experience.* I wanted to incorporate some interrogation of these aspects of the interviewee's culture into my interviews and I borrowed a little from the framework of Amir and Williams in considering how to do so.

Children might bring to the interviews beliefs, whether religious or superstitious, governing their attitude towards chance and their readiness to engage in probabilistic thinking. By asking interviewees to consider what they thought might occur next at various points in the interview, I was inviting them to express their beliefs. However, I was aware that some beliefs might not be articulated without encouragement. I needed to value the interviewee's ideas and to invite them to give more than one-word answers to questions such as "What might happen next?" by following up with something like "Why do you think that?"

I have already discussed the need to be aware of the limited probabilistic vocabulary that some children might have available to them in discussion of the interview tasks. My response to this issue required me to be careful in using words such as 'random' or 'probability' in my questioning. But I also needed to listen closely to interviewees' as they expressed ideas that might be unfamiliar, to ensure that I was able to discern as closely as possible the sense that they were trying to convey.

Finally, I planned to ask interviewees about their prior experiences of playing with or using random generators. I expected that most would have had extensive experience of playing games with dice or packs of cards, but that there might be some who had not. I also wanted to ask what experiences interviewees had had of using random generators in mathematics lessons. I was particularly interested in seeing whether they remembered exploring long run behaviour, and possibly linking this with their formal knowledge of probability.

5.2.5 Analysis of interview data

Analysis of interview data requires close attention to the detail of the exchange in the dialogue. This detail resides not only in what is said, but also in the behaviour of the parties and in the circumstances surrounding the exchanges. In the context of my interviews, I needed to take account of the outcomes generated at each stage from the random generators used in the process tasks, and I needed to watch and listen closely to the reactions of the interviewee. For example, if an interviewee seemed to express surprise, then I needed to check whether this was indeed the case and look closely at what might have been the trigger for this reaction.

In a sense, some analysis is almost inseparable from the interview process itself. During the course of any interview, I was listening and watching closely for signs of reaction on behalf of the interviewee to the successive outcomes. Sometimes – when I inferred that the interviewee had shown unease, surprise or some other reaction – I was able to prompt the interviewee to speak about this by asking a question. However, this was never going to be possible all the time, so the later stages of transcription and more detailed analysis of the text were vital to the development of a more detailed and contextualised account of the data.

In order that the later processes of transcription and the creation of a reflective commentary for each interview should be most fruitful, it was important that they were done soon after the interview. My own recall of the detail recorded in the audio-tapes and in the record of the outcomes from the random generators was an important resource in providing the detail to the reflective commentary on the interview.

During the analysis, when studying each extract of the interview transcript, it was important for me to consider the precise question that each interviewee considered they were addressing. The nature of the tasks meant that, while sometimes an interviewee's attention might be on the question of whether the process of rolling the die and recording the outcome was random, while at other times the issue might be whether or not the die was biased. The difference between these two questions is slight, but is potentially significant for some people. It is possible that a die might be recognised as biased, but the outcomes might still be considered to be random.

5.2.6 The next few chapters

In Part 2 of this study, the next chapters report on what happened in the first stage of the investigation. In Chapter 6, I report on the initial informal stage of the investigation, during which I tried out ideas for data collection and gathered an initial sense of the range of ways in which various groups of people spoke about randomness. Chapter 7 contains a detailed description of the first stage interviews, of the tasks used in them, and reports on the individual interviewees. Then, in Chapter 8, I discuss the data that arose from the first stage interviews; I describe how I went about the analysis of the interview transcripts and I report on the themes that emerged from my analysis. A major theme to emerge from the first stage was the interviewees' shifting perspectives between randomness and distribution, and this is described in detail in Chapter 9.

In Part 3, the theme of shifting perspectives formed the basis for the development of a new task for the second stage of interviews, which is described in Chapter 10. A detailed description of a revised interview structure used in the second stage interviews, and reports on the individual interviewees, are given in Chapter 11. In Chapter 12, I discuss the data from the second stage interviews and show how the themes of shifting perspectives developed during stage 2. At the end of Part 3, Chapter 13 contains an analysis that draws on the discussions in Chapters 6, 8, 9 and 12.

PART 2:

Stage 1 of the Study

Chapter 6: Background to Stage 1 Interviews

In the first few months of this study I made several informal investigations of ways in which people think and speak about randomness. At this initial stage I made little attempt to be rigorous or exhaustive, and the people used in this stage were groups of people who were readily accessible and who were willing to answer a few questions or to take part in a few short activities. My intention was to gain an initial sense of the variety of ways in which ideas about randomness were expressed.

In this chapter, I report on data gathered from three different settings. I report on the data from each setting in turn, and then summarise the lessons I learned from this preliminary stage of the investigation. In each setting, one of the tasks was designed specifically to elicit a description of what people thought would be meant by 'randomness'.

At the end of the chapter, I have included an account of an incident that led me to discover some unexpected uses of the word 'random' in everyday contexts. I have included it here because it occurred during this preliminary phase of my work, and it influenced the way that I approached the interviews later.

6.1 A level Students

The first setting that I investigated was a small group of three A level students, all girls, in the final two months before they sat their A level examination in Statistics. I was the tutor for these students for the final two months of their A level course after their regular teacher had to go into hospital. These three girls were the only students in their sixth form who were taking this statistics module. Remarkably, two of the three girls happened to be identical twin sisters! The school is a non-selective mixed RC 13-18 school. During the lesson, I asked students to produce concept maps for randomness, but there was some discussion about an examination question first.

Over the weekend before this lesson, the students had been asked to work through a specimen paper, and all of them had experienced difficulty with a particular question. The students did not know how to respond to part (a) and therefore could not access the question. The twins had each made a written attempt, but the third student had been unable to start. The students were not used to discussion in mathematics classes and they were all very reluctant to offer any suggestions in class to tackle the question in part (a).

6.1.1 The question

Question 2 from Edexcel GCE AS/A Statistics, Unit S2, Specimen paper (2000):

> A piece of string *AB* has length 12 cm. A child cuts the string at a randomly chosen point *P*, into two pieces. The random variable *X* represents the length, in cm, of the piece *AP*.
>
> a) Suggest a suitable model for the distribution of *X* and specify it fully.
>
> b) Find the cumulative distribution function of *X*.
>
> c) Write down $P(X<4)$.
>
> (Edexcel, 2000)

The question in part (a) is worded in an open way, and appears to invite students to speculate and be inventive. In reality, of course, this is a timed examination; the examiner has in mind an 'obvious' response and expects a short answer.

6.1.2 The discussion and concept maps

In the passages of dialogue quoted below, the twins are D and S, the third student is R and the tutor is T (me in this case).

The dialogue initially revolved around how to solve the question. I invited the students to focus upon the words 'at a randomly chosen point *P*', and to consider what this tells us about the Point *P*. I was hoping for some thoughts about the likelihood of the point lying at different places on the line *AB*, as well as the obvious impossibility of the point lying outside *AB*. The response was extremely slow.

D: (*after a long silence*) It could be anywhere.

R: (*after another long silence*) It's not affected by anything else.
It's gone somewhere. It's gone where it's gone.
(*pause*) I suppose it's independent of itself.
(*pause*) I mean independent from anything else.

T: Do you have a picture in your head?

Silence.

I had hoped that a mental image might encourage them to externalise their ideas about whether the point *P* was more likely to lie in some parts of *AB* than in others. R described the following image, and I invited her to draw it on the board.

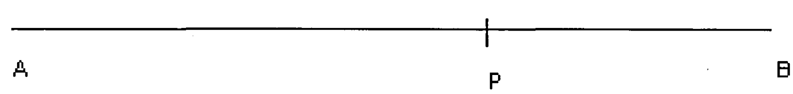

Figure 6.1: A randomly chosen point P on a line segment AB

The twins had each drawn a similar diagram in their written attempt at the question. I asked R if the position of *P* was fixed in her image. She said it was not, but was free to float along the line *AB*. When invited to comment on this, S said she had a similar image, but that *P* was fixed at where it had been chosen by the child. It seemed to me that R had expressed a dynamic sense of the randomness of the position of the point *P*, while S had not. My own image was of the point *P* floating freely along the line *AB*, not fixed until I needed to fix it.

At this point I introduced the idea of concept maps. As well as wanting to try getting people to draw concept maps about randomness, I hoped that this activity might also help these students to see some connections.

I invited them to place 'randomness' in a bubble at the centre of the page and then to think of all the related ideas they could and write these on the page around the central idea, showing how these ideas were related. R produced about 8 ideas, S about 3 and D about 6. Their maps were very simple with links radiating out from the central 'randomness' and almost no links across ideas.

In order to try to extend and develop the ideas they produced and the connections they made, I asked them to change pens to a different colour and to consider the ideas I gave them. They decided whether each idea should be included on their map and if so where and how. Then I gave them a collection of ideas:

> Dice, coins, spinners, random number generators,
> Chance, probability, distribution,
> uniform, binomial, Poisson, normal,
> sample, random sample, population,
> hypothesis test, significance level,
> risk, accident, Potters Bar.

I asked them to add in any further links or ideas that occurred to them, and then I gave them some more ideas to consider.

> Likely, Equally likely,
> Random variable, discrete, continuous,
> probability function, probability density function, cumulative distribution function

I collected the resulting maps and studied them later to consider what I could learn from this experience. In the lesson, the pupils returned to discussion of the question, particularly to the diagram on the board. With considerable help they were eventually able to arrive at a solution.

6.1.3 Reflections on concept maps

After the lesson, it occurred to me that perhaps the problem for these students arose, not from understanding 'randomness', but rather from understanding the term 'model' in the question. However, my experience of trying to elicit concept maps had showed me how important it was for subjects to understand what a concept map is. As noted in the discussion about concept mapping in section 5.1.4, Williams (1998) gave her participants careful instruction in concept mapping, and encouraged them to look at examples of concept maps of other concepts before she asked them to produce their own maps of 'function'. I had not given my students enough idea of what a concept map might look like and what purpose it might serve. In particular, I had not encouraged the students to consider the relationships connecting the ideas they had shown on their maps, and to look out for clusters of related ideas. As a result, the maps I elicited from the students were not very informative.

I considered trying McGowan's approach of subjects writing their ideas onto 'post-its' before rearranging them on the concept map (McGowan, 1998, discussed in section 5.1.4), but concluded that, while it might be effective, it would be too time-consuming. I also considered trying to get subjects to produce concept maps as one activity within a one-hour interview session. I concluded that, if I did use concept mapping, this would need to be done outside the interview setting, possibly with several subjects in the room together, although each working independently. That way, the instructions could be delivered to several subjects at the same time. In the end I did not return to this method again.

6.1.4 Words and phrases to describe 'random'

The concept maps produced by the three students were not at all sophisticated. They each had a single bubble at the centre containing the word 'random', surrounded by a number of single lines leading to a variety of words and phrases. None of them had any connections to ideas at a second level. However, I listed the words and phrases used by the students in their concept maps, ignoring items that they had included from the list of suggestions that I had given them. The list I constructed from the students' concept maps is shown in Table 6.1 below.

When I examined this list, two things struck me as echoes of ideas from my reading. Firstly, several of the ideas were phrased in the negative, describing 'random' in terms of what it is not. There were also suggestions of this in the short dialogue quoted earlier. Secondly, I felt that some of the words and phrases described what might be observed about 'random outcomes', indicated in Bold in Table 6.1. Others were much more about a 'random process', shown in Italic. The remaining three descriptions, I considered might have been about either process or outcomes. Apart from 'unpredictable', the words that were in the negative were all words that I considered applied to the outcomes. This distinction between randomness of process (primary randomness) and randomness of outcomes (secondary randomness) had been in my mind from my reading.

Without pattern
Without structure
Irregular
Unorganised
Unpredictable
Variation
No specific logic
Equal chances
Happens
Thought of
Arrived at

Table 6.1: Words and phrases to describe 'random'

6.2 Key Stage 3 lesson for Able Pupils

This second setting arose because I was teaching a series of classes for a group of 20 able pupils in Years 7, 8 and 9, from neighbouring schools. These pupils had all been nominated by their schools to receive an invitation to attend one of a number of different courses for able children, and they had then chosen to follow this course of six sessions, entitled Exploring mathematics with ICT. During one session, I introduced them to randomness, but before inviting them to explore computer-based random simulations, I asked them to try some activities away from the computer. I had two aims in using these activities: first, I hoped to encourage the pupils to think about the behaviour they expected to see from some random processes before they looked at simulations of these; and secondly, I hoped to learn something about the range of ideas about randomness that were represented amongst this select group.

I used a sentence completion task to elicit pupils' ideas about randomness. Pupils wrote the start of a sentence "I think randomness is …" and were invited to complete it. After they had attempted this, they were given a second sentence to complete: "I would know it is random because…". Finally, they were invited to write down words and phrases that they would associate with 'randomness'. I was aware that in the first task I was taking the risk of presenting the pupils with a word whose meaning they might not know. I hoped to get a sense of how unfamiliar this word was to able children aged 11 to 14. From the second sentence, I hoped pupils would suggest how they might recognise a random sequence. Finally, I wanted to explore whether the pupils were aware of situations in which randomness might be used as a model.

6.2.1 The data

There were sixteen responses to the three tasks. The final task appeared to give some insight into the pupils' awareness of the word 'random', since even amongst those pupils who were unable to say much in response to the first two tasks, some were able to associate the word with other relevant terms. Twelve of the sixteen appeared to associate some understanding of the concept of randomness with the word 'random', mentioning words such as chance, luck, lottery or probability in response to the third task. However, the other four showed no mention of such terms and their sentences showed little understanding of randomness.

The responses of the sixteen pupils to each task are shown in Table 6.2 below. As in Table 6.1, I have used Bold to indicate ideas which say something to me about outcomes and Italic font to indicate ideas that are more about process. The words struck out were also struck out in the pupil's script. It is worth noting that, although I did not encourage the pupils to collaborate over their responses, there was some surreptitious exchange of ideas. This may lie behind the fact that the responses, which are listed in the table in the order in which I collected them in from the pupils, show some clustering. For example, numbers 13, 14, 15 and 16 list similar responses in the words and phrases response, and they were probably seated near each other in the classroom since their scripts came in to me one after the other.

	Randomness means:	*I would know it is random because:*	*Words and Phrases*
1	The same as random	If there is any number picked out of a hat at 'random'	Chance, Luck, At any time
2	A selection of a group in different orders	The numbers will be in a different order	Chance, randomness, probability
3	I don't know what it is	You can't tell	Random, lottery, chance
4	It is where you change an algebraic letter to a random number to see if a formula works	I randomly chose it out of all the numbers in my head	Random, luck, chance, probability
5	Any item (in number) which **does not follow a sequence**	The items would not follow a sequence at all	Non-sequential, chance
6	The selection of numbers (or something similar) with **no preference to the selection**. Every number has an **equal chance of coming up**, so any number is as likely to come up as any other. This is best done with the vibrations of a radioactive atom, or something else with absolute randomness.	The numbers would follow no pattern whatsoever	Radioactive, chance, probability, lottery
7	Making a chance, and **you don't know the outcomes**	I don't know, you can't tell.	Luck, anything, chance, probability
8	I don't know	I don't know	Luck, chance, probability
9	If something is a random order it is in **any order that happens**	There is no specific order or pattern	Approximate, chance, probability, estimation
10	As an example, is to **pick any number** or an **educated guess**	It was not associated	Probability, estimate, approximation, chance
11	**Picking** something at random **without knowing what it** is	I would not know which one is which	Chance, luck, probability, estimate, guess
12	I don't know what it is	I didn't choose it. I just picked it straight away.	Probability, chance, luck, estimate, guess
13	(~~When the numbers keep going and don't stop with the number you think is right~~) Don't know	~~I have read all about~~ I have a number sequence	Relationships, groups, sequences
14	a group of data that **has no relationships**	If it was in a sequence that doesn't repeat itself	Relationship, sequence
15	I don't know what it is	It's just random	Sequences, odd and even numbers
16	I think it means something that is taken	Don't know	Taken, estimate

Table 6.2: Responses of pupils to sentence completion tasks

The following list of bullet points summarises the kinds of response received from the sixteen pupils to the first task.

Words and phrases used by KS3 pupils

- A selection of a group in different orders
- Any item which **does not follow a sequence**
- The selection of numbers with **no preference to the selection**
- Every number has an *equal chance of coming up*, so any number is as likely to come up as any other
- Making a chance, and **you don't know the outcomes**
- A random order - in *any order that happens*
- To *pick any number*
- An *educated guess*
- **Picking** something at random **without knowing what it** is

- A group of data that **has no relationships**
- I think it means *something that is taken*

Within these responses are further examples of randomness being thought of in terms of what it is not: 'not follow a sequence'; 'no preference'; 'you don't know the outcomes'; 'no relationships'. Inevitably these responses are less clear and precise than the responses given by people in the other two settings.

A second theme within these responses is that randomness is to do with selection or 'picking'. I guess that children hear many uses of the word 'random' that involve randomly selecting or choosing an item from a set of similar items. This may also help explain why some people identify randomness with equal likelihood. The classic 'urn' model of randomness, which lies behind the idea of 'picking at random', carries the idea of 'equal likelihood' implicitly in the context.

Finally, a surprising number of these pupils (I think seven) suggested interpretations of randomness that referred to algebra, or sequences of relationships. Whilst it would clearly be foolish to emphasise the detail of these responses without questioning the pupils further about what they wrote, there are suggestions here that some pupils had a restricted understanding of the word 'randomness' and a few were unable to interpret the word at all. Where people apply restricted meanings to a word, they may be drawing upon a restricted range of examples of the use of the word. While I have no evidence to suggest this was the case here, I can imagine that pupil 4 in Table 6.2 might have recalled a particular example of the use of the word 'random' in a mathematics lesson, where perhaps he had been asked to choose a 'random' value to substitute for a particular letter in an algebraic expression. The problem with this usage is that the selection of a value to substitute is unlikely to be at all random, and is best thought of as an 'artful' selection, either to reveal some property of the expression or to test a conjecture. Pupils with such a restricted sense of the word random might experience difficulty in interpreting the dialogue in a lesson on probability.

6.3 Colleagues in a Workshop

The final setting was a meeting of colleagues in an informal seminar during which I ran a workshop to try out some activities for eliciting ideas relating to randomness. One of the tasks that I asked colleagues to do was to write down, in just a minute of two, how they would describe 'randomness'. The data surprised me as I had not anticipated the contradictions that emerged in what was written. It is important to note that these contributions came from people who work in mathematics education, and they are therefore not typical of a wider population. However, there are ideas in these words and phrases that resonate with things that arose in the previous two settings.

The following list summarises the words and phrases used in these brief descriptions.

- Unstructured, uncoordinated parameter
- Unpredictable dimension
- The result of a process that has no history, hence not possible to predict what will happen next
- Without pattern
- You think something is happening at random until a pattern emerges
- An ideal against which we can compare things or a model
- Weather is unpredictable, but not random!
- "Variability" from a statistical norm

As before, there are attempts to think of randomness in terms of what it is not: without pattern, unstructured and unpredictable. However, it was surprising to see the suggestion that something could be unpredictable but not random. Another interesting idea here was the suggestion that 'randomness' is a provisional model for thinking about any situation "until a pattern emerges". There was also a hint in the idea of an emerging pattern, and in the idea of randomness as "variability from a norm", of a global view of randomness, involving a sense of an underlying distribution. I was curious to see whether such views might also be held by learners in school.

Although there were some suggestions in this list that appeared more sophisticated, I found this list more remarkable for the similarity of many of the suggestions to those in previous lists.

I tried a version of the dice tasks with colleagues in this third setting and was encouraged by the dialogue that was generated. I distributed dice to colleagues working in pairs in the workshop. I asked people to roll their die a few times and then consider what they thought might happen next. People quickly engaged in discussion about possibilities, and some became quite animated as disagreement emerged within pairs about what to expect next.

6.4 The Incident of the Random Stapler

The colloquial usage of the word 'random' was drawn to my attention while I was working on this initial phase of my research, when I heard someone call out in frustration: "Has any one seen a random stapler?" I was curious about this phrase, and decided to explore what might be understood by it. I could envisage three different possible meanings that might be inferred from the idea of the 'random stapler'.

- A stapler that releases staples in unpredictable places, at unpredictable times and / or in an unpredictable fashion.
- A stapler that is to be found in an unpredictable place – a stapler that is lost!
- A stapler chosen at random from some large set of staplers – the sample space of staplers for this purpose.

Looking at these ideas now, I see that my own view of 'randomness' was strongly inclined towards the idea that 'randomness is unpredictability'.

When I asked two colleagues of my own generation what they thought might have been meant by a 'random stapler', there was some disagreement. One preferred the second possibility, but the other thought it could be either the first or the third, as the context did not provide enough information for him. Later, I had an email conversation with an undergraduate student, who informed me that the word 'random' was widely used in this way amongst her acquaintances who were aged from 19 to 21. She did not agree with any of the alternatives I had proposed and suggested in her email the following.

> "A 'random' stapler would, for me, simply be a spare stapler, or one not needed at that time. I would take the 'randomness' of the stapler to come from the fact that, until my arrival, it was not needed and so was 'randomly' lying about."

This was different from what I had considered, but nonetheless, I could recognise the idea. She went on to suggest further examples of ways in which the term might be used by her friends.

The first example was when people open a conversation or introduce a new topic into a conversation with a phrase such as "This is a bit random, but…" or "I've got a random question." She suggested that this usually also implies that it will be a brief conversation. I have since become aware of this usage appearing in my own discourse, particularly with my children or with students.

The second usage she identified was that her contemporaries might speak of "random people", or "randoms", when discussing who was at a party or other event. So a typical usage might be the following.

> "What was the party like?"
>
> "Good but full of randoms"

She described the idea of 'random people' in the following extract.

> Random people are most often people that you don't know, that infringe on your day in some way – being a student, I'll often hear about a 'big group of random boys' who cause a scene in a pub. They are random to an individual because they have no knowledge or control over the presence of these people.
>
> Random people can also be people that you already know but who you see in circumstances you wouldn't expect. In this situation I might say I saw 'some random physics people', the 'randomness' coming from the incongruity of seeing people where I would not expect to.

This idea of 'random people' as unknown or unexpected people who enter my experience seemed to me to go beyond the idea of the 'random stapler' from which I had started. I also thought that these uses of the word were significantly different from the formal use of 'random' in mathematics and statistics. These examples illustrate new uses of the word, with which I was not previously familiar. I suspect that this is an example of how use of the word 'random' is changing even as I am engaged in this study.

6.5 Summary

In this chapter, I have my preliminary investigations, before the first stage of interviews. I considered informal evidence from a variety of different settings involving groups of people with different levels of experience. In this summary I discuss some of the themes that emerged and which later became significant in the interviews that formed the second and third stages of the investigation.

Several themes resonated with ideas from my reading about randomness, and some were seen in more than one setting. However, there was a striking similarity across the three sets of responses from these very different groups of people. In each setting some responses were either meaningless or obscure, illustrating the difficulty of explaining or defining randomness.

Common across the three settings was a tendency to think about randomness in terms of what it lacks: order, pattern, control, prediction. This was not a surprise and it may indicate how natural it is to speak of randomness in these terms. I expect these ideas to feature in my interviews

In the first two settings, some suggestions appeared to identify randomness with equally likely outcomes. I had expected this idea to be confined to younger learners, but I later met the same suggestion, only much more strongly expressed, from a colleague in a seminar to discuss my research. My colleague was adamant that, if the outcomes were not equally likely, then the situation was not random.

The idea of randomness as variability or variation occurred in the final two settings, although not in the first. Perhaps the concept of variability is less well developed in the mathematics curriculum at age 14. The Mathematics Framework (DfEE, 2001) from the Key Stage 3 Strategy does not mention measures of variation such as range, interquartile range, or standard deviation until Year 9.

Key Stage 3 pupils were the only group to suggest that luck was associated with randomness. The fact that it was not suggested by either of the other two groups may suggest that increasing education (whether in mathematics or statistics or neither is not clear) tends to reduce the individual's reliance on luck to account for chance phenomena. However this is a very tentative suggestion.

The group of able pupils in Key Stage 3 showed a variety of views about the meaning of randomness. A surprising number expressed ideas that had more to do with algebra and sequences than with randomness, suggesting that they shared a different understanding from my own. Association of the word 'random' with the act of picking or choosing was another theme in the responses from these pupils. This was particularly interesting, as it indicated that some pupils might hold a concept of randomness that was both apparently correct but also possibly incomplete.

I expected that colloquial uses of the word 'random', like those described in section 6.4, would be important in interviews. I intended to look for evidence that people used the word in a different sense from that usual in a mathematics classroom.

The experience of using concept maps in the first setting did not encourage me to try them again. I was not sure that the time investment was fruitful. However, the experience gave me an indication of how difficult the topic of randomness is, even for A level students.

The sentence completion tasks were interesting but seemed only to touch the surface of people's thinking about randomness. After these preliminary studies I was sure that I needed to develop 'dynamic tasks' and talk to individuals interacting with these as their ideas evolved.

In the next three chapters I describe what happened when I used my dynamic tasks in interviews with individual pupils.

Chapter 7: Stage 1 Interviews

In this chapter, I provide a description of the processes involved in the conduct of the interviews in the first stage of the investigation. I describe the stimulus tasks used in the interviews, the participants and how they came to be selected, and the processes involved in the transcription and analysis of each interview.

7.1 The Interviewees

There were nine interviews in stage 1. Table 7.1 lists the interviewees in the order in which they were interviewed and gives some information about each. A fuller description of each interviewee follows later in this chapter.

	Name	Gender	Year	Age	Maths set
1	Lara	F	12	16y 10m	AS
2	David	M	9	14y 1m	set 1/6
3	Alex	F	10	15y 3m	Set 2/6
4	Ben	M	10	15y 7m	Set 2/6
5	Dom	M	8	13y 5m	Set 2
6	Nick	M	10	15y 6m	Set 2
7	Joe	M	8	13y 5m	Set 1
8	Abby	F	12	17y 7m	AS
9	Belle	F	10	14y 11m	Set 1

Table 7.1: Interviewees in stage 1 interviews

7.2 The tasks

Three kinds of tasks were used in the first stage interviews. First were three process tasks, using three different dice, which were described in section 5.2.3, and which I discuss further in 7.2.1. Secondly, there were two coin tossing tasks, which are described in 7.2.2. Finally, I asked interviewees to consider what other situations they were aware of in which ideas of 'chance' and 'randomness' might be used.

7.2.1 The dice tasks

Three dice were used in the first stage interviews: a biased die, a spherical die and a cracked cubical die.

The biased die looks like a standard cube, except that it has two faces labelled 5 and no face labelled 3. However, it also has a weight in the face labelled 1, resulting in it being heavily biased in favour of 6.

The spherical die is a hollow sphere containing a small bead. The hollow interior is so shaped that the weight of the small bead contained in it causes the sphere to come to rest in one of six different orientations when it is rolled on a flat surface. The surface of the sphere is marked symmetrically with the numbers 1 through 6, in the same manner as a standard cubical die. Thus the spherical die behaves like a standard cubical die, in that it always comes to rest with one of the six numbers uppermost, and if the die is correctly balanced, each of the six possible outcomes should be equally likely.

The cracked cubical die is damaged. It has a split running across the face labelled 6, and spreading partway across the faces labelled 2 and 5. I expected the crack to raise in the interviewee's mind the possibility that the die might be biased, especially as they met this die after their experience with the first die which is heavily biased. I expected that it would be slightly biased, but it is definitely not a serious bias and I do not know which faces are favoured. Whether the cracked die is in fact biased is not my concern, and I have not tested it.

In each of the three dice tasks, the interviewee was first given the die, asked to comment on its appearance, and then to consider how they expected it to behave when rolled several times. When they had commented on the die, they were invited to roll the die a few times, perhaps six times, before commenting on the outcomes they had observed and considering what they might see if they rolled it again. Throughout these tasks I encouraged the interviewees to talk about their thinking. I watched their behaviour closely as they observed successive outcomes. If the interviewee appeared to be bothered by a run of outcomes, or even by an individual outcome, I invited them to explain what they were thinking.

The three dice used in the dice tasks were chosen to provoke awareness in the interviewee of some of the issues that had been identified in the literature review and in the preliminary investigation described in Chapter 6. For example, I hoped that the use of the biased die and of unfamiliar dice, such as the spherical die and the potentially biased cracked die, would unsettle the interviewee enough to cause them to become more consciously aware of, and possibly question, their unspoken assumptions about how they expected dice to behave. As they became more aware, I hoped that interviewees would be able to externalise their assumptions. In particular, I hoped that these tasks would provoke the interviewees to be able to talk about the idea of equally likely outcomes and whether these were a necessary condition for them to consider the die to be behaving 'randomly'.

7.2.2 The coin tasks

The second set of tasks consisted of two tasks relating to coin tossing. One was a generation task in which I mimed handing the interviewee an imaginary coin, explaining what I was giving them as I did so, and asked them to toss it 20 times and record the outcomes in the order that they occurred. My intention here was to observe the interviewee's behaviour as they performed this task, and then to talk to them afterwards about their experience. I wanted them to discuss what they had experienced and what they had been paying attention to as they made up the successive outcomes.

Immediately after the coin generation task I moved on to a coin tossing task, which I conducted as a process task in the same manner as I had conducted the dice tasks. Once there were twenty outcomes from the real coin to consider, I asked the interviewees whether they considered that the made up sequence from the imaginary coin was visibly different from the sequence that arose from tossing the real coin. This comparison of the two sequences enabled further discussion of what the interviewee had been looking for in a random process.

7.2.3 The review task

In the final task I was concerned to broaden our discussion beyond coins and dice as random generators. I asked interviewees to suggest other situations or circumstances in which they might describe things as occurring randomly. Recognising that this might be a difficult question, particularly for the younger students, I was prepared to suggest possibilities to them if necessary, such as spinners, playing cards, road accidents, results of football matches. Although this might be seen as feeding ideas to the interviewee, I felt it was important to elicit some response to ideas beyond dice and coins, even where the interviewee was unable to offer suggestions of their own.

In closing each interview, I asked about the interviewee's mathematical background, their previous experiences of studying probability and randomness, and especially practical work using random generators.

7.3 The selection of participants

Interviewees were school pupils aged between 13 and 17, and came from two sources. Some were children of neighbours or colleagues. I approached their parents directly, and their parents invited them to participate in the study. Interviews with these learners took place at weekends, during early evenings or during school holidays, in my home or in the interviewee's own home.

Others were pupils at a local mixed 14-18 comprehensive upper school. I approached an Assistant Head Teacher at the school, who was also a mathematics teacher. She agreed that pupils could be invited to participate, and she sent a letter to parents of selected pupils requesting parental consent. I had asked the teacher to select pupils in Years 9 and 10 who would be able to express their ideas confidently. These interviews took place in a small office on school premises during mathematics lessons.

7.4 The interviewees in detail

In this section I give a brief description of each interviewee in turn. The interviewees were all either in the top two maths sets out of 6, or in selective schools, so they were all above averagely successful at mathematics. The descriptions in this section are drawn from my interview notes made shortly after each interview, as I transcribed the audiotapes.

7.4.1 Lara

Lara was in Year 12, aged 16.10, in a mixed comprehensive school, and was studying AS Mathematics, including a module in Statistics. She was described by her teacher as hard-working and conscientious in school. However, she claimed that she lacked confidence in Statistics, because her teacher did not appear confident in understanding the material. She needed constant encouragement to express her ideas in the interview, and was very softly spoken. This made the audiotape difficult to transcribe.

Lara's was my first interview and I did not use the biased die, but started with the spherical die, followed by the cracked die. The interview took place during an evening in my home.

Although I had emphasised to Lara at the start of the interview that I wanted to hear her talk about each of the activities as she worked on them, she was often reticent. Thus the interview did not contain as much contribution from Lara as I had wanted.

7.4.2 David

David was in Year 9, aged 14.1, in a mixed comprehensive school, and was in the top set for maths. His teacher considered him to be unusually articulate, but he sometimes pursued ideas in unusual directions not typical of pupils of his age. The interview took place in a private office in the school, during a mathematics lesson.

During the interview with David, I tried to avoid using the word random until later in the interview. Therefore during the early activities I framed my interventions in terms of 'chance'.

7.4.3 Alex

Alex was in Year 10, aged 15.3, in a mixed comprehensive school, in set two out of six for maths. She was positive about mathematics, and was very happy to talk about her ideas. However, according to her teacher, she sometimes found the subject difficult. The interview took place in a private office in the school, during a mathematics lesson.

7.4.4 Ben

Ben was in Year 10, aged 15.7, in a mixed comprehensive school, in set two out of six for maths. He claimed that he enjoyed mathematics and he appeared very confident in the interview. The interview took place in a private office in the school, during a mathematics lesson.

7.4.5 Dom

Dom was aged 13.5 and was in Year 8, in set 2 for mathematics. Dom was not very articulate, and I found it difficult to get him to talk much in this interview. I began by telling him that I was interested in people's ideas about randomness and asked him if the word meant anything to him. He replied with a monosyllabic "yeah". This seemed to set a pattern for much of the interview. The interview took place in my home.

7.4.6 Nick

Nick was in Year 10, aged 15.6, and in set 2 out of 8 for mathematics. In my opening remarks, I emphasised to Nick that I wanted him to tell me what he was thinking about the tasks as he worked on them. The interview took place in my home.

7.4.7 Joe

Joe was aged 13.5, and attended a selective independent school. He was in Year 9, in set 1 for maths, having been advanced one year in his schooling. He was highly able and articulate. The interview took place in my home.

7.4.8 Abby

Abby had been in Year 12 in a mixed comprehensive school for part of the autumn term, before she had to be absent from school for a prolonged period. She had studied AS Mathematics, including a module in Statistics, and was now hoping to return to start Year 12 again in the next September. She gained A* in Mathematics at GCSE. Aged 17.7, she was highly articulate in the interview, and needed no encouragement to express her ideas. The interview took place one day in the summer term at her home.

This interview lasted much longer than others: Abby's enthusiasm to express her ideas without prompting meant that this interview diverted frequently into discussion of other examples offered by Abby herself.

7.4.9 Belle

Belle was in Year 10, aged 14.11, at an independent girls' school. She was in the top set and is one of the best mathematicians in her year group. She was highly articulate. Although she was aware that I was investigating children's ideas about randomness, she did not know in advance what I was going to ask her to do.

7.5 The conduct of the first stage interviews

The first stage interviews were planned to last about 50 minutes. In the event, some interviews lasted only about 40 minutes, but a few lasted as long as 90 minutes. At the start of each interview, I told the interviewee that I was interested in how they thought about situations involving chance. I emphasised that, in the activities that followed, I would be interested in what they were thinking as they worked; it was therefore important that they tried to tell me as much as possible about what they were thinking as they worked.

Almost all interviews followed the same structure and used the same tasks. All interviews except one opened with the dice tasks as described above, using each of the three dice in turn: the biased die, the spherical die, and the cracked die. The exception was the interview with Lara, the first interview that I did, in which I omitted the biased die and the coin tossing. At the time that I interviewed Lara, my plans for the tasks to use were still evolving and I was concerned that the biased die might be too heavily biased. After Lara had spent a long time discussing the coin generation task, I was concerned that she was becoming bored so I went straight on to the review task.

Throughout the interviews, especially when interviewing younger students, I was concerned to use the word 'random' very cautiously. In some interviews, where I felt that the interviewee did not appear to understand me, or where they appeared unsure of what was required of them, I reworded questions using the words ‘by chance’ instead of ‘random’. This may raise some difficult issues for the analysis and interpretation of the interviews, and the interviewee may infer a quite different meaning when they hear the term ‘by chance’ from that they would infer from hearing the word ‘random’. Where I became aware of some such difference during the interview I tried to follow this up, but I cannot be certain that I succeeded in spotting every case where this occurred.

Each interview was recorded on audiotape and later transcribed. During the interviews, I made some notes of particular issues that caught my attention, and kept a record of the outcomes from the dice-rolling and coin-tossing activities.

7.6 The analysis

The initial analysis of the interviews was in several stages. I began by transcribing each interview in detail. I then went through each transcript and made a second account of the interview consisting of my commentary woven around substantial quotations from the dialogue. The third stage was to work through each account, identifying significant themes, particularly any which recurred in more than one interview. In identifying important themes, I drew initially upon the issues identified previously from the literature review.

I began by making a full and detailed transcription of the interview from the audiotape. This enabled me to recall the detail of each interview and I started to consider issues and themes to follow up. As well as the audiotape, I also drew from a paper record of the outcomes observed in each of the dice and coin tasks, and any notes I had recorded by hand during the interview. I tried to make the full transcription within a few weeks of the interview in order to capture from memory the details of the interviewee’s behaviour and any other events, such as the way the die rolled, that would supplement the audio record. It was not always possible to transcribe sooner than this as I sometimes recorded as many as four interviews in a week, and each transcription took two or three days to complete. The transcription was a complete record of all that was said throughout the interview, and by whom, including a record of any pause of 5 seconds or more.

