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MAXIMAL ℓ_p-REGULARITY FOR DISCRETE TIME VOLTERRA EQUATIONS WITH DELAY

CARLOS LIZAMA AND MARINA MURILLO-ARCILA

Abstract. In this paper we investigate the existence and uniqueness of solutions belonging to the vector-valued space $\ell_p(\mathbb{Z}, X)$ by using Blunck's theorem on the equivalence between operator-valued ℓ_p-multipliers and the notion of R-boundedness for the discrete time volterra equation with delay given by

$$u(n) = \sum_{j=-\infty}^{n} b(n-j)Au(j) + \sum_{j=1}^{k} \beta_j u(n-\tau_j) + f(n), \qquad n \in \mathbb{Z},$$

where A is a closed linear operator with domain $D(A)$ defined on a Banach space X and $b \in \ell_1(\mathbb{Z})$ verifies suitable conditions such as 1-regularity. We characterize maximal ℓ_p-regularity of solutions of such problems in terms of the data and an spectral condition and we provide optimal estimates. Moreover, we illustrate our results providing different models that label into our general scheme such as the discrete time wave and Kuznetsov equations.

Keywords: volterra equations, maximal ℓ_p-regularity, R-bounded, discrete wave equation, discrete kuznetsov equation

Mathematics Subject Classification (2010): 45D05, 35R09, 65Q10, 39A06.

1. Introduction

In this paper we analyze the existence and uniqueness of solutions in vector valued $\ell_p(\mathbb{Z}, X)$ spaces for discrete time formulations of the following integro partial differential equation with delay,

$$u(t) = \int_{-\infty}^{t} a(t-s)Au(s)ds + \beta u(t-\tau) + f(t, u(t)), \quad t \in \mathbb{R},$$

where A is a closed linear operator defined on a Banach space X. This equation models viscoelastic fluids, heat conduction with memory and electrodynamics processes with memory [4, 28]. Typical models that are included in this article correspond to different discrete versions of the multidimensional wave and Kuznetsov equations

$$u_{tt} - c^2\Delta u - \nu\epsilon\Delta u_t = f(t), \qquad t \in \mathbb{R}. \tag{1}$$

The analysis of qualitative properties of Volterra type equations has been considered by various authors, see [10] and all the references therein. Moreover, numerical methods for the resolution of Volterra equations have been studied among others in [6, 7, 9, 19, 26, 27]. On the other hand, the study of maximal regularity for discrete systems that belong to the Lebesgue space of vector-valued sequences since the pioneer work of S. Blunck [5] has experimented a great development as it can be seen in the recent papers

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[13, 21, 20, 22]. This study is strongly connected with the necessity of optimal $\ell_p - \ell_q$ time-space estimates for the corresponding linearized problem [2, 11, 13, 16, 14, 18, 17, 24, 25].

In our work, we succeed characterizing maximal ℓ_p-regularity for the following abstract model

$$u(n) = \sum_{j=-\infty}^{n} b(n-j)Au(j) + \sum_{j=1}^{k} \beta_j u(n-\tau_j) + f(n), \qquad n \in \mathbb{Z}, \tag{2}$$

where $f \in \ell_p(\mathbb{Z}, X)$, A is a closed linear operator with domain $D(A)$ defined on X and $b \in \ell_1(\mathbb{Z})$. It is worthwhile to observe that, for instance, model (2) includes among others the discrete Kuznetsov equation (1) taking $A = -\Delta_{d,N}$, the multidimensional discrete Laplacian, $b(n) = -(c^2 + \nu\epsilon)\delta_0(n) + \nu\epsilon r\delta_0(n-1)$, $\beta_1 = 2r, \tau_1 = 1$ and $\beta_2 = -r^2, \tau_2 = 2$.

This paper is organized as follows: in Section 2, we first recall the notions of UMD-spaces, R-boundedness, ℓ_p-multipliers, sectorial operators and the discrete time Fourier transform defined on the space of distributions. Moreover, we recall the well-known Blunck's Fourier multiplier theorem [5] for operator-valued symbols on UMD-spaces that establishes the equivalence between ℓ_p-multipliers and R-boundedness.