The second stage of the analysis of each interview was to prepare a commentary around extracts from the transcript. In writing the commentary, I highlighted the actions and decisions made by me and by the interviewee. I omitted unnecessary detail from the transcript that was not relevant to my work. My commentary on each interview began with a brief description of the interviewee, their age and gender, experience of learning mathematics and probability, and their apparent disposition during the interview. I also described the setting in which the interview was conducted. Within my commentary, I provided accounts of key moments and incidents from the interview. I described the interviewee's response to each activity. I have drawn upon excerpts from the commentaries in Chapters 8 and 9.

In the course of analysing the first stage interviews, a problem emerged, which applied to all the interviews in stage 1. I had repeatedly asked, "What do you think will happen next?" during the process tasks. My intention had been to encourage the interviewee to consider the impact of the observed outcomes on her preconceptions of how she expected the die to behave, and to say how she felt about the "randomness" of the die. I did not want the interviewee to feel compelled to give a precise prediction of the outcome that she thought would occur next, but I hoped that she would be encouraged to use probabilistic language to describe the likelihood of different outcomes. In fact the question produced a narrower set of responses than I had hoped for, and in too many of the responses the interviewee tried to give a precise prediction.

In the third stage of the analysis, I identified common themes that occurred in more than one interview, often on several occasions. In looking out for these themes, I initially had in mind the issues identified in both the literature review in Chapter 2 and in my preliminary investigations reported in Chapter 6. I subsequently sorted and refined my list of key themes, connecting together themes that I considered to be related under a single heading. The refined list of key themes became the framework by which I structured my account of the findings from the first stage on the investigation, reported in Chapter 8.

A particular theme highlighted in the literature review in Chapter 2 was the distinction between the randomness of the generating process (primary randomness) and the randomness of the outcomes generated (secondary randomness). As I pondered this idea in the interview commentaries, I became aware that many interviewees sometimes appeared to focus upon a small number of particular outcomes that they had observed, perhaps six or seven, and to try to make inferences from these outcomes. At other times their attention was more upon the generating process, how it worked and why it might be thought of as random, and they had much less regard for what the particular outcomes were. It appeared that the interviewees were not necessarily aware of how their attention was shifting from moment to moment. This shifting of the interviewee's viewpoint during the interview became an object of study for me. However, this theme was different from the themes I had identified initially in that it did not show up in isolated incidents within the interviews, but rather had to be tracked through a sequence of interactions over a longer period of time. The analysis of these sequences required attention to longer passages from the commentaries. I have written more extensively about this idea, which I have termed 'shifting perspectives', in Chapter 9.

7.7 Summary

In this chapter, I have described the tasks that were used in the stage 1 interviews. These tasks were designed to embody the dynamic view of randomness outlined in Chapter 3. Individual interviewees working on these tasks engaged with ideas about randomness and this provided a focus for discussion in the interviews.

I have described briefly each of the interviewees. Excerpts from the interview transcripts are included in Chapters 8 and 9, in which I discuss the themes that emerged from the interviews.

Finally I have described briefly the process of analysing the interview transcripts and the detailed commentaries that were written around each transcript.

Chapter 8: Themes in Stage 1 Interviews

In this chapter, I present the themes that emerged from the analysis of the stage 1 interviews. Themes emerged in the early analysis and were organised under three headings:

- Interpretations of the idea of randomness
- Strategies for recognising randomness
- Shifting perspectives

Each of these three headings is described in the following sections, before being discussed in greater detail and illustrated with extracts from the interviews.

8.1 Interpretations of the idea of randomness

In the course of discussion about the tasks presented in the interviews, interviewees spoke about what they expected to see in these situations, and of their interpretation of what they observed. I have chosen to refer to the ideas under this heading as interpretations rather than meanings, since it seems to me that the interviewee's ideas here were often emerging and evolving in the moment while trying to make sense of successive outcomes from the tasks. I have grouped the interpretations discussed in this section into three clusters. The first is discussed in 8.1.1 and deals with interpretations relating to equally likely outcomes and to the unpredictability and lack of pattern in the short run sequence of outcomes. Fairness, unpredictability and irregularity (lack of pattern) are three of the local meanings of randomness that were identified by Pratt (1998), which were discussed in Chapter 3. The second cluster of interpretations, described in 8.1.2 relate to the idea that the outcomes are caused or brought about by some outside agency. This second cluster has something in common with the idea of 'unsteerability', the fourth of the local meanings of randomness identified by Pratt (1998). Some of the interpretations considered in these first two clusters are related to the strategies considered under the next heading. The final cluster is presented in 8.1.3 and relates to the interpretation of randomness as a model for situations of incomplete knowledge, as discussed in Chapter 2. This interpretation expresses the idea that I can consider randomness to be a model for any situations in which I have insufficient knowledge to describe a deterministic process.

8.1.1 Equally likely, fair, unpredictable

In this section I present a selection of interview extracts from three interviews, with Alex (age 15.3), Nick (age 15.6) and Joe (age 13.5). These three interviewees illustrate contrasting interpretations that were assigned to the term 'random'. Alex considered that randomness, in the context of dice and coin tasks, implied that the outcomes were both equiprobable and unpredictable. She also needed to be able to trust that the outcomes were chosen 'without cheating'. Nick considered that randomness could express uncertainty, and did not require either equally likely outcomes or fairness. For Joe, randomness of the die simply meant that the outcomes did not follow any pattern.

8.1.1.1 Fair: Alex

For some interviewees, the idea that a die could be described as 'random' was closely associated with the idea that the die should be 'fair'. Alex displayed this association very strongly. The following extracts from the interview with Alex illustrate this. In these extracts Alex was working with the dice rolling and the coin tossing tasks.

When I asked Alex if the biased die was 'random', she initially appeared to be extremely puzzled, as though she did not understand the word 'random'. However, when I rephrased the question in terms of chance, she made clear that she considered the behaviour of the biased die was *not* due to chance.

When she was introduced to the spherical die, Alex appeared to be able to incorporate chance into her account as a way of accounting for when the die stops – "It'll stop whenever it wants to" (line 55) – "it stops because the ball-bearings are weighing it down in a particular place" (line 60).

In spite of apparently having been confused by my use of the word 'random' earlier, Alex was later able to use the word for herself when talking of the first four outcomes from the spherical die: 4 5 1 4.

I: OK. So, if you throw it again, what do you think you might get next?

Alex: Um. I can't tell, because… the numbers have been *random* so far, apart from the 4, which has come up twice.

(Lines 61-64)

Her caveat "apart from the 4" referred to the fact that she had seen two 4s and only one of each of the other observed outcomes. She was pointing out that she saw the possibility of an emerging evidence of bias in the occurrence of the second 4. There is evidence here that Alex is using a strategy of predicting and pattern-breaking, like that described in section 8.3.2.

After a number of attempts to predict future outcomes from emerging patterns in the list of observed outcomes, Alex became more circumspect and cautious. She concluded, "I can't really make a prediction, because it's like not going in any pattern now" (line 96). Thus the outcomes from the spherical die were, for her, unpredictable.

When tossing a coin, Alex found it very difficult physically to spin her coin in the air, and very often it only managed half a turn. On one occasion it did not spin at all, and she insisted on throwing it again, claiming that she would know what the outcome would be so it was not chance.

Alex: Well, I threw it up flat, so that it would come down and then I would turn it over and I would know what one it would be.

I: So, that's not random? (*Pause 5 seconds*) That's not chance?

Alex: No that's not chance, that's kind of like, (*laughs*) me cheating!

(Lines 352-353, 356-357)

Note that, again, she appeared to be uncomfortable with my use of the word 'random'. On the other hand, she was happy to suggest that 'chance' behaviour implied 'unpredictable'. The idea that 'chance' produces an outcome that she cannot know is important for Alex. If she were able to know the next outcome then that would be 'cheating': controlling the process.

Alex: It has to be fair, like where you can't cheat at it.

(Line 367)

Alex seemed to express the idea that, for the coin to be 'fair', she needed to trust it. This is of course at the core of the difficulty with 'randomness', since a truly random generator is essentially like a 'black box': you cannot inspect the process to predict what will happen next, since if you could it would not be random. Neither can you know for sure whether it is random, for although 'random' is defined in terms of not showing pattern, even a truly random process might produce a pattern for 'a while'!

To summarise Alex's view, the idea of randomness in a die was associated with the idea that the outcomes should be unpredictable. She also emphasised the idea of 'fairness' although, for her, this meant more than that the outcomes were equiprobable. There was apparently an element of being able to 'trust' the die or the coin to choose the outcomes fairly – 'without cheating' – as she suggested when tossing the coin.

8.1.1.2 Unpredictable: Nick

The next set of extracts comes from the interview with Nick, who was also working with the dice rolling activities. In contrast to Alex, Nick showed a different understanding of 'randomness'. When he rolled the biased die, he observed, after only four outcomes (6 2 2 6), that the die seemed "a bit biased towards 2 and 6" (line 30). At this point he predicted that he might get a 2 or 6 when he rolled again. After a total of 12 outcomes (6 2 2 6 6 6 4 6 6 6 6 6) he stated, "It's mainly getting sixes, so maybe the other two were just accidents, and it's just biased towards sixes" (lines 56-57). Having heard Nick suggest that the die was biased, I asked him directly if he thought that the die was random. This question caused him to pause and consider, and his reply was thoughtful and cautious, with many hesitations and breaks.

Nick: Yeah, maybe. Yeah, could be. (*Pause 5 seconds.*)

I: What do you mean when you say it may be? What's going on for you? What are you thinking?

Nick: (*Pause 5 seconds.*) ...I'm not really sure... (*Pause 4 seconds*) Yeah, cos it keeps... it could be just random between six, four and two. So... we haven't had three, five and one... (*Pause 5 seconds.*)

I: So, when you say it could be random between six, four and two, what do you mean?

Nick: Those three could come up at any time... (*Pause 6 seconds.*) ...It's... biased towards those three... and it's just chance which ones it comes out as.

(Lines 59-70)

He dealt first with the fact that he had seen 6, 4 and 2 as outcomes, suggesting that the die could be random between these three outcomes, and he noted that he had not seen any occurrences of 3, 5 or 1. When I prompted him to explain what he meant by "random between 6 4 and 2", I had wondered if he would suggest that these three outcomes were equally likely. He did not, but avoided any discussion of the relative chances of these outcomes. His phrase, "it's just chance which ones it comes out as", was as close as he got to suggesting equiprobability.

It was interesting that Nick made no attempt to look for a physical explanation for the bias in the die. Before I took the die back, I invited him to look closely at the face labelled 1, where the weight had been inserted. He spotted the weight, and agreed that the die was heavier on that side. When I asked him if the presence of the weight would affect his judgement of whether the die was random, he acknowledged that the situation seemed different, but he still considered the events to be random.

Nick: I think it's not as random. I think … the chances are that you're going to get six, but there are random events that occur… that make it different.

(Lines 84-85)

Nick's suggestion that "it's not as random" had a hint of the idea of 'degrees of randomness' (see section 8.1.3). His precise meaning was unclear as he had made a comparative statement, but had not specified what he was comparing with what.

Later, when he was working with the spherical die, Nick enlarged on his view of what it means for a die to be biased.

Nick: It's not as biased as the last one. It changes… more frequently. … Cos you get… less of the same… score.

(Lines 106, 115).

When Nick considered whether the die was biased, his attention appeared to be on the extent to which successive outcomes were different from one another. After twelve outcomes from the spherical die, Nick became aware that he had still not seen a 6, but he said he was not concerned, claiming "It could just be random that we haven't had a six" (line 186). When I questioned him further about this after just one further throw, it became apparent that his opinion was rather unstable.

Nick: "I would have expected to get a six about now, so if it didn't come up in a couple of throws, I would think it was a bit biased against six."

(Lines 207-208)

After three more throws, he concluded hesitantly, "Actually, I think it is biased against six, so … it makes a six roll away from…" (lines 224-225). After a total of twenty throws, and still no 6, he was sure: "Twenty throws and we haven't had a single six, so… I think it was pretty biased against six" (lines 237-238).

When I asked him whether his conviction would be shaken if he got a 6 on the next throw, he said that his view would not change as one 6 in twenty throws would be "just a flukey chance" (line 241)! However, when I asked if he thought the die was producing random outcomes, Nick was still in no doubt that it was "…random except for the fact that it's not got sixes. But the rest, I think it is random" (line 243).

When handed the cracked die, Nick noted immediately that it was cracked, but was sure that this die was otherwise normal. He stated, "It doesn't feel weighted at all," and he went on to suggest, "I expect that any number could come up at any time, and that it's purely chance what the… result is" (lines 251, 255-256). When he was asked to suggest what might be the outcome of the first throw of the die, he again stated, "it could be anything; it could be any one of these, cos it's just random … whatever number it is, it's just totally random" (lines 259-262). At this point he was clearly expecting that the behaviour of this die would be characterised as unpredictable.

For Nick, randomness expressed any uncertainty in predicting the outcome of rolling the die, or tossing the coin. Fairness was not necessary for randomness, and neither was equiprobability. Even when the physical cause of the bias was apparent to him, he still reasoned from the element of unpredictability in the observed sequence of outcomes to conclude that the die was random. I take this to be an expression that there was some randomness, since he also suggested "It's not as random…"

At a deeper level of thinking about the spherical die, and again later with the cracked die, Nick did look for each of the possible outcomes to be represented in relatively short sequences of only between 12 and 20 outcomes. However, he did not pay attention to the relative frequencies of the outcomes.

Nick gave the impression that his primary focus when considering the randomness of a die was whether or not the successive outcomes were predictable. To the extent that he was able to conclude that they were not, the die could be described as 'random'.

8.1.1.3 Fair / Unpredictable: Joe

Joe seemed to hold a view of randomness that was both very simple and yet also comparatively sophisticated. When he used the biased die, he suggested after seeing only one outcome that this die was weighted, would always land on a six, and was therefore not behaving randomly. He said, "No, it's not random, because it can't land… it can't do anything, if you see what I mean" (lines 29-30). He experimented with balancing the die on an edge and demonstrated that it could only settle onto the six if it was carefully positioned on an edge so that it could not fall in any other direction and could not bounce once it had fallen.

When Joe rolled the spherical die, he rolled 6 5 6 4 and suggested "that one is… random, because it can land on anything" (lines 44-45). It was remarkable how quickly he arrived at this conclusion, given that he had seen only four outcomes, all high scores. When I asked him to explain what he meant by "random" in this case, he replied, "I mean that it doesn't… follow a pattern… Whereas the other one always was a six unless you placed it, this one is going where it likes…" (lines 53, 61-62). A little later, when he was considering the cracked die, Joe stated again what he meant by "random" in this context.

Joe: Something that's not definite, that doesn't follow a pattern… you can't ever predict random… I couldn't roll a normal dice and say for definite, I'm going to get a six, because it's random.

(Lines 104, 107, 109-110)

Again he emphasised that, for him, a die was considered random if the outcomes did not follow a pattern and were unpredictable. A little later, he stated something a little different, suggesting that he did not believe the cracked die was random because it was not fair.

Joe: ...It's not random, but it's not fixed, it's not definite.

(Line 127)

Even though the outcomes from the cracked die were not completely predictable, the process was not considered to be random. Joe's understanding of 'random' seemed to have changed from line 110 to line 127, in only a few seconds.

8.1.2 Agency, luck, lack of control

Alex was unusual in appearing to ascribe to some 'outside agency', real or imagined, much of what she considered to be 'chance'. In our discussion of 'chance' events in everyday life, she discussed many examples which each carried this idea.

Finding money on the pavement was ascribed to a lucky chance. (Lines 373-374). She considered that she had been lucky that the teacher was in a good mood "when she told me off. I didn't make her happy. She just happened to be happy" (lines 385-390). She considered it could be chance to win on the lottery "if it's your lucky day" (line 418).

She described how some people are just lucky, and she gave the example of her *lucky* auntie who 'always' won when she played the lottery (lines 427-432). She described chance as being like a game.

Alex: It's kind of like a game. It doesn't matter whether you've done something or not. It's whether it chooses to happen. So that... we didn't make it happen, but the thing which you're deciding on, that makes it happen... You're just kind of like the force which pushes it and makes it come like, move, but when it stops, unless you're cheating and stop it yourself, it – you can't like tell for certain unless it's got all the same number or all the same things on whatever you're deciding on the chance about.

(Lines 435-440)

When she considered the behaviour of a rolled die, she said that the die stops where it does because of gravity, but you don't know where it will stop "unless you're psychic"! (line 449). I found it interesting that she believed some people are psychic (lines 450-470), since such a belief seemed consistent with her beliefs that chance could be 'caused' by some agency.

When she discussed accidents, she said they could be considered as 'chance' if the people affected did not do anything to cause them! (lines 487-494). On the other hand, if she got a puncture, this would not be chance because something caused it, unless it is caused 'by chance'! (lines 500-504). Football results are not chance, because they are the result of many causal factors (lines 509-516). Similarly, the weather is not chance because "it's whatever nature decides to do" (line 520). There are so many factors contributing to the weather being as it is at any time (lines 522-524).

Alex seemed able to identify something as a chance phenomenon when she could see no other factor causing the outcomes. In this sense, she appeared to see chance as the cause when nothing else was identifiable. Thus, she seems to see chance as the cause, the active instigator, of events that occur 'by chance'. There was a sense throughout this discussion that Alex saw 'chance' as capricious and unpredictable, but also as an independent agency acting on real events. Thus, she was also able to see some things, such as getting a puncture, as though they were chance events, when the outcome was essentially unpredictable, in spite of there being a clear cause. I found her remarks suggestive of the blind Goddess Fortuna, whose intervention was considered by the Greeks and Romans to be necessary to govern the occurrence of unusual or rare events.

8.1.3 Model of incomplete knowledge

When I asked David (age 14.1) to think of examples, other than dice and coins, where things happen 'by chance', his first suggestion was 'football matches', although he went on to qualify this example by excluding factors such the skill of the teams.

David: Football matches. Because even if it was ... someone like Bradford playing Manchester United, everyone would say Manchester United are going to win, cos they're the best team. But although it is sometimes in football down to skill, a lot of it can be chance and good luck, so I'd say they could still have a chance to win. It's still just two teams and one of them has to win. So you couldn't ever know definitely which one's going to win.

(Lines 458-462)

Unlike Alex, David saw the result of a football match as being partly down to chance or luck, not as an external agency, but as a model to account for the remaining uncertainty about the result after taking account of what he knew about the skill of the teams. This informal way of thinking is reminiscent of the way in which a statistician using analysis of variance will think of accounting for a proportion of the variability and look on the residual variability as due to 'chance'.

A few pupils were able to articulate quite sophisticated ideas about the application of randomness. For Ben (age 15.7), when discussing what he could consider to be random, the very large number of factors that affect the outcomes was the critical element in making something random. However, when he tried to apply this reasoning to an imagined person involved in a recent rail crash, for which no cause was yet known, he stated that the outcome was strictly not completely random because there are contributing factors that affect the outcome, and these could be predicted. Nonetheless, the number of contributing factors appeared to be so large that it became difficult or impossible to predict the outcome, and so the situation could be considered random.

I: If I ask you now to tell me in your own words, what does it mean to say that something is random, what would you say?

Ben: Well it would have to be something that occurs because of … millions and millions of variables that just can't be controlled… just sort of controls themselves… they just change for no reason, or they have sort of a reason, but can be affected… The reasons could be other variables, which would make it even more complicated…

(Lines 589-594)

Here Ben was quite clearly suggesting that randomness could be a useful model for events where the causes are unpredictable or unknown.

8.2 Restricted use of the word 'random'

When interviewees chose to use the word "random", their use often echoed the meanings identified from the literature in Chapters 2 and 3. "Randomness" was often described in terms of not being something, as in unpredictable and without pattern.

While I was conducting the first round of interviews, I became aware that, for some interviewees, the word 'random' had meanings that I had not foreseen. Almost all the interviewees made some use of the word 'random' during their interview. Even Alex (discussed in the previous section), who had initially appeared not to understand the word, used it later in discussion. However, the word was sometimes used in ways that were surprising.

Two distinct categories of use of the word 'random' were identified from the interviews. The first focuses on the contrast between the mathematical meaning of the term and the more colloquial usage that was emerging amongst young people in the late 1990's. The second considers a more subtle distinction between, on the one hand, seeing randomness as an absolute – a situation is either random or it is not – and on the other hand, seeing it as a matter of degree.

8.2.1 Colloquial and technical uses of the word 'random'

The example in this section illustrates the way that use of the word 'random' may not match the accepted mathematical use. In this example David appeared to use the word 'random' in a restricted sense.

When I asked David to identify examples of situations in which events could be described as occurring 'by chance', he readily identified some. The first was the result of a football match and the second was trying to open a book at a particular a page, but being unlikely to get it exactly right. However, when I asked him if he saw the word 'random' as the same as 'by chance', or whether he would distinguish these two ideas, his reply surprised me. He restricted the meaning of 'random' to situations in which someone had to choose an item from a set. He used the word in the context of 'picking at random' and used the word 'guess' to describe what he means. Thus he could say that he guessed the outcome of a football match, by picking a result at random, but he would not speak of the result of the match as random.

David: Random – something you pick out of free will basically. Something you choose. Because when you do something at random or pick something at random, it's more like a guess. ...And it's your will saying, "Oh I think it might be that one", but ... like with a coin if I picked Tails at random, there's still a 50-50 chance it could land on Heads, or Tails. So random is just... your free will basically, thinking of it.

I suppose, a lot of the time, for instance, like with the coin, picking one at random... there's a 50-50 chance it can land on it, so you've got quite a high probability. But if, like a page, if you picked one at random, and then said to your friend, "which one did I just pick", and it's a 600-paged book, then, if they just picked one at random, they're probably not going to get it. So it's just a guess really, a guess of what you think.

(Lines 477-486)

It is interesting to note that David has used the example of choosing a page from a book to illustrate his explanation. He could not use the example of the football match.

In contrast, David used the word 'chance' to describe whether something can happen and sometimes how likely it is to happen. His use of the word 'chance' seemed to merge into the idea of 'having a chance' as in 'having an opportunity'. If something has a chance, then it can occur.

I: What would you say is the meaning of chance? If things happen by chance, what are we saying?

David: That everything has a chance. I mean… you wouldn't try something that didn't have a chance. But I would say if they had a chance… for instance… if a man fell off a cliff, there is still some chance, it might not be very high, there is still some chance that he would survive. And it's within a lot of things. Chance is involved. …I mean, people say "chance" every day, usually, don't they? …There's a chance you'll get higher than me in the test, or there's a chance that you'll score from here, or… there's a chance that the keeper will accidentally drop it in. … Everything has a chance, usually.

I: So is there a difference between those two words? 'Random' and 'Chance'?

David: Yeah. Because 'Random' is a guess of your own free will, and 'Chance' is how much chance have you got of either getting that or not getting it. So I would say there is a difference.

(Lines 487-498)

The somewhat restricted meaning that David assigns to the word 'random' could lead him to experience difficulties understanding the use of the word in mathematics when he has to interact with the school mathematics curriculum. The difference between his meaning and a more generally accepted mathematical meaning is subtle, but important.

8.2.2 Degrees of randomness

In reflecting on these interviews, I saw that important terms were often used loosely and that this led to some confusion between related ideas, for example, between randomness and fairness, randomness and distribution, distribution and probability. Fairness was a term used by many interviewees, as well as by me, to describe 'equiprobability', and yet it was sometimes used to mean 'random'. A 'fair' die is seen as being 'more random' than the biased die. This leads to the idea that some people think about 'degrees of randomness': some processes are seen as random, but less random than others. I was surprised to discover that the term 'degree(s) of randomness' is used in various contexts on Internet websites, including some academic sites in fields such as geography, mathematics and linguistics. This may indicate that the term is emerging in wider use of language, and it might not be surprising to find that children are using the underlying idea in their discourse about randomness.

This theme is remarkable for the surprising differences of opinion that I found relating to it. In an early seminar in which I presented my developing ideas to a group of colleagues, I commented that some pupils, to whom I had spoken about their ideas of chance and randomness, had indicated to me that they considered some situations were more random than others. I suggested that this seemed to me to be quite a sensible idea. My suggestion provoked a vigorous response from a colleague in the meeting, who was convinced that 'randomness' was a black and white concept: something was either random or it was not!

The idea of 'degrees of randomness' really relates to a fundamental philosophical issue about the nature of randomness. If randomness is unpredictability, then a probability distribution may be considered to be a measure of the degree to which an outcome is predictable. Considering a situation in which there is a finite number of possible outcomes, the situation has maximum unpredictability if the outcomes are equally likely and this situation might therefore be considered to be the most random. The extent to which the biased die is biased is the degree to which it is predictable, and in that sense bias is the antithesis of randomness. On the other hand, if randomness is considered to be absolute – the opposite of deterministic – then any situation in which the outcome is not completely determined might be considered to be subject to a random effect. There were hints of each of these views in the interviews.

Several interviewees compared the degree of randomness shown by the three dice in the dice tasks. David (age 14.1) used this idea after he had finally recognised the bias in the biased die (after 23 throws): he commented that he still thought the behaviour of this die was 'chance', "but not as well as a normal dice" (line 134). This seemed to be a reasonable comment in view of the fact that the outcomes from the biased die were certain. Later, when considering the spherical die, he commented "this dice is more chance than the other one, because… it was high numbers then low numbers" (lines 187-188). Ben (age 15.7) said something similar after ten outcomes from the spherical die, when he commented that this die appeared to be "a lot more random than the other one… although we're getting lots of 1s and 5s" (lines 60, 62), referring to the biased die. He meant by this that it didn't "seem to be so biased" (line 64). Note that neither Ben nor David were saying that the spherical die appeared to be completely random, as both were still concerned about aspects of the outcomes that they had observed so far.

Soon after this, when discussing the cracked die, Ben seemed to suggest that he could control the degree of randomness in the outcomes by the way he chose to roll the die.

Ben: If you pick it up like that, and then roll it… off your fingers… it's more likely to get… a number on one of these faces, so long as it'll land straight, and doesn't bounce. And then if you just… pick it up and chuck it, then… that's more random, rather than just keep rolling it all the time, which is what I'm doing on this table. Because if I just… chuck it, it would bounce around on the carpet a bit and go away. So you're more likely to get… the same type of numbers recurring over again. Well… it might not be the same numbers, but it will be less random because, if we're always picking it up in the same way and rolling it in the same way, then it should start occurring in a pattern rather than… just completely random.

(Lines 311-320)

He suggested that, by rolling the die in controlled manner along the table, he could restrict the outcomes, whereas if he were to "chuck it, it would bounce around on the carpet a bit…"

Finally, Ben thought sporting results (football, or horse-racing) were essentially determined by factors that could be known, and were to some extent predictable by people in the know. However, he still insisted that these events had an element of chance.

B: They're not completely random, but there will be some anomalies that just… turn up. So, I'd say that they're almost not random, but they sort of are as well.

(Lines 552-553)

He mentioned very rare events, such as "getting knocked over by a bus if you walk outside" (line 554), and commented "it's such a small chance, hopefully, that… you just don't treat it as anything random at all – it's just not going to happen" (lines 555-556). This remark suggests that he may have been thinking of probability as a measure of the extent to which something is not determinate, and hence 'random'.

These examples from sport are complex situations in which there are clearly causal factors, but Ben was able to see that, in spite of these, a random model could be applied appropriately.

8.3 Strategies for recognising randomness

Because the tasks presented in the interviews were dynamic, and had been selected to encourage the interviewee to be aware of what they expected to see as outcomes from a typical random generator, interviewees used a variety of strategies to try to make sense of their observations. Three strategies are discussed under this heading: pattern breaking, representativeness and physical characteristics. I make no claim that these three strategies are exhaustive - indeed, there were indications within some interviews that other strategies were also used - but these three appear to be distinct, they cover most observed cases and they give an indication of the variety of strategies.

8.3.1 Ben: No pattern / Pattern-breaking / Representative

Having worked on all three dice activities, Ben was unusual in identifying three features that he looked for in the context of rolling a die. No other interviewee summarised their approach in quite this way.

Ben: Numbers occurring in…no particular… I can't really see a pattern that much but … if something's random it couldn't have any sort of pattern at all.

And … something that was completely random would have a bit of everything in it… and that sort of has a bit of everything…

It starts off with only 5s, 2s and 6s, and then… moves about a bit…. If it was completely random, you would expect that to happen a couple of times. Else it wouldn't be completely random. ... You just sort of expect all the numbers to appear in absolutely no order whatsoever.

(Lines 329-337)

His reply identified three features that he looked for in a dice context: no particular pattern ("if something's random it couldn't have any sort of pattern at all"); a bit of every different possible outcome ("something that was completely random would have a bit of everything in it"); and some bits of pattern that then break down ("It starts off with only 5s, 2s and 6s, and then sort of …moves about a bit"). This is a more sophisticated explanation than he offered after the spherical die activity.

Later, when discussing the coin generation task, Ben did something similar. He identified three features to which he had paid attention: there should be runs of each kind, Heads and Tails, but not usually longer than three, and there should be balance between frequencies of Heads and Tails. These three features are closely parallel to two of the three features that he identified for the dice activities.

Ben's summaries encapsulate both the pattern-breaking strategy and the desire for the random outcomes to be representative of the underlying process.

8.3 2 Pattern-Breaking

The coin generation task provided an interesting insight into the ways in which people recognised randomness. When I asked David what he had been paying attention to as he made up twenty outcomes from tossing a coin, his reply showed a surprising awareness of randomness.

David: I was trying to forget about the number I had put. Every time I put one, like a heads or a tails, I would go to underneath and I would try to forget the one I put before. And I would just think of one in my mind to see what one came and popped up first.

I: Good. That's interesting. Why were you doing that?

David: Because I thought if I thought about the one that were a heads or a tails before, then I would try and make a more interesting sequence than just random. Because it's just a probability, one out of two chance.

(Lines 308-314)

I had not expected such a clear articulation of the idea of independence. David was clearly aware of this property and that it would be difficult to reproduce in this generation activity. Nevertheless, it was interesting to note that his sequence had no runs longer than four, and had a high degree of switching between outcomes.

When I asked him if he would be able to tell the difference between his made up sequence and one produced by tossing a real coin, he was sure he would not. However, he recognised that a particularly difficult time at which to 'forget' previous outcomes was immediately following a run of three or four of a kind.

I: You were trying to forget what you had written down for the previous one. Were there times when you found that difficult to do?

David: Yeah. When I had… lots of heads in a row. And I just… thought, "Oh well, I'll forget that one", and another head came up, and I thought oh there's four in row, and I thought, "oh no, I don't want to think that cos I just want it to be probability".

I: Tell me more about that.

David: I think, because I knew there were three Heads in a row, and it would be… quite peculiar... for a coin to have four Heads in a row, I kept on… The one that popped up first was a Head, because I was thinking of it, at the time – so I think that changed it from what it would have been if I'd've just done one and then got a Tails afterwards.

(Lines 320-329)

David has expressed the tension that many people feel between 'knowing that the next result is independent of what has come before' and the salience of recent history in their memory. Any run that is experienced in a random process is 'expected' to change soon. People feel excitement waiting for it to change, and may experience relief when it does.

8.3.3 Representativeness

The interviewees who demonstrated this strategy did so most clearly while working on the dice throwing and the coin tossing activities. They were looking for evidence that all the possible outcomes occurred within a relatively short sequence. Where one outcome was not present they expressed concern, either anticipating its occurrence in the next throw (negative recency), or, perversely, predicting that the missing outcome would not occur at all (a kind of positive recency). Some interviewees shifted between these two ideas, as Lara (age 16.10) did when working on the spherical die.

Lara was very reticent throughout her interview. In the first six throws, Lara had obtained 1 2 2 6 4 3; this represented at least one of each possible outcome except 5. When I asked her what might happen next, she suggested it would be a two.

Lara: A two… Because I've had two 2s already.

(Line 34)

The next throw produced another 1. Lara was not surprised, but when asked what might happen next, she again chose the outcomes which had occurred more than once, keeping to the positive recency strategy that she had used before.

I: What do you think might happen next time?

Lara: Two or a one... Because I've had them more.

(Lines 43-44, 46)

Although Lara used the frequencies observed so far as a guide to what might occur next, she said she was not strongly committed to her prediction and she stated she still believed the die was unbiased. However, she changed her strategy after the next throw, a 6, and stated that anything could happen. She now agreed that the outcomes so far were not a reason to modify the unbiased assumption.

After the next outcome, a 3, Lara emphasised again her view that the die looked 'normal'. However, when invited to say what might happen next, she reversed her previous strategy and opted for 5, the only possible outcome that had not yet occurred. This strategy supposes the die to be fair, and that short sequences will be representative of this 'fairness'.

I: So what do you think might happen next time?

Lara: A five? Just because no fives have come up…

I: You think there might be a 5 next?

Lara: Well probably not, (*laughs*)

(Lines 72-75)

Even so, Lara was not committed to this strategy, as the extract above illustrates. After the next outcome, a 4, Lara had observed ten outcomes, but had not seen a 5. She noticed this and suggested that the next outcome might be anything except a 5, reverting back to her original strategy.

Lara: Anything but a 5.

I: Anything but 5. Does it feel like a fair dice?

Lara: No.

(Lines 86-88)

Lara had now expressed more concern than before about the absence of a 5. When I asked her to explain why she was concerned, she simply repeated her concern.

Lara: Because, you would have thought there would have been a 5 by now.

(Line 92)

When the next throw produced a 1, Lara said that she felt the absence of a 5 was odd, but she did not feel strongly about it. The twelfth throw produced the missing 5, and Lara laughed. When thinking about what might happen next, Lara stuck to her original strategy of picking the outcome that had occurred most often.

Lara: One? ...There's more ones so far.

(Lines 112, 114)

Lara's preferred strategy for considering what might happen next appeared to be to select the most frequent outcome. This assumed that the die might be biased. In addition, the spherical die clearly looked unusual. In spite of this, she still felt that the spherical die did not behave very differently from an ordinary die.

Lara: ...It's not like an ordinary dice.

I: In what way?

Lara: Well, it's round. (*Laughs*)

I: OK. Does it feel as though it behaves differently?

Lara: No.

(Lines 121-125)

After the next outcome, a second 5, Lara expressed more clearly why she felt that this die was unbiased. She was looking at the frequencies of the various outcomes and noting that they were fairly even. In fact, this particular sequence of thirteen outcomes had been remarkably even, with exactly two occurrences of each possible outcome, except 1, which had occurred three times. However, Lara did note that she had "not rolled it that many times yet" (line 136).

Throughout her work with the spherical die, Lara's attention was almost entirely focused on whether the short sequence of observed outcomes included each of the six possible outcomes. She expected the outcomes to be representative of her view that the die was unbiased. Often, in her ideas about what might happen next, she appeared to see the hypothesis of equal likelihood as one to be challenged and suggested that the next outcome would be one of those which had occurred most often. Occasionally she reversed this argument and suggested that the missing outcome would come up next. However, all along, she did not feel strongly about these suggestions and she held to her original belief that the die would be fair.

8.3.4 Physical characteristics

When asked if the spherical die was like a normal die, Alex (age 15.3) expressed some concern about its unusual physical behaviour. Eventually she concluded that the outcomes were equiprobable.

Alex: Well, the chances of it coming up on the same number are… one in six… or, it depends…

(Line 115)

However, in spite of this, she was not happy to consider it a 'fair' die, because the moving ball inside "makes it go on a specific number" (line 183) and that meant "that dice is unfair" (line 189). For Alex, the moving ball inside the sphere was causing the behaviour of the spherical die, which meant that she did not consider the die to be fair.

Later, when she considered the cracked die, Alex was again more concerned about the physical characteristics of the die than whether the outcomes were equally likely. Initially she was concerned with where the crack lay: "…it's got more cracks in the 6. Then it's got some in the 2 and the 5" (line 193).

Later she concluded that the outcomes were occurring fairly.

Alex: It's… come up a fair amount of each different number. So, it's had 1s, it's had 2s, it's had 3s, it's had not many 4s, but it's had some, so I would be happy to use that dice in a game.

(Lines 238-239)

However, when I asked her if she thought this die behaved by chance, she discussed the physical characteristics in terms of symmetry and seemed to suggest that it was symmetry that made it fair and hence behave 'by chance'.

8.4 Different perspectives

The ideas considered under this heading relate to the phenomenographic nature of this study. The analysis revealed distinct 'ways of seeing' the phenomena represented by the interview tasks. Some of these 'ways of seeing' were clearly related to one another, and interviewees were seen to shift between these as they attempted to make sense of the emerging outcomes.

This theme of distinct perspectives, to which interviewees appeared to hold lightly and between which they moved rapidly, began to emerge when I studied the interview transcripts more closely and attempted to identify the focus of the interviewee's attention as the outcomes emerged. Changes in the focus were particularly apparent when interviewees were working with the dice tasks and were unsure whether a die was showing a bias. At first, I saw this constant shifting of position as an expression of an interviewee's uncertainty about how to interpret their experiences. As I examined the transcripts, I recognized that the interviewee's shifting position seemed to be related to the way that their attention moved between different features of the randomness of the die as more outcomes were generated and observed.

The most distinctive collection of ways of seeing was what I have termed *local, global*. In this section I describe local and global ways of seeing and how they differ. My intention here is to try to clarify the distinction between these two perspectives on randomness. In Chapter 9, I present a more extended discussion of the ways in which interviewees were seen to shift between them.

I begin in 8.4.1 by illustrating how the two perspectives are present in the way interviewees approach the activity. I present a brief description of Lara's work on the cracked die. Later sub-sections present further examples of each of the local and global perspectives.

8.4.1 Lara: two perspectives

Lara was not much interested by either the biased die or the spherical die. When she started to work on the cracked die, her initial reaction was that it was a normal die and she was sure that the crack would not affect its fairness. She appeared quite dismissive of my questions about what might happen next. However, after nine outcomes she noticed that she had not yet seen any occurrence of 1 or 3, and when this was still the case after twelve outcomes, she expressed suspicion about the fairness of the die. Her attention was drawn into the task as she noticed something that she did not expect and began to anticipate whether this would change or persist. The thirteenth outcome was 1, but she had still not seen a 3 after 28 outcomes. She became certain that the die was biased, and she became sure that 3 was less likely to occur than any other possible outcome. Then, the 29th throw was a 3. On the way from the 12th to the 29th outcome, Lara's way of viewing the situation changed frequently, as did her approach to predicting what might happen next. She became increasingly unsure, and asked me on two occasions to tell her whether this die was really fair. I could not tell her, because I did not know, and I told her so.

When Lara's attention was first drawn into the activity by the absence of some possible outcomes, she was using the strategy of representativeness, as she had done earlier when working on the spherical die (see section 8.3.3). Her attention was in the local perspective, as she was concerned with the individual outcomes unfolding in front of her. This changed between the 12th and the 29th outcomes, and when 3 occurred on the 29th throw, her attention was more focused in the global perspective of the aggregated distribution of outcomes.

8.4.2 The local perspective

When Dom (age 13.5) worked with the spherical die, he worked almost entirely within the local perspective, although initially, he was concerned more with the physical aspects of the die. When he first rolled the die and noted a 4, he sat and thought for several seconds, preoccupied by the motion of the moving weight. He stated that the die was 'random' and cited the movement of the weight inside as the die rolled.

Dom: Well, this is random, because the weight moves. So, depending on which way you roll it, is going to determine where the weight lands, and which number… is facing down or up.

(Lines 72-74)

He rolled the die a further five times. The first six outcomes were 4, 4, 3, 2, 3 and 1. He stated that he had "no idea" what might happen next but, when I asked him if he had any preference, he argued from the list of outcomes seen so far ("because there's most of them") that he might get a 4 or a 3. However, he also recognised that the die might 'land on something completely different'.

Dom: It could land on… it might land on a 4 or a 3. Because there's the most of them. But probably it's going to land on something completely different.

(Lines 83-84)

Although Dom's reply suggested that he considered no outcome was any more likely than any other, the form of the argument that he used, reasoning from the sequence of six outcomes observed so far, suggests that he was working in the local perspective.

He rolled the die twice more and got a six each time. Dom still thought this die was random but, when I invited him to say more about what he meant, he still reasoned from the mixture of outcomes in the short sequence observed.

I: What do you mean when you say that that is random?

Dom: That it can land on absolutely anything, every time you throw it.

(Lines 89-90)

Although Dom's view of randomness here was like saying that outcomes are equiprobable, he was not reasoning from any aggregation of outcomes and so his reasoning appears to be still within the local perspective. It is notable that he arrived at this conclusion after only eight outcomes. The sequence of outcomes observed was 4, 4, 3, 2, 3, 1, 6 and 6, which does not yet include any 5, but it may be that he had not noticed this fact. Insofar as Dom was using the data, and reasoning from the observed results, he was reasoning in the local perspective.

The local perspective was also important within the coin generation task. This is illustrated in Dom's discussion of his experience of generating twenty outcomes from the imaginary coin. While he was aware at the end that the numbers of heads and tails were approximately the same, he suggested that he had not monitored the frequencies while generating the outcomes.

Dom: Well, there's almost the same number of heads as of, the same number of tails. Because there's only two different things you can put down… so if you tossed a coin fifty times, you would be probably likely to get about 25 heads, and 25 tails, or close to that. (*Pause 8 seconds.*)

I: So, were you watching for that as you wrote down your sequence? Were you monitoring how many heads and how many tails?

Dom: No, I was just… seeing what came into my head first, head or tail.

(Lines 157-163)

However, he commented that he had sometimes been aware of changing the next outcome in his head to avoid continuing a run.

Dom: Sometimes if I put… about… 3 tails, it… would change to a head. It got too many of one thing.

I: So, were you consciously thinking it shouldn't be another tail?

Dom: ... Sometimes. If I... didn't want to put too many of one number, of one tail or one head.

(Lines 166-170)

Dom's description of his uncertainty about how the coin will behave on any one throw expressed the essence of how I saw the local perspective on randomness at this point. Although he was aware of the idea of a prior distribution, his description of the behaviour focused on the uncertainty of the next few outcomes. He was not concerned with any aggregation of the data.

Dom had concluded that the spherical die was fair after only eight outcomes and without appearing to move out of the local perspective. Several interviewees appeared to reach conclusions in this way, but this was most common when working on the biased die. Because the bias was quite strong, people were sometimes drawn to conclude after very few outcomes that the die was not fair.

For example, Belle (age 14.11), looked surprised as the first five outcomes unfolded and, after the third 6, she expressed this aloud.

Belle: OK (*rolls*) 3. (*rolls*) 6. (*rolls*) 6. (*rolls*) 6. What on earth... (*rolls*) 6. OK. Again?

(Lines 36-42)

After the fourth 6, I invited her to consider what might happen next. She already suspected strongly that the die was biased, and that she was likely to get a six next, but she was perturbed about the three on the first outcome.

I: What do you think might happen next?

Belle: Six.

I: Why?

Belle: ... Biased dice, maybe... which confuses me, because the first one was a 3.

(Lines 43-48)

It may be that Belle was trying to express a global perspective here to account for the run of sixes, but that it clashed with the evidence of the 3 in her local perspective. Belle rolled the die again and got another 6. She continued to believe that the next outcome was likely to be 6, but she tried to explain the first outcome of 3 as an aberration caused by her doing something wrong. She appeared to need a reason for the first outcome being different.

I: Do you think it'll be a six again?

Belle: Yeah. And probably it was only a 3 the first time because I did it at the wrong angle, because the three's next to the six.

(Lines 50-52)

Fortuitously, the next outcome was a three, causing Belle to feel quite uncomfortable. She even suggested that I might be controlling the outcomes with my mind! However, when I asked her directly, she did not believe that I could be doing so.

Belle: If I didn't know better, I'd think you were controlling this one with your mind...

I: Well. Do you think I am?

Belle: (*laughs*) No. It would be a bit hard, too complicated for that to happen.

(Lines 54-57)

The occurrence of the second 3 shook her attempt to express a global perspective and left her still in the local perspective. She speculated that the die might be biased.

Belle: So threes and sixes. Which implies that it's biased at about between the 1 and the 4... if at all.

(Line 58)

However, she did not appear to examine the die to find a source of bias.

Belle's reasoning is not entirely within the local perspective, but insofar as she reasoned from the outcomes 3 6 6 6 6 6 3 6 she was reasoning locally.

8.4.3 The global perspective

When Belle later found the weight in the biased die, she became much more confident in her expression of the bias.

Shortly before she found the weight, she continued to throw sixes. She appeared to be getting quite frustrated, and did not know what else she could say about the die.

Belle: What else is there to say? Six, six, six, six...with the occasional threes.

(Lines 67, 70)

The statement has a hint of the global in it, but it could still be interpreted as reasoning about short run behaviour. After several more throws, each generating a six, I suggested that she pause and look at the die. This time she spotted the weight in the face labelled one. Initially, she thought that this was a magnet, and suspected that the tray on which she had been rolling the die was metal.

I: Take a look at it.

Belle: In what way? Like that bit? Some holey magnet style thing. Is this (*indicating the tray that she was rolling the die onto*) made of metal? (*Taps the tray*) No. So it can't be a magnet.

I: No the tray's not made of metal.

Belle: It looks like something's been done to the dice. Something been stuck on it or something. On the one, just opposite the six. I not quite sure how the threes came up, unless that was just because it landed at the wrong angle and turned to tip that way. OK (*Laughs*)

(Lines 86-94)

Seeing the weight in the face gave Belle a rational explanation for the sequence of outcomes that she had observed. However, she still struggled to account for the occurrences of 3. She tried again to suggest the 3s were an aberration because the die 'landed at the wrong angle', rather than accepting that such a run of outcomes might occur by chance. However, she was now clear that the die was biased. I was interested to see whether she still saw the biased die as being random, since this might enable her to account for the occasional outcome of 3. She was clear in her response that the die was not random as she could predict the outcomes.

I: Is the outcome random?

Belle: No. Because you can predict what the outcome's going to be.

I: And what is the outcome going to be?

Belle: Most probably a six, maybe a three.

(Lines 99-102)

In spite of her view that the outcomes were not random, Belle still used informal language of probability in describing what the outcome of rolling the die could be: 'most probably a six'. I was interested in her use of these words and I asked to explain further what she meant. Her response conveyed that she was aware that there were other possibilities than six, but these came up rarely. Again she implied that these other possibilities occurred under strange circumstances: 'it could land at a weird angle'.

I: ...I'm just interested by 'most probably'. What does that mean?

Belle: Well, it means that once or twice it came up as a three, so it could... be a three. It could land at a weird angle and end up... ...if it is a biased dice, which... is almost certain as far as I am concerned, then even biased dice you don't necessarily get the number it's biased to, because it depends on how it lands.

(Lines 107-113)

In this last exchange Belle was beginning to express informally a view of a probability distribution of the outcomes. I suggest this is the beginning of a global perspective on randomness, even though Belle herself has not recognised these outcomes as random.

Joe (age 13.5) showed understanding of the global perspective when he discussed his view of the cracked die. He was convinced that, while any outcome was possible, the relative chances of each outcome could be affected by the asymmetry of the cracks in some of the faces.

Joe: It should... land on any number, if you roll it. I dunno what... But it could be influenced by the cracks.

(Lines 77, 79, 81)

Later, he repeated his view that this die might not have equally likely outcomes because of the cracks. He explained his understanding of the relationship of long run frequency to probability – a view which approaches the global perspective.

Joe: ...This one... has the capability of landing on another number ... the crack seemed to affect it in a way, like we haven't actually got a single one or a six. And that might be because of the cracks, or it might not...

I: How could you find out?

Joe: Well, you could roll it a number of times and record your results and see... whether it seemed to favour one side... But then that could be just luck or chance. So there's no definite way of finding out, really. But you could, if you rolled it a number of times, say ten times, say a hundred times, and you didn't get a single six or a single one, then... it would probably be in your mind that six and one... they had less chance of being rolled. So... it's not random, but it's not fixed, it's not definite.

(Lines 113-120, 122-127)

Joe appeared to be aware that rolling the die "a number of times... say a hundred..." would tell him something about the underlying possibilities; if one outcome failed to occur in so many trials then he might conclude that outcome was less likely to occur than the others. However, he did not go as far as suggesting the idea of the law of large numbers: that with increasing number of trials the relative frequency of an outcome approaches its probability. Thus Joe appeared to be working in the global perspective, insofar as he was discussing what he would see if he aggregated the results of a large number of trials, and he was expecting that this would enable him to say something more than he was able to say from the local perspective.

8.5 Summary

The themes identified from the stage 1 interviews are broadly in four categories.

First, interviewees demonstrated several interpretations of randomness, when they talked about what they believed was random in the dice activities. These interpretations are similar to the local meanings identified by Pratt (1998). For some interviewees, random outcomes needed to be approximately equally likely, as in Pratt's local meaning of 'fairness'. For others it was enough that the outcomes have an element of unpredictability. However, a few interviewees moved between interpretations of randomness, as was the case with Joe in 8.1.3. Interpretations are not fixed, but appear to change and adapt to circumstances in subtle ways. However, I have no sense that an individual can control these changes of interpretation.

Two different ways of interpreting the word "random" were considered in this chapter. David seemed to adopt a narrow sense of the word, possibly arising from having heard the word used in restricted contexts. This is different from the colloquial meaning discussed in Chapter 6, but might also have arisen from particular social interactions. The second idea was 'degrees of randomness'. Some interviewees considered some dice to be more random than others, while others considered a die was either random or it was not. It seems plausible that the idea of degrees of randomness might be a reverse of believing that randomness requires equally likely outcomes.

In the third category I considered three strategies that interviewees appeared to use to identify whether outcomes were random: pattern-breaking, representativeness, and examination of the physical characteristics of the system. People using pattern-breaking looked for short patterns in the sequence of outcomes and expected them to be quickly broken. Looking for runs of the same outcome was a special case of pattern-breaking, which emerged in the coin generation task: runs of length three were expected, but not longer. Pattern-breaking may be related to Pratt's local meaning of "irregularity", in which a child might refer to an experiment as random if there is no evidence of patterned sequence in the observed results. A person using representativeness expects a random process to produce all the possible outcomes within a short sequence. This strategy seems closest to Pratt's local meaning of "Fairness". The third strategy of looking at the physical characteristics of the system differs from the other two. I identified in Chapter 2 the distinction between seeing randomness in the process and in the outcomes. Whereas the other strategies focused on looking at outcomes, this one examines physical aspects of the generating process.

Finally, I have distinguished two perspectives on randomness: local and global. The local perspective is concerned with short sequences of outcomes, and seeing local disorder, unpredictability and irregularity. At the global level, an individual becomes aware of an emerging frequency distribution. In Chapter 9, I look more closely at the relationship and movement between these two perspectives.

Chapter 9: Shifting perspectives in Stage 1 Interviews

9.1 Randomness and Distribution

When used in a mathematical sense, the terms 'random' and 'randomness' have quite precise meanings (although there are considerable differences between the meanings and interpretations used by different mathematicians – see Chapter 2 for examples). In contrast, when used by non-specialists, and particularly children, these terms are much less precise. However, the interpretations and meanings given to these words in everyday contexts form an essential part of the context in which any learner is working when they are introduced to these terms in school mathematics lessons.

In both mathematical and everyday contexts 'randomness' is often thought as without pattern and without order. Yet, in order to develop ideas of probability and distribution, the pattern and order that emerge from the long run behaviour of random phenomena are critically important. This apparent contradiction, between 'randomness' as without pattern, and 'distribution' as a pattern that emerges from random behaviour in the long run, may cause difficulties for learners.

In Chapter 8, I described two contrasting perspectives –local and global – which were apparent in discussion about the outcomes from the dice tasks, and which appear to correspond to seeing disorder in the local perspective and seeing order in the global perspective. In this chapter, I examine the relationship between these perspectives in the interview data from the first round of interviews and I describe the ways in which individual interviewees appeared to shift between these perspectives.

I discussed in Chapter 2 the idea of distribution as order that emerges from the aggregation of disordered outcomes in the long run (Wilensky, 1997). The aggregation of observed outcomes from a random process produces an emergent pattern, which becomes clearer as the process generates more outcomes. This pattern may be described as the distribution of outcomes. To the extent that a person has intuitions about the nature of this distribution, they will have expectations about the outcomes that will arise from the process in the short run.

Before I started this study, I thought of the development of the ideas of randomness and distribution as being hierarchical: I considered that the individual would first be aware of the disorderliness of random outcomes, with the notion of distribution developing in the individual learner much later. A person gains experience of randomness in the short run from everyday interaction with random phenomena. Later, as experience builds, the learner may begin to discern patterns that emerge in the long run. Such discernment may be encouraged if the learner is able to interact with computer-based random simulations, which allow collation of results from large numbers of trials in a short time. The concept of distribution develops in the learner with experience in a similar way to that in which a distribution emerges empirically from examination of collated long run behaviour. In thinking this way, I was seeing distribution both as an emergent phenomenon (Wilensky, 1993 and 1997), and, in parallel, as an emergent concept. As a result I expected that any shift of perspective between local and global would be in one direction: from local to global.

In a person with a sophisticated understanding of stochastics, the two ideas of randomness and distribution constitute two different, but complementary, perspectives. They are two ways of seeing and thinking about a stochastic situation. The perspective of randomness is appropriate for the short run, for examining the situation on a micro level; and distribution is more useful for reasoning at a macro level. Even here, I had been thinking of the shift in perspective as being from randomness to distribution. I was ignoring the possibility of shifts in the opposite direction: from distribution to randomness.

When I began to explore what it might mean to shift one's perspective from a distribution at a global level to the randomness of a small number of outcomes at a local level, my first thought here was of sampling from a distribution: of seeing 'randomness' in the outcomes in a small sample from a known distribution. Here, I think, is a natural shift from global to local.

The shift from global to local can be seen to be possible in very young children too. An individual in a class might see the 'distribution' of the heights of all pupils in the class, and then consider their own height as a single value within the distribution.

However, I found further examples in my interview tasks (described in Chapter 7), where I invited interviewees to consider what they expected to happen next. This required them first to form an idea of the 'distribution' that they expect (a *prior* distribution) and to reconsider this in the light of the random outcomes that emerged.

The idea of distribution has two distinct forms. There is the empirical frequentist idea, which is the emergent phenomenon referred to by Wilensky (1993), and there is the theoretical *a priori* distribution, with which my interviewees were experimenting when they worked with the biased die. The prior belief about a distribution is also global in a sense, but in a different sense from the emergent frequency distribution. Distribution is essentially a probability concept and, in this chapter, I am discussing the shifts between the perspectives of randomness and two different approaches to probability, theoretical and frequentist.

The shifting of perspective from local to global and back again seemed a useful way to examine the interview data. When I began to do so, I recognised that I needed to treat the two forms of distribution as distinct global perspectives. In the extracts discussed below I have selected passages to illustrate what I see as 'shifting perspectives' between the *local*, the *global frequentist* and the *global prior*. These changes in perspective emerged in the interviews over time, so it has been necessary to discuss a few passages at length, to show how the interviewees' expression of their ideas changed.

I observed interviewees using random outcomes to evaluate their preconceptions about the expected distribution. An interviewee, when handed a die, usually had a preconceived idea of what the distribution of possible values 1, 2, 3, 4, 5, 6 would be. For example, when I gave Nick the cracked die, he expressed his view that the six possible outcomes were equiprobable. He stated that he thought the outcome would be 'random' and explained, "I expect that any number could come up at any time, and that it's purely chance what the… result is" (lines 255-256). When I invited him to predict the first outcome he enlarged on this: "It could be anything, it could be any one of these… It would have been randomly selected" (lines 259-261).

When an interviewee rolled the die and considered the observed outcomes, they evaluated their expected distribution against the observed sequence of outcomes. When Nick observed 6, 3, 5, 2, 5, 5 from rolling the cracked die, he noted that the cracked faces of the die were 5, 6 and 2, and then commented.

Nick: I notice that 5, 6 and 2 came up quite… regularly… There was only one other number that wasn't a 5, 6 or 2, which was a 3.

(Lines 266-268).

After only six outcomes he had observed that the partition of the outcomes into 'cracked faces' and 'non-cracked faces' showed an imbalance of five to one.

In section 9.2, I discuss the interview responses from a learner, David, in Year 9, who displayed a developed understanding of these three complementary perspectives.

9.2 David's shifting perspectives: the biased die

My first example of shifting perspectives between local and global is from my interview with David (age 14.1) when he worked with the biased die. In the following account, I show how David's attention moved between his prior beliefs about the probability distribution of the six possible outcomes and the emerging sequence of observed outcomes as he rolled the die.

David initially stated that the faces were equiprobable, even after observing two sixes and a four in the first three throws.

David: It's all the same probability, but it gave me more sixes than any other number.

(Line 17)

He went on to generate two further sixes and expressed some disquiet.

David: It seems to always land on six.

(Line 25)

Here his attention was shifting from his prior view of the probability distribution to the local level of the sequence of observed outcomes. On the seventh throw, David obtained a five, and he tried to articulate a pattern that he thought was emerging.

David: It seems to land on higher numbers than lower numbers.

(Line 31)

David's attention was still at the local level on the short sequence of observed outcomes, but he was now attempting to create a new global view from the pattern he saw in this short sequence (6, 4, 6, 6, 6, 6, 5). However, he had still not given up on his prior view that the outcomes should be equiprobable, as was shown by what he said next, referring to the number of sixes.

David: …It's just chance.

(Lines 35)

Two further sixes did not shake his view that the large number of sixes observed could be "just chance". He adopted a 'negative recency' argument (Kahneman and Tversky, 1973) to suggest that 'chance' would correct the imbalance by producing something lower.

David: I think it'll be a lower number next ...because there's been too many higher numbers. It could be any – it could be any of the six numbers. I think it might be a lower one.

(Lines 38, 40-41)

David was still holding on to his prior belief about the equiprobable distribution, but he tried to modify the behaviour of a 'chance' process to be self-correcting. On observing yet another six, David adopted an extreme version of this prior view, choosing the lowest value available as his prediction to maximise the degree of correction.

David: (*Silence for 10 seconds*) I think it – might be a one ...because it's landed on 4, 5 and 6 and on each of the sides it's close to 2, 4 and 1.

(Lines 44, 46)

David seemed to be quite uncomfortable with the idea that the outcomes were equiprobable. The long pause before he commented suggests he was unsure how to respond.

David had now observed 10 outcomes: 6, 4, 6, 6, 6, 6, 5, 6, 6, 6. On the eleventh throw he obtained a 5, and changed his prediction completely in favour of one based on 'positive recency'.

David: (*Throws*) Five... I think it might land on another six. (*Laughs*)

I: Why have you changed your mind?

David: Because ... I don't know actually. Because it seems to always land on high numbers, and I'm not sure why, it just always seems to land on a high number. It hasn't landed on any under four has it? See if it will land this time. (*Throws*) Another six.

(Lines 48-52)

In his explanation, David seemed to accept that the die was behaving in a biased way. From here onwards, he appeared to view the die as biased, although he had not yet articulated this explicitly. His local view of randomness through the lens of the emerging sequence of observed outcomes had become so much at variance with his prior belief that he began to feel unable to continue in that belief.

However, even after two further sixes, he still struggled to explain his new view. When I asked him if he still held to his previously expressed prior view, that the next outcome could be any number, he restated, rather hesitantly, the idea that the outcomes were equally likely.

I: You started off by saying that it could be any number... Do you still think it could be any number?

David: I think it <u>can</u> be any number yeah. But it's just... cos there's a one in six chance of getting every number there.

(Lines 61, 63-65)

David went on to examine the die closely, looking for an explanation for the global distribution that was emerging. He spotted that the die was incorrectly labelled – it had an extra five in place of the three – but he recognised that this did not explain what he was observing.

David: Yeah but that means it should land on 5 more, but it doesn't. (*Silence 9 seconds*) ...but there is still one in six chance of getting a 6. (*Throws*) Another 6. (*David picks up the die to examine it.*)

(Lines 77-81)

Again, his language in predicting the next outcome reverted to equiprobability, although he seemed to be actively seeking a reason to suggest that the die was biased. The disparity between the evidence of the observed outcomes at the local level and his prior belief in equiprobability was now driving his search for an explanation, but he had not yet abandoned his prior belief.

When I asked how many sixes he had observed, he counted 13 sixes in 16 throws. Even now, he clung to the idea that the faces *should* be equally likely and, at the same time, that *chance* would correct the imbalance of outcomes.

David: I'm not sure… there should be a one in six chance of getting a 6. …I'm hoping it's going to land on a low number. (*Laughs*)

(Lines 90, 92)

After throwing another six, he eventually expressed a global view that the die was biased.

David: I think the probability of getting 6 is higher now. Because just of the outcome. I'm not sure why.

(Line 94)

David was now convinced that the die was biased but he still had no explanation for his new global view. From now on, he spent time examining the die after each throw. Eventually, he spotted the metal weight in one face and quickly proposed an explanation for the observed global distribution of outcomes.

David: (*Examines the die again*) Is it cos that bit there – is metal? So, it's going to… put more outcome onto 6, put more chance onto it… Well, I'm not sure cos… there's two 5s, but it would have to… go on the other side of that… metal part. And that only happens when it slides across. But every time I actually roll it, it always lands on a 6. So I think it might be that, it's heavier, so it's landing down further, and there's more force going down, so it's going to stick down on it.

(Lines 123-124, 128-132)

At last, David was able to construct, in his own terms, a global view of the 'probability' distribution that he could reconcile with what he observed at the local level. Because he had observed 6 so often, he argued that the weight "is heavier" and the die is "going to stick down on it". Until he found this explanation, he could not be comfortable about rejecting his prior belief, and he could not reconcile the observed outcomes with his global sense of distribution.

David's response to the biased die was unusual in that it was so protracted. He appeared to have been deeply committed to his prior idea that the die should be fair and he continued to seek justification for this view, even when he had observed 13 heads in 16 throws. The manner in which he attempted to justify that his observations could arise from a fair die indicated that he was aware that a random process might not produce outcomes representative of the long run. However, he carried this to an extreme, possibly because he was not aware of how much variability to expect in the outcomes of a fair die.

9.3 David: shifting perspectives and unpredictability

I saw further examples of David shifting between the prior, the local and the global views when he was working with the other activities. However, these three perspectives supported one another in the spherical die and the cracked die, since the observed frequencies were not very different from his prior belief that the outcomes would be equally likely. As he considered these dice, it became apparent that David was also looking for the sequence of outcomes to be unpredictable.

9.3.1 The spherical die

The first seven outcomes that David observed from the spherical die were: 6, 6, 3, 5, 5, 3, 5. He expressed concern that he had not seen a 1, although he could see that the spherical die was behaving quite differently from the biased die. In his discussion of the sequence of observed outcomes, he focused on the 'alternating' behaviour of the sequence, and discussed whether each successive outcome was higher or lower than the previous one.

David: We still haven't had a 1, but it's changed from... Well this dice is more chance than the other one, because... it was high numbers then low numbers. But it still hasn't been ... what I would like from a high then a low, like that... but I still say it's chance, this dice.

(Lines 187-189, 191)

The fact that the sequence had shown "high numbers, then low numbers" was evidence for him that this die was "more chance than the other one" (the biased die). Here he was focussing on the local level, looking closely at the sequence and trying to reconcile it to his global prior that the six possible outcomes were equally likely. In spite of the fact that the sequence was only seven outcomes, David was concerned that he had not yet observed a 1.

Shortly after this, David threw a 1 and became increasingly convinced that the spherical die was "normal". The task of predicting what might occur next when the die was apparently fair was difficult. Although I had not intended him to make precise predictions, he tried to do so, and was drawn into basing his predictions on short sequences of the most recent outcomes.

David: Looking at how it's been, in about the last five throws, I'd say it will land on a high one, 5 or 6. (*Rolls*) …a 3! (*Silence 6 seconds*) …I think this one will just be down to chance, because 3 is in the middle… I would say this is quite a normal dice, quite fair. I think it will land on a high one though. (*Rolls*) 2.

I: Two. So, it didn't land on a high one.

David: No… I this is a fair dice, cos it's harder to predict.

(Lines 207-214)

David based his predictions around whether the outcome would be high or low. He suggested that the outcome would "be down to chance" and yet he still attempted to predict "it will land on a high one though". When it did not, he quite reasonably suggested that this appeared to be "a fair die" because it was "harder to predict".

9.3.2 The cracked die

In a similar way, when he worked with the cracked die, David quickly adopted the prior view that this was a fair die and that the crack was not going to affect the behaviour of the die. This was shown in his first comment.

David: Well it just looks normal. It's got a crack in it, but I don't think that'd affect it.

(Line 250)

A few moments later he restated this view.

David: …I think this, at the moment, is a fair dice. It's just down to chance. I've already had four throws.

(Lines 259-260)

David clearly believed that the outcomes were equiprobable. After seven throws (6, 3, 6, 5, 4, 2, 5) I asked him what he thought might happen next. His reply showed further his commitment to the prior belief that this was a fair die. However, when he suggested that the next outcome would be more likely to be a 1, he was looking for the 'fairness' of the die to be represented in the short sequence of outcomes.

David: I would say anything could happen… I'd think it would be more likely to be a 1, because that hasn't showed up yet… This, I think this is a fair dice though, so I think it would just be a one out six for every number, cos I think that seems fair.

(Lines 265-267)

David had almost simultaneously expressed two apparently 'contradictory' views. He played with the idea that "it would be more likely to be a 1" but appeared to reject it because he thought "this is a fair dice" and therefore the probability should be "a one out six for every number". He was testing his prior view of the distribution against the short sequence of observed outcomes.

David obtained a 1 on the eighth throw and he predicted a 5 for the ninth, arguing in the same way as he had done with the spherical die from the pattern of the observed sequence. When he threw another 1, he was not put out, but reiterated his prior conviction that the die was fair. In explaining why he believed the die was fair, David again used the idea that the sequence was unpredictable.

David: Because it's all down to chance. You can't – I mean you can always predict what's going to come next, but you can't ever really know, as with the first dice. I sort of knew it was going to be a 6 or a 5 most of the time, because that was just all that it landed on. But this one – I think it's just a fair dice… It's landed on every number in about 10 throws, so I think that's about a fair dice.

(Lines 276-280)

As he had done with the spherical die, David argued that the observed sequence had not enabled him to predict outcomes. He saw this as evidence that the die was random ("it's all down to chance"). However, the fact that he had seen the die land "on every number in about ten throws" was seen as evidence supporting the idea that the die was "fair".

9.3.3 Coin generation and coin tossing

I described, in 8.3.2, the remarkable insight that David showed into the independence of the successive outcomes during the coin generation task. He had also explained graphically the tension he experienced in deciding the next outcome after a run of 3 successive heads.

Therefore, when he started the coin tossing activity, David was aware of run length as an issue. After five outcomes, he had obtained HTTTT, and I expected that the 4 successive Tails would make this an interesting point at which to explore further the difficulty David had experienced in the generation task. His response suggested that he thought the coin more likely to correct the imbalance in the observed sequence, although he understood that it had no memory and that successive outcomes were independent. Here is another example of two contradictory views being expressed simultaneously.

David: …I think it would, it could land on any of them. But I'd say a Heads, because there have been 4 Tails in a row.

(*Toss*) Heads.

It could just be any one.

(*Toss*) Tails.

It's thrown more <u>Tails</u>, but I think more Heads <u>will</u> come.

(*Toss*) Heads.

(*Toss*) Tails.

(*Toss*) Heads.

(Line 354-362)

David had now seen three successive alternations of TH in the last six outcomes, so I intervened again. This time, his response showed that his attention was on the length of runs rather than the alternating pattern that I had noticed. Instead, he noted that there had been no run of two of a kind recently.

David: I'd say another Heads, because there haven't been more than one Head in a row. So, I'd say another Heads.

(*Toss*) Heads

I'm not sure. It could be anything.

I: OK. Go on.

David: (*Toss*) Tails.

I think it will just be the same, just anything, cos...

(Lines 364-370)

After sixteen outcomes, David had seen four successive heads. He paused in silence for a while when he recognised, without my intervention, that this situation deserved comment. The run of Heads lengthened and David became increasingly uneasy. He knew that the chances of getting a Head or a Tail remained the same but, as he said, "It's hard!" Nevertheless, he reaffirmed that the chances were equal, even after eight successive heads.

David: (*Silence 8 seconds*) ...I don't think it'll make a difference that there's been 4 heads in row, because it's still... just a coin. The chance of getting a Tail is still one out of two – the chance of getting a Head is one out of two. I think that's just chance of getting 4 Heads in a row. So, I'm not sure what will come next.

(*Toss*) Oh it's another Heads.

(*Silence 7 seconds*) That's just the same – I'm not sure – there could be another Heads, but it could be a Tails as well.

(*Toss*) a Head.

(*Silence 9 seconds*) I think it will just be the same. Cos it <u>can be</u> a Tails and – when you just look at it you think – Oh it's obviously going to be a Tails, cos there's been six Heads in a row, but it's still just a one out of two chance to get a Tails. Same with Heads.

(*Toss*) Heads.

(*Toss*) Heads again.

I: Hmm. What next?

David: (*Silence 7 seconds*) It's – Ah – It's hard. (*Silence 4 seconds*)

(Lines 375-389)

David's final comment before the run of heads ended showed how good an understanding of randomness he had: if you look at a long enough sequence then you will see long runs of Heads and of Tails.

David: …it is just the same chance. I think – ah yes, it is chance that I've got so many heads. Because that could, if I kept on going, if I did a million tosses, I think that would happen for Tails as well.

(Lines 395-397)

In this passage, David appeared to refine and develop his understanding of what can happen in a random sequence of Heads and Tails to take account of the long run of Heads with which he was faced. Although he struggled, his belief that tossing a normal coin should be fair, and his knowledge that successive outcomes were independent, led him to recognise that long runs do occur with a fair coin.

9.4 Alex: random means unpredictable

My next examples of shifting perspectives come from my interview with Alex (age 15.3). I have described, in Chapter 8, Alex's view of randomness as unpredictability and fairness, and her need to trust that the process would choose fairly. However, she also demonstrated shifting perspectives.

Alex was not comfortable about the spherical die. Her initial discussion revolved around the physical mechanism inside the sphere. She saw a similarity with a set of jumping beans that she had owned long ago. After four throws, she stated that she could not comment on what might happen next as "the numbers have been random so far". Here she was concerned only with the local level of the apparent disorder in the sequence.

Later she began to make reference to patterns that she saw emerging in the sequence. After she had observed 4, 5, 1, 4, 5, she confidently suggested that the next throw would give a 1. When it was actually a 6, she laughed and commented on the 6 having thrown the sequence "off a pattern so far".

Alex: Well the 6 has come up throwing it off a pattern so far, so I might get 4.

(Line 79)

Alex's attempt to find pattern in the sequence can be seen as trying to impose order on the apparent disorder of the 'random' sequence. She was trying unsuccessfully to move towards an organising structure for the sequence. This might be seen as an attempt to move towards a global view, albeit unlikely to be productive.

The seventh throw produced a 4, matching Alex's prediction. She predicted 5 for the eighth throw, and was correct again: two successes in a row. I did not ask her about this in the interview, but her predictions appeared to be an attempt to follow the pattern that she had discerned earlier. The sequence {4 5 1 4 5 6 4 5} showed a recurrence of 4, 5.

Asked what she might get next, she was more cautious, opting for two outcomes, 6 and 4. However, her justification for this referred to the 6 that had "just kind of popped up on its own".

I: What are you going to get next?

Alex: ...Maybe a 6 or another 4... Because the 6 just kind of popped up on its own. (*Rolls.*) 2.

(Lines 86-87, 91)

Alex became more cautious in her predictions after this, using phrases such as "can't be sure" and "it's not going in any pattern now" to indicate her sense that the outcomes were not predictable.

She compared the physical properties and behaviour of the spherical die to a normal cubical die. As she looked for similarities she eventually recognised that the probabilities of getting each different outcome on the spherical die could be considered as equal.

Alex: Well, the chances of it coming up on the same number are... one in six, or it depends...

I: So you think this one has the... equal chances for each face as you would get...

Alex: Yeah, the same probability.

I ...as you would get with a normal die?

Alex: Yeah.

(Lines 115-119)

This was the first time Alex had expressed a global view about a probability distribution for the six outcomes from the spherical die, and she did not refer to it again in this activity.

A few throws later, Alex observed and commented on a run of three 5s in succession. When I asked her whether she thought the chance of getting a 5 on the next throw was more or less than 1 in 6, she replied without any strong commitment either way.

Alex: There's kind of like a recurrence in the 5s. I keep on having them.

I: And what's going to happen next?

Alex: ...It will probably either be a 5, or it will drastically change and...

I: Do you think it's more likely to be a 5 than it was before, or less likely to be a 5 than it was before, or just the same.

Alex: ...The same, because you can't really tell.

(Lines 168-173)

This non-committal reply seemed to convey the idea that the outcome was unpredictable. She seemed to stay with her earlier expression of the probability distribution.

In summary, Alex expressed a local perspective and looked for patterns to bring order to what she saw. After several unsuccessful attempts to follow patterns, she expressed a belief that the outcomes were unpredictable, and linked this to a statement of her prior view that the outcomes were equally likely. However, she continued to search for patterns.

Throughout the interview, Alex rarely show awareness of 'distribution', unlike other interviewees, but there was still evidence that she tried to find a global organising view to counter the disorder of the 'randomness' at the local level. Alex clearly understood that something that behaved randomly, or 'by chance', should be unpredictable.

9.5 Ben

Ben (age 15.7) had spotted the weight in the biased die after only four throws. I have included here an account of nearly all Ben's work with the spherical die, as it shows his struggle to reconcile the frequency distribution he expected from a fair die with the disorder that he observed in the short sequence of observed outcomes.

9.5.1 The spherical die

When he began to work with the spherical die, Ben was sure he had never seen anything like it before. His first two throws each showed 5, which led him to be mistrustful, until the third throw showed a 1. As Ben put it, he "thought it might be another weighted one". The strange movement of the die, due to the moving weight inside, fascinated him.

After seven throws (5, 5, 1, 4, 1, 6, 1), I asked Ben what he thought might happen next. His initial response was to look at the 'pattern' in the outcomes so far. He noted that there had been no "3s or 2s" and appeared to use a 'negative recency' argument to suggest that the next outcome would be not a 1.

Ben: Well, we haven't had any 3s or 2s, so it could be one of those, but it looks – well, it'll probably be another number than a 1.

I: Why?

Ben: Just from following the pattern. If it wasn't a die, that's what I'd say.