In Section 3, we prove our main result, namely, if $b \in \ell_1(\mathbb{Z})$ is 1-regular, $\hat{b}(t) \neq 0$ for all $t \in \mathbb{T}$ and

$$\left\{ \frac{1 - \sum_{j=1}^{k} \beta_j e^{-it\tau_j}}{\hat{b}(t)} \right\}_{t\in\mathbb{T}} \subset \rho(A).$$

then the following assertions are equivalent:

(i) For all $f \in \ell_p(\mathbb{Z}, X)$ equation

$$u(n) = \sum_{j=-\infty}^{n} b(n-j)Au(j) + \sum_{j=1}^{k} \beta_j u(n-\tau_j) + f(n), \qquad n \in \mathbb{Z},$$

has a unique solution in $\ell_p(\mathbb{Z}, [D(A)])$;

(ii) $M(t) := (1 - \sum_{j=1}^{k} \beta_j e^{-it\tau_j} - \hat{b}(t)A)^{-1}$ is an ℓ_p-multiplier from X to $[D(A)]$;

(iii) The set $\{M(t) \ : \ t \in \mathbb{T}\}$ is R-bounded.

Observe, that our result demands 1-regularity of the kernel sequence $b(n)$. We introduce this concept for the first time in definition 3.2 and it corresponds to the discrete counterpart of the notion of 1-regularity introduced in [15]. Furthermore when X is Hilbert we simplify the previous result by replacing the condition (iii) above by an easier computable condition

$$\sup_{t\in\mathbb{T}} \|M(t)\| < \infty.$$

We also ensure optimal estimates for model (2) under any of the above conditions, that is, the following estimate also holds

$$\|u\|_{\ell_p(\mathbb{Z};X)} + \|b * Au\|_{\ell_p(\mathbb{Z};X)} \leq C\|f\|_{\ell_p(\mathbb{Z};X)}.$$

Finally, in section 4, we prove, as an application of our characterization, the existence and uniqueness of $\ell_p(\mathbb{Z}; \ell_q(\mathbb{Z}^N))$ solutions for time discretizations forms of the wave and

Kuznetsov equations in terms of the data of the problem as it can be seen in theorems 4.2,4.3 and 4.4. In addition, we obtain maximal $\ell_p - \ell_q$- estimates for such models.

2. Analytical framework and notation

In this section, we present some results that will be needed throughout the paper.

Let X be a Banach space. We denote by $\mathcal{S}(\mathbb{Z};X)$ the space of all vector-valued sequences $f : \mathbb{Z} \to X$ such that for each $k \in \mathbb{N}_0$ there exists a constant $C_k > 0$ satisfying $p_k(f) := \sup_{n\in\mathbb{Z}} |n|^k \|f(n)\| < C_k$ and when $X = \mathbb{R}$ we denote $\mathcal{S}(\mathbb{Z})$. We write as $C^n_{per}(\mathbb{R};X), n \in \mathbb{N}_0$, the space of all 2π-periodic X-valued and n-times continuously differentiable functions defined in $\mathbb{R}$. In what follows, we will denote $\mathbb{T} := (-\pi,\pi)$ and $\mathbb{T}_0 := (-\pi,\pi) \setminus \{0\}$. The space of test functions is the space $C^\infty_{per}(\mathbb{T};X) := \bigcap_{n\in\mathbb{N}_0} C^n_{per}(\mathbb{R};X)$. When $X = \mathbb{R}$ we simply write $C^\infty_{per}(\mathbb{T})$.

For each $f \in \ell_p(\mathbb{Z};X)$ we can define the map

$$T_f(\psi) := \langle T_f, \psi\rangle := \sum_{n\in\mathbb{Z}} f(n)\psi(n), \quad \psi \in \mathcal{S}(\mathbb{Z}), \tag{3}$$

and we have $T_f \in \mathcal{S}'(\mathbb{Z},X) = \{T : \mathcal{S}(\mathbb{Z}) \to X \ : \ T \text{ is linear and continuous}\}$.