(Lines 39-42)

By working initially from patterns in the sequence of outcomes, Ben was working at the local level. When I encouraged Ben to take account of the fact that this was a die, he said that the next throw could be anything, showing that he recognised that all outcomes were possible. This may also indicate that Ben understood at some level that the outcomes of successive throws of the die would be independent, in spite of claiming "it'll probably be another number than a 1".

Ben actively looked for an explanation for the absence of 2s and 3s, and wondered whether he might be rolling the die in the same way each time.

Ben: It might be the way I'm throwing it though. Or when I picked it up, I'm throwing it the same way. Or it could just be chance.

(Lines 58-59)

Ben was keenly aware that the sequence of outcomes so far was still short, and this awareness led him to look for a structure to the generating process. I see this as an attempt to create a global level understanding of the process.

Without my asking him, Ben commented after ten outcomes that this die appeared to be "a lot more random than the other one… although we're getting lots of 1s and 5s" (lines 60, 62). I asked him what he meant by "more random".

Ben: The other one just had to be 6s, but this one doesn't seem to be so biased. Just seem to get a lot of 1s and 5s. Could be just the weight inside jumping.

(Lines 64, 66)

However, out of concern that he had not seen any 2s or 3s, he checked carefully that the die was correctly labelled. On the fourteenth throw he got a 3 and cheered.

For the next few throws, Ben showed more interest in the physical factors affecting the outcomes than the distribution. He played deliberately with the die between rolls, and considered the shape of the weight moving inside the sphere.

Because he still appeared to be concerned, I asked him how he would know whether this was a fair die. In his reply he showed that he understood that a large number of trials would be needed, and he expressed for the first time his prior belief about the probability distribution.

Ben: A fair dice?... You just have to keep rolling it. It should in the end even out if it's a fair dice. If it's not a fair dice it'll… keep on staying away from the 2s and 3s, like it is at the moment.

I: Are you worried about it at the moment, being fair?

Ben: …No, not really. …It could just be one of those things. Just probabilities and stuff like that.

I: So it could just be…

Ben: It could just be chance. If there's a 1 in 6 chance of getting each different number... I just haven't got a 2 yet, which is strange. Although I'll probably get a 2 now, if I roll it...

(Lines 121-128)

Ben seemed to switch between two contrasting views about this die. On the one hand, while his attention was at the local level on the sequence of outcomes, he was looking for the first occurrence of a 2, and he expressed concern that he had not seen it after fifteen throws. On the other hand, when his attention was at the global level on what he expected of a fair die, looking for a frequency distribution that matched his prior belief, he still accepted that "It could just be chance". By changing the focus of his awareness he adopted two quite different perspectives on the process and arrived at two different explanations for the absence of 2 from the sequence of outcomes.

Ben's next two throws each produced a 5, and he became very quiet. In spite of what he had just said about the probability distribution, when I suggested that he was still disturbed that he had not yet had any 2s, he agreed. He experimented with the die, very gently rolling it around in his hand without talking for 13 seconds, before commenting on the fairness of the die.

Ben: It seems pretty fair. But it depends what happens when you roll it.

(Lines 139-140)

This comment again brought into focus the tension Ben felt between the apparent 'fairness' of the generating process, and the imbalance he detected in the outcomes.

He rolled the die and got a 6. In order to resolve the tension that he was experiencing, he now very much wanted the die to show a two. It was as though he wanted to remove the anticipation of waiting for a two to occur, and by experimenting with the way he rolled the die he was trying to make it happen. On the very next throw, he succeeded and was very pleased.

Ben: (*Rolls it gently in his hand again*) …oh land on a 2. (*Rolls*) Eee!

I: A 2!

Ben: There you go.

(Lines 144-147)

Ben's experimentation with his method of rolling the die had been associated in time with the occurrence of a rolling a two. This fact lent support to his idea at the local level that the way he rolled the die was affecting the outcomes. He reflected on the implications of his idea, noting that both 2 and 3 had occurred only once each, and that his theory that the axis of rolling affected the outcomes did not entirely explain this as 2 and 3 were not opposite to each other on the die.

I: What are you thinking?

Ben: Just seeing… if I was always rolling it in a way so it only lands on 6, 5 1, 4. But that wouldn't work, or make sense… But so it stays away from 2s and 3s, but it won't cos they're not next to each other – but they are.

(Lines 150-153)

He continued to look for an explanation for the relative rarity of 2 and 3, and suggested that the sphere might be heavier on one side, between the 2 and the 3, than the other.

Ben: It might be weighted more heavily on the 2 and the 3, on the inside, I was just thinking. If the weight's heavier there it will be less likely to turn that way round.

(Lines 155-156)

His next throw was a 1, and he remarked that the die now seemed "more random". He appeared to hold two ideas in tension – randomness (by which he means equiprobability) and bias – and he expressed them alternately. These two ideas seemed to be conflicting global interpretations for Ben – the one his prior belief and the other, a global frequentist view, possibly emerging from the aggregation of the observed outcomes. As soon as he had expressed the idea of randomness, he reverted to discussing the bias.

Ben: It seems to be more random now. The more you do it, you know, the more different... But at the start it was all the same. So the more you do it the better the results you get, I suppose. (*Rolls a 5*)

I: What do you mean, the more you do it the better results you get?

Ben: Well. It should, unless it's weighted, be completely random. But at the start it just seemed to be 5s and 1s. But then it... just got a lot more mixed as it went down, so I suppose it... it's just... more and more of a dice and, sort of less chance that the odd number will count for so much. You got a couple of 5s at the beginning, then, later as you go on, you'll get more of the other numbers as well. In theory, I think.

I: OK. Try again?

Ben: Although I haven't got that many 2s still. (*Rolls another 5*) But I have got quite few 5s I think.

I: Another 5!

Ben: But that could just be the way that I'm rolling it.

(Lines 160-174)

In the next few throws, Ben's concern about bias diminished as he obtained more 2s and 3s. He remarked again on the apparent randomness.

Ben: Maybe it's just... I suppose it could just be a completely fair dice...

I: So you think this could be a completely fair dice?

Ben: It could be, yeah. It does have quite a few 5s, but that just might be me rolling it, rather than the dice would be weighted or something.

(Lines 182, 185-187)

Ben's external expression of his ideas appears to have been affected by short run behaviour of the die. When the sequence of recent outcomes did not include one or two of the possible outcomes, he tried to explain the apparent bias. When the missing outcomes had appeared once or twice, he described the behaviour of the die as "random". Ben appeared to be trying to explain patterns in short sequences of outcomes, and he reverted to describing the sequence as "random" when he had no alternative explanation. Sometimes Ben seemed to think of "random" as "the absence of pattern", and this cue seemed to be switched on and off by short-term changes in the sequence of outcomes.

When I invited Ben to consider what were the factors affecting the fairness of the die, he mentioned again the way that he rolled it and agreed that he could change both the speed and the direction of rolling.

Ben: ...Well, sort of the speed and the direction it's going in... I can change the, yeah I can change the way I roll it, yeah. If I sort of shake it before I roll it and stuff each time, I should get different numbers.

(Lines 191-194)

I encouraged Ben to use the experience of this activity to think more generally about what he would look for in something that was truly "random". He readily did so, although the way he expressed himself was a little unclear.

Ben: ...If something's really random, I wouldn't expect the same number to come up so many times. Because, although it was completely random, it seems that 5 will come up more than the other ones. Even if it is a fair dice that wouldn't necessarily happen but it just does seem to do that, which is strange.

(Lines 197-200)

A difficulty for Ben appeared to be that he did not have any clear sense of how much variability to expect from a fair die. For example, he did not know for how many throws he might reasonably expect to need to wait until all six outcomes had appeared at least once. And he did not have a sense of how often he could expect the most commonly occurring outcome to appear in the first n throws of the die. He therefore had no way of judging whether the number of 5s he had observed was unreasonable, or whether the waiting times he observed before the first 3 and the first 2 occurred, were appropriate in a fair die. In order to refine his judgement of whether a die was fair, he needed to develop his intuitions about variability. An understanding of variability, and how this varies with the number of trials, is an important ingredient in reconciling the local and the global views of randomness.

To encourage Ben to think about this I asked him to count the number of 5s and the total number of outcomes so far. Unfortunately, he initially miscounted the number of observations as 24, when it was in fact 29. He decided that he wanted a round number of outcomes, 25, so he threw once more. When I asked him how many 5s he would expect in this total from a fair die, he correctly stated that he expected about four (recognising then that the arithmetic would have been easier if he had kept to 24). He thought that ten or more occurrences of a 5 in 25 trials would be unacceptable.

He found he had in fact seen eleven occurrences of a 5, but when he checked the numbers of occurrences of the other outcomes, he recognised that he had miscounted the total. There were now thirty recorded outcomes in total.

Ben: Oh right. Oh in that case there should be five of each number, shouldn't there. So, yeah, it does kind of make sense. ...It's just a lot of 5s, which is just...strange.

I: So eleven compared to five that you would have expected. It seems all right does it?

Ben: Yeah. So looking at that, you'd say that... 5 is... got a 1 in 3 chance instead of a 1 in 6 chance, and the other ones have got a lot less. Like 2 and 3 and 6 have only got a 1 in 10 chance of coming up. And they should all each have a 1 in 6 chance. Just, I suppose, like, just from rolling it. I suppose if you rolled it a load more times it would even out probably – eventually.

I: OK. So at the moment you'd be happy to use that die again would you?

Ben: Yeah, pretty much. 5s just must ... have been the way I was rolling it I suppose, that made 5 come up more than the others.

(Lines 225-234)

By the end, Ben seemed to have decided that eleven 5s in thirty trials is a bit unusual, but not completely unreasonable. He even suggested that the imbalance might even out eventually with a larger number of throws. In fact eleven or more of one outcome in thirty trials of a fair die has a probability of only 0.7%, so is quite unusual. However, Ben was arguing intuitively, and he lacked experience by which to evaluate how unusual was eleven in thirty. My view is reinforced by the fact that finally he reverted to justifying the number of 5s he had observed by the way he had rolled the die.

This account of Ben's attempts to make sense of his experiences of the spherical die shows that his struggle to construct a stable global perspective was complicated by his uncertainty about what would be an appropriate model. Is the die biased, or are the outcomes equally likely? His repeated attempts to articulate a global perspective, and to explain to himself the physical process by which the process might be biased, were constantly interacting with the latest outcomes from rolling the die. Initially he tried to work intuitively with an impression of the observed frequency distribution, but later, with encouragement from the interviewer, he counted the number of observations of each possible outcome. However, he was unable to resolve the uncertainty that he experienced, possibly because he did not have a sense of how far the observed frequencies might be expected to depart from the expected frequencies.

9.5.2 The cracked die

When I gave him the cracked die, Ben considered that the crack might affect the symmetry of the weight distribution, but he was confident that any effect on the fairness of the die would be very slight. After the first six outcomes (5, 2, 6, 6, 6, 2) his comment was that all the outcomes so far had been cracked faces, focussing on the asymmetrical feature of the die.

Ben: The numbers that have come up so far are only the ones with cracks in the faces. So … it could be something to do with the cracks that's making those numbers come up.

(Lines 255-256)

I had seen that the three successive sixes had surprised him as they occurred, but when I asked him about this, he seemed to have decided that the crack was the important factor.

Ben: I thought it just might be the way I was picking it up each time and throwing it in the exact same way, so it would bounce the same way. But I think it seems pretty fair.

(Lines 258-260)

His focus here is on the local perspective, and he is attempting to explain his observations by pattern spotting (the run of sixes) and by recourse to the physical attributes of the die. When I asked him to consider what might happen next, he argued that another 6 was most likely, using the features that he had considered so far.

Ben: Well, well the 5s, 6s and 2s have all appeared, but the 6 has appeared more than any of them and that's got the biggest crack in it. So I'd say that the 6 is going to appear most often and the 5 and 2 next often, from what's happened so far.

(Lines 264-266)

Ben rolled the die a further 18 times, with occasional comments, and at the end of this he commented on the manner in which he threw the die. However, his subsequent explanation showed that he was aware of the emerging balance between the frequencies of the six possible outcomes, and that he was still monitoring the frequencies of the cracked faces.

Ben: I think this dice is different from the other dice, because, if you pick up… it's got to be exactly the same way each time… the same way each time. And you get the same… It depends on… If you roll it that way, and it hits straight like that, it might bounce straight over to a 6 or anything. And that other one's a bit different because it would depend on how hard you rolled it, rather than what angle it lands on when you drop it…

I: OK

Ben: But this one…I see I've got lots of 6s and 5s… (*Counts the throws so far*) …18, so 12 out of 18 throws, were 6s, 2s, and 5s. That's just over, not really that much over half really. So it might not be a biased dice at all. …I'm just thinking, because at the beginning it was just 6s, 2s and 5s, and then the other numbers sort of just started creeping in. It's just like on that one there; it started off being all 5s and 1s, and then the other numbers, sort of started appearing the more you threw it.

(Lines 289-299)

Here he was moving from the local into the global perspective, seeing the frequencies as representing the distribution to some extent. His explanation of why this die might not be biased showed awareness of the need to consider a longer run of outcomes. His phrase that the other numbers 'started creeping in' was particularly evocative of the gradual revelation of the nature of the process. Even at this stage, he was still holding two ideas in mind: the physical attributes of the process were still being monitored as shown by his remarks about the way in which he rolled the die. But now the frequencies of the cracked and the non-cracked faces, and indeed the frequencies of each of the six outcomes, were also being monitored.

After a few more throws, Ben stated he was confident that this die was sufficiently fair to use in a game. His comment on this refers to a gradual revelation "as you got more numbers".

I: So, do you think this is a fair die that you would be happy to use in a game?

Ben: Yeah, pretty much. At the start, it didn't seem so, but as you got more numbers… it seemed more fair. Yeah, I think that's a pretty fair dice as well.

(Lines 323-325)

He had been gradually moving towards expressing the global perspective with greater conviction, and in this quotation he appeared to have reconciled the global with the local to his own satisfaction.

I commented at the end of section 8.1 that Ben held quite sophisticated ideas about the application of randomness. His ability to reconcile the local and global perspectives in his work with the spherical and the cracked dice was similarly effective.

9.5.3 Coin tossing

Before he started the coin tossing activity, but after he had completed the coin generation task, Ben discussed an idea that he learned about coin tossing from a discussion in a lesson.

Ben: I remember a while ago we were doing something… if you flip a coin up and catch it, and … you get a tail or whatever, … you find that if you keep flicking the coin up with the same strength, and it turns the same amount of turns, you catch it and you put it down, so either it always stays a head or it always stays a tail, or it just keeps alternating, almost exactly.

(Lines 374-379)

He appeared to have learned from this that it was possible to control the repeated tossing of a coin to produce either a sustained sequence of the same outcome, or an alternating sequence of heads and tails. If one really wanted the coin to be 'random', then Ben's view could be a problem.

When I asked him how he could get around it if he needed to make the coin behave randomly, he appeared to be fascinated with the idea.

Ben: …completely randomly? Well you have to throw it up at different strengths each time, but then you're controlling how hard you throw it. I would have to say you were controlling how many turns it had, so if you're practised in flicking coins, then you can probably control how many turns you want it to do.

I: Are you practised at flipping coins?

Ben: No. I was just saying that you could. … You'd have to… just guess, try and do it… differently each time, try and do a… different method, sometimes flicking it and sometimes throwing it…

(Lines 390-396, 398-399)

When I gave him the coin for the coin tossing activity, he was influenced by the idea of needing to make the coin behave "completely randomly". He tossed the coin hard each time and had little control over where it might land, let alone which way up! However, the sequence of 20 outcomes that he produced was remarkable for having no runs longer than 2, and it alternated frequently between heads and tails, so much so that the outcomes suggested the possibility that Ben had been controlling the outcomes by striving so hard to make each throw different from the one before.

After seven outcomes, there appeared to be an alternating pattern emerging (HHTTHHT). Ben noticed this and talked about the next outcome. In noticing and thinking about the alternating pattern, Ben was working in the local perspective: trying to follow a repeating pattern in short sequence of outcomes.

Ben: Head or Tail next. (*Looks at the list of outcomes so far*) Head, no it should be another Tail. It seems to be in groups of twos. Well, it shouldn't be another Tail. It should be either a Head or a Tail, but it looks like from that, it should be a Tail.

I: So do you, in your head do you still think it's going to be equally likely a Head or a Tail next time?

Ben: No. I think it'll be a Tail. Well, now I've said that, it'll probably be a head, but if I was going to bet on a Head or a Tail, I'd say a Tail. It just looks like it's going to do that. (*Toss*) Oops. It's a head. (*Laughs*)

(Lines 424-430)

He seemed to switch between selecting the outcome to continue the pattern and choosing the other to break the pattern. In the end, he opted for continuing the pattern.

When he had tossed the coin twenty times, I invited Ben to comment on the sequence he had produced. Again he commented at the level of local pattern.

Ben: Well, it does look like it follows a pattern. It's gone Head Head Tail, Head Head Tail, Head Head Tail, Head!

(Line 458-459)

Ben agreed that there were more patterns in this sequence than there were in the sequence that he had made up earlier, but his justification for this surprised me. He attributed the patterns to the way in which he had been flicking the coin! After the lengths to which he had gone to ensure that his coin tossing was 'completely random', I had not expected him to suggest that nevertheless this had caused the patterns. He elaborated by suggesting that his attempts to make the process random, such as flinging the coin across the room, were the reason why the patterns had not continued!

Ben: I was trying to make it more random. But still, me trying to make it random… changes whether it's going to be a head or a tail, depending on what I want to do… Once I'd thrown it across the room, then… it's… like me deciding to throw it across the room, rather than me just flicking on my hand each time. Because if I was flicking it in my hand each time, I'd get... a pattern… Head Tail Head Tail Head Tail. But then if I throw it across the room on purpose, or whatever… then that pattern… it's less likely to continue, if I start throwing it about. But that's me making it more different… I think.

(Lines 471-478)

It seems that Ben was aware of having controlled hiss coin tossing to break any run or alternating pattern that he noticed. Any control would have been incomplete because of the wild manner in which he was throwing the coin in different directions and with varying degrees of spin. However, he seemed to believe that he had made the outcomes 'more random' than they would have been if he had tossed the coin in a more conventional fashion. In this analysis he seemed to work solely in the local perspective.

Ben: Yeah, the coin is… generally, pretty random. … It's just that these patterns just sort of occur. If you were to do… a lot more times, then it would get other patterns. … It's just sort of strange that the Heads… comes up with Head Head, and there's only one Head by itself the whole way down. And there's never, there's only one time when Tails come together.

I: So the heads have always appeared in pairs…apart from that once in the middle.

Ben: Yeah. And the tails have always appeared singly, apart from at the very beginning… which is just one of those things. Strange. It's not strange, it should happen, but it's just in that little list it looks different.

(Lines 480-483, 485-488, 490-491, 493-494)

Ben appeared to recognise that some patterns were inevitable in a random sequence. He recognised that this sequence was "pretty random" and that if "you do more times" then "other patterns" would occur. Nonetheless, his ideas showed elements of two contradictory views being held almost simultaneously. For example, he continued to be surprised by the fact that Heads nearly always occurred in pairs, while Tails did not, saying it was "strange", and yet "it should happen".

9.6 Abby

Abby (age 17.7) was highly articulate in the interview, and needed no encouragement to express her ideas.

9.6.1 The biased die

Abby had recognised that the biased die was weighted as soon as I gave it to her. She rolled the die a few times, obtaining six on most occasions and expressing mock surprise. However, on one throw she got a four and appeared to be more genuinely surprised.

Abby: It's weighted! (*Laughs*) It is! It's a weighted dice.
(*Rolls*) Oh, I've got a six (*mocking tone*).
(*Rolls*) Oh, I've got a six. How surprising!
(*Rolls*) Oh, I've got a four! (*surprise*) Ooh!

(Lines 35-38)

Although Abby had perceived the biased nature of the die very quickly, her mocking tone when she first threw the die – "Oh I've got a 6, how surprising!" – may indicate that she had expected the outcomes to be entirely determined. Thus, her third throw, provoked a tone of genuine surprise.

She rolled the die a few more times, obtaining mostly sixes, but also a five. She then examined the markings on the faces and noted that there were two faces marked five and no face marked three. When I asked her whether the behaviour of the die was 'random', her reply was hesitant. She appeared to have some difficulty in deciding how to answer my question, but eventually settled on describing it as only "a bit random".

Abby: Well, it's not completely random. It is a bit. I don't know. It's really complicated…

I: You say it is a bit?

Abby: (*Laughs*) It's a bit random. Well, you're more likely to get a six than you are to get a one, statistically. But the likelihood of you getting a 2 or a 4 is equal. And a 5 is the same but twice. (*Pause*) ...It's not very random. (*Laughs*)

(Lines 76-80, 82)

Her remarks about this being "really complicated" seem to have arisen from her recognition that the die was incorrectly labelled, as a result of which she was trying to apply reasoning based on symmetry to determine the probabilities of the various faces.

9.6.2 The cracked die

When she saw the cracked die, Abby commented that it was "broken". She expected this to make some difference to the fairness of the die, although she did not expect the die to be as unbalanced as the biased die had been.

When she rolled the die a few times, she soon described the effect of the crack on the shape of the faces and speculated about possible effects on the likelihood of each outcome. Abby switched from the local view to the global, attempting to link the observed outcomes to the physical attributes of the die as she did so.

Abby: Right, the break is mostly on the side where the 6 and the 5 are, and it's also down where the 2 is. ...So it's mostly where the 6 is. So it makes that side bigger. So it's not an even shape the whole way round. Because the 1 and the 4 and the 3 are smaller.

(Lines 112-118)

When I asked her if she thought the outcomes from this die would be 'random', her reply was hesitant. She recognised that the numbers were not equally likely, yet the outcomes were clearly not determined.

Abby: No! Yes! I don't know. Well it's a bit random. It's not completely.... There isn't the same chance of you getting every number, but ...I don't know whether there's more chance of getting the broken numbers or more chance of... I think there's more chance of getting the broken numbers.

(Lines 121-124)

Abby was able to use her reasoning about the effect of the crack to consider a modification to the uniform distribution. She was able to articulate a global view of the distribution of the possible outcomes. She expected the cracked faces to be more likely than the others. However, it seems from her initial hesitation in responding to my question that she may have held a conception of 'random' that required 'equally likely outcomes'. This global view is really a prior view, being largely based on her view that the die would be only a little unbalanced, and not really using observed frequencies.

When I asked how she could find out whether broken faces were more likely, she replied in two ways, possibly indicating some tension between two different ways of thinking about the outcomes of rolling the die. The first part of her reply suggested the classic long run relative frequency approach to determining probability. Abby had previously articulated a view based on a modification of equally likely outcomes, but here she showed that she was aware of a global frequentist view. However, her second comment, "you could ask a physics person" indicated a sense in which the 'deterministic' laws of physics might govern the outcomes of a die rolling experiment.

Abby: By doing it hundreds and millions of times. (*Laughs*) Well, you could... Or you could ask a physics person.

(Lines 126-127)

Abby considered the surface area of each face in more detail. She suddenly recognised that each spot on the die was in fact a tiny depression in the surface and that the different number of spots on each face might affect the weight on each face of any die. She was trying to relate the physical features of the die to her global prior view of the distribution.

Abby stated again that, because of the crack, the faces were not exactly equally likely. At this point, she appeared to resolve for herself the dilemma of whether 'random' implied 'equally likely'.

I: Right, so you would feel that that one would not be…

Abby: It's not exactly the same chance for every number.

I: So is it random?

Abby: It is **random**. But it's not **equal**.

(Lines 136-139)

Abby had now explicitly separated the ideas of randomness and equiprobability. Previously she had said that the outcomes from the die were "a bit random", but here she stated unequivocally "it is random".

I tried to use this discussion to move on to a more general discussion of the concept of randomness. Abby's reply was long and included many examples to illustrate her meaning. She began by thinking of random as 'unpredictable' and, to illustrate this, she chose an extreme example of an elephant falling on the die! The problem with this example is that it is unpredictable really because it is so unlikely!

I: So the question now is 'what do you think randomness is?'

Abby: (*Pause 5 seconds*) …It's a word in English. …It's when you can't predict what's going to happen. Well, you know that something's going to happen, but… I mean, when you roll a dice, what's the chance of an elephant falling on the dice? That would be very random, if an elephant fell on the dice. But… You know that it's likely that one of six things is going to happen, or it's going to fall off the table or something like that… but you don't know which of those six things is going to happen.

(Lines 144-150)

Unfortunately, the idea of random as unpredictable led her into a difficulty when she considered the outcome of rolling a die, since the outcome was (almost) certain to be one of the six possible outcomes. Abby was not yet thinking of randomness as an idealised model of the real situation, but was trying to see reasons why an *a priori* model with a defined finite set of outcomes did not perfectly apply.

In the next sentence, Abby acknowledged the need to simplify the situation when she said "let's just pretend…" As she considered this further, she moved to thinking of random outcomes as impossible to influence.

Abby: …Which one happens out of the six things that are likely to happen…? Well, let's just pretend that they're certain. I mean, because, you could get some bird come down and eat the dice before it lands or something like that, but that's not likely. But say that these six things are certain to happen. One of the six things is certain to happen. But you don't know which is going to happen, and you can't - with a normal, a fair dice - you can't… influence which one's going to happen – well you could if you dropped it down – but, no, if you do it properly then you can't influence it. So there are six things… one of them is going to happen, and you can't influence which one's going to happen and you can't know. And one of them, whichever one, is going to happen.

(Lines 152-158, 161-162)

Abby had defined an idealised outcome space for the process of rolling a die, but she had stopped short of specifying that the outcomes were equally likely. Although something could occur to prevent one of the six outcomes occurring, she proposed to model the situation as though the only possibilities were the six outcomes.

Abby went on to draw a parallel between the randomness of rolling a die and the randomness of the lottery. Initially, she focused on describing the physical process by which the lottery balls are selected, emphasising the idea that the process was "up to chance completely". She emphasised that each ball was equally likely to be chosen and she appeared to state that this made the process random. I reflected that statement back to her, which led her to identify again a distinction between 'random' outcomes and 'fair' (equally likely) outcomes.

Abby: It's like on the lottery—if it's not fixed, which I haven't worked out whether it is or not yet—the balls, they're just thrown around, and one falls down, and one happens to go into the little dent before all the rest. And because all the balls are supposedly the same weight, and they're being blown at the same force, and all these things that you learn in GCSEs about doing a fair test in science… then it's up to chance completely which one of—how many is it? 49 balls? —falls into the little gap.

And, they always bring up the statistics about "this ball has appeared 966 times this week," and umm, that doesn't matter. If it's completely fair, it doesn't matter how many times it's done it before, because the chance of that ball, or the other ball, coming down is the same, so it's random.

I: Right. OK. So, it's because all the chances are the same that makes it random.

Abby: No. But that makes it fairly, fairly random. Not fairly as in quite, but fairly as in… fair.

(Lines 163-173)

So, Abby seemed to suggest that 'equally likely' outcomes were those 'random' outcomes that were also 'fair'. She had drawn a distinction between situations in which random outcomes were equally likely, which she called "fairly random" and she believed were exemplified by the lottery, and those in which outcomes were random but not equally likely, as she thought might be the case with the cracked die. Her understanding of randomness seemed to be evolving as she spoke.

She contrasted her idea of "fairly random" with the example of the weighted die that she had looked at first. In that case, outcomes were not 'equally likely', the die was clearly not 'fair', and yet outcomes were still somewhat random, since each possible outcome had non-zero likelihood. As she said, "it's random, but it's not as random as having six things that are the same likelihood". Abby seemed somewhat surprised by her conclusion that some things are more random than others.

Abby: But, like, the weighted dice - I don't know the exact numbers, because I haven't learned it. But your weighted dice - it's most likely to fall on a six… But it could fall on a one, probably - I didn't manage to make it do it, but it probably could - or it could fall on a five, a four, or a two. So there are five things that could happen. And the likelihood of them is different: one is probably quite a small likelihood; six is a high likelihood; five is higher than four and two; and four and two are the same likelihood. So, it's random, but it's not as random as having six things that are the same likelihood.

(Lines 176-182)

During her examination of the cracked die, Abby appeared to have developed her ideas about randomness. While she began with an idea that the die was "a bit random", by the end she had articulated the idea more clearly by providing an analysis of the likelihood of different outcomes. She had also clarified her understanding of the relationship between 'equally likely' outcomes and randomness.

Abby's remarks about the biased die, and her early remarks about the randomness of the cracked die, indicated that she initially thought of randomness as requiring equally likely outcomes. Even the cracked die was initially described as "only a bit random". Her later discussion of the parallel between rolling a fair die and the lottery draw seems to have helped her to a different way of expressing the relationship between randomness and equiprobability, in which she used her idea of "fairly random".

As she worked within the local perspective on the observed outcomes, she began to articulate a global view. She expected the die to be slightly biased by the crack, and she tried to articulate this expectation by modifying the standard model of a fair die. However, when asked how she could discover the extent of any bias, it became apparent that she was trying to deal with two different global perspectives. On one hand she saw the relevance of the frequentist approach, and on the other hand she saw the behaviour of the die as subject to the laws of physics and therefore as a phenomenon that could be analysed and described deterministically by an expert ("a physics person"). This dual view might be required in order to understand randomness as a model for a deterministic situation in which the person making the judgement has incomplete knowledge.

9.7 Summary

The extracts discussed in this chapter illustrate how the attention of interviewees moved frequently between a prior perspective on randomness and a local perspective. In the local perspective, interviewees were acutely aware of every successive outcome and whether it matched their expectation according to their prior belief.

The interviews also provided insight into the process by which an individual begins to reconcile the local perspective with an emerging global view. Where there was a genuine mismatch between a strongly held prior belief and the long run frequency distribution, the tension within the individual was seen to mount with each successive outcome. David's work with the biased die and Ben's struggle with the spherical die both stand out in this regard. The interviewee's attention shifted rapidly between the three conflicting perspectives, and the mounting anxiety seemed to fuel the level of uncertainty experienced by the interviewee about what to attend to.

The shift to a truly global perspective requires awareness and understanding of distribution. Without a developed sense of distribution a person would struggle to express either prior or frequentist views. However, the extracts from the interview with Alex give some insight into this difficulty. While it is not possible to state precisely what understanding Alex had of the idea of a distribution, her reluctance to make statements about a frequency distribution suggests that she might have had difficulty with the idea. However, she was able to express the simpler idea of equiprobability as a prior belief. To the extent that she saw the outcomes as unpredictable, she saw her prior belief as supported.

While the shifts between prior, local and global perspectives have been most prominent in this chapter, I am also aware of other shifts. For example, Abby's view of randomness as requiring equiprobability changed so that, by the end of the interview, she was expressing "degrees of randomness".

The dice and coin tasks that I have used in the stage 1 interviews have all invited the interviewee to express a prior belief based upon a set of equally likely outcomes. In that sense these tasks were all very similar and I needed to consider the development of a different task for the next stage of interviews. The next chapters cover stage 2 and, in Chapter 10, I address the development of a new task.

PART 3:

Stage 2 of the Study

Chapter 10: Background to Stage 2 Interviews

In the course of examining shifting perspectives in stage 1, it became apparent that each interviewee approached the dice activities with preconceptions about the distribution they expected to see when they rolled the die. In the case of the biased die, that preconception was usually that the die would be fair, although a few interviewees were alerted to the possibility of bias in the first throw, possibly by the way the die moved as it bounced on the surface. In the later dice activities, interviewees sometimes showed suspicion of me, and of the die that it too might be biased, but the predominant expectation was still that the die would be fair. In reviewing the findings at the end of stage 1, I became more aware of the significance of preconceived prior distributions in the dynamic of the shifting perspectives. In Stage 2, the focus of my attention moved towards investigating the interviewees' shifting attention as they worked to interpret the emerging frequency distribution.

The emergence of the phenomenon of 'shifting perspectives' raised the need to design an additional task to explore this phenomenon. The reasons are set out in section 10.1. Then, in section 10.2, I describe how the idea for an appropriate task was developed.

10.1 Review of stage 1 tasks

The first three process tasks used in stage 1 were described in Chapters 5 and 7. In these tasks, the interviewee rolled three different dice and considered what they expected to occur next. The nature of the task and my questioning led the interviewee to work mainly within the local perspective. In many cases, it was apparent that the interviewee held a prior conception of the distribution of outcomes that they expected to see when they rolled the die. The fact that the first die was heavily biased caused some interviewees to question their assumption that the six possible outcomes were equally likely. When working on the cracked die, some interviewees were concerned that the crack would bias the die; the preconception of 'fairness' was sometimes modified to an expectation that one or more faces would be favoured over the others. As the list of observed outcomes grew, interviewees were sometimes able to compare a preconceived 'prior distribution' with the observed frequency distribution they saw in the emerging sequence. These preconceived ideas of an expected distribution seemed to exist before the interviewee had observed any outcomes, and the ideas evolved in interaction with observation of increasing numbers of outcomes. The ideas express a 'global prior' perspective on randomness, which is different from perspective given by the emerging frequency distribution as observed outcomes are aggregated.

Thus the dynamic of shifting perspectives between only two perspectives – the local and the global – could be seen to be an over-simplification. The global perspective existed in the interviews in two forms: the global frequency distribution emerging from the aggregated outcomes, and the global 'prior' or expected distribution that was an expression of the interviewer's beliefs about the task before they observed any outcomes. I therefore became aware that there could be shifts between three perspectives, rather than only two.

In each of the dice activities, the interviewee was presented with a theoretical prior distribution in the context of an apparently fair die. Interviewees used this as a starting point when they tried to interpret the outcomes. Similarly, the coin tossing activity gave them a uniform prior distribution to work from. In the dice activities and in the coin activities, it was natural to work from an assumption of equiprobability. None of the tasks in stage 1 presented the interviewee with a context in which they needed to work explicitly from the observed data to form a global perspective. A new task was required to probe the relationship between the local perspective and the global frequentist perspective. This new task would need to cause the interviewee to express explicitly their ideas about the emerging frequency distribution, and enable the interviewee to interact with the generating process to alter the frequency distribution.

10.2 Task ideas

Several practical considerations were important in deciding what task was possible. There was limited time available for the development of the physical resource, and I did not want to introduce unintended biases in the process of building a physical resource.

After discussions with colleagues, I focussed on the idea of using two random generators and requiring the interviewee to make some comparison between them. The idea was to develop a task that would involve matching an unknown distribution in one random generator with another random generator in which the distribution was controllable. I also needed to consider the form of generator to use. All the previous tasks had used dice or coins as generators. The tasks were also the precursor to a more general discussion in the interviews about the variety of contexts for which the interviewee saw randomness as a suitable model. I therefore considered there might be advantage in presenting the new task using a different kind of generator, such as an urn or a spinner, as this might open the interviewee's mind to a wider variety of possibilities.

Some early ideas arose in discussion with colleagues and the consideration of these helped to form a clearer sense of what was required in the new task. An initial idea was to construct a spinner with adjustable regions, and a sampling box with the contents hidden. The task would be to sample repeatedly, with replacement, from the sampling box in order to try to estimate the distribution of the contents (2 or possibly 3 different outcomes in different proportions), and to adjust the regions on the spinner so that the distribution of outcomes from the spinner would be the same as the distribution of outcomes from the sampling box. A mechanical refinement would enable the user to inspect a large sample from the sampling box to obtain more precise estimates of the distribution.

There are some disadvantages to using a spinner as the generator with a known and adjustable distribution. First, it is difficult to ensure that a spinner is well-balanced and therefore 'fair'. Spinners are often biased in favour of stopping at or near a particular position, especially when a spinner is home-made, as this one was going to be. Secondly, creating a spinner with easily adjustable regions might not be easy, especially bearing in mind the first difficulty.

Another issue was that the user needed to be able to convince themselves that the random generators in the task had the properties that they expected, such as unpredictability, or independence. The aim of the research at this stage required not only that the interviewee should be able to pay attention to the local perspective as much as they needed to, but also that their attention should be drawn to the global perspective to consider the emerging frequency distribution.

Consideration of these issues led me to consider two different possibilities. One was to use a computer simulation of the real spinner and urn to lead the interviewee to turn their attention to the global perspective. A computer simulation would allow the user to generate the results of a large number of trials in a short time, and display these as a distribution. This would give the interviewee ready access to the global perspective on randomness and would enable them to see directly the effects on the distribution of changes in the underlying process. However, I needed such a simulation to enable the user to experiment with individual trials in order to examine the process from the local perspective. It was important for my examination of the relationship between global and local perspectives that interviewees should be required to think at both local and global levels and that they should be able to move easily between different perspectives. In a computer simulation it is easy to move from simulating individual outcomes to replicating many outcomes in succession, but it is not natural to move in the other direction. I was concerned that a computer simulation might produce a separation between local and global in the way that the user interacted with them, and so remove the need for the user to reconcile in their own mind the potential contradictions between local and global perspectives. I was also concerned that the user might need to be convinced that the simulation truly represented a spinner and/or urn. With these concerns in addition to those that I discussed in Chapter 3 (section 3.13), I was therefore still uncomfortable with using pseudo-random generators in this research to explore people's perceptions of randomness.

The second approach was to dispense with the spinner entirely, use two much simpler urns containing only a small number of beads of two different colours, abandon the requirement to be able to view the results of many trials quickly, but use instead the fact that the contents of the urn can be inspected at the end. This second approach became the basis for the design of the counters task.

10.3 The proposed task

In the new task, the interviewee is presented with a supply of black and white counters, and two sampling bags (or urns), A and B, each containing a total of ten counters, some black and some white. The content of each bag is hidden and can only be inspected by sampling with replacement one counter at a time. The task is to sample from bags A and B and then to adjust the contents of bag B by adding counters to replicate the estimated contents of bag A. Only at the end of the task can the contents of each bag be inspected to make known the distribution of black and white counters in each bag. To help the interviewee manage the two bags, the interviewer looks after bag A, draws counters from it when required, and keeps a record of the outcomes. The interviewee manages bag B, drawing counters and recording outcomes, and then adjusting the contents when appropriate.

Many variations of this task are possible. Some of these are listed below.

- The number of counters in each bag (ten in the version above) could be unknown.
- The total number of counters in bag A might differ from the number in bag B.
- The contents of bag B might be adjusted only by adding single counters, either black or white.
- The content of bag A, the one that is not to be adjusted, might be made known at some stage.

These were all considered and some were tried in early trials of the activity, but they were not adopted in the task as used in the interviews. The task needed to be kept simple to avoid interviewees misunderstanding and to prevent any difficulty of mastering the task from masking the discussion of randomness that was the primary purpose.

A decision had to be made about what proportion of Black and White counters to place in each bag at the start. It was decided that the two bags should initially contain different proportions, but not so obviously different as to be apparent after 10 draws. The proportions used were 7:3 in bag A and 5:5 in bag B. These proportions were arrived at from experimentation with Excel simulation, in which it appeared that the differences between the two were almost always suggested but not totally clear after about 20 draws. It is important to remember here that the purpose of this task is to provoke discussion of randomness, and particularly to create the uncertainty that encourages awareness of randomness at the micro level. If the differences between the proportions in the two bags were too clearly apparent, then the interviewee might spend little time dwelling in uncertainty.

A final consideration was whether to motivate the interviewee to see their decision as important by attaching some kind of reward to a successful match between the two bags. I felt that to do so would change the focus of the discussion from the phenomenon of randomness to the interviewee's decision-making strategies and behaviour. However, in order to encourage the interviewee to look for differences I decided to tell them from the start that the proportions in the two bags were different.

Early versions of the task were trialled with two Year 7 pupils and, later, in a workshop attended by eight colleagues, who worked in two groups. As a result of the trials, I was prompted to consider more carefully how I worded the instructions for the task. The key lessons from these trials are summarised in the following bullet points.

- Stating the aim of the participant in this task as "To match the proportions in the other bag exactly" was not helpful, since it is essentially impossible to be sure that the proportions are exactly matched. Stating the aim of the participant as "To match proportions as closely as possible" was better.
- The word "proportion" was not understood in this context by one of the Year 7 pupils, whereas the word "probability" was. I decided to use "probability" in the first place, but to be prepared to reword the instruction using "proportion" if "probability" was not understood.

There is a delicate balance to be struck in the design of this task between keeping the task open to stimulate ideas of both randomness and distribution, and stimulating the ideas of distribution so much that randomness is not considered. Early informal trials of the activity suggested that this balance was struck, in that responses included some indications of the 'shifting perspectives' that had been observed in previous tasks.

10.4 Presentation of the counters task

In presenting the task in an interview, I decided that this should be the first task, to ensure that all interviews in the second stage would address it fully. However, it appeared from the trials with Year 7 pupils that this task was more difficult than the dice activities had been, so it would be important to ensure that the initial interaction with the sampling bags was made accessible.

The task was presented in three parts.

In part 1, I presented to the interviewee the sampling bags and a supply of extra counters, both black and white. I explained the initial purpose with words such as:

> We have two sampling bags here, each containing 10 of these counters, some black and some white, but in different proportions. So there is a total of 10 counters in each bag at the moment. We shall sample one counter from one bag, have a look at it, record the colour and replace it. We then sample one counter from the other bag, look at it, record it, put it back. We shall repeat that a few times and then I shall invite you to talk to me about what we've seen. Later, I shall give you a task to do in response to this.

One sampling bag was held and managed by the interviewee, who drew the counters, recorded the outcomes and replaced the counters. The second bag was held, managed and recorded by the interviewer (that is, by me). After recording ten outcomes from each bag, I invited the interviewee to consider the outcomes recorded from each bag and to comment on these.

Part 1 was relatively easy to understand. I expected that it would provide a gentle introductory activity for the interview. I hoped to begin discussion of the interviewee's interpretation of randomness when they commented on the first ten outcomes observed from each bag.

In part 2, I explained to the interviewee the task of matching the two bags using words something like:

> Your task now is to change the probability of drawing a black counter from your bag by adding counters to your bag, until you think that the probability of drawing a black counter from your bag is the same as it is from my bag. You can only find out what is in each bag by sampling one counter at a

> time, record the colour and replacing it. You are allowed to add more counters to your bag if you wish, in order to change the probability of drawing a black counter until you think the probability is the same in your bag as in my bag.

I invited the interviewee to ask any questions and attempted to clear any confusion. Then I explained to the interviewee the choices for what to do next.

> You might choose to collect more data, from one or both of the bags.
>
> Or you might choose to add some counters to your bag to try to match more closely the probabilities of drawing a black.

If the interviewee chose to collect more data, I asked them which bags they wanted to sample from, and how many observations they wanted to record from each bag. If the interviewee chose to add some counters to their own bag, I asked them to tell me how many they wanted to add of each colour. In either case, I also asked them to explain how they had arrived at their decision.

This second part of the task was complex and the interviewee needed to get a clear idea of what was required of them. I expected that I would need to answer questions and clarify the explanation of this part. The process of achieving what the interviewee considered to be a good enough match between the probability of drawing a black counter from Bag A and that from Bag B, might require several iterations of sampling, adding counters, and sampling again.

In part 3, when the interviewee was satisfied that they had got the probabilities of drawing a black from each of the two bags as closely matched as they could, I invited them to inspect the contents of the two bags, to see how well they had done. In this part I expected the interviewee to count the numbers of black and white counters in each bag, and then possibly look back through their data and their decisions at each stage to see how well their inferences about the contents of each bag matched what was actually there.

10.5 Summary

The new counters task was developed specifically to probe more deeply the relationship between the local perspective on randomness and the global frequentist view, without clouding the picture with prior beliefs. While it is not possible to remove prior beliefs completely from any activity that I might use in interviews to explore randomness, the counters tasks does not encourage the interviewee to focus awareness on any such prior beliefs.

The stage 2 interviews were planned to begin with the counters task and move on to some of the previous tasks afterwards. Chapter 11 sets out in more detail how the stage 2 interviews were conducted.

Chapter 11: Stage 2 Interviews

The process involved in the interviews in Stage 2 was broadly similar to the process in Stage 1, with the addition of the Counters Task at the start of the process. In this chapter, I describe the presentation of the Counters task alongside the other tasks. I describe the participants in stage 2 and how they came to be selected, and the processes involved in the transcription and analysis of each interview.

11.1 The Interviewees

There were nine interviews in stage 2. Table 11.1 lists the interviewees in the order in which they were interviewed and gives some information about each. A fuller description of each interviewee follows later in this chapter.

	Name	Gender	Year	Age	Maths set
1	Linda	F	11	16y 1m	Set 1
2	Bernice	F	10	15y 0m	set 3/6
3	Claire	F	10	15y 0m	Set 1/6
4	Hannah	F	10	14y 8m	Set 3/6
5	Rory	M	9	14y 5m	Set 4/7
6	Mosaab	M	11	16y 11m	Set 5/6
7	Andrew	M	9	13y 11m	Set 1
8	Guhan	M	8	13y 3m	Set 1/6
9	Assim	M	12	17y 2m	AS

Table 11.1: Interviewees in stage 2 interviews

11.2 The selection of participants

Interviewees were school pupils aged between 13 and 17, and they came from three sources. Some were children of neighbours or colleagues as in stage 1.

A second source was a group of 20 able pupils in Years 7, 8 and 9, from neighbouring schools, who attended a series of Saturday morning classes taught by me. I sent letters home to parents of all pupils and interviewed those pupils who replied. Interviews with these learners from these two groups took place out of school time: at weekends, during early evenings or during school holidays in my home.

Others were pupils at a mixed 13-18 Upper School, with a comprehensive and ethnically diverse intake. I had approached the Head Teacher, who had agreed that pupils could be invited to participate, and a letter requesting parental consent was sent home to parents of pupils selected by the head of the mathematics department. I wanted to broaden my sample of learners, and so I asked the head of department to include some learners from ethnic minorities, and some pupils for whom mathematics was more challenging. However, I requested that learners selected should be willing and able to express their ideas in discussion. I wanted to try to avoid selecting learners who were unwilling to speak in front of me and the microphone. The selected pupils were from various sets in Years 9, 10 and 11. These pupils were interviewed on school premises in a small office during their mathematics lesson.

11.3 The interviewees in detail

In this section I give a brief description of each interviewee in turn. Most of the interviewees were relatively able and successful pupils, as in stage 1. However, three of the interviewees in stage 2 were from middle or lower sets and experienced difficulties with some aspects of mathematics. The descriptions in this section are drawn from my interview notes made shortly after each interview, as I transcribed the audiotapes.

11.3.1 Linda

Linda, aged 16.1, was a highly able and articulate pupil in the top set of Year 11 at an independent selective girls' school, and was studying GCSE Mathematics. She was the daughter of a family friend and was known to be an extremely conscientious and hard-working student. She did not see herself as mathematically able and appeared to lack confidence in her mathematical ability.

Linda's interview was the first one that I did in Stage 2 and so was the first time that I had tried the counters task in an interview. The interview took place in my home during a weekend.

At the start I emphasised to Linda that I wanted her to think aloud about the tasks and to externalise what was going on in her mind. Initially, as I briefed her about the first task, she appeared to be quiet and reserved, giving monosyllabic responses, but as the interview developed she was able very effectively to 'think aloud', expressing both her uncertainties about how to interpret what she saw, and her developing ideas.

11.3.2 Bernice

Bernice is a Year 10 pupil, aged 15.0, of moderate attainment (set 3 out of 6) in a mixed comprehensive 13-18 school. The class teacher selected Bernice for this interview as a pupil who would be willing to talk about her thinking. The interview took place in school in the maths department office during as mathematics lesson.

I began by explaining that I would be giving her some tasks to do and that it was important that she should talk about her thinking as she was working on the tasks. I explained that I wanted her to try to think out loud. Bernice was quiet at the start of the interview, but she spoke more as she relaxed.

11.3.3 Claire

Claire was a highly able Year 10 pupil, aged 15.0, in the top set in a comprehensive upper school (13-18). She was rated by the Head of Department as a 'star mathematician', but she also excelled in other areas of the curriculum (literacy, languages and science). She was highly articulate and expressed her ideas clearly and effectively; she had also read widely. The interview took place in her school, in the maths department office, during a maths lesson.

I discovered in the first few moments of the interview that Claire is the daughter of a family friend, whom I had seen very little in the past five years. She had recognised my name on the permission letter that she had been given to take home for her parents. In the interview, Claire was exceptionally coherent and articulate.

11.3.4 Hannah

Hannah, aged 14.8, was a highly articulate pupil in Year 10 at a comprehensive upper school, in set 3 out of 6 for mathematics. She was described by her maths teacher as being of moderate ability, but ‘quite talkative’. The interview took place in the school maths office during a morning mathematics lesson. At the start Hannah was somewhat apprehensive, and said little, but as she relaxed, she began to say more, and by the end she was explaining confidently her ideas and thinking about the tasks.

11.3.5 Rory

Rory, aged 14.5, was a boy in Year 9 at a comprehensive upper school. His teacher described him as a likeable lad but not particularly good at mathematics. He was in set 4 out of 7. The interview took place in the school mathematics department office during a morning mathematics lesson. At the start of the interview, Rory appeared shy. He followed instructions carefully and was cautious about taking action until he was quite sure what I wanted him to do. As the interview progressed, he began to relax and was more forthcoming with ideas and comments.

11.3.6 Mosaab

Mosaab was a young East-African, who had only been in the UK for a few years. He was working in year 11 towards GCSE mathematics in a lower set. Aged 16.11, he was in fact a year older than others in Year 11. He was very polite and keen to please, but sometimes he did not appear to understand what to do. The interview took place in the school mathematics office during a mathematics lesson. In this interview, I had to pay particular attention to my language. Mosaab was working in his third language and he had a limited vocabulary. He also had a strong accent, which made transcription difficult in places.

11.3.7 Andrew

Andrew was an exceptionally able child in Year 9, aged 13.11, who had just started at a local selective independent school. He had particular strengths in mathematics, science and technology. The interview took place at Andrew's home just after the end of term.

I had worked with Andrew about nine months earlier when he was a pupil on the series of Saturday morning classes that I taught. He was a very thoughtful child who tended to think deeply before expressing himself. I was a little concerned that he might be reluctant to express himself in the interview so, when I explained to Andrew that I would be talking to him about a variety of tasks, I emphasised that I wanted to hear how he was thinking about them. It was therefore important that he should try to think aloud.

11.3.8 Guhan

Guhan was a very able boy, aged 13.3, in the top set in Year 8 in a local Middle School. I had taught him nine months earlier when he was a pupil on the series of Saturday morning classes that I taught. He had impressed me as a highly intuitive thinker, who processed ideas very quickly. The interview took place in my home one afternoon after the end of term.

11.3.9 Assim

Assim was a Year 12 student, aged 17.2, studying for AS level mathematics at a comprehensive upper school. He was due to take a module in statistics, but he had not yet started it. I had an additional reason for interviewing him in that he was interested in the possibility of programming a simulation of the counters task in Flash. I took him through the counters task first so that he would understand what was involved in it, and interviewed him about. The remainder of this interview was concerned more with the practicalities of building a computer simulation. Therefore, this interview deals only with the counters task and the review task.

11.4 The conduct of the second stage interviews

The second stage interviews were conducted in a similar manner to stage 1 and usually needed to be completed within 50-60 minutes. However, the introduction of the counters task as the first activity changed the timing of the remainder of the interview, and in most cases one or more of the other tasks was omitted. The tasks used in the interviews in stage 2 are summarised in Table 11.2. The interviews are listed in the order in which they were conducted.

	Interviewee	Counters Task	Biased Die	Spherical Die	Cracked Die	Coin Generation	Coin Tossing	Review task
1	Linda	1	1	1	1			1
2	Bernice	1	1	1	1			1
3	Claire	1	1	1	1	1		1
4	Hannah	1	1	1	1	1	1	1
5	Rory	1	1	1	1	1	1	1
6	Mosaab	1	1	1	1			1
7	Andrew	1	1	1	1			1
8	Guhan	1	1	1	1			1
9	Assim	1						1

Table 11.2: Tasks used in Stage 2 Interviews

Where there was not time for all the tasks I omitted one or both of the coins activities, and occasionally also omitted one of the dice activities. In each interview I kept time for a discussion of the review task.

During the analysis of stage 1 interviews, I was concerned about some of the questions I had asked. In particular, I felt two questions were being overused.

Firstly, I had often asked questions such as: "Do you think that was random?" or "Was that chance?" In planning my questions for the stage 2 interviews, I aimed to ask more open questions, such as: "What do you make of what you have seen so far?" or "What do you think of this die so far?"

Secondly, I was too often asking: "What might happen next?" This question had been introduced to elicit the interviewees' beliefs about the random generator that they were working with. I planned this question less often in the stage 2 interviews.

11.5 The analysis

My approach to the analysis of the interviews was similar to that in stage 1. Once I had prepared a detailed commentary on each interview, I worked through each account, identifying the themes that had been found significant in the first stage interviews, and paying particular attention to the idea of shifting perspectives between the local and the global views of randomness. Occasionally, I found the need to identify a new theme which seemed to occur across several of the second stage interviews.

At the end of this process, I reviewed the list of themes again, checking the ways in which I had clustered related themes under a single heading, and reducing areas of overlap between themes as far as possible.

11.6 Summary

The format for the stage 2 interviews was essentially the same as was used in stage 1. The first task was the new counters task, and some of the original stage 1 tasks were cut as required by the constraints of the time available for the interview.

The process of analysis was also similar to that applied to the stage 1 interviews. The intention was to build upon the themes and issues identified in stage 1, to refine and develop these ideas, and to add to them where appropriate.

In Chapter 12, I set out a clearer sense of shifting perspectives that has arisen from analysis of stage 2 interviews, particularly in the counters task, in the light of the findings from stage 1.

Chapter 13 brings together the findings from stage 1 and stage 2, drawing on the material in Chapter 8 and 9 as well as data from stage 2.

Chapter 12: Shifting perspectives in Stage 2 Interviews

The counters task was developed for use in the second round of interviews and differed fundamentally from the dice rolling and the coin tossing tasks. Its main purpose was to look specifically at the relationship between local and global perspectives on randomness, which were identified in the first round of interviews. In contrast with the original tasks, the counters task required the interviewee explicitly to infer the nature of the underlying distribution from consideration of the local perspective.

In this chapter, I present examples of interviewees' working on the counters task, and I focus particularly on some shifts of attention that I have seen. I describe some features of the circumstances of these shifts that may contribute to their occurrence. Shifts of attention were evident in all the tasks, but the counters task has highlighted some particular issues. For most interviewees, the task highlighted the question of how many counters would need to be sampled to produce a sufficiently reliable estimate on the probability of drawing a white counter from a bag. Some interviewees changed their view about this issue as the task progressed. An example was Linda, whose work is described in section 12.1. Typically, the interviewee's attention shifted between specific outcomes from the interviewee's bag and the interviewer's bag. This was evident in all the interviews and is exemplified in the work of Bernice in section 12.2. In some instances, this shift occurred almost item by item, at each draw. There was also a shift of attention from the sequence of outcomes, often focusing only upon either white or black but not both, to the unseen contents of the bag. Bernice's work in 12.2 demonstrates this.

Finally, in section 12.3, I describe a shift from looking for evidence of randomness or non-randomness in the outcomes to looking within the generating process. I had not discerned this shift in previous interviews, but have found evidence of it in almost all interviews within the counters task. Section 12.3 examines extracts from five different interviews.

12.1 Linda: sample size in the counters task

In this section, extracts from the interview with Linda (age 16.1) are used to illustrate the responses of interviewees to the counters task. As with the dice and coin activities in stage 1, some subjects showed confidence in their judgements on the basis of very few outcomes. Linda did so from a very early stage in this task: for example, after only 3 outcomes (B W B), Linda stated "there are probably more black ones" (line 26). As she continued to work on the task, she kept returning to this issue of sample size, encouraged by the structure of the task. How many outcomes needed to be observed from each bag before a judgement could be made?

After the sixth draw from each bag, Linda had observed 3W and 3B from Bag A and 2W and 4B from bag B, and suggested that Bag B was not likely to contain equal numbers of black and white counters. After the seventh draw from each bag, W from Bag A and B from Bag B, she restated her view with more confidence.

Linda: In your bag it looks like there are more black counters, I think, than white ones, because there have only been 2 white ones pulled out in 6 turns, no… 7 turns. And in my one … there could be equal numbers of each, I think. I'm not sure though.

(Lines 60-63)

Linda wanted to estimate the proportions in the bags on the basis of the first 10 outcomes. My intention that the counters task should cause the subject to begin with their attention rooted in the distributional perspective seemed to work in this case. However, it seemed that Linda was concerned to identify the exact numbers of black and white counters in each bag rather than to estimate the probabilities of drawing black and white. The difference here is subtle, as the probabilities should be dictated by the relative numbers of black and white in the bag. Linda's intent, however, appears to be to infer from her sampling the precise numbers of black counters and white counters in the bag. The counters task, as it was set up in these interviews, did not enable me to explore the difference between seeking to identify the contents and seeking to estimate the probability, since I had fixed the total number of counters in the bag at the start (ten) and had told the interviewees what that was.

Linda: You need to have more trials where you pull them out before you get a good picture of what's in the bag. Because, now we've had 10 goes, you get a kind of clearer picture, whereas when we just had 3 goes… I couldn't really tell… how many white ones or how many black ones there might be.

(Lines 85-88)

Linda's awareness of the issue of sample size seemed to be evolving. She was already aware that, although ten outcomes gave a better picture of the contents of the bag than three outcomes had done, she really needed to see more before she could make her estimate of the contents.

I explained to Linda what she was required to achieve in the remainder of the task. I emphasised that each bag had begun with ten counters, that the number of black counters in each bag was different and that she was only allowed to add counters to her own bag. So far, she had observed exactly five of each colour in the first ten outcomes from each bag, so she had no basis for adding any counters. However, Linda was uncertain how to respond to the situation. In the next extract, she reviewed the list of outcomes again and noted that the proportion of black counters observed after only four draws had been the same for each bag, and again after five draws. Her attempts to learn more from sub-sequences of the set of outcomes observed showed that she was unsure what she should pay attention to in the list of outcomes. She then sought clarification about the rules of the task.

Linda: … I don't quite know. I mean, if I … just look at the first 4. That indicates that … there's the same proportion in each bag. But if I look at the first 5, that indicates that the proportions are the same. And so… I'm not really sure what to do. But I guess I could take more out of… No. (*Laughs*) … Am I allowed to sample them from both bags, or just one, just mine?

I: You can sample from both bags, one at a time.

(Lines 122-128)

In the end, she decided to collect more data from each bag, but she found it frustrating when, for each of the next three draws, the outcome from the second bag matched that from the first bag. After completing a second batch of ten outcomes from each bag, Linda chose to combine the first and second batches and to look at the numbers of black and white in the first twenty draws. She concluded that the observation of 10B:10W from her bag and 9B:11W from mine was clear evidence that there were more white counters in my bag than in hers. However, she had trouble deciding how many counters to add to her bag.

Linda: OK. So then that indicates that there are more white ones in your bag. So if I was to match that, I would have to add… No… How many would I have to add to my bag? I would probably have to add one white... no, 2 white ones... no, that wouldn't work… I don't know!

(Lines 147-150)

Given that there was already so much uncertainty about the contents of each bag arising from the variability in the proportion of whites in a small sample, Linda seemed unduly concerned to be precise about how many white counters to add. She seemed to have mistaken the recorded numbers of black and white outcomes for the actual contents of each bag. When I reminded her that there were exactly ten counters in each bag, she appeared to be even more confused and uncertain, concluding that she did not know how many white counters to add. When I suggested that she might wish to collect more data, she decided she would do so. However, Linda had little idea of how much more data she might need, or even whether any further data could be relied on to help.

I: … you might want to collect some more data?

Linda: … I think so, but I don't know if it would make any difference… Can we collect like 5, do 5 more turns. And that might help.

(Lines 161-163)

Her decision to collect only five more observations from each bag was surprising, as I had expected her to be more aware now that small samples were not reliable. Once again, the outcomes from the second bag matched exactly those from the first. This meant that there was still little difference between the outcomes observed from the two bags – 11B:14W from my bag and 12B:13W from hers. In spite of the small difference between the two sets of data, Linda decided that this confirmed there were more whites in my bag. But she was still uncertain how many whites to add to her bag, and became anxious that she was "doing it wrong". I reassured her that there was no right or wrong approach to this problem, and I emphasised again that I was interested in how she thought about it rather than whether her answer was 'right'.

Linda: So… that definitely… that kind of confirms that there are more white ones in your bag. I'm just not sure how many I would need to add… if I double the ratio 11 to 14, then you'd have 22 to 28. So what I could do is add… 12 black counters to mine and 18 white counters. But that would be a bit too many wouldn't it? … Or would it? I don't know. Can I do that?

(Lines 172-177)

Linda had invented an algorithm to determine what to add, although she appeared to have taken the outcomes from her bag as 10B:10W instead of 12B:13W. She decided to proceed with adding counters, but carried out one last 'check' of her calculations.

Linda: I think I'll do that then. I'll just check the ratio once more though, to check I'm doing it right. So, I've got 9 black ones, 10 11, yeah, 11 black ones. 11, 12, 13, 14, so yeah, it's definitely 11 to 14. So 22 to 28 ... I don't know if that would work. But… yeah… I'll add 12… no… yes…12 black ones and 18 white ones.

(Lines 182-185)

Unfortunately, in her check, she only counted the results from my bag, but did not recheck the calculation from her own bag, so she did not spot her earlier error. Linda had established that her estimate of the ratio of black to white in her own bag was 22:28.

Taking my bag as containing 10 Black and 10 White, she added 12 Black and 18 White to give 22B:28W.

However, Linda's intention here was clearly to treat the aggregated observations from sampling the bags as an indication of the underlying distribution of black and white counters in each bag. In that sense, she was trying to work in the global perspective, treating the aggregated results as indicative of a long term pattern.

Having added 12 Black and 18 White counters to her bag, Linda chose to collect more data from each bag. She decided to collect the same number of outcomes as before, 25 from each bag, because "it might be easier to compare the results".

Linda: I mean, you could do fewer trials, I guess, but it might be easier to compare the results if…you use the same number.

(Lines 199-200, 202)

The result of the next 25 outcomes showed 15 white and 10 black from my bag, and 14 white and 11 black from Laura's. The difference between the proportions of black counters drawn from each bag was small, but she was not sure how much difference would be expected if the probability of drawing black was the same for each bag. She was aware that some uncertainty should be present due to the 'randomness' of the situation, but was not sure how to relate this to her inference that the underlying probabilities were the same in each bag.

Linda: In mine I've got 14 white ones and 11 black ones. And in yours, (*counts*) 15 white ones and that would be 10 black ones. So, the results are very similar because you've got 15, 15 times the white counter occurred and in mine it occurred 14 times. Yours, the black one occurred 10 times and in mine 11 times. So, well… that would kind of show me that… I don't know... I don't know if you need to have identical results in order that… in order to show you that the ratio is the same in each bag. Because the ratio could be the same in each bag… and… perhaps these results are just not what you would expect.

(Lines 212-219)

Linda was aware of the possibility of variation in the sample proportion, but she did not have a sense of how much variation to expect.

Towards the end of the counters task, she tried to relate her reasoning about the proportion of blacks in each bag to the fact that she knew the total number of counters in each bag at the start had been 10. With bag A, whose contents had not changed, she could not convert the ratio 11:14 into a ratio $n:(10-n)$. This difficulty with calculating the ratios appeared to cause her some anxiety and interfered with her discussion about 'randomness'.

Linda: ... if the ratio was 11 to 14, then that's a bit hard to... Oh I don't know. ...11 to 14. So there are more whites... there would be more whites in your bag. So perhaps there would be 6 white counters and 4 black counters, because when you double that ratio you get 8 and 12... No... that's not very much like 11 and 14. ...You would need to divide... No. Oh, I don't know. ...If you divided... No. (*Laughs nervously*). I don't know. I think there would probably be 6... There would definitely be more white counters in the bag, so there would probably be either 6 white counters and 4 black counters or perhaps 7 white counters and 5... no that doesn't work... 7 white... 7 white counters and... no that's not enough. OK. I think there would be 6 and 4 - 6 white and 4 black.

(Lines 289-299)

Linda was struggling to hold in her mind simultaneously the various constraints on the ratio calculations. However, her goal appears to have been to identify the contents of my bag. The ratio of 6W:4B seemed more acceptable to her than the incompletely articulated ratio of 7W:3B.

Again the discussion returned to the question of how much data Linda would need to see to be able to be confident that she had matched the ratios in the two bags. It was clear from what she said that she felt some frustration about her inability to explain the differences in the ratios that she had observed, and she went on to attempt to explain the differences by suggesting that she really needed to have collected more data.

Linda: ... So that would suggest that we didn't do enough trials, really. In fact, you need to do more trials before you get a better idea of ... how many counters there are...

(Lines 318-320)

It was interesting that she did not see the need, this time, to qualify her response by noting that, however much data she collated, there would always be some uncertainty. However, her ideas about how much data would be required clearly changed during her work on the counters task: early on she had generalised on the basis of only three outcomes from each bag, and later she had thought that about 20 outcomes would suffice, but now she was looking for many more!

I: So 25 isn't enough. Do you have any sense of how many you might need to do?

Linda: Perhaps, a hundred? Or more? Perhaps 200? That might give you a better idea.

(Lines 321-323)

This last observation emphasises again the need for learners to have experience of interacting with live random processes. This student is highly able and articulate, and academically successful. Yet her experience of working with live random processes has been quite limited. She suggested there had been one or two maths lessons that had involved rolling dice, but no more. However, she appears to have learned quickly; her suggestion that "perhaps 200" observations might be required in order to estimate the probability of drawing a black counter is quite different from her apparent willingness to estimate on the basis of only 3 outcomes at the start.

The issue of variability in a sample statistics is present in both the local perspective of disorder in randomness, and in the global distributional perspective. Within the local perspective, variation is the essence of the disorder and the unpredictability of the successive outcomes, but it is not measured. In contrast, the global perspective brings awareness of a distribution and hence of a sense that there may be bounds on the variation, that the variation may be one parameter of the distribution and that it is related to the sample size. Linda appears to have some awareness of distribution, that there may be a bound on the variation, and that the variation is related to sample size, but she does not understand how much the variation is affected by the sample size.

12.2 Bernice: Local and global perspectives in the counters task

In the early stages of the interview with Bernice (age 15.0), she was reluctant to talk, but as she relaxed she became more articulate. As she worked on the counters task, her attention shifted rapidly between the outcomes from the two bags.

The first time that Bernice was invited to reflect on what she had seen was after we had drawn ten counters from each bag and I had told her about the task. She had observed 9W and 1B from her own bag, and 4W and 6B from my bag. Bernice chose initially to add a black counter to her bag "because there's more black ones in that bag" (line 50). In this justification her attention was focused on comparing the number of blacks in each bag, which she inferred by comparing the number of blacks drawn from each bag. It is possible that she was prompted to think in this way by my remark in the initial briefing in line 17: "… we'll just sort of compare as we go along".

Given the large disparity between the observations from the two bags, Bernice's strategy was not surprising, but I might have expected her to have chosen to add more than one black counter. In fact, she had misunderstood the brief and wanted to add some whites to my bag as well. When I explained again that she was only allowed to add counters to her own bag, she chose to add three further black counters to her bag, making four extra black counter in all. Having added the counters, she began to draw again. After four draws, she had drawn four black counters in succession from her own bag, to which she had added the black counter, and I had drawn WBBW from my bag. When I asked her to comment on these outcomes, she simply noted: "It's obviously coming out more black" (line 75), again focusing her attention on comparing the number of blacks drawn from each bag.

When I asked if she thought she might have it right, she was more cautious.

I: So, do you think you've got it right?

Bernice: (*Pause 7 seconds.*) Yeah, kind of.

(Lines 76-77)

We continued drawing until we had each drawn ten counters. Bernice had drawn seven black counters in the sequence: BBBBWBBWWB. I had drawn WBBWBWWBBB, giving six black counters. Bernice was staring at the lists of outcomes, so I asked her what she was thinking. Her reply surprised me and showed a clear shift of attention.

Bernice: Umm. Some of them are right, but the ones I got out are a long way opposite of what you've got out. Does that make sense?

I: The ones that you're pulling out are the opposite to mine?

Bernice: Yeah. Some of them.

(Lines 81-84)

She seemed to be looking for a one-to-one match of individual outcomes from each bag. When the sequences are paired, only three pairs match. As Bernice had said, "Some of them are right".

I did not comment further on this at this time, but asked her if she wanted to add more counters to her bag, or to collect more data. She chose to add some white counters. Her explanation showed that she was looking at the outcomes from the first ten and the second ten and considering what she had added last time.

Bernice: Cos, ... (*Pause 4 seconds*) just because to start with there was more white and I added black, but there was already black in the bag. And if I add more white then I'll make it more even.

(Lines 90-92)

This seems to indicate that her attention had reverted to comparing the number of a kind drawn from each bag, in this case the whites. She had seen nine whites in the first ten, added four blacks, and then seen three whites in the second ten. She felt that she had added too many blacks the first time and now needed to add some white to "make it more even" (line 92). She chose to add just two white counters and then started to collect a third set of ten outcomes from each bag.

After three draws each, we had both drawn WWB, an exact 'match'. When I invited her to comment, Bernice was cautious, possibly because the sample was so small, but I inferred from her comment that her attention might have switched again to matching the individual outcomes.

I: What do you think so far?

Bernice: ... Equal. Quite.

(Lines 110-111)

The fourth draws did not match: Bernice drew white and I drew black. Bernice still considered this was OK, but when the same thing happened on the fifth draw, Bernice said, "Help!" (line 116). Her search for matching individual outcomes was not being successful.

At the end of the third set of ten draws, Bernice had drawn WWBWWBWWWW, and I had drawn WWBBBWBWBB. This gave her eight whites and me four whites. Bernice's next comment seemed to be an attempt to reconcile the two different approaches that she had been trying with the outcomes that she had observed. She was very hesitant about this suggestion, but she wanted to attribute the difference in the number of whites drawn from each bag to the uncertainty about which particular counter was selected at each draw.

Bernice: I think this depends on... how... much you shake it about... and which ones you pick out... It's about the same in each, but they're just... it depends which ones you pick out.

(Lines 120-121, 123, 125)

In making this suggestion, Bernice had moved her attention from looking only at the outcomes to a consideration of the generating process. Of course, her motivation for doing so was to account for the outcomes that she had observed, but this was nonetheless a new shift from looking at the randomness in the outcomes to looking at the process.

Previously, Bernice had attempted to make statements at a local level about the patterns, or lack of them, in short sequences of outcomes, but she had also shifted her focus to the global level when she attempted to make statements comparing the proportions of white counters in the two bags. Those were the shifts between local and global that I had noted previously in other interviews. This shift from looking only at outcomes to considering the process was different.

Following these remarks, Bernice's attention appeared to shift again towards the ratios of black and white counters in her bag, possibly encouraged by my next question. However, she seemed to be looking only at the sample from her bag when she suggested that the probability of drawing a white from her bag was slightly higher than that of drawing a black. Whereas a few moments before she had been trying to compare the individual outcomes from each bag, now she was looking at the outcomes from only one bag and she was attempting to make a global statement about the probability of drawing a white counter. Her intention here seemed to be to compare the ratio of black and white in her bag to 50/50.

I: Do you think the probability of getting a white is about the same in each bag now?

Bernice: … Yeah, I think so.

I: Roughly, or…

Bernice: Yeah. Actually, I think more…white.

I: So you think…?

Bernice: The probability of getting white is more high…

I: From… your bag?

Bernice: Yeah.

(Lines 126-134)

When I asked her to consider my bag, her attention seemed to be the same as before: comparing the ratio in one bag to 50/50.

I: What about in this bag?

Bernice: … I think getting a black in your bag is a bit higher.

I: Yeah? Why do you think that?

Bernice: Because there's more Bs in there (*indicating the record of outcomes from I's bag*).

(Lines 135-138)

I wondered if she had considered the evidence from the previous two samples, and I invited her to note that the number of black counters in each of the three samples seen so far was six, which had been very consistent. I asked her to consider the number of blacks in each of her samples. She noted the number of whites in her first sample and then went silent for a long time.

I: And in yours?

Bernice: I've got, in the white, nine. (*Pause 27 seconds.*) The, it's… I think it just depends…

I: On?

Bernice: On… (*Pause 5 seconds.*) No. I'm not sure. I think if I didn't add the two white ones to that, that it would have been lower…

(Lines 144-148)

The number of whites in each sample from Bernice's bag had varied greatly: nine, three and eight. After the first sample she had added 4 black counters, and after the second sample, two white counters. She did note that the addition of the two whites after the second sample might have helped produce the increase from three in the second sample to eight in the third. However, the variations between the samples were larger than the numbers of counters she had added, and so were larger than she had expected to see and she did not know what to make of it. She suggested that "it just depends…" but could not suggest what the factors might be. I suggest that she was simply saying that she did not know what to make of the evidence.

Bernice did not have enough experience of variability to understand that the kind of variation that she was seeing between the three samples was not really very unusual. Because she did not understand that such variation was to be expected, she did not know how to interpret the results and was unable to comment. She also seemed at this stage to be unaware of the principle that a larger sample might be expected to show a smaller variation, as every time she sampled exactly ten outcomes from each bag, and she made no attempt to combine the samples from my bag for which the contents had remained constant throughout.

In order to try to make the probabilities equal for the two bags, she decided to "add some more black", and added just one black counter this time. This time she drew BWWBBWBWBW, making five whites. My sample was WWBWBBWWWW, having seven whites. Initially, Bernice's attention was on the balance on Black and White counters in her own bag. She noted the equality of the white and black counters in this sample with some relief.

Bernice: That might have evened out.

(Line 165)

When I invited her to look at both samples together she said she was "not sure". Her attention seemed to shift again as she tried to explain the mismatch between the number of blacks in each sample this time by reference to the extra black counter that she added before taking this sample.

Bernice: ... I think because I added that extra black one, that's evened out the numbers that are in there. That makes it higher, the probability for... to get black or white...

I: A higher probability to get black...?

Bernice: ...Yeah. Cos there was more white in before.

(Lines 173-176)

Bernice suggested that the extra black counter that she had added had made the chances of getting black (or white) out of her bag "more even" because "there was more white in before", and she indicated results from her second sample in saying this. She was making a global statement to account both for the counters that she had added to her bag and for the outcomes she had seen subsequently. It is important to note here that Bernice has not suggested here that the probability of getting a black was the same for each bag. Indeed when I asked her if that was what she had meant, she was able to make a clear global statement comparing the probabilities that she inferred for each bag. She stated clearly that she thought the probability of getting a black was lower for my bag.