Remark 2.1. By this mapping we identify $\ell_p(\mathbb{Z};X)$ with a subspace of $\mathcal{S}'(\mathbb{Z};X)$. When convenient and confusion seems unlikely, a function $f \in \ell_p(\mathbb{Z};X)$ is identified with $T_f \in \mathcal{S}'(\mathbb{Z},X)$.

There also exists a natural mapping that identifies $C^\infty_{per}(\mathbb{T};X)$ with a subspace of $\mathcal{D}'(\mathbb{T};X) = \{T : C^\infty_{per}(\mathbb{T}) \to X \ : \ T \text{ is linear and continuous}\}$ which assigns to each $S \in C^\infty_{per}(\mathbb{T};X)$ the linear map

$$L_S(\varphi) := \langle L_S, \varphi\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(t)S(t)dt, \quad \varphi \in C^\infty_{per}(\mathbb{T}),$$

and we have $L_S \in \mathcal{D}'(\mathbb{T};X)$.

Definition 2.2. The discrete time Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{Z};X) \to C^\infty_{per}(\mathbb{T};X)$ is defined by

$$\mathcal{F}\varphi(t) \equiv \widehat{\varphi}(t) := \sum_{j=-\infty}^{\infty} e^{-ijt}\varphi(j), \qquad t \in (-\pi,\pi]$$

and the corresponding inverse transform is given by

$$\mathcal{F}^{-1}\varphi(n) \equiv \check{\varphi}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(t)e^{int}dt, \quad n \in \mathbb{Z}, \tag{4}$$

where $\varphi \in C^\infty_{per}(\mathbb{T};X)$.

This isomorphism, allows us to define the discrete time Fourier transform (DTFT) between the spaces of distributions $\mathcal{S}'(\mathbb{Z};X)$ and $\mathcal{D}'(\mathbb{T};X)$ as follows:

$$\langle \mathcal{F}T, \psi\rangle \equiv \mathcal{F}(T)(\psi) := \widehat{T}(\psi) \equiv \langle T, \check{\psi}\rangle, \quad T \in \mathcal{S}'(\mathbb{Z};X), \quad \psi \in C^\infty_{per}(\mathbb{T}), \tag{5}$$

whose inverse $\mathcal{F}^{-1} : \mathcal{D}'(\mathbb{T};X) \to \mathcal{S}'(\mathbb{Z};X)$ is given by

$$\langle \mathcal{F}^{-1}L, \psi\rangle \equiv \mathcal{F}^{-1}(L)(\psi) := \check{L}(\psi) \equiv \langle L, \widehat{\psi}\rangle, \quad L \in \mathcal{D}'(\mathbb{T};X), \quad \psi \in \mathcal{S}(\mathbb{Z}).$$

We finally present a technical lemma introduced in [22] which will be necessary througout the paper. We first need the following definition.

Definition 2.3. Given $u \in \ell_p(\mathbb{Z}; X)$ and $v \in \ell_1(\mathbb{Z})$ the convolution product between u and v is defined as

$$(u * v)(n) := \sum_{j=-\infty}^{n} u(n-j)v(j) = \sum_{j=0}^{\infty} u(j)v(n-j), \quad n \in \mathbb{Z}.$$

Moreover, the convolution of a distribution $T \in \mathcal{S}'(\mathbb{Z}, X)$ with a function $a \in \ell_1(\mathbb{Z}_+)$ is defined by

$$\langle T * a, \varphi \rangle := \langle T, a \circ \varphi \rangle, \quad \varphi \in \mathcal{S}(\mathbb{Z}), \tag{6}$$

where

$$(a \circ \varphi)(n) := \sum_{j=0}^{\infty} a(j)\varphi(j+n).$$

Lemma 2.4. *Let $u, v \in \ell_p(\mathbb{Z}; X)$ be given and $a \in \ell_1(\mathbb{Z}_+)$ which is defined by 0 for negative values of n. The following assertions are equivalent:*

(i) $a * v \in \ell_p(\mathbb{Z}, X)$ and $(a * v)(n) = u(n)$ for all $n \in \mathbb{Z}$.