Bernice: No. I think it's lower in that bag (indicating I's bag).

(Line 178)

She gave as a reason for this conjecture the number of whites (seven) in my most recent sample. However, she did not choose to take account of the outcomes from previous samples, in each of which there had been only four whites from my bag.

Bernice: Because you've got more whites in each one. I think. Or in the end one.

(Line 182)

When I drew her attention to the fact that the contents of my bag had not changed throughout the activity, and that the previous samples had also come from the same contents, Bernice hesitated, and seemed not to know how to deal with this information. Then she suggested that she thought my bag contained the same number of black counters as white counters.

Bernice: (*Pause 5 seconds.*) Then I'm not sure.

I: You're not sure what?

Bernice: What I'm thinking about it.

I: What do you think is in this bag?

Bernice: The same amount of… in that bag of black and white.

I: You think there's the same of each? Yeah?

Bernice: Yeah.

I: So five of each?

Bernice: Yeah.

(Lines 185-193)

It is helpful here to look more closely at the outcomes that Bernice had seen. These are summarised in Table 12.1.

Sample number	**Bag A**		**Bag B**		**Added to bag A**
	Black	**White**	**Black**	**White**	
1	1	9	6	4	4 Black
2	7	3	6	4	2 White
3	2	8	6	4	1 Black
4	5	5	3	7	

Table 12.1: Bernice's observed outcomes

I cannot tell what process Bernice went through to arrive at this estimate, but it seems possible that she may have taken the average number of white, or black, counters across the four samples, to arrive at five per sample. I do not think she was taking account of the previous samples until I suggested them to her. She was not able to explain her reasoning.

When I asked her to estimate the number of black counters in her own bag, Bernice hesitated before suggesting that there would be about eight of each colour in her bag. The processes that might have been involved in arriving at such an estimate are many, and Bernice gave little clue as to how she had got there. I tried reviewing with her the counters that she had added during the task, reminding her of the single black that she added before the final sample. This alone seemed to be enough to alert her to the fact that there should now be seventeen counters in her bag, and she immediately modified her estimate of the numbers of black and white to make this total.

I: And what have you put in? You added one black there didn't you?

Bernice: Oh. About nine… white and eight black.

(Lines 199-200)

I was still curious to know more about how she had arrived at this estimate. When I asked her, she reviewed the outcomes of the first and second samples, and her explanation indicated that she had reasoned informally.

Bernice: … to start with … I pulled out more white than I did black. And then when I added the four black, more blacks came out. And… then when I added the two whites, it come slightly evened out. But not much. There was more white there. Then it came out equal the last one.

(Lines 204-205, 207-208)

She had overcorrected the number of blacks after the first sample, and possibly overcorrected the number of whites after the second sample, and the numbers of black and white were probably now approximately even, as was suggested by the final sample.

Bernice was now satisfied that she had done enough to make the proportions in the two bags equal, and she wanted to see what was inside the bags. As it turned out, Bernice had estimated correctly the contents of each of the bags: her bag contained nine white and eight black and my bag contained five of each colour, as she had suggested.

When I asked her what would be the probability of drawing a black counter from her bag, her reply showed that she did not necessarily understand formally how to calculate a probability. However, she was able to use informal language of 'odds' to express her ideas approximately.

Bernice: Umm. About 40 60.

I: How did you work that out?

Bernice: No. I don't know. Umm. (*Starts to count the counters again.*) It's about 40 60 because there's one less black than there is white.

(Lines 231-234)

I cannot tell from what she said whether she knew what calculation to do, or whether the difficulty for her lay in the particular values involved. However, her strategy of making adjustments from 50:50 is evident.

Looking at Bernice's work on the counters task, there is an indication in her attempts to infer probabilities from frequencies that she has not developed an understanding of the law of large numbers. Her attempts to use the observed frequencies to make statements about the probability of drawing a white were all based upon single blocks of ten outcomes. Thus her attempts to adopt a global perspective in this activity had limited success because she could neither account for nor deal with the variation that she saw from one sample to the next.

For both Linda and Bernice, their attention shifts rapidly as successive outcomes emerge from the sampling bags. The interviewee's attention moves in different ways: sometimes from outcomes from one bag to outcomes from the other bag; sometimes from considering the sequence of outcomes to considering the contents of the bag; sometimes from comparing the outcome sequences from the two bags to comparing the contents of the bags. Once for Bernice, the attention shifts clearly from considering outcomes to looking at the generating process. For each of these interviewees the rapid shifting of attention indicates uncertainty. They don't know what to focus on, so they follow their attention, sometimes influenced by the interview question and sometimes by an outcome.

At one point, Linda appeared to have confused in her mind the samples of outcomes from the bags for the actual contents of the bags. This might be seen as a moment when the shifting of perspective between in this case the observed outcomes and the content of the bag fails to occur completely. There is a 'slip of attention'. The person's attention shifts but her awareness of what she is attending to does not.

Both Linda and Bernice had shown little understanding of the relation between sample size and variability of the sample mean. Each had seemed to develop awareness during the activity that, in order reliably to estimate the contents of a bag, they needed to work with larger samples than they had assumed initially.

12.3 Sampling the same counter again

On a number of different occasions an interviewee attempted to explain some perceived anomaly in the observed sequence of outcomes from drawing counters out of the bag by suggesting that the process of sampling without replacement was in some sense 'unfair' or otherwise unsatisfactory. Sometimes their explanation mentioned the 'danger' of 'sampling the same counter again', and at other times it was more a concern that the counters had not been sufficiently mixed up at the start, or that the counters had not been shaken after each replacement. One example of this was Bernice, whose work was discussed in 12.2.1.

Each of these incidents can be interpreted as exemplifying a new shift of attention. The interviewees were usually looking at the tasks through the sequences of outcomes that were generated. In these incidents, where the interviewee noted the idea that the same counter could be sampled more than once, the interviewee's attention has moved to looking at the task through the generating process. I suggest that this is a shift from looking for randomness in the outcomes from the process to looking for randomness within the process itself.

12.3.1 Andrew

Right at the start of the counters task, after only the first ten draws, Andrew's first comment on considering the ten outcomes observed was that they might bear little relationship to the contents of the bag since he might "have picked the same one out multiple times" (line 36). His attention had moved quickly from the outcomes to the process.

Andrew: Well, although you've got ten there, it's not necessarily all the ten that were in the bag, because you could have picked the same one out multiple times, or... You may have picked all ten out, but ... there's no way of telling because ... there could only have been one or two, because if there's one of ten, then you'd know if you'd picked all ten out.

I: Right. OK ... Can you draw any conclusions at all from what we've seen so far?

Andrew: ... From this, you could draw the chance of picking something, but ... that wouldn't necessarily be correct for that bag. ... It could only be correct for those ones you've drawn.

(Line 35-36, 38-45, 47)

Later, after he had added some counters to his own bag and completed sampling to his satisfaction, he went on to inspect the contents of the two bags. As he counted the contents of each bag he reconstructed an account of the decision he had made to add two black counters. This led him to express a similar idea about the need for the bag to be effectively shaken.

Andrew: Inside we had... seven whites and... 1 2 3 4 5... blacks (*recording the totals as he speaks.)*

(*Andrew then counted the contents of the second bag*)

Andrew: Five and... five. So you had an equal probability because you had five of each.

And... this one... (*indicates the first bag again)* to begin with I had... three blacks and seven whites. ... So I added two to it, so now, in my bag, I've got the same amount of... blacks as you have. But I can never have the same amount of whites because I'm not allowed to remove them... or add them to you, so I can't make it the...

I: You could have added more counters.

Andrew: Well, yeah. But... then you'd get to the point where... you've got so many counters, that increase the probability by so much. So although it may still be fifty-fifty, it depends... because you've got where your hand chooses to go... to pick up the counter which could be a concentration of black or white. So... with the low number then you've got... although it's the same probability, it's where your hand goes. So if you've got lots of counters, you could have a bit pocket of white up one side of the bag, because it hadn't been shook properly, or a big concentration of black, whereas if

> you've got... a smaller number then it's easier for them to move, so it's more likely to be a fairer probability.

(Lines 186-188, 192, 194-196, 198-202, 204, 206-211)

Andrew thought that the counters were more likely to be properly mixed if the total number of counters was smaller. Inspecting the contents of the bag and reconstructing what had happened during the experiment had drawn Andrew's attention firmly to the process by which counters were selected from the bag.

Although he did not use the word random in this discussion, he expressed the goal that he had in mind as "a fairer probability". He was trying to ensure that, at each draw, each counter was equally likely to be chosen from the bag. In trying to consider the randomness of this experiment, Andrew focused on the generating process.

12.3.2 Claire

When Claire worked on the counters task, she decided after seeing only ten outcomes from each bag, and as soon as the task was explained to her, that she would add two black counters to her own bag. She was sure that her bag currently contained more white counters than black, and that the reverse seemed to be true of my bag. By adding two black counters to her bag she hoped to 'even it out a bit'.

Claire: ... I think there's more whites than blacks in this bag... and... it seems the reverse is true, there's more blacks than whites in that bag. So I'm going to try and even it out a bit more.

(Lines 74, 76)

She was unsure how many to sample; after some thought she decided to draw five, but only from her own bag. On observing 4W and 1B, she chose to add two more black counters to her bag, and drew a further five outcomes. This time she drew WBWBW and commented that the outcomes seemed to be 'getting more even'.

Claire: So it seems to be getting more even... However, I'll add just one more black to this one, and then do five trials from each.

(line 98)

Claire added the extra black counter and began to sample. She appeared to have decided that she had added enough counters to her bag and she now wanted to get on with collecting more data from each bag. However, I was surprised by her willingness to work with such small samples.

Claire drew WWBBW and I drew BBWBB, still showing more white counters in her bag. This time Claire paused for a while, and seemed less sure of how to respond. The trend she had observed earlier - the proportion black counters increasing in her sample of five each time she added more blacks to the bag - was now halted. After some thought she concluded that the bag still seemed to contain a greater proportion of white counters than black, and she elected to add two further black counters, and to look at five more draws, but only from her own bag.

Claire drew WWWBW. This was an even more anomalous result than the previous one and she appeared to find it difficult to interpret. However, she did suggest that the sample she had seen was still small, and that a larger number of outcomes would give more reliable (accurate) results.

Claire: ... I can't get very accurate results from these things, because I keep putting them back in. (*Speaking very slowly.*) But the more I do there's probably going to be more accurate results. (*Pause 8 seconds.*) There seems to be about 7 or 8... out of ten blacks in bag B... (*Pause 6 seconds*) ...and still less blacks than whites in bag A.

(Lines 133-136)

Claire seemed to be trying to rationalise to herself the apparent contradictions between the unreliability, disorder and unpredictability of the individual outcomes, and the underlying distribution that she was trying to infer. She attempted to do this by appealing to the fact that she was sampling with replacement, and could therefore choose the same counter more than once. The contradictions between the local and the global perspectives led her on this occasion to move her attention from the outcomes to the process.

After she had made four additions to the contents of her bag, Claire wanted to inspect the contents of the bags as she was satisfied that she had got the probabilities as close as she could. In the process of sampling she had seen some variability between the ratios observed from successive samples. Before she inspected the contents I asked her if she had been surprised by the variability between the ratios observed in successive blocks of five trials. In her answer she suggested that the method of sampling with replacement was a "not very accurate way of testing at all... because you keep putting it back in" (lines 193, 195). Again her way of accounting for the unexpected variability in the outcomes was to suggest that the process was in some sense inadequate.

12.3.3 Guhan

Having seen the first ten outcomes from each bag, Guhan had added one black to his bag and drew a second sample of ten counters from each bag. The second samples each contained six black counters, but Guhan was not satisfied that the proportions were now equal as he knew that his bag now contained 11 counters, while my bag only contained 10 counters. He therefore added a further 9 counters to his own bag, and drew 5 more outcomes from each bag to test this. Unfortunately, this time the proportions of black counters in the two samples were not alike: 4 blacks in 5 out of his bag, and 2 blacks in 5 out of my bag. Guhan used the fact that he was sampling with replacement to account for this, and suggested it might be better to sample without replacement:

Guhan: ...either I have too many blacks, or that was just because we put them back in. So, maybe if you <u>take</u> them out and <u>keep</u> them out on the side...

(Lines 116-117)

Guhan was faced with a set of outcomes that did not behave as he had expected, and his response was to move his attention to the process by which the outcomes were generated and to suggest that this process was not good enough.

12.3.4 Hannah

Hannah observed 7 whites in 10 draws from her own bag and commented that she thought her bag had more whites than blacks. She then noted that my bag also had shown 7 whites in 10 draws, and she commented: "That might have more whites… Because it would be a coincidence if you'd picked out the same ones" (lines 95, 97). She seemed to be stating the obvious that my bag is most likely to have more white than black counters. Her subsequent comment could be interpreted to mean that she had assumed that each counter in my bag had been sampled exactly once in the sample of ten observations, but that she recognised that the same counter could have been sampled more than once. Indeed, if the true ratio of white to black in my bag had not been 7:3 then some counter would have to have been sampled more than once. Thus for the true ratio in my bag to be other than 7:3 would require a coincidence to have occurred, whereas if the true ratio were 7:3 there is no requirement that a coincidence has occurred. However, the remark shows a shift of attention from the outcomes to think about the generating process.

Hannah struggled to understand what she was required to for this task when I explained it to her at this point, as she did not understand the word 'probability'. However, after a discussion, she decided to draw more counters since both bags had shown the same proportion of white in the first ten draws. After a further 6 outcomes from each bag, she noted that there were more whites coming out of my bag (4 out of 6) than she observed from hers (1 out of 6). She recognised that she might choose to put a white counter into her bag, but commented:

Hannah: I could put … another white in. You can never tell though really. Cos… it's just like a chance. You could be picking out the same one over and over again. But if you weren't, this says that you've got more whites than what I have.

(Lines 185-188)

Here Hannah seemed to be struggling to know what to pay attention to. She did not have a sense of which of the competing influences of which she is aware (variability of the samples, sample proportion represents the proportion inside the bag) was most important to attend to in the moment. Her attention shifted briefly to the process in her remark "You could be picking out the same one over and over again", and then reverted to using the outcomes to infer a global statement "you've got more whites than what I have".

12.3.5 Linda

Linda did not mention the idea that the same counter might be sampled again while she was working on the counters task. However, towards the end of the interview when she was reflecting on the nature of randomness and reviewing the activities that we had worked on in the interview, she referred back to the dice tasks as well as to the counters task.

During her work with the spherical die and the cracked die, Linda had experienced significant uncertainty about whether the outcomes she had observed could really be considered to have arisen from a fair die. For example, with the spherical die, she had experienced a lower proportion of fours than she had expected. After some vacillation she eventually decided that both dice could be considered fair, but that the unevenness of the observed frequencies was "probably due to the way I'm throwing it" (line 713), rather than to chance.

I was surprised to find such uncertainty in a pupil of Linda's age and ability and I wondered if she had lacked experience of interacting with dice. She acknowledged that she did not often use dice, but again referred to her concern about how to roll the die. This appeared to be an aspect of the activity that she felt insecure about. She even speculated about ways of designing a dice rolling experiment which did not involve her, by using several people to roll the die, or even avoiding a human altogether by using a machine to be 'sure that it's random'.

Linda: … I think that people normally shake it don't they? Shake it and then throw it. And I've just been kind of tipping (*rolls the cracked die*) it onto the table and letting roll and then stop. So I think that's probably something to do with it. See. I just threw it then and I got a four. So I think, I don't know, maybe… if you were going to do this more accurately, you might get several people to throw the dice because they're going to throw it in different ways… Or you might get some kind of machine to roll it so that it stops and you're sure that it's random.

(Lines 716-722)

Linda's concerns about how to account for the variability that she had observed in the frequencies of the six possible outcomes had led her to move her attention to the process of rolling the die. This is very similar to what was observed in other interviews.

The extract above was the first time that Linda had used the word 'random' in the interview. When I asked her to explain what the word meant to her, she found this very difficult and initially suggested the idea that something 'random' is "not influenced by anything". This idea has appeared in other interviews. However, she also suggested a new idea that "it's not… chosen; it's just generated". For Linda this was illustrated in a powerful way by the idea of the random number generator in a computer or a calculator.

Linda: Oh, 'random'? … I don't know. … (*Laughs.*) Well I kind of know what it means. It's hard to define isn't it? It means that the outcome was not really influenced by anything. It's a completely… I don't know… Well, if you've got like a random button on the calculator that… that just brings up this random number that's just been generated by a computer, and… It's just been selected at random so <u>that</u> means that… you don't choose it. You just… Oh I don't know. It's just a bit hard to explain. … I would just say that it's not influenced by any other outcome. It's not selected or chosen; it's just generated.

(Lines 724-732)

Linda's view of randomness here appears to be based in the outcomes: "just been generated" and "not influenced by any other outcome". However, she still held a process dimension to this, as was shown when I asked her she would describe the outcomes produced by the cracked die as 'random'. Linda was clear that these outcomes were not random, because the frequencies were unbalanced, and again she ascribed this to the way she rolled it, and again drew the parallel with the outcomes from the spherical die. In a sense she seemed to be saying that the outcomes of the cracked die could not be 'random' because they appeared to be influenced by the way she rolled the die.

Linda: ... No. I don't think so. Because... because more fours and threes and twos are coming up more often, again I think that's just down to probably the way that I'm throwing it. Either that or it is biased. ... I'm not sure. But because... it the fits the... pattern of the last dice - that suggests to me that it's not biased, because... I think it would be hard to bias two die in the same way ...

(Lines 736-740)

When I asked the same question about the outcomes from the spherical die, Linda was even more explicit that the die did not produce random outcomes. She asserted, "You can never be sure that they're random results" (line 745), and a "person can (not) create completely random results because you've got all these factors influencing it..." (lines 751-753). She cited the counters not being adequately mixed in the bags experiment, and rolling the die in a manner that is systematically biased.

Linda: ... No, I don't think so. I think that neither dice produced random results, because, you can never be sure that they're random results. Because... it's like the counters in the bag: you put the counters in the bag and you're not sure if you're shuffling them, or, you know, you might shuffle them around different times. Or because you've taken one out, you put it in and it's kind of on the top, so you might be more likely to pick it next time. Or, on the other hand, when you put one back in, it might get shuffled around to the bottom, so you're less likely to pick it again. So ... I don't think that a person can... create completely random results, because you've got all these... other factors influencing it, like... shuffling the counters or rolling the dice in a particular way... So...

> No. I don't think the results can ever be completely "random". I think the only way you can get completely random results is to use a calculator or a computer or something.

(Lines 744-756)

Her idea, that "you can never be sure that they're random results", was one I had not heard elsewhere. Linda was strongly aware of the many other factors that were affecting or influencing the outcomes in our experiments. In the dice rolling tasks, this included the way in which she rolled the die, as illustrated in her earlier concern that she, in some way, was not rolling the die correctly because she did not shake it first, or blow on it. In the counters task, she was concerned that she had not been sufficiently rigorous in shaking and mixing the contents before each successive draw, and that she needed to place her hand into the bag in different ways each time. *She seemed almost to be looking for ways in which to 'out-fox' some deterministic gremlin that might be imagined to be trying to produce systematically biased outcomes.* And yet, in the end, she seemed to suggest that, because the outcomes were influenced by so many different factors, they could never be completely 'random'. This seems to express the idea that randomness is neither caused nor determined nor influenced.

So how did that relate to her initial idea that "you can never be sure that they're random results"? I think that because she was so aware of the multiple factors that might influence the outcomes, and because she could never be sure that she had adequately controlled them (out-foxed the gremlin!) then she could never be sure that the outcomes were 'completely random'. The phrase 'completely random' seems to be synonymous in her mind with unbiased or fair, and seems to carry with it the prior belief that, if the process were unbiased, then each of the possible outcomes would be equiprobable.

Linda's final assertion about the only way to get random results was remarkable. She saw the computer generation of random outcomes as removing the 'human influences' on the outcomes. In this way, computer generated random outcomes may be seen as 'more random' because they are detached from reality and therefore not influenced by 'reality'.

12.4 Conclusion

Two distinct kinds of shifting perspectives have been discussed in this chapter relating specifically to work on the counters task. The first relates to rapid shifts between the local and the global perspectives. The primary intention behind the development of the counters task had been to examine this shift from a different direction: starting from the perspective of the global frequency distribution. It is clear that the task prompted interviewees to work with their ideas of frequency distribution to a greater extent than the dice activities did, but that the counters task did not incorporate an opportunity to articulate a prior belief about the distribution of white and black counters in the bags before any data was collected. Therefore the shifts seen in the counters task were almost entirely between local and global frequentist, with little evidence of the global prior perspective that was seen in the dice activities.

However, a new kind of shift has been identified in seven of the stage two interviews: five of these were discussed in section 12.3, and a sixth, Bernice, was described in 12.2.1. A seventh example, Rory, is presented briefly in Chapter 13. This new shift is from looking for evidence of randomness (or non-randomness) in the outcomes, to looking instead inside the process itself. In each case, the motivation for the shift seems to have been a concern that the outcomes were not behaving as had been expected – usually a concern that the outcomes showed more variability than expected. Looking inside the process might be seen as an attempt to identify a cause for the perceived anomaly in the outcomes. In that sense there might be a relationship between this shift of attention from outcomes to process and the vacillation that was observed in some interviews in stage one between whether a perceived bias in the outcomes from rolling a die had a cause or not.

Several other themes were present in stage 2 interviews, but these are reviewed in Chapter 13, alongside related themes from stage 1 to provide an overview of the ideas that have emerged across the two stages.

Chapter 13: Themes across the interviews

In this chapter, I revisit the ideas presented in Chapter 8, but drawing this time on data from both rounds of interviews. This chapter is organised in four sections, each addressing a question relating to how learners perceive randomness.

- What do people think randomness is?
- How are ideas about randomness expressed?
- How do learners recognise randomness?
- Is "randomness" caused?

There is some correspondence between these questions and the three headings used to organise the themes in Chapter 8. The first two of the questions above correspond broadly to the idea of "Interpretations of randomness" that formed section 8.1. The other two headings in Chapter 8 – "Strategies for recognising randomness" and "Shifting perspectives" – are different aspects of the answer to the third question. The fourth question raises a theme that relates to one of the interpretations of randomness discussed in Chapter 8, namely the ideas of Agency, Luck and Lack of control that were discussed in section 8.1.2.

While the ideas discussed in this chapter echo those addressed in Chapter 8, my intention here is to show how these ideas have been developed and refined as a result of the second round of interviews.

13.1 What people think random is

As in the first stage interviews, when interviewees were trying to describe or talk about the phenomenon of randomness, or about a process which might be considered to be random, it was important for me to be cautious about how I introduced the word 'random'. The word was sometimes not part of an interviewee's vocabulary. In this section, I give some examples of interviewees' attempts to speak of randomness in the second stage interviews. Although my focus here is upon the ways in which these interviewees chose to express themselves, I have also reflected upon the circumstances in which they were speaking.

13.1.1 Linda: random is chance

For some, randomness is 'just chance', as it was for Linda (age 16.1). She had already worked on the counters task and puzzled over the biased die before she discussed the cracked die. She threw the cracked die in blocks of six throws and she referred to one block of six outcomes as "a trial". After each trial, she paused to discuss the outcomes. Linda was bothered by the fact that, in successive trials (blocks of six throws) of the cracked die, she had 3 successive twos and 3 successive sixes.

Linda: Six occurred three times out of six, which again is quite unusual. I would expect, umm, the probability of it falling on say a four, or something, on any number, would be one in six. So it's quite unusual that you've got three twos in the first trial, and then three sixes in the next one.

(Lines 494-498)

However, on reflection she concluded that the occurrence of these successions was "just chance". She thought that a greater number of trials (than 12) would be needed to get a sense of whether this die was biased.

Linda: I think it was just chance that that happened. That there were 3 twos and then 3 sixes. I don't think that's realistic to what would normally happen when you throw this dice. I think you would need to do a… maybe a lot more trials, and then you would get a better idea of whether it's… umm. Yeah. You would get a better idea of the probability…

(Lines 500-504

This incident also illustrates the difficulty that people experience when they interact with events in a situation where there is some reason to question the assumption that the outcomes are equiprobable, even though the outcomes might still be fair. The crack in this die was enough to sow the seed of doubt, so that when Linda saw an unusual sequence of outcomes, she needed to reflect upon whether these outcomes could be evidence that the die was not fair. Linda showed some insight in recognising that runs of three identical outcomes might occur "by chance".

13.1.2 Bernice: random is unpredictable

In contrast, some interviewees did choose to use the word 'random'. For example, Bernice (age 15.0) used the word late on in her interview when discussing the behaviour of the spherical die. She had observed a sequence of eight outcomes (3, 2, 6, 1, 5, 6, 3, 2) and had been using a pattern seeking strategy to think about what might happen next. When this strategy failed, she suddenly announced that she thought that this die was 'random', which surprised me.

Bernice: I reckon it's random.

I: That's an interesting word.

Bernice: (*Rolls.*) It could be anything.

(Lines 398-400)

Bernice's description of the die as "it could be anything" could suggest that she was thinking of the word 'random' as meaning 'unpredictable'. She went on to roll two sixes in succession and a two. When I asked again about what might occur next, she simply replied "Anything."

I: Tell me what might happen next.

Bernice: Anything.

I: Anything. Because…?

Bernice: It's a random dice. (*Rolls.*)

(Lines 405-408)

Here again she was connecting the word 'random' with the idea of unpredictable outcomes.

When she later went on to consider the cracked die, Bernice rolled it six times, getting 4, 3, 1, 3, 2, and 1. She initially used the word 'random' as before to describe the behaviour as unpredictable. Soon afterwards, however, after throwing the die a few more times, Bernice became aware that she had not observed any sixes from the cracked die, and she questioned whether the die had been drilled.

Bernice: Has it been drilled, so it can't land on a number six?
(Line 442)

I assured her that it had not, to my knowledge, and I asked why she thought it might have been. She answered that she suspected that it might never produce a six. Thus, for her, the idea of 'random', when applied to a die, held within it the sense that every outcome should be represented.

After fourteen outcomes Bernice had still not observed a six. Even when, on the fifteenth throw, she observed a six, Bernice did not return to the idea that the die was 'random'. Instead she expressed surprise and then tried to find alternative explanations for the lack of sixes in the first fourteen throws, attributing it to "how you throw it" (line 466).

Rather than to accept that it is not uncommon to observe a run of fourteen throws in which only five of the six possible outcomes occurred, Bernice's instinct was to look for an explanation for what she perceived to be non-random behaviour of the die. However, after a few more outcomes, she observed more sixes, and concluded that the die could "go on anything" (line 476). This was similar to her earlier statement that the die was 'random' because it was unpredictable and each possible outcome was represented (lines 398-400).

When I went on to ask Bernice if she thought that the counters task could be described as 'random', she hesitated before agreeing. Although the word 'random' could be transferred to this different context, she seemed to feel that the transfer was not easy.

Bernice: No… Because, if it was to… No! Yeah, I think it is, kind of… randomly… choosing a counter. They could be out of the two… randomly choosing out of two. See what happens.

(Lines 496-497, 499)

So, for Bernice, the counters task involved 'randomly choosing' one out the two possible outcomes. She noted that, if the bag had contained only white counters, then the outcome would be determined, and she suggested that the outcome would not then be random. Her idea of 'random' here seems to require that the outcome is indeterminate.

Bernice: Well, if the whole bag was white, then… you wouldn't be able to get black.

I: Right. So, it would depend on what was in the bag.

Bernice: Yeah.

(Lines 503-505)

Although she had used the phrase "randomly choosing", Bernice found this very difficult to explain, and after several hesitations she gave up.

I: OK. What does "randomly choosing" mean?

Bernice: (*Pause 10 seconds.*) …Choosing out of something that could be anything… (*Pause 4 seconds.*) I think… (*Pause 8 seconds.*) Choosing… (*Pause 4 seconds.*) No, I'm not sure.

(Lines 508-511)

The main view articulated by Bernice in these extracts was that random outcomes are *unpredictable*, but she also expected them to be *representative* of the possible outcomes. These ideas she expressed quite easily with reference to the dice activities. She also saw the counters task as producing random outcomes, but she found it much more difficult to explain what this meant beyond the idea that outcomes were randomly chosen.

13.1.3 Rory: random is luck

Rory (age 14.5) did not use the word 'random' at all, and yet he wanted to express such an idea quite early in the interview. He had begun the counters task by drawing a sample of 10 counters with replacement from each bag. When he was told his aim for the remainder of the task, his immediate response was to discount the kind of evidence he would have to work from as 'just luck'.

Rory: I know, but... But that's just going to be luck isn't it.

(Line 50)

I was surprised and I asked him to tell me more about this. He indicated that he was worried about the process of sampling individual counters with replacement.

Rory: Because, like... well you could be just lucky that you just keep picking out the same white one. Or the same black one.

(Lines 54-55)

Rory's use of the word 'luck' and 'lucky' here is slightly different from the way it has been used by some other interviewees. Very often 'luck' is used to refer to a situation where the outcome is good, as in winning a prize on the lottery, or bad, as in bad luck. In this example, Rory appears to be value neutral in his use of the words.

Rory did not appear to have a word for the concept of 'randomness'. He seems to have attributed 'random' and unpredictable outcomes to 'luck'. Indeed, his use of the word luck is in some cases a replacement for the missing word 'random' that he does not have. In this sense, his reference to the idea that you could "just keep picking out the same white" counter could be seen as Rory's natural sense that the randomness of the sampling activity lay in the process itself, and not simply in the outcomes. Rory used the word 'luck' and its derivatives many more times than any other interviewee. Other interviewees sometimes used the word 'chance' to express ideas related to randomness, but Rory used the word 'chance' only twice. Some interviewees used words probable, probably and probability to express uncertainty, but Rory did not use these words at all.

In spite of his lack of vocabulary to discuss the ideas contained in the tasks in this interview, I was impressed with Rory's expression of his thinking. He seemed to have an understanding of the concept of randomness that encompassed process with outcomes, and he did not seem to have the same difficulty that others had in integrating the local and the global views of randomness.

Rory went on to use the word 'luck' again about the counters task to describe the idea that he did not expect a sample from the bag necessarily to match the contents of the bag.

I: Would you expect it to be equal, if the probabilities were the same?

Rory: Mmm, no, not really cos… I mean it's just luck init?

(Lines 139-141)

In these early uses of the word, Rory seemed to be using word "luck" as a synonym for "random". Later however, he spoke more about luck in the usual sense, and of people "being lucky". I shall return to a discussion of Rory's ideas about luck in section 13.4.

13.1.4 Mosaab: random is not expected

Another interviewee who had no knowledge of the word 'random' was Mosaab (age 16.11), who was a year older than other students in his year group. He had only been in the UK for a few years and he spoke English with a strong African accent. Towards the end of my interview with Mosaab, I asked him if he was able to think of any words that described the ideas that we had been talking about in the counters task and the dice tasks. He suggested "probability" and "chance", but when I asked him if he had ever heard the word 'random', he said he had not. However, when I asked him if he could think of situations where he might think of things happening by chance, he was able to suggest a situation from everyday life very quickly.

I: What other kinds of… situations do you think things happen by chance?

Mosaab: It's like when you think of something… I don't know… it's not expected to happen.

I: OK. Like what? Can you think of something?

Mosaab: ... like you drive with the car and the stoplights stop you, ...to get in time, ...or it's getting busy in the road so ...you're not expecting to get there.

(Lines 561-564, 566, 568, 570)

In spite of his difficulty in finding a word for the idea of randomness, I suggest that his suggestions above indicate that Mosaab had a concept of randomness, in that he had the idea that some events occurred unexpectedly. However, possibly because of his difficulties with the language, he had experienced serious difficulties in working on the interview tasks and explaining his ideas.

13.2 How randomness is expressed

Some interviewees showed clear evidence that their understanding of the word 'random' was constructed informally from social settings. That is, they had built their understanding of the word from interactions with other people, not necessarily in a classroom context. Some examples were the restricted uses of the word 'random' in stage 1 interviews, described in Chapter 8, section 8.2.

In this section I describe Claire's use of the word "random", which drew upon colloquial usage that has become current among teenagers, as was discussed in the incident of the 'random stapler' in Chapter 6.

13.2.1 Claire: colloquial use of 'random'

Claire (age 15.0) had discussed the dice activities fluently and confidently. In order to find out if she saw any relationship between these tasks and the wider topic of randomness, I told Claire that I was particularly interested how people think about randomness, and I asked her if she saw a connection between the tasks she had been doing – drawing counters from a bag and rolling dice – and randomness. She replied that she saw these tasks as related to probability, but when I prompted her further about the connection between probability and randomness she showed that she was unsure of the meaning of randomness.

Claire: Look. What do you mean by randomness? Or … is that what you want me to tell you? I don't really know. … (*Pause 5 seconds.*) I think random is quite a strange word. And you can use it for lots of different things. I'm not sure what the true meaning would be.

(Lines 334-337)

When I asked her how she would use the word 'random', she noted that the word has become commonly used in teenage conversation. She suggested that this modern colloquial use might be obscuring the true meaning and had little to do with probability. She tried to describe how the word is used by using the word 'random' to describe itself.

Claire: …It's just used for anything. Just randomly become a random word. (*Laughs*) It's become used quite regularly, so I think that kind of obscures its actual meaning. …I probably wouldn't use it that much in anything to do with probability… Although I suppose they do go together, I wouldn't think of it like that.

(Lines 341-345)

When I asked Claire now to try to describe what she thought the true meaning of the word might be, she found this very difficult. She suggested that the word 'random' had something to do with "a strange event… that hasn't really got a meaning" or "an event that you wouldn't expect" (lines 352-353).

In view of the difficulty she was experiencing in describing the meaning of the word, I suggested that we should avoid it for now. I asked Claire to think again about the process of drawing counters out of the bag and to consider what was happening to her thinking. However, when she tried to do this, she quickly returned to linking her ideas about the counters task to the idea of 'randomness'. It seemed that she was fascinated by the word and wanted to explore it. She spoke of the process of sampling counters from the bag as "a random way of testing" (line 374) and, after a short pause, gave an example of what she meant by a "strange event".

Claire: A lot of people talk about... you've heard about 'random acts of kindness'. And that means, I think, that's using it to mean unconnected to anything... Without really a meaning, just for the sake of it.

I: So, for you a random act of kindness might be somebody acting kindly towards somebody else without any reason for it...

Claire: Yeah. I think, I think they use it to mean... to strangers mostly. Like someone goes into a coffee shop and pays for the next 20 coffees that any one goes in... that anyone buys. They don't know the people... they're not really doing it for anything, just to be kind. And I think that's... that is quite random because... it's not connected to anything. They haven't got a reason to do it.

(Lines 375-384)

In Claire's example, the person committing the random acts of kindness needed to be a stranger. This seemed to emphasise the lack of rational explanation, the lack of meaning, for the person's actions, as well as the idea that these 'random' actions are "not connected to anything", and particularly not connected to the person judging the act as 'random'. For Claire the task of drawing counters out the bags may have showed some of the aspects of 'unconnectedness' lack of meaning or reason, that she described in her example. However, she did not explain how she saw her example of randomness in "random acts of kindness" as being related to the counters.

Claire's use of the word "random" in these extracts was unlike anything seen in the stage 1 interviews, but was clearly based in the colloquial usage common amongst her peers. It involved an event "unconnected to anything" and "without meaning" and without reason: it carried the contradiction of the rational human being apparently acting irrationally, without reason or without meaning, and it clearly matched her earlier idea of a random event as "strange". This was very similar to the modern colloquial usage that I discussed in section 6.4, where I reported an undergraduate student who had used the word "randoms" to mean "random people", "people that you don't know, that infringe on your day in some way".

In the light of Claire's illustration of her interpretation of the word 'random', it is now worth returning to reconsider her earlier statement that the process of sampling counters with replacement was "a random way of testing". The following extract sets this in context.

I: As you were doing it... what were you thinking about what you saw... some outcomes come up... and you started to make a list?

Claire: It was affecting... my idea of how many counters were in the bag and proportion of black to white... But I knew that... that it wasn't very accurate, because of the way... because only picking one out at a time, I could be picking out 2 counters... there could be just 2 counters that I was picking out each time... Do you want me to link it to randomness?

I: If you can, yes.

Claire: (*Pause 6 seconds.*) I don't know whether I can. (*Pause 4 seconds.*) I suppose... it's a... (*pause 4 seconds*) random way of testing.

(Lines 362, 364, 366-374)

A teacher listening to this exchange, without the benefit of the illustration that followed it, might assume that Claire was using the word 'random' in a standard sense, to describe the sampling as random. However, Claire's illustration of 'random acts of kindness' suggests that what she really meant here was that it was a strange way of testing that had no meaning and appeared to be irrational. This mismatch between the way that Claire understands the word 'random' and the meaning a teacher would intend in classroom activities might cause some confusion.