(ii) $\langle u, \check{\varphi} \rangle = \langle v, \check{(\varphi \cdot \widehat{a}_-)} \rangle$ for all $\varphi \in C^\infty_{per}(\mathbb{T})$,

where

$$\check{(\varphi \cdot \widehat{a}_-)}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{a}(-t)\varphi(t)e^{int}dt, \quad n \in \mathbb{Z}.$$

We recall the notion of R-bounded sets and ℓ_p-multipliers in the space $\mathcal{B}(X, Y)$ of bounded linear operators from X into Y endowed with the uniform operator topology.

Definition 2.5. Let X and Y be Banach spaces. A subset $\mathcal{T}$ of $\mathcal{B}(X, Y)$ is called R-bounded if there is a constant $c > 0$ such that

$$\|(T_1x_1, ..., T_nx_n)\|_R \leq c\|(x_1, ..., x_n)\|_R, \tag{7}$$

for all $T_1, ..., T_n \in \mathcal{T}$, $x_1, ..., x_n \in X$, $n \in \mathbb{N}$, where

$$\|(x_1, ..., x_n)\|_R := \frac{1}{2^n} \sum_{\epsilon_j \in \{-1,1\}^n} \Big\| \sum_{j=1}^{n} \epsilon_j x_j \Big\|,$$

for $x_1, ..., x_n \in X$.

For more information about R-bounded sets and their properties see [1, Section 2.2] and [8]. We next recall the following notion.

Definition 2.6. [22] Let X, Y be Banach spaces, $1 < p < \infty$. A function $M \in C^\infty_{per}(\mathbb{T}, \mathcal{B}(X, Y))$ is an ℓ_p-multiplier (from X to Y) if there exists a bounded operator $T : \ell_p(\mathbb{Z}; X) \to \ell_p(\mathbb{Z}; Y)$ such that

$$\sum_{n\in\mathbb{Z}} (Tf)(n)\check{\varphi}(n) = \sum_{n\in\mathbb{Z}} \check{(\varphi \cdot M_-)}(n)f(n) \tag{8}$$

for all $f \in \ell_p(\mathbb{Z}; X)$ and all $\varphi \in C^\infty_{per}(\mathbb{T})$. Here

$$\check{(\varphi \cdot M_-)}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{int}\varphi(t)M(-t)dt, \quad n \in \mathbb{Z}.$$

We now recall the following Fourier multiplier theorem for operator-valued symbols given by S. Blunck [5, 1]. This theorem provides sufficient conditions to ensure when an operator-valued symbol is a multiplier, and allows to establish an equivalence between ℓ_p-multipliers and the notion of R- boundedness for the UMD class of Banach spaces. For more information about these spaces see [3, Section III.4.3-III.4.5].

Theorem 2.7. [5, Theorem 1.3] *and* [22] *Let $p \in (1,\infty)$ and let X, Y be UMD spaces. Let $M \in C^{\infty}_{per}(\mathbb{T}_0, \mathcal{B}(X;Y))$ such that the sets*

$$\{M(t) : t \in \mathbb{T}_0\} \quad \text{and} \quad \left\{(1-e^{it})(1+e^{it})M'(t) : \ t \in \mathbb{T}_0\right\},$$

are both R-bounded. Then M is an ℓ_p-multiplier (from X to Y) for $1 < p < \infty$.

The converse of Blunck's theorem also holds without any restriction on the Banach spaces X, Y as follows:

Theorem 2.8. [5, Proposition 1.4] *Let $p \in (1,\infty)$ and let X, Y be Banach spaces. Let $M : \mathbb{T} \to \mathcal{B}(X;Y)$ be an operator valued function. Suppose that there is a bounded operator $T_M : l_p(\mathbb{Z};X) \to l_p(\mathbb{Z};Y)$ such that (8) holds. Then the set*

$$\{M(t) : t \in \mathbb{T}\}$$

is R-bounded.