Claire quickly began to take on the mathematical use of the term 'random' in a way that was new for her. Towards the end of the interview, in her discussion of the relationship between luck and randomness in the context of tossing a coin, she said, "It's just random whether it's Heads or Tails" (line 480). I suggested that she was now using the word 'random' more often and I asked whether she was using it in the sense that she had done earlier or in a different sense. Her reply was clear.

C: I think I'm trying to use it in what its true meaning probably is, rather than the everyday usage.

(Lines 498-499)

Claire's emphasis on colloquial use the word random was unusual, but it was remarkable how rapidly she appeared to take on and seek to use the mathematical sense of the word within the 40 minutes of the interview.

13.2.2 Degrees of randomness

The expression of degrees of randomness involves an understanding beyond seeing randomness as only equivalent to fairness. It also seems to require a facility with language relating to probability and randomness. In the stage one interviews, the two interviewees who expressed this idea most clearly were Ben (age 15.7) and David (age 14.1) (see 8.2 2), both of whom also showed that they were able to think of randomness as a model for incomplete knowledge in various real world contexts (see 8.1.3).

In the stage two interviews, Claire expressed a sense of degrees of randomness when she tried to explain the relationship between randomness and probability. In view of the fact that she had previously used the word 'random' only in a colloquial sense, this was a difficult task for her. She was able explain probability as "the likeliness of an event" (line 389) and she recognised that, given a bag containing 7 black counters and 3 white ones, she was most likely to draw a black.

Claire: ... You could get either, quite easily. But there's more chance of getting a black, because there's more counters.

(Lines 398-399)

When I then asked her if the word random had connection with the idea of probability her reply contained the unusual phrase "almost random chance".

Claire: It's almost random chance whether you'll get a black or a white… because… it could, it could be either… so the fact that you've got a black is random, or… (*Pause 5 seconds.*) And the same for the dice, it could be any number, so it's just… getting a six is random.

(Lines 404-407)

This has echoes of the idea of 'degrees of randomness' where the situation is 'more random' when the chances are more equal. The situation I had described for her had 30% white and 70% black and so she described the chance as 'almost random'. Here she seems to be using the word 'random' to mean that the chances of all the possible outcomes are equal, and again a little later when she commented on the chance of getting a six from rolling a fair die. But her usage of the word shifted when she described the fact of getting a black on the first outcome as 'random'. When applied to the observed outcome she seems to intend 'random' to suggest that the outcome was 'unconnected to anything else', or in simpler and more direct terms than those used by Claire, it was unpredictable. Although Claire did not ever use the word 'unpredictable' as a synonym for 'random', her use of the word 'random' here seems to suggest that she may have been thinking in those terms.

In the stage 1 interviews, Abby came to the interesting conclusion that "randomness is personal"! She had been discussing the idea that the outcome of rolling a die was unpredictable, but that if she knew enough about the way it was rolled, how much it weighed, and enough about the physics, then it could become predictable. She went on to consider whether this would still be random.

Abby: (It) depends on what randomness means. Because if it does mean something that is unpredictable, then… that depends what you mean by unpredictable, because I can't predict the weather, but the Met office can, to a point. ... So that all depends on what is predictable and what is random. I see random as something that's unpredictable, but … I've all of a sudden got a bit stumped as to what predictable and unpredictable mean.

Abby: Perhaps random is a personal thing.

(Lines 307-309, 311-314)

She seemed to be creating for herself a new way of understanding 'randomness' as she spoke. The degree of randomness was related to the knowledge of the person making the judgement. She returned to this idea again later in the interview and gave further examples. It seems to me that this was an example of how the idea of degrees of randomness could be extended to form the more complex idea that randomness can be a model that depends on the knowledge of the individual.

13.3 How randomness is recognised

In this section I consider two different aspects of recognising randomness. The first is more obvious and straightforward and examines the strategies that interviewees used to decide whether the generators they worked with were behaving randomly, in a manner expected. The second aspect of the recognition of randomness considers the different and shifting perspectives adopted by interviewees to consider what they observed, and the ways in which attention appeared to shift, sometimes rapidly and frequently, between these alternative perspectives.

The strategies for recognising randomness, which are discussed below in 13.3.1, were usually exercised within the local perspective and tended to focus on the outcomes rather than the process. However, there is a sense in which the strategy of seeking representativeness in a sample of outcomes, which was discussed in Chapter 8, may provide a bridge between the local and global perspectives through awareness of the importance of sample size. I discuss this idea below in 13.3.1.2.

In circumstances where sequences of outcomes were clearly not representative of the underlying process expected by the interviewee, as when working with the biased die, the interviewee's attention often switched to consider the symmetry of the generating process.

Thus, the outcome of the search for representativeness sometimes led to a switch from attending to outcomes to the generating process. Some examples are considered in 13.3.2.

13.3.1 Strategies to recognise randomness

A wide variety of strategies has been observed during the interviews in this study. In Chapter 8, I described pattern seeking and pattern breaking as a particularly significant set of strategies and I consider that again here. I also considered two other strategies in Chapter 8: representativeness and physical characteristics. I suggest that representativeness can be seen as related to pattern seeking. While the search for pattern in a sequence of outcomes from a random process is bound to fail in the long run, and the strategy of pattern seeking is overtaken by the expectation of pattern breaking, the search for representativeness is a strategy that is open to modification to make it more successful. While a short sequence of outcomes might not be representative of the underlying features of the random process that generated the outcomes, the degree of representativeness increases as the sequence of outcomes considered becomes longer. It is this feature of representativeness that has led me to consider sample size as the second significant strategy in this chapter. Awareness of the sample size required to make appropriate inferences about probabilities is difficult to acquire. Part of the reason behind the tendency observed in Chapter 8 for interviewees to expect representativeness even in short sequences was a lack of awareness of the importance of sample size.

13.3.1.1 Pattern seeking, Pattern spotting

When interviewees were working on the dice activities and the coin generation and tossing activities, they tended to focus on looking for non-randomness in the outcomes. In these circumstances one of the most widespread strategies was that of pattern seeking. Rather than focussing on finding evidence for randomness, the search for pattern is focused on instances of possibly non-random behaviour. When a pattern is spotted, if it does not break quickly, then it is deemed to be evidence that the outcomes are predictable. An example from the first stage interviews was presented in Chapter 8, section 8.3.2.

In the counters task, this strategy was rarely seen. However, it may have been behind Bernice's attempt to reinterpret the task as matching the successive outcomes from Bags 1 and 2, discussed in section 12.2. Focussing on the successive outcomes from each bag, she spotted a sequence of three outcomes in which the outcome from Bag A matched exactly the outcome from Bag B, and she began to look for this pattern to continue.

The strategy of pattern seeking was much more commonly observed in the dice and coin tasks, being encouraged by the nature of these tasks: interviewees were encouraged to speculate about the future outcomes.

Related to the pattern-seeking strategy was the strategy of monitoring the lengths of runs of the same outcome. In the dice activities, this was seen mainly when people were working with the biased die, as it was mainly here that they observed runs on longer than three outcomes. Indeed, spotting such runs was one of the features that seemed to alert some interviewees to the possibility that there might be something wrong with the die. Guhan (age 13.3) observed a run of four successive ones when he was rolling the cracked die: he laughed nervously as he rolled the third, and held his breath as he rolled the die for a possible fifth successive one.

However, run length seemed to be a much more significant feature when interviewees worked on the coin activities. For example, when Hannah (age 14.8) was working on the coin tossing activity, she observed THHHHTTTTH in the first ten outcomes and noticed the runs of four heads and four tails. She commented that this was quite different from what she had generated earlier from the imaginary coin.

Hannah: Yeah. There seems to be more Heads and Tails clumped together than like spread out. Cos on mine I put Tails Heads Tails Tails Heads.

(Lines 824, 826)

She went on to suggest that such runs were another example of the 'unpredictability' of outcomes from a random process, saying, "You just can't tell". When I asked her if she had been surprised by the occurrence of four heads in succession, Hannah acknowledged that she had been and said that she had been avoiding such runs when performing the coin generation task.

Hannah: … I didn't think you'd get them all together like that.

(Lines 834-835)

A long run of the same outcome is a particularly simple pattern to spot. In an activity like the coin tossing activity, in which a person needs to observe the successive outcomes closely, a run of consecutive Heads or Tails might challenge the individual's expectation that a run of more than three is unlikely to occur in twenty throws of a fair coin.

I was surprised to find that run length did not emerge as a concern of interviewees during the counters task, as this was also a situation in which there were only two possible outcomes and I had expected the interviewees to try modelling these outcomes as equally likely at some points during the activity. However, the nature of the task focused interviewees' attention on the task of estimating the proportions of white and black counters in the bag, and they very rarely paid any attention to the sequence of outcomes that they observed. I found no comments in any interview on any run of black or white outcomes in the counters task.

13.3.1.2 Sample size

Many interviewees showed a tendency to draw conclusions or make generalisations on the basis of very few observations. Inevitably such generalisations were very unstable as successive observations would often suggest something different. The work of Linda, which was described in section 12.1, was a good example of this. Another example was Andrew, also working on the counters task.

When I outlined the counters task to Andrew, he quickly understood the problem, and decided that he wanted to collect more data before he decided what counters to add to his bag. In the first ten draws he had seen 3 black counters from my bag and 4 black counters from his own. I was surprised when he asked to sample only five more outcomes and only from my bag. He explained that he was not yet sure which bag contained the greater proportion of Black counters since the number of blacks observed from each bag differed only by one.

The number of further outcomes that he proposed to examine was really too small to be sure of providing much more clarity. Also, I could not understand at this point why Andrew chose only to look at outcomes from the one bag. In the event, Andrew observed a succession of 4 black counters followed by only a single white. After a pause he decided that he therefore needed to collect a further 5 outcomes from his own bag. Andrew drew WBBWW and recognised that, for each bag, he could combine the results from the second five with first sample of ten.

Andrew: But then I can add these to my previous results, and find out how many whites I've got from both. (Lines 98-99)

He found nine whites and six blacks from his bag, and eight whites and seven blacks from mine. Thus the difference of only one that he had observed in the first sample of 10 was now reversed to a difference of one in the other direction. Andrew was therefore still uncertain how to respond and spent some time considering what to do next. The small samples had produced unstable estimates of the proportion of black counters in each bag;

this combined with Andrew's concern that he should 'get it right' to produce his indecision.

After clarifying the rules he decided to collect more data, but again he elected to take only 5 outcomes, although this time he sampled from both bags at the same time. He observed BWBWW from my bag and BBWBB from his own and he combined the new data with what he had already to give a total of ten black observed from his bag and only nine black from mine. Yet again the difference in the number of blacks observed from each bag had reversed.

After a pause, Andrew decided to add some counters although he now had little more evidence than he had had to start with. His decision to add two black counters to his own bag, which had produced more black observations, appeared irrational. I do not know why he decided to do this. When asked what he was going to do next, he first explained his goal in his own words to check that he had understood correctly.

Andrew: … I've got to make it so that they're both an equal chance of getting a black?

I: As equal as you can get it. Yes.

Andrew: OK.

(Lines 153-156)

Satisfied that he had understood correctly, Andrew elected to draw a further block of 5 outcomes from each bag. This time, he observed two black and three white counters from his bag and the same from mine. He commented that, while the observed proportions were equal now, he would need to repeat the experiment to be sure.

Andrew: That's the final one and I got 2 blacks and 3 whites, and you got 3 whites and 2 blacks. So I'll do that… I'll repeat that again, because they're equal now, but I need to repeat it to make sure.

(Lines 163-165)

In spite of his reliance on very small samples for evidence, Andrew was aware that he really needed to see more evidence to judge that the probabilities of drawing black were approximately equal in each bag. However, again he chose to draw only five more outcomes from each bag. This time he observed 2 white and 3 black from each bag. Again the numbers were equal from each block, although not the same result as previously. Andrew seemed to be satisfied that these results indicated that he had made the proportion of black counters the same in each bag.

Andrew: … They're closer than they were before now, because out of the two results we've got… an equal chance from both bags of getting black and an equal chance from both bags of getting white… and that's because, from the two, after adding… two, we've got 5 blacks from each bag, and… five whites… from each bag, so it's fifty-fifty from this set of results.

(Lines 172-174, 176-178)

There is a suggestion in his use of the phrase "from this set of results" that Andrew has recognised that the results observed from samples such as these are inherently variable. However, he did not appear to be aware that this high variability in the proportion of black counters observed in samples of only five or ten outcomes could mean that his observation of equal proportions from each bag was unreliable. It is possible that the fact that his latest samples, when taken together, also indicated that the proportion of black counters was equal to the proportion of white counters in each bag, was a further encouragement to Andrew to rely on the results he had seen.

Andrew's reliance on small samples was typical of many interviewees. However, he also recognised that, when the contents of the bag remained unchanged, he could combine the results of successive samples to give larger samples. He seemed to recognise that larger samples were more reliable, but he did not have a sense of how much the reliability would increase with a given increase in sample size.

Few interviewees were able to suggest sample sizes that would give reliable conclusions in the tasks. However, there were some exceptions. In the second stage, Linda showed evidence of having learned from the experience of drawing counters and trying to make estimates. She stated at the end of her work on the counters task that she wanted to look at a sample of "perhaps two hundred" in order to make a good estimate of the probability of drawing a white. I did not see such a change in any other interviews. In stage one, Joe (age 13.5), one of the youngest interviewees, showed greater awareness of the relationship between variability and sample size than most interviewees had done. The extract discussed here was also discussed at the end of Chapter 8, section 8.4.3, as an example of working in the global perspective. Towards the end of his work on the cracked die, Joe had become convinced that the die might be biased by the crack, but he was aware that, although he had not seen any occurrence of a 1 or a 6, the evidence he had collected from about fifteen rolls was not conclusive.

Joe: … the crack seemed to affect it in a way. …We haven't actually got a single one or six. And that might be because of the cracks, or it might not…

(Line 115-117)

When I asked him how one would be able to tell if the outcomes from rolling this die were affected by the crack he confidently suggested that you would need to roll the die many times and record the outcomes.

Joe: Well you could roll it a number of times and record your results and see… whether it seemed to favour one side… But then that could be just luck or chance. So there's no definite way of finding out, really. But you could, if you rolled it a number of times… say a hundred times, and you didn't get a single six or a single one, then… it would probably be in your mind that six and one were… less, they had less chance of being rolled.

(Lines 119-120, 122-126)

The level of caution expressed in Joe's conclusion was unusual: after observing no ones or sixes in one hundred rolls of the die, you still could not be certain that the die was biased, although you might have in mind that one and six "had less chance of being rolled". A sample of one hundred rolls of a die was larger than had been suggested in other interviews. Joe was aware that sample size was important, but he could not judge the reliability of his conclusions.

Another interviewee who showed awareness of the value of basing conclusions on larger samples was Abby, whose work on the dice activities was discussed in Chapter 9. Abby had commented that one way of determining whether the cracked die was biased would be to roll it "hundreds and millions of times" (line 126).

It is interesting that these few examples of people who recognised the need for larger samples are also people who adopted a global frequentist perspective on randomness, looking for evidence from a frequency distribution. I suggest that this connection between awareness of sample size and the ability to work comfortably within the global perspective on randomness indicates that experiments to explore the variability of estimates based on different samples size may play an important role in the development of a secure integration of the local and the global perspectives.

13.3.2 Shifting perspectives

My thinking about the different perspectives that I saw interviewees adopting and shifting between during these interviews has evolved significantly during the analysis of stage 2. When I first became aware of interviewees' changes of attention, during my analysis of stage 1, I noticed movement between what I then saw as local and global perspectives. However, I was strongly aware in most interviews that the global perspective took two distinct forms. In Chapter 9, I have described what I saw as two forms of the global distributional perspective: the *prior* and the *frequentist*.

In this sub-section, I describe how my thinking about these three perspectives has evolved. I also discuss briefly a second kind of shift – between outcomes and process – which I introduced in Chapter 12. Alongside these discussions, I present data indicating how often these different perspectives and shifts were identified in each interview and within each task.

13.3.2.1 Local, Global (prior and frequentist)

A person is working within their *prior* perspective when their attention is on what they believed about the distribution of outcomes before they saw any outcomes. When interviewees were first give a die in a dice activity, most believed initially that the die would be fair. However, in the case of the biased die, the observed outcomes challenged this *prior* belief. Some interviewees continued to hold onto their prior belief until they had seen and recorded a significant number of outcomes. An example was David, described in section 9.2. These interviewees became aware of the emerging long-run distribution of observed outcomes alongside their *prior* belief. In these few cases, the interviewee was clearly working with two distinctly different global perspectives – their prior belief, and the emerging frequency distribution. Each of these perspectives – the prior and the frequentist - was clearly global, in that they each carried meanings about the long run behaviour of the process. In particular, in the language of the first of the global meanings of randomness identified by Pratt, the proportion of outcomes for each possibility in the long run is to some extent predictable (Pratt, 1998, p142). Thus a mismatch between the proportions seen in these two global perspectives created a conflict within the individual that needed to be resolved.

The distinction between global prior and global frequentist perspectives is not always either necessary or helpful. For many interviewees working with the spherical die, their prior belief was that this would be a fair die, and they observed an emerging frequency distribution that supported them in this belief. The only means by which I could infer what perspective the interviewee had adopted to think about the situation at any particular moment was to listen closely to what they said and to watch their behaviour. Because there was no clear difference between what the interviewee perceived from their prior belief and from the frequency distribution, it was sometimes impossible to discern from the data which global perspective the interviewee was working in at any moment. In such cases the distinction was unnecessary.

The counters task in stage 2 did not require the interviewee to articulate any prior belief about the probability distribution of black and white counters before sampling any counters. Thus the *prior* variant of the global perspective was never expressed, and it was as though it did not exist for the interviewee working on this task. However, each interviewee needed to work within a global frequentist perspective in order to make inferences about the proportions of black counters in each bag and so decide how to adjust the contents of their own bag. This task encouraged the interviewee to express their beliefs about when a global frequentist perspective would become available to them; this was why the discussions about sample size that were reviewed in 13.3.1.2 came to the fore.

13.3.2.2 Outcomes and Process

In all the tasks, interviewees' attention was usually on the outcomes generated. However, I described in Chapter 12, several interviewees who became concerned while working on the counters task that the procedure of sampling with replacement was in some way unfair, or uninformative. I interpreted the shift of attention from the observed outcomes to the manner in which counters were selected as a shift from outcomes to process. A similar shift of attention can be seen in the dice activities when a person attempts to account for the observed outcomes in terms of the physical features of the die. This is very clear in work with the biased die, where the conclusion that the die has a weight in one face provides a clear account for the process that generated the observed outcomes. However, it can also be seen in the work with the spherical die, where the presence of the tiny moving bead inside the sphere provided for some interviewees a reason to consider the generating process to be fair, while for others it was a reason why the process might be biased.

For example, Hannah (age 14.8) experienced a long period of uncertainty before concluding that the biased die was really biased. When she worked with the spherical die, she quickly suggested that it seemed "OK", contrasting the way in which the biased die (the square one) had not rolled with the way this one did.

Hannah: I think with this one you've got a… a pretty good chance, because with the square you couldn't exactly roll it because it was square, but this one it can just keep rolling. Or it… could keep rolling if it hadn't got weight in it, but… I dunno. I think… I like this dice.

(Lines 571-574)

While it is likely that her judgement had been influenced by the outcomes that she had observed, which were much more variable than those from the biased die, she did not refer to these observations in her explanation. Rather she constructed an explanation in terms of the physical rolling of the die. She seems to have been reassured by the fact that this die would not stop easily at any particular outcome, unlike the biased die, which bounced and stopped quite quickly on each throw. For Hannah, the roundness of the spherical die suggested that it was fair, as she confirmed a little later.

Hannah: … I think this one is chance, actually.

(Line 581)

Similarly, a shift of attention from outcomes to process can be discerned in some work with the cracked die, where the asymmetry of the crack is sometimes used to account for a perceived asymmetry in the emerging frequency distribution. In this case, there seems to be a suggestion that the shift of attention from outcomes to process can be triggered by a perceived mismatch between the two global perspectives: prior and frequentist.

When he was working on the cracked die, Mosaab (age 16.11) rolled the die seven times quickly, without comment, getting 6, 6, 4, 6, 3, 4 and 6. Then he sighed deeply and commented, "I would say that it's not fair." He justified this remark, not by reference to the outcomes he had just obtained, but talking about the physical attributes of the die.

Mosaab: Because here is quite wider, and makes the bottom heavier. Like... want to break and it will stick (*rolls*) there. Like you see, here.

(Lines 527, 529)

Mosaab appears to have thought the seven observed outcomes were unrepresentative of a fair die, and he has treated them as indicative of a frequency distribution biased in favour of sixes. He has sought to justify the observed outcomes in terms of the asymmetry of the crack – an asymmetry of the generating process.

In many cases the shift from outcomes to process is associated with a desire to account for the outcomes. This is closely related to the question addressed in section 13.4: "Is randomness caused?" An interviewee who is able to account for the observed outcomes in terms of luck as an external force or agency might be unlikely to examine closely the generating process to account for asymmetries in the observed frequencies.

13.3.2.3 Occurrences of shifting perspectives

In this sub-section I summarise the occurrences of the two kinds of shifting perspectives described above. It is difficult to count the number of instances of a shift shown by an interviewee within an activity, since the beginning and end of any particular shift are not clearly defined. However, it is possible to examine each task separately, and consider whether an interviewee demonstrated a particular shift of perspective at some time while working on that task.

In Table 13.1, I have indicated the tasks during which each interviewee showed evidence of a particular shift. A shaded cell indicates that an interviewee did not undertake a task. Evidence of the process/outcome shift is indicated by a "P", and the local/global shift by "G". A blank cell indicates that no shift was demonstrated.

The table gives an indication of the prevalence of the two shifts identified in this study. While the process/outcome shift has been observed slightly more frequently than the global/local shift, the difference is not great. It is remarkable how often the two shifts were both seen in the same task-interview combination.

Instances of shifting from a focus on outcomes to a consideration of the process were sometimes brief, and shown only in a passing remark, such as "Perhaps it was just the way I rolled it" (Linda, line 380). Typically, an occurrence of the process/outcome shift in a task produced fewer remarks than a local/global shift did.

A few interviewees showed some shifts during their discussion in the review task, but these were unusual.

Name	Stage	Counters task	Biased die	Spherical die	Cracked die	Coin generation	Coin tossing	Review
Lara	1				G			
David	1		G P	G P	G		G P	
Alex	1		P	G P	P		P	
Ben	1		P	G P	G P		G P	
Dom	1		G P	P				P
Nick	1			G	G P			P
Joe	1		P	P	G P	P	G P	
Abby	1		G P	G P	G P	G P	G P	G P
Belle	1		P	P	G P	G	G	P
Linda	2	G P	G P	G P	G P			P
Bernice	2	G P	P		G P			P
Claire	2	G P	P		G P			G P
Hannah	2	G P	G P	G P	P		P	
Rory	2	G P	P	G P	P		P	
Mosaab	2	G	G	P	P			
Andrew	2	G P	P	G P	G			
Guhan	2	G P	P		G			
Assim	2	G P						

Table 13.1: Shifting perspectives, by task and by interviewee

Every interviewee except Lara showed some evidence of each shift. However, Lara's interview was the first one, and the tasks were still being refined. Lara did not really relax during her interview, and I was aware that my conversation with Lara had not worked as well as later interviews.

There were some interviewees who demonstrated the local/global shift only once. Alex was reluctant to consider a frequency distribution, as noted in Chapter 9, and her only reference to a global distribution was brief and based on a very small sample. Similarly, Dom showed only a fleeting reference to the global perspective when he was discussing the results from the biased die.

13.4 Randomness and causal factors

A few interviewees demonstrated a belief in luck as an agency influencing the outcomes from an otherwise random process. Others showed a rather different view of causes lying behind the supposedly random outcomes, and viewed a random process as one which could be thought of as very complicated, with many factors affecting the outcomes. In this section, I describe some evidence about causality from the stage 2 interviews. The ideas relating to 'luck' are discussed in 13.4.1, and the contrasting views relating to many factors are considered in 13.4.2.

13.4.1 Luck and randomness

There were more incidences of the use of the word "luck" in the stage 2 interviews than there had been in stage 1. The only interviewee in stage 1 to discuss at length her ideas about luck was Alex, whose ideas were reported in Chapter 8, section 8.1.2. In the stage 2 interviews, four interviewees discussed luck: Hannah, Rory, Claire and Andrew. While the first two were believers in luck, the second two took essentially rational positions, either that while some people perceive themselves to be lucky or are perceived to be so by others, they are no different from everybody else, or that luck does not exist in any objective sense.

13.4.1.1 Hannah

When Hannah (age 14.8) was working with the biased die, she continued to believe that the die was a fair die for a long time. She struggled to account for the long run of sixes that she observed and expressed confusion and frustration. After sixteen outcomes she had seen ten sixes, four fives and two ones. As she struggled to find some explanation, she drew an interesting parallel with the counters task, suggesting that she considered that both the counters task and rolling the die should produce unpredictable outcomes.

Hannah: You can't tell ... It's like... with the bag and the... white and black things, you know, you couldn't tell which you were going to get. ... it's like you're just taking a chance really isn't it?

(Lines 389-391, 393)

Hannah went on to restate that this die was not behaving in the way that she would expect a die to behave, and she said that she could not understand why "there's so many different numbers on there and all I'm getting is a five and a six" (lines 396-397). This showed a shift of perspective between her prior belief that the die should have six equally likely outcomes, and the contrary empirical idea suggested by the unfolding frequency table in which nearly every outcome is either a five or a six.

She struggled to account for what she saw, and eventually suggested that she was just "being lucky".

Hannah: ... I'm being lucky.

I: You're being lucky?

Hannah: Yeah. (*Laughs.*) I don't know. It's weird.

I: Have you ever been lucky before?

Hannah: No, actually. When I want to get a six, I don't get sixes, and now I'm getting sixes all the time. I think it's this dice. I think I should take this dice home with me.

(Lines 435-441)

There is a suggestion in her remark, "I should take this dice home", that she had begun to suspect that the die was to blame, but she tried to explain it to herself as a lucky die. She rolled a few more times, observing 6 6 6 6 2 6, and then cried out in anguish!

Hannah: I give up! I'm not going to get another number. I'm not though. It's just going to keep coming up sixes. So, I'm not going to get another number.

(Lines 455, 457-458)

I was not sure whether this remark indicated that she thought the die was biased so I asked her again if she thought she was lucky. She reiterated that she must be lucky, and restated an earlier argument that the outcome of rolling the die was "just chance".

Hannah: Yeah, I must be lucky. It's just a chance isn't it? You don't know when you roll a dice which number is going to come up. There's just a big chance I'm going to keep getting sixes.

(Lines 462-464)

However, there was an ambiguity here: her statement "there's just a big chance I'm going to keep getting sixes" sounded remarkably like the idea that the die was biased. To explore this distinction further, I asked her if she thought this kind of thing might happen if she were using this die at home or with her friends to play a board game.

Hannah: I don't know. You can't tell can you? It probably wouldn't, knowing my luck. But... (*laughs*). No. You can't tell really, cos, you know... it's just a dice. It's like... six other... like sides on there. So, I don't know. I don't think so though. I don't think it could happen.

(Lines 467-470)

Her answer made the kind of distinction that I had hoped for. If she had believed that this die was biased, she would have had no problem in thinking that these outcomes would be replicated in a different setting. Hannah did not consider this possibility, but used again her argument about the chances of the sides, implying again that the six had "a big chance". I interpreted her conclusion, that she did not think these outcomes would be replicated in her home setting, to imply that she still considered that the outcomes were just a consequence of her being 'lucky'.

Hannah continued trying to hold on to the idea that the die was not biased, that it was as it appeared, a fair cubical die. In trying to reconcile this view with the data that she had collected from rolling the die, she appeared to use two approaches. The first was to consider that she was being lucky: a luck that might run out at some time. The second was to consider that the die was just showing a chance run of sixes that would stop at some time. These two views were not necessarily mutually exclusive. However, it seemed to me that the "I'm just lucky" viewpoint was stronger for her, together with the sense that the 'luck' would eventually 'run out'.

Hannah continued in this vein until she had seen 36 outcomes, at which point she finally began to suspect that the die might be weighted, and eventually asked me outright if it was. After admitting that the die was weighted, I talked to Hannah about the experience that she had just had. She said she had suspected earlier that the die might be weighted, but she had strongly wanted it to be lucky.

During the review task, Hannah had shown signs of switching between using the words "chance" and "luck", so I wanted to explore whether Hannah saw a distinction between these two ideas. When she told me that she was accident-prone, and was always hurting herself, I asked if she ever thought of such accidents as luck or chance. She was reluctant to call it "luck" since she saw luck as involving good fortune – "it's not very lucky when you hurt yourself". However, she was comfortable with using the term "bad luck" to describe her sporting accidents. When I asked if she thought that these sporting accidents could be described as 'chance', she spoke of them in a different way, focusing on the chance outcome of a tackle, and considering which player would come off worst and potentially be hurt.

I: So is it chance?

Hannah: Well. It's because I play a lot of football. And it's like a chance I could go in for a bad tackle and hurt myself. You can never know. It's a fifty-fifty... who's going to come out worst.

(Lines 885-887, 889)

It seems clear from Hannah's replies that she saw luck as having a positive outcome, and that the 'lucky die' that kept producing sixes was the epitome of this. She interpreted the word "chance" as more like a probability or one of a set of possibilities.

13.4.1.2 Rory

In the review task Rory (age 14.5) was keen to emphasise that all the tasks involved 'luck', so I asked what he meant by luck. His reply suggested a perception that outcomes which occurred in his favour would be called luck

Rory: Just, when... you're not actually thinking about it, you're just having a guess, and then... it just seems to happen to go your way.

(Lines 579-579)

Earlier in the interview, Rory's use of the word 'luck' had appeared to be value-neutral. Here he was thinking of luck as when things "happen to go your way", a much more clearly positive view of 'good luck'. Rory stated clearly that he believed some people are lucky, so I asked how he would recognise someone who was lucky. His reply suggested that a lucky person can only be recognised by the fact that something good has happened to them, such as they have "won the lottery". Rory was unwilling to describe the person as lucky before the good fortune has occurred to them. As he said: "You wouldn't know" (line 593).

I asked Rory to identify what determines whether or not a person is going to be lucky. He found this difficult and his answer was not coherent. Luck could be "the same as the dice", which I took to mean 'unpredictable'. He had trouble identifying games that involved luck, stating that there was "no game that's just luck", but after some hesitation he mentioned Monopoly, which he thought was "a bit luck… you have the dice that's lucky but nothing else. You have to just… think" (lines 615, 617-618). He was able to identify that the luck came into Monopoly through the dice, but noted that the game also depended on thought.

When I asked Rory to think of other things that involved luck, he constructed two intricate scenarios. The first was about a fight in which there was "a lucky escape", and the second about a building disaster from which there was "a lucky escape". These seemed to draw inspiration from electronic games of some kind.

I: … does luck only relate to things that involve dice and tossing coins and pulling things out of bags? Are there any other things that are lucky?

Rory: Yeah. Like … something was going to happen like people were having a fight, and… there was just this little gap and they said you would have to stay in the middle. Just this tiny little gap. And that's luck that you got through without damage cos that little gap there. Or a building was going to… something was going to happen to a building. And… there was just a little hole for you to get through, and that's lucky.

(Lines 619-625)

These scenarios seemed quite different from dice or coins, but I wanted to be sure that Rory did not imagine some connection. I asked him if the lucky escape felt a bit like rolling a die: roll a die and get a six, and you have a lucky escape. Rory was sure this was quite different because these scenarios involved being scared.

Rory: No, because… you're so scared you're just trying to look for <u>any</u>thing. …It's just luck that you actually saw the hole. …But it won't happen in different things. Like… when you're scared, it's like… it won't happen. If you're scared… when you're just trying to look for somewhere… some way out. And a different scared is like when you're just… hoping that they go away, and if <u>they</u> just fell over then that's <u>luck</u>, and if you run away then that's luck as well. If you got away that counts.

(Lines 634, 636, 638-642)

These two scenarios, and the idea of being scared, seemed to refer to events that were perceived as just beyond his control; when the outcome went his way, then that was luck. In this sense, the idea of luck that Rory was using was a little like the local meaning of randomness identified as "unsteerability" by Pratt (1998).

I asked Rory if he would consider himself unlucky if he had an accident. His answer did not engage in a discussion of whether the event of the car crash itself would be considered lucky or not. Rather he was concerned with the outcome of the accident for a particular individual: escaping the worst effects, or suffering serious injury.

I: Are you unlucky if you have an accident?

Rory: (*Pause 5 seconds.*) Luck. Cos luck is hard to get, so unlucky must just be normal then. If you hurt yourself... if you... if you're in a crash, in a car crash, and everyone else went through the windscreen and you were still sitting there, then that's lucky. ...But if you were the person who went through the windscreen and they were still sitting there, then you would say you were unlucky... like the person in hospital is unlucky...

(Lines 645-648, 650-651)

For Rory, being lucky in a car crash was to benefit from a good fortune that goes beyond the normal, to escape from something bad. Then to be unlucky was to be the opposite, in the sense of being left behind with the bad consequences. The person who goes through the windscreen is unlucky, and the person in hospital is unlucky. In this sense, to be unlucky is the norm: if you are in an accident, then the norm is to be injured. Rory's statement that "luck is hard to get" was striking, with its image of fighting against the odds to benefit from a bit of good fortune.

As he talked about this, Rory suddenly switched to thinking about football. He considered the football manager who says, "We were unlucky not to score".

Rory: … I don't see why … football managers say you're <u>un</u>lucky. If they did score, they wouldn't say "you're lucky", they'd say "that's good", and it is.

(Lines 652-654)

Because Rory was trying to see unlucky as the opposite of lucky, he suggested that to fail to score would only be lucky if to score would be considered lucky. When I asked Rory if he thought that there was luck in football, I expected him to say that there was. I was surprised when he said that it wasn't luck, but just a "lot of hard work" (line 658). When I pressed him further on this, he admitted that here might be luck, in that luck can occur anywhere. However, the qualifications that he put in to what he said strongly emphasised the idea that luck is unpredictable and "it sometimes never happens" (line 661).

As we ended the interview, Rory said several times, "it's strange," and then commented. He was still considering what was lucky and what was unlucky. Indeed, the way he expressed himself suggested to me that, during the interview, he felt that he had come to a different and more critical understanding of 'luck' and what might be meant by saying someone or something is 'lucky' or 'unlucky'.

Rory: I was just thinking about luck. And if my friend says, "oh you're lucky", I would think it was just normal that I'd actually experienced different things about luck, so… If someone says, "you're lucky" now I would just think, "why am I lucky?" Why is that "unlucky"?

(Lines 672-675)

During the interview, our discussion had moved from an awareness of randomness in the activities, without using the words 'random' or 'chance', towards a discussion of the idea of luck. In some ways Rory used 'luck' as a way of speaking about 'randomness', but there were some differences arising from the way in which the word 'luck' is used colloquially because the word is overlaid with ideas of good or bad, beneficial or not. Only occasionally did his use of the word "luck" convey a sense of an outside agency controlling the outcomes.

13.4.1.3 Claire

During the review task, Claire (age 15.0) suggested that the outcome of rolling a fair die was "complete luck". When I invited her to reflect further on this idea, she seemed to use "luck" in place of "random", and suggested an idea similar to degrees of randomness, contrasting rolling a weighted die (less luck) with rolling a fair die (more luck). She suggested that the outcome of the weighted die involved less luck, but that when the outcomes were equally likely (it's "a fair test") they were not affected by anything else and the outcome would be "just luck". She went on to suggest that luck might be related to randomness.

Claire: … Luck. What a difficult word… Luck's random… Oh yeah. That's what I should be saying. It's just random whether it's Heads or Tails… Luck… Yeah. I think it probably is quite… very closely linked to randomness.

(Lines 479-482)

As she thought further about the differences between luck and randomness, Claire recognised that in some circumstances there were differences. She noted that "being lucky" had connotations of having good things happen but seemed to consider that this might be a colloquial distortion of the "true meaning". She considered that the "true meanings" of luck and randomness were "probably very similar".

Claire: Luck can also be like… to do with getting good things… That probably… because luck gets used quite often…that probably obscured its meaning a bit as well. … But I think, luck and randomness, their true meanings probably are very similar, because it's just unconnected with anything else.

(Lines 484-488)

Claire had suggested that some people were considered to be "lucky" so I asked her whether she considered that people were lucky. I was interested to discover that she thought not. She suggested that some factors might contribute to making someone appear to be lucky, but she did not elaborate on what these might be. On the other hand she considered that whether or not something good happened to a person (that is, they appeared to be lucky), as opposed to something bad happening (which would be unlucky) could be seen as "just random".

I: You say, "if you're a lucky person". Are some people lucky?

Claire: No, I don't think they are ... I think factors can contribute to make someone seem lucky. ... I think that, yeah, it's just random whether they... something good might happen to them, rather than something bad.

(Lines 489-493)

Claire seemed to have used luck as a synonym for random, but had then recognised that the word "luck" carries a value judgement that the outcome is good. However, she was sure that people only appeared to be lucky when good things happened to them. Appearing to be lucky did not make them different from everybody else, and in that sense she did not consider that they were 'lucky' in an objective sense.

13.4.1.4 Andrew

In contrast to the previous examples, Andrew (age 13.11) did not believe in luck!

Andrew: ... luck is just something that people associate with... if someone were to appear lucky by like winning a game three times in a row, it might not mean that they're any good at it, it might just mean that for some reason they did win. So I don't believe that there's such a thing as luck.

(Lines 543-546)

He accepted that one might refer to someone as being lucky – for example if they had won the lottery – but he was in no doubt that their win was due to chance, not luck!

Andrew: ... you could call someone lucky for some reason, like they've won the lottery, which is like a one in whatever million chance... But it's... nothing that they've done, like they've decided "I'm so lucky, I'm going to go out and play the lottery because I'm going to win". It's more, they've played it and they've just happened to win because of chance.

(Lines 552-553, 555-557)

Andrew was also able to explain how people tend to rationalise to themselves after the event that they were lucky.

Andrew: I mean they could have like drawn conclusions from, like … "I almost got run over by a lorry but I decided to step back at the last moment, so I'm lucky." …But that's… just something that happened, not necessarily that they're born lucky or that they're luckier than everyone else.

(Lines 559-560, 562-563)

Andrew's attitude was strikingly in contrast to that of other interviewees who had discussed luck, in that he was so clear in rejecting the idea of luck.

13.4.2 Many factors

The question of whether a process affected by causal factors could be considered to be random was one which produced a variety of responses from interviewees. As mentioned in the discussion of shifting perspectives in 13.3.2, there was indication that some interviewees were able to hold more than one view about this, and were able to move from one view to another during the interview. In the following extracts I present three different ways of thinking about the relationship between random outcomes and causality. In the first, Hannah spoke of the die in animist terms, and found it difficult to see chance as a model for real world experiences, preferring to look for causal explanations. In the second, both Linda and Abby suggested that situations that were influenced by many factors could not be considered to be random. Finally, Ben and Andrew started from positions similar to those of Linda and Abby, but went on to express a sense that, where the process is complex and unpredictable, involving many factors, it can be still considered to be random.

13.4.2.1 Animism and causal explanations

When Hannah (age 14.8) was working with the biased die, she had observed three fives and seven sixes. Commenting on this, she seemed to attribute an independent will to the die: "this dice obviously don't want to give me a one two or a three" (line 359). She could not understand what was going on, but she knew that it was not as she expected it to be.

Later, in the Review Task, because Hannah seemed to play a lot of football, I asked if she considered that the result of a football match could be thought of as 'chance'. She agreed that it could be, but her consideration of a football game seemed to have two distinct elements. First, she referred to not knowing which would be the better side "until you start playing".

Hannah: Yeah… you don't know who's the better side until you start playing. You don't know whether you can score or not. It all depends really.

(Lines 893-894)

In saying this, Hannah seemed to be suggesting that the result could be considered random if she did not know which was the better side, and implied that one side being better than the other would be a factor determining the result. The second element emerged when I asked her if the better side would always win. She agreed that this was not always the case since the better side (her own in her account!) might drastically under-perform against a poor team, and so lose. However, in her eyes, this was not 'chance', but "just stupidity" (line 900)! Thus, she seemed to have identified two factors determining the result of a match: which was the better side, and whether the better side played as well as they could. She had agreed that the result could be considered to be chance, but she had continued to have causal factors in mind at the same time.

13.4.2.2 Many factors mean not random

In an extract discussed in 12.3.5, Linda (age 16.1) discussed the counters task and asserted that no person could generate truly random outcomes by sampling counters with replacement because factors such as mixing the counters or rolling the die in a standardised random manner could never be fully controlled. She went even further and stated that "the only way you can get completely random results is to use a calculator or a computer or something" (line 755-756). This final assertion was remarkable in that it seemed to suggest that randomness required a degree of purity removed from human influence.

Abby (age 17.7) had seemed to argue in a similar way in the stage 1 interviews. During the Review task, she discussed the idea that whether or not her plants would flower could be considered to be a random event, but this was less random because there were identifiable factors affecting the outcome.

Abby: Most things are affected by other things. Like whether my plants are going to flower or not is affected by whether I water them…

I: So, does that make it **not** random?

Abby: Not quite **so** random. And I mean it's affected by the quality of the soil… and (*laughs*) whether it's got tomato spot disease.

(Lines 557-560, 562)

Later she considered the idea of programming a machine to water her plants automatically to avoid the possibility of her forgetting to do it for herself.

Abby: … you can get a machine to do it, and programme it in to water the plant, but what if the machine breaks?

(Lines 581-582)

I had expected that Abby would see a breakdown of the machine as 'unpredictable' and therefore as 'random'. However, she suggested that this could not be considered to be random.

Abby: It would be due to something being wrong with the machine. …Machines are generally not very random. … Machines, you can make them do random things because they don't have other influences. The less… something is influenced by other things, I believe, makes it more random.

(Lines 585, 587, 590-592)

Abby seemed to be saying that the less the outcome is influenced by other things, the more random it is, which is similar to the idea suggested by Linda. Later in the interview, Abby made a similar claim relating to accidents, noting that when the factors causing the accident are predictable, the accident itself is more predictable – that is, it is less random.

She listed various factors that might contribute to causing a road accident, before finally summarising her claim.

Abby: You know, it's all the factors that influence it. And things being influenced by a factor that is predictable makes it more predictable, obviously. Like a hole in the road is a factor that means that you can predict that it's more likely that an accident will happen there. Or a more busy road is a factor that makes it more likely that a car is going to crash, because, you know, if there's more cars, then one's likely to crash.

(Lines 640-644)

The final factor listed by Abby, that an accident is more likely on a busy road, is slightly different in that it is related to the idea a particular outcome is more likely to be observed when a larger sample of cases is inspected. It is interesting that, in spite of Abby's claim that the accident with many factors causing it is 'less random', she was still using a probabilistic term - "more likely" - to express her argument.

13.4.2.3 Many factors mean random

During the Review Task, Ben (age 15.7) began by expressing himself in a similar way to Linda and Abby. He was trying to think of examples of random phenomena, but was finding this difficult because other factors were involved.

Ben: It's quite a hard thing to think of something that's just completely random though. There's always factors involved in it ...

(Lines 510-511)

Later however, Ben expressed the idea that a situation could be considered to be random if there were "millions of factors" so that the outcome could not be predicted. I have discussed this extract from Ben's interview in Chapter 8 (8.1.3). Ben's view of randomness seemed to change from a sense that he could not consider a situation to be random if there were causal factors involved to a more sophisticated view that randomness could be used to model a situation in which there were many uncontrollable factors working together in a complex and unpredictable system.

In the second stage interviews, Andrew (age 13.11) showed a similar shift in his view of the role of influencing factors during his interview. During his work with the spherical die, he was concerned by the absence of fives and sixes from the sequence of the first twelve outcomes. However, he said that he could not be sure that this was sufficient evidence to be concerned about the fairness of the die because "there are too many variables for it to just be that one thing" (line 337). When I asked him to explain this remark, he mentioned many other factors that were not controlled in this experiment.

Andrew: Well, there could be differences in the surface, or something, which could cause it to slow down earlier or later. ...If you were to have a completely perfect surface, so there couldn't be as many variables, then you might get the same result all the time, because... there would be less things to interfere with the dice, so it could just be left down to the person throwing it, and the speed they throw it with. ... And the way it's made, you can't be sure that... it will always be perfectly fair... because of the process its made in. There could be like defects between dice to dice.

(Lines 342-348)

Andrew was suggesting that many factors might have contributed to the absence of 5 and 6, besides the behaviour of the die itself or how hard it was rolled. In particular, he considered the features of the surface he rolled onto. Andrew's remark about the die that "you can't be sure that... it will always be perfectly fair" seems to suggest that at this moment he saw 'fairness' as a perfect ideal that cannot in reality be attained. This seems to me to be a sophisticated idea akin to seeing randomness as an idealised model for some situations.

Later, when he was working with the cracked die, Andrew had been concerned about the fact that he had not seen any sixes in the first twelve outcomes. When he saw a six on the thirteenth throw, and another on the fifteenth, I had expected that he would have been reassured and would have stopped the experiment. However, he continued throwing with great interest a further nine times. When I asked him why he had done so, he explained that he had wanted to see if he would get any more sixes. He suggested that he was concerned that the occurrence of sixes might have been dependent on the way in which he threw the die. In the event, he had not seen any further sixes in the extra nine throws, so I wondered whether he was still bothered about this. He stated clearly that he was not and explained why.

Andrew: Because… you could go for… an entire game or something not getting a six, and then get a load… from straightforward throws. It's just… there's so many things that could affect it… it doesn't really make a difference.

(Lines 489-490, 492, 494)

On this occasion, Andrew was reassured about the fairness of the die by the fact that there were so many factors that could affect the outcome.

Both Ben and Andrew had shown signs that, while they were aware that factors influencing the outcomes in a process might be used to provide a causal non-random explanation for the outcomes, they could see randomness as a useful model in the presence of complex systems of uncontrollable or unpredictable factors. This awareness seems to encompass the idea that randomness can be a model for complex deterministic systems, in which the details of the interactions between the factors affecting the outcomes are too complex for the judge to know or to want to know. It was interesting that, unlike many other interviewees, both Andrew and Ben were able to think easily of examples of situations in everyday life that might be modelled using randomness and the associated ideas of probability. They also did not show any sign of thinking in terms of luck; indeed Andrew had explicitly denied the existence of luck (see 13.4.1.4).

13.5 Conclusions

In this chapter, I have reviewed several different themes across the two stages. It now seems possible to speculate about ways in which some themes may be related to one another.

People often hold contradictory 'beliefs', and 'claimed beliefs' are not always compatible with observed behaviour. For example, pattern-seeking behaviour often seemed to coexist with a clear belief that the outcomes were unpredictable. Such behaviour might be particularly found where learners are struggling with ideas which demand that their attention be focused in one place yet they naturally find their attention attracted elsewhere. Thus people's sense of *randomness* is confusingly reflected in the apparently 'random' way in which their attention seems to shift, especially where they seem to lack vocabulary for what they want to express, or are not sure what it is they want to (or the interviewer wants them to) express.

The words used by interviewees to speak of randomness, and the sometimes surprising ways in which the word 'random' has been understood by interviewees, seem to relate in subtle ways to the ideas that interviewees express. For example, Rory's use of 'luck' as a synonym for 'random' may have encouraged him to slip from expressing apparently rational views about randomness to a more superstitious expression. The words used, and the networks of meaning that they express for the speaker and listener, can enable clear expression of ideas, and yet at the same time they seem to constrain and limit expression and understanding. Claire's initial preoccupation with a colloquial sense of 'random' influenced her expression, and probably her thinking, throughout her interview. Abby's realisation that she could think of the judgement of randomness as a 'personal judgement' was a revelation for her, and it seemed to move her thinking forward.

The strategies used by interviewees to discern randomness may be related to the developing awareness of aspects of randomness. Pattern seeking may develop into representativeness and lead to recognition of the importance of the sample size. These strategies in turn appear to be related to the growing awareness of the global perspectives, particularly the empirical perspective.

Expressions of superstitious belief sometimes seem to act as a bar to people questioning when they perceive departures from disorder and chaos that they might expect to see in the local perspective on randomness. Ideas such as animism (the die has a will of its own) and luck (this die is lucky) provide accounts for outcomes from the biased die, and these seemed to remove any need for interviewees to look away from the outcomes to inspect the generating process.

When interviewees were able to think of randomness as a model for situations in which outcomes were 'caused' by many factors in a complex system, they showed subtle and delicate shifts between speaking of the outcomes as random and determined.

PART 4:

Concluding the Study

Chapter 14: Conclusions and Implications

In this final chapter, I summarise the results of this study of learners' perceptions of randomness, referring back to the aims set out in Chapter 4. I discuss the limitations of the study and the implications of the findings for pupils' learning.

In sections 14.1 and 14.2, I summarise the findings and relate these to the aims and to key references that were discussed in the review of literature in Chapters 2 and 3. Section 14.1 deals with findings relating to the understanding of randomness, while 14.2 is about shifting perspectives between different ways of looking at or seeing randomness. Section 14.3 contains a discussion of the limitations of the study and considers the aspects of the study that constrain my ability to elaborate on the aims. Implications of the study for future research are presented in 14.4. Finally, in section 14.5, I discuss some implications of the findings for teaching and learning statistics and probability.

14.1 A summary of the findings

In the aims, I proposed two sets of questions that needed to be studied. Firstly, what learners believe about randomness:

- What do learners believe randomness to be?
- What do learners expect from a random generator?
- How do learners recognise what they believe to be random and discern the random from what they consider to be non-random?

The second set of questions relates to the idea that randomness is an important model to be applied to describe a process.

- To what situations and circumstances do learners consider randomness to be an appropriate model?

- What is the range and variety of situations and circumstances for which learners consider that randomness is an appropriate model?
- How are learners' responses to this second set of questions related to their responses to the first set?

The bulk of the results of my study relate clearly to the first set of questions, but analysis of learners' beliefs about randomness also reveals some insights about the ways in which learners consider that randomness can be applied.

Learners' beliefs about randomness need to be considered at two different levels: understanding of the word, and understanding of the concept.

14.1.1 Understanding of the word 'random'

Some learners have a knowledge of the word 'random', but it is clear that learners' usage of the word 'random' may not always match the meaning given to the word in mathematics lessons. Not all learners are comfortable with the word.

- In stage 2 interviews, five out of nine interviewees did not use the word at all.
- One interviewee, David in stage 1, used the word 'random' in a restricted sense.
- There is some evidence, from the interview with Claire and from the preliminary study, that the modern colloquial use amongst young people of the word random, and its derivatives, has spread and could lead to some misunderstanding between teachers and learners in classrooms.
- In several interviews, the word chance was used as a synonym for randomness, and occasionally the word 'luck' was used.

There is indication in the evidence referred to above that many interviewees are not confident in their use of the word 'random'.

14.1.2 Understanding of the concept of randomness

Understanding of the concept seemed to fall into various categories. In this section I summarise the evidence under five headings:

- Agency: situations are governed by an outside agency or by luck.
- Randomness is unpredictable
- Restricted randomness: random outcomes must be fair
- Degrees of randomness
- Randomness as a model for incomplete knowledge

In the following sub-sections I consider each of these headings in turn.

14.1.2.1 Superstition

A few interviewees spoke of the outcomes in the interview tasks as being caused by some agency or by luck. I see this as possibly being different from the use of the word 'luck' as a synonym for 'random', such as Rory's described in 13.1.3. The difference emerged in the way that an interviewee spoke of some people as being lucky, or implied that their behaviour might be affected by their view of being lucky or unlucky. Alex (age 15.3, set 2/6) displayed this understanding in stage 1 as described in section 8.1.2, as did Hannah (age 14.8, set 3/6) in stage 2 (section 13.4.1.1), and Rory (age 14.5, set 4/7) in later passages of his interview (section 13.4.1.2). A related view was the animist view shown by Hannah (section 13.4.2.1), in which she ascribed to the biased die a will of its own. In essence this category of understanding is characterised by an element of superstition.

These views are similar to those reported by Kath Truran (1998), and Amir and Williams (1999). In this study, evidence of superstition was not found amongst the most able pupils. However, out of the eight interviewees in this study who were not in top sets or in A level classes, three have demonstrated some evidence of superstition. Such beliefs may indeed be widespread amongst secondary pupils, as was suggested by Amir and Williams (1999).

14.1.2.2 Randomness is unpredictable

Evidence has been presented of learners interpreting randomness in terms of what it is not. By far the most common in the interviews in this study was the idea that randomness is characterised by unpredictability. Many interviewees spoke of outcomes as unpredictable, particularly in the dice activities. It seems likely that these tasks lead the interviewees to express themselves in this way, as they consider the outcomes seen so far and think about what might happen next. One interviewee, Mosaab in stage 2, spoke of outcomes as not expected. This was a similar idea to unpredictable.

Related to randomness being unpredictable was the idea of pattern-breaking. Patterns that were seen to emerge in the sequence of observed outcomes were expected to break and not to persist. I return to this idea in the discussion of strategies in 14.1.3, since it formed the basis of a significant strategy for recognising randomness.

The idea that unpredictability is interpreted as randomness was present in the first of the local meanings of randomness identified by Pratt (1998).

14.1.2.3 Restricted interpretations of randomness

A few interviewees (for example, Alex in section 8.1.1.1) spoke of the idea of randomness as being equivalent to fairness, or as requiring that the outcomes should be equally likely. This is a restricted view of randomness and it might be expected to limit the individual's ability to take on the more flexible idea of degrees of randomness.

Pratt (1998) identified fairness as the fourth of his local meanings of randomness, suggesting that where the outcomes occurred with approximately symmetrical frequencies, pupils interpreted this as evidence of randomness. However, there is evidence in this study that some people take this further and believe that if the outcomes are not equally likely then the process is not random. It is not surprising to find this belief being expressed since it was identified in Chapter 2 as one of the meanings of the word 'random' listed in two commonly used dictionaries, and this sense of the word is also sometimes used by statisticians when speaking of a 'random sample'.

I consider that the significance of this view of randomness for learners lies in its antithetical relationship to the idea of 'degrees of randomness', which I see as an important idea for developing an appreciation for the power of randomness as a model for everyday situations.

14.1.2.4 Degrees of randomness

Interviewees in stage 1 who spoke in terms of randomness being a matter of degree, and who were able to think of some things as being more random than others, included David (age 14.1) and Ben (age 15.7) in Chapter 8, and Abby (age 17.7) in Chapter 9. In stage 2, this idea was expressed by Claire (age 15.0). I see this idea as critical in the development of an understanding of the applicability of randomness as a model for a variety of situations in the real world.

I also see the idea of degrees of randomness as part of the solution to the paradox of seeing randomness as equivalent to fairness. Once randomness is seen as unpredictability and a probability distribution as a measure of the degree of predictability, then the learner can be seen to possess the resources to construct a global perspective on randomness – that is to construct the idea of a probability distribution.

Interviewees who expressed the idea of degrees of randomness were also able to see a variety of applications of randomness to the weather, to sport and to chance events in everyday life. Interviewees who appeared to see randomness as an absolute concept, without a clear expression of degrees of randomness, such as Hannah or Alex, did not appear to perceive this variety of applications.

14.1.2.5 Randomness as a model for incomplete knowledge

A more powerful view of randomness, discussed in Chapter 2, arises from the definition of randomness proposed by Kyburg (1974): randomness can provide a model for a situation in which the judge has incomplete knowledge, even though the situation might be strictly deterministic. Such a view of randomness was clearly expressed in stage 1 by David and Ben, in 8.1.3, and by Abby when she spoke of randomness as 'personal' in 13.2.2. In stage 2, Claire also seemed to express this idea, but much less clearly. It is interesting to note that these individuals are precisely those who also expressed the idea of 'degrees of randomness', and that, with the exception of Claire, the extracts showing this idea of randomness as a model for incomplete knowledge are from different parts of the interviews from the places where they expressed 'degrees of randomness'. This suggests to me that these two ideas might be linked.

Ben and Andrew also expressed the idea of randomness as a model for incomplete knowledge in their suggestions that randomness might be a model for situations were affected by many factors (13.4.2).

The example of Abby is particularly interesting as her idea that randomness was a personal judgement appeared to have developed within the interview.

14.1.3 Strategies for recognising randomness

Three significant strategies are described here, drawing upon the discussion in Chapter 13.

14.1.3.1 Pattern Seeking

The primitive strategy of pattern seeking was described in Chapter 3 as a fundamental human activity (in section 3.9). Most interviewees showed some evidence of pattern seeking in their attempts to interpret the behaviour of the random generators presented in the interview tasks. This strategy is clearly most fundamental in making the judgement as to whether or not the outcomes generated are predictable. However, many interviewees were able to take this further, and expected to see some patterns emerging in the sequence of observed outcomes, but also expected these patterns to be short-lived. A difficulty was to decide how long a pattern might be expected to continue before it became too long and constituted evidence that the sequence was not random.

14.1.3.2 Representativeness

I suggested in Chapter 13 that use of representativeness as a strategy for recognising randomness might be seen as a special case of pattern seeking. Instead of looking for patterns in the ordered sequence of observed outcomes, as in the pattern seeking strategy, a person using representativeness is looking at an aggregation of the observed outcomes, and in this sense the use of representativeness can be seen to mark a step towards recognising a frequency distribution. Thus, when rolling a die that is believed to be fair, the person is expecting each outcome to occur roughly the same number of times. However, as discussed in Chapter 13, many interviewees experienced significant difficulty in knowing how many outcomes were needed before the aggregation could be regarded as representative. Nearly all interviewees tended to make their judgement on the basis of sequences of outcomes that were too short to provide reliable evidence. Thus, the judgements that they made were frequently changed, which seemed to increase the uncertainty expressed by the interviewee.

14.1.3.3 Sample size

A few interviewees recognised that, to improve the reliability of the judgements they made, they needed to increase the size of the sample they considered. I have tentatively suggested that the development of an awareness of importance of sample size may be related to the development of a global frequentist perspective.

14.2 Shifting perspectives

In the course of transcribing and analysing these interviews, I have become increasingly aware that the interviewee's attention appeared to shift between different aspects of the task, or different ways of looking at the task. Often, the focus of the interviewee's attention would be apparent from what they said and from the way in which they responded to outcomes they were considering. In Chapter 9, I gave examples from the first round of interviews relating particularly to the two ideas of local disorder and distribution. The counters task was designed to explore this issue further. However, during the analysis of the second stage interviews, I have seen a more complex picture emerging in which issues other than the local and global perspectives appeared to play a part.

Writers such as Gattegno (1987) and Marton and Booth (1997) have suggested that learning is strongly directed by the learner's awareness of features of the phenomenon that is being considered. However, awareness does not tell the whole story; at any instant a learner may hold one or more features of the situation in their attention and reflect on them. The shifting of attention between different, and sometimes apparently conflicting, perspectives on a problem may be a precursor to the learner bringing each of the perspectives into awareness simultaneously.

Within these interviews, the apparently different viewpoints that learners adopted, and between which they were seen to shift their attention, could often be classified as local or global. A similar dichotomy, albeit a little different, which was discussed in the literature review, is that between seeing randomness as either the underlying process or the sequence of outcomes generated. I have observed a third, and rather different shift between seeing the 'random' outcomes as having been 'caused' by some agency or as having no cause. For some interviewees, the shift between not caused and caused was similar to a more fundamental shift between seeing the outcomes as random or not. Random outcomes were sometimes seen as having no cause, whereas outcomes that were caused by some agency could not be considered as being random.

14.2.1 Local and global

At the local level of what will happen next there is unpredictability. In the short run, sequences of outcomes typically lack sustained order and pattern. Although on many occasions some pattern may appear by chance, such patterns are typically not sustained when the process is allowed to continue further. However, these illusory patterns can be a significant distraction for a person interacting with the outcomes, especially if they are trying to make some judgement either of what outcome might occur next, or more generally about whether the process may be thought of as random.

In the dice activities and the coins tasks used in this study, typically the interviewee's attention was initially focused at the local level and in the short run. Subjects looked for patterns in the sequence of outcomes, as they tried to predict what might happen next. When they observed patterns emerging in the sequence of outcomes, they looked to see those extended and tried to use them to make predictions. When they considered that outcomes were unpredictable, they accepted that the process was 'chance' or 'random' (or sometimes 'luck').

At the global level, in the longer run, there emerges an order and pattern in the distribution of the observed outcomes across the space of possible outcomes. This is the emergence of distribution.

Sometimes the interviewees consciously tried to 'control' the process to produce an outcome that had not been seen in a sequence of outcomes. If the process is allowed to proceed, then the outcomes are experienced as uncontrollable.

The shift of attention is between, on the one hand, the unpredictability of the next outcome (what will happen next) and the lack of order and pattern in the short run (in a short sequence of outcomes), and on the other hand, the order and pattern of an empirical probability distribution. In my interviews, the interviewee's attention was often particularly unstable when the samples, from which they were trying to infer the distribution, were too small, and produced apparently conflicting information. This can be seen, for example, in the accounts of Linda in 12.1 and Andrew in 13.3.

Within the counters task, when interviewees saw the distribution as being controlled by the adjustments that they made to the contents of the bag, this awareness was often accompanied by beginning to adopt a global perspective on the randomness of the outcomes. The nature of the counters tasks encouraged interviewees to express awareness of the distributional aspect of randomness.

14.2.2 Two variants of global: prior and empirical

When a person interacts with an apparently familiar random process, such as rolling a die, tossing a coin, or drawing counters from a bag, they bring to that interaction their prior beliefs about how the process will behave, derived in part from past experiences. Such prior beliefs are often naturally expressed in terms of the distribution of outcomes that is expected to be observed. For example, when rolling a cubical die, most people would tend to expect the six different outcomes to be equiprobable, and they might therefore expect that, in a number of trials, the relative frequencies of the six outcomes would be approximately equal. However, many people have little understanding of how much the variation of the observed relative frequencies from the expected relative frequencies of each outcome will depend upon the total number of trials.

The fact that such prior beliefs exist means that sometimes the person's attention will be on their prior belief, and at other times they will be attending to the emerging empirical distribution, as they look for a match between the two. This shift of attention between prior beliefs and emerging distribution was not seen in the counters task as the task did not highlight for the interviewee a prior belief about the probabilities of drawing a black counter or a white counter from each bag; rather the task focused the interviewee's attention on inferring probabilities from outcomes. This shift was much more commonly observed in the dice rolling and the coin tossing tasks, where there were clearly identifiable prior beliefs, based on equiprobability, that were highlighted in the tasks.

I suggested in Chapter 13 that both the prior and the empirical perspectives can be seen to be variants of the global perspective, since each represents a deterministic meaning of the long-run behaviour of randomness. Therefore, movement between attending to the prior and the empirical might be more profitably seen as part of the local-global dimension, in which there are really three different perspectives: local, prior and empirical. Then the empirical can be identified with the global frequentist idea of distribution. As the total number of outcomes considered becomes larger, so the empirical distribution becomes much more clearly visible in the aggregated data. It can be considered by the learner as a phenomenon subject to external factors, and can be 'controlled' by changing those factors in a controlled manner.

The manner in which learners appear to switch their attention rapidly between different aspects of and perspectives on randomness, may indicate something of the uncertainty and anxiety that learners experience about how to interpret random phenomena, especially in the context of mathematics lessons. Such uncertainty about what to attend to may shed some light on the failure identified by Metz (1999) to integrate understanding of the uncertainty of individual outcomes and small samples with the long run pattern and determinism of distribution. Metz described two failures amongst learners. The first was a tendency to over-state the deterministic aspects of a distribution, and the second, a reverse tendency to under-rate the information given by a distribution. I now see each of these failures as arising in part from a shifting of perspectives. Resting in the global perspective might lead to overstating deterministic aspects of distribution, while reasoning from the local perspective might produce the inability to recognise uncertainty separately from distribution. Behind each of these lies a failure to understand the nature of variability and its relationship with sample size. The learners' failure to understand may be prolonged by their inability to focus productively on either the global or the local perspective. For example, David, Beulah and Hannah each experienced prolonged periods of uncertainty when working with the biased die, during which they tried to account for the extreme imbalance in the outcomes in terms of luck or chance, before finally recognising that the die was biased.

14.2.3 Process or outcomes

The shift of attention between process and outcomes refers to two contrasting ways of seeing randomness, either in the sequence of outcomes observed, or focussing on the dynamic process that generated the outcomes. In some interviews, the interviewee's attention shifted from seeing individual outcomes to commenting on the process by which outcomes were generated.

For example, in the counters task, several interviewees suggested that replacing the sampled counter after each draw was not satisfactory because the same counter might be sampled in each successive draw. Sometimes, this suggestion was used to account for a set of outcomes that had not come out in the way the interviewee had expected. Hannah (in section 12.3.4) used such an argument to suggest the possibility that the observed relative frequency of black counters might not represent the proportion of black counters in the bag. For Rory (in section 13.1.3), concern about possibly sampling the same counter again and again led him to suggest that the counters task was not really possible, dismissing it as "just luck".

Some interviewees tried, *post hoc*, to account for a relative surplus or deficiency of a particular subset of possible outcomes through the way in which the die was rolled, or the asymmetry of the crack in the cracked die, as described in 13.3.2.

It may be seen from the examples above that the shift between process and outcomes seemed to be sometimes due to concern about "causation": that is, to account for how a particular subset of outcomes were 'caused' to appear.

There is a possible association between the shift between local and global perspectives and the shift between seeing randomness in outcomes or in process. In the local perspective, attention is naturally upon the disorder of individual outcomes. A person working in the local perspective may view individual outcomes closely and be concerned either to describe or to account for the randomness of the unfolding sequence of outcomes. On the other hand, in the global perspective, the individual discerns a long-run distribution of outcomes, which may be controlled by manipulating the underlying process. This awareness may bring the user into a stronger awareness of the generating process.

14.3 Limitations of this study

The intention of this study was to provide detailed case studies based upon evidence from 18 interviews with learners of various ages from 13 to 17 years. Therefore the sample of learners was not controlled for age, ability, gender or cultural background, although each of these variables might affect an individual's ideas about randomness and how these are expressed. Age and gender have been recorded for each interviewee, and a broad indication of ability is given by the nature of the school if selective, or by which set the learner is in for maths. The sample as a whole contains a disproportionate number of more successful learners. This is largely a consequence of the fact that I set out to interview pupils who would be able to express their ideas in interview. It is likely that there are further differences to be observed in how learners perceive randomness, chance and luck, particularly amongst pupils who are less successful in school.

While the sample of interviewees is small, each interview consisted of several tasks and lasted between 40 and 90 minutes. The data in the interview transcripts is rich and varied. However, the interviewees' responses to the tasks are likely to be sensitive to the interventions from the interviewer. Indeed, there are strong indications in the interviews that this is the case, particularly in the differences between the interview transcripts from stages 1 and 2. This not surprising; indeed flexibility was built into the methodology for the interviewer to interact with interviewees as they worked on the tasks. It was this flexible response to the dynamic nature of the tasks that enabled me to follow interviewee's attention as it shifted between different ways of looking at and seeing randomness. However, it would be valuable to develop further tasks founded upon a dynamic view of randomness, and focused on particular features of learners' perceptions, that could be used in further interviews.

Time has been an important constraint in this study. The time required to transcribe and analyse these interviews is considerable. This has limited the number of interviewees that I could manage at each stage, and the number of tasks that could be developed and used in the interviews.

My concern with validity in this study has focused on ensuring trustworthiness and credibility. By providing a detailed account of my interviews and of the interventions that I made, I have aimed to ensure that my account is trustworthy and hence that my interpretations are credible with respect to the interviewees in my study.

14.4 Issues for future research

I see two important issues for further research arising out of this study. One relates to the controlled exploration of the perceptions of randomness highlighted here, to see the extent to which these vary with age, ability, gender and social and ethnic background. The second is to explore further the transition from local to global perspectives, particularly the development of the empirical distributional perspective. I discuss each of these in the following sections.

14.4.1 Controlling for learner variables

The sample in this study has shed light on the variety of perceptions of randomness that learners bring to their studies. However, there is much still to be learned about how these perceptions develop within the individual as their understanding develops in other areas. Many of the perceptions shown by an individual may be related to the learner's previous experiences of randomness. It would be helpful to learn more about how perceptions develop within an individual and about how the variety of perceptions is related to age, gender and abilities, and to social and ethnic backgrounds.

A study of how perceptions vary with age and abilities might lead to greater understanding of the development of perceptions of randomness within individuals. However, I am particularly interested to learn more about how prevalent are the ideas that I have termed 'pre-random', such animism and luck. I should like to understand how such ideas are formed or acquired, and how they interact with the ongoing development of mathematical ideas about randomness and probability. I strongly suspect that these ideas may be related to family circumstances in which gambling plays a part; for example, where discussion of and participation in activities such as the National Lottery is usual. Researchers such as Wagenaar (1988) have long taken an interest in gambling behaviours, but I am not aware of any study which attempted to research the impact of gambling by older family members on children's perceptions of randomness.

14.4.2 Transition from local to global

I see one of the most significant issues to have emerged from this study as being the shifting of attention in the mind of the learner between the local and the global perspectives. The emphasis that I have placed within this study upon randomness as a dynamic concept has been an important element in enabling me to see the way people's attention has moved in the interview tasks. However, my emphasis upon tasks that do not use computer simulation has restricted my work in this study to comparatively small numbers of trials. Interviewees have not generated more than about 40 outcomes in any task, and their ability to aggregate the data has been mostly restricted to simple frequency tables. In experiments with only two possible outcomes, the frequency distribution is just beginning to settle down after 40 trials. For example, in the counters task, where bag A began with 7 black and 3 white counters, and bag B had 5 black and 5 white, the observed frequencies of black and white from each bag might be expected to show some difference after 40 trials. However, to acquire more certainty requires more outcomes. In an experiment with a slightly biased die, inequalities in the frequencies might require many more than 40 outcomes. It has been clear in this study that learners have little sense of how many outcomes they need to observe in order to make reliable judgments about probabilities.

Computer simulation can enable learners to run an experiment many times and view aggregated data from large samples very quickly. Learners can then change factors within the experiment and explore the effect upon the aggregated data. There is a need to discover more about how learners' interact with such software, and how their interactions affect the development of their perceptions of randomness. However, there is still a need to address the concerns about pseudo-random generators discussed in Chapter 3, and to discover whether learners' understanding of pseudo-randomness is different from their understanding of other random generators. Pratt's (1998) study looked at some aspects of these issues, but did not look closely at the shifts of attention between global and local or between outcomes and process.

It is important now to explore how learners can develop awareness and understanding of the relationship between sample size and variability. I am sure that computer simulation will play an important role in the necessary research into the development of understanding, and in providing learning environments. However, the rapidly shifting perspectives displayed by learners suggest to me that there is deep uncertainty about where to place attention, what to attend to and how to attend. There is a need to develop clearer understanding of how to structure a learner's attention productively to develop awareness of variation and how it is affected by the degrees of randomness as well as sample size.

14.5 Teaching and learning

The findings of this study suggest very strongly that learners do not have sufficient experience of interacting with random outcomes and of building towards the concept of distribution. Throughout all my interviews, I had the clear impression that the interviewees had not previously spent time in the 'no-man's-land' between the local perspective of observing a few (perhaps five or six) individual outcomes and the global perspective of viewing and interpreting a long-run frequency distribution. The notion of probability expressed by interviewees was usually based on an *a priori* interpretation of probability, but occasionally moved to a frequentist approach. However, when a frequentist idea of probability was expressed, there was rarely an understanding that the underlying probability might take many hundreds or thousands of outcomes to emerge with the required degree of precision.

The emphasis in the Cockcroft Report (DES, 1982) upon practical work in mathematics, and more recently in the National Curriculum for Mathematics (DfEE, 1999) upon experimental work in statistics and probability, may be appropriate but alone it is seriously insufficient. I suggest that such practical experimentation needs to be accompanied by structured questioning from the teacher, and focused discussion between learners. Learners need guidance to help them to focus their attention productively upon aspects of randomness, but they also need guided discussion to help them develop both the language and the underlying concepts with which to think productively about randomness. Here I am drawn back to my discussion in Chapter 3 of learning from constructivist and Vygostkian perspectives. In my study, some interviewees did not appear to have discussed experiments with random generators with their peers. Their understanding of the language of uncertainty, and particularly their use of words like random, fair, chance, luck, seemed unfamiliar and rather vague. Concepts such as variation and distribution were only rarely mentioned. Learners need time to experience the uncertainty that arises when they need to make judgements about random processes based upon small samples, and they need to be encouraged to discuss their experiences with the teacher, and later with one another, using mathematical and statistical words to describe the underlying concepts.

Finally, I suggest that the findings of this study have implications for classroom activities with random generators. It is dangerous to make assumptions about how much pupils understand about randomness, probability, and about random generators. Several interviewees in this study showed surprising lapses of understanding: for example, Lara, who was studying statistics at AS level, lacked confidence in using the language of uncertainty, and Claire did not know the mathematical meaning of 'random'. There is a natural assumption that a physical random generator will be symmetrical, but, where the process was asymmetrical, interviewees often did not know how to take account of this. For many interviewees, there was an important period of interaction between collection of data about outcomes and inspecting the physical generating process, during which they speculated about the effect of the physical characteristics of the process upon the observed outcomes.

I suggest that, in pupils' early experiences of experimenting with random processes, it is important that learners are allowed and encouraged to inspect the generating process, to consider whether a model of randomness is appropriate, and to discuss their view with peers. It is also important that pupils are introduced, at an appropriate time, to some generating processes that cannot be easily modelled by a probability model based upon a set of equally likely outcomes. It is through consideration of such 'unfair' generators that learners can begin to develop an awareness of degrees of randomness, and to speak about this. From an appreciation of degrees of randomness, pupils can begin to consider ways of applying their mental model of randomness to a wider variety of contexts, and to consider more carefully the relationship between deterministic contexts and randomness. Classroom discussion about whether and under what circumstances the event of scoring a winning goal in the final minute of a critical football match could be modelled using randomness could allow learners to consider the role of causal factors in the predictability or unpredictability of such an event. Then perhaps an appropriately designed computer modelling tool might be introduced to enable learners to build a representation of their random model for this event. Of course, even in a computer simulation, the random generator should be open to inspection, discussion and amendment by the learner.

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