We now provide some notions concerning sectorial operators. Let $\Sigma_\phi \subset \mathbb{C}$ denote the open sector

$$\Sigma_\phi = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \phi\}, \quad 0 < \phi \leq \pi.$$

We denote by

$$\mathcal{H}(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic}\}.$$

and

$$\mathcal{H}^\infty(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic and bounded}\}.$$

$\mathcal{H}^\infty(\Sigma_\phi)$ is equipped with the norm

$$||f||^\phi_\infty = \sup_{|\arg \lambda| < \phi} |f(\lambda)|.$$

We further define the subspace $\mathcal{H}_0(\Sigma_\phi)$ of $\mathcal{H}(\Sigma_\phi)$ as follows

$$\mathcal{H}_0(\Sigma_\phi) = \bigcup_{\alpha,\beta<0} \{f \in \mathcal{H}(\Sigma_\phi) : ||f||^\phi_{\alpha,\beta} < \infty\},$$

where

$$||f||^\phi_{\alpha,\beta} = \sup_{|\lambda| \leq 1} |\lambda^\alpha f(\lambda)| + \sup_{|\lambda| \geq 1} |\lambda^{-\beta} f(\lambda)|.$$

Definition 2.9. A closed linear operator A in X is called sectorial if the following conditions hold:
(i) $\overline{D(A)} = X, \overline{R(A)} = X, (-\infty, 0) \subset \rho(A)$;
(ii)$||t(t+A)^{-1}|| \leq M$ for all $t > 0$ and some $M > 0$.

A is called R-sectorial if the set $\{t(t+A)^{-1}\}_{t>0}$ is R-bounded.

The class of sectorial (resp. R-sectorial) operators in X will be denoted by $\mathcal{S}(X)$ (resp. $\mathcal{RS}(X)$. Set $\mathcal{R}_A(\phi) = R(\lambda(\lambda + A)^{-1} : |\arg\lambda| \leq \phi\}$. If $A \in \mathcal{S}(X)$ then $\Sigma_\phi \subset \rho(-A)$ for some $\phi > 0$ and

$$\sup_{|\arg\lambda|<\phi} ||\lambda(\lambda + A)^{-1}|| < \infty.$$

We denote the *spectral angle* of $A \in \mathcal{S}(X)$ by

$$\phi_A = \inf\{\phi : \Sigma_{\pi-\phi} \subset \rho(-A), \quad \sup_{\lambda\in\Sigma_{\pi-\phi}} ||\lambda(\lambda + A)^{-1}|| < \infty\}.$$

Definition 2.10. A sectorial operator A is said to admit a bounded $\mathcal{H}^\infty-$calculus if there are $\phi > \phi_A$ and a constant $K_\phi > 0$ such that

$$||f(A)|| \leq K_\phi ||f||^\phi_\infty \text{ for all } f \in \mathcal{H}_0(\Sigma_\phi). \tag{9}$$

The class of sectorial operators A which admit a bounded $\mathcal{H}^\infty-$calculus is denoted by $\mathcal{H}^\infty(X)$. Moreover, the $\mathcal{H}^\infty-$ angle is defined by

$$\phi^\infty_A = \inf\{\phi > \phi_A : (9) \text{ holds } \}$$

When $A \in \mathcal{H}^\infty(X)$ we say that A *admits an R-bounded $\mathcal{H}^\infty-$calculus* if the set

$$\{h(A) : h \in \mathcal{H}^\infty(\Sigma_\theta), ||f||^\theta_\infty \leq 1\}$$

is R-bounded for some $\theta > 0$. We denote the class of such operators by $\mathcal{RH}^\infty(X)$. The corresponding angle is defined in an obvious way and denoted by $\theta^{R_\infty}_A$.

Remark 2.11. If A is a sectorial operator on a Hilbert space, Lebesgue spaces $L_p(\Omega)$, $1 < p < \infty$, Sobolev spaces $W^{s,p}(\Omega)$, $1 < p < \infty$, $s \in \mathbb{R}$ or Besov spaces $B^s_{p,q}(\Omega), 1 < p, q < \infty$, $s \in \mathbb{R}$ and A admits a bounded $\mathcal{H}^\infty$ calculus of angle β, then A already admits and $\mathcal{RH}^\infty$ calculus on the same angle β on each of the above described spaces (see Kalton and Weis [12]). More generally, this property is true whenever X is a UMD space with the so called property (α) (see [12]).

Example 2.12. *Well known examples for general classes of closed linear operators with a bounded $\mathcal{H}^\infty$ calculus are: normal sectorial operators in a Hilbert space; m-accretive operators in a Hilbert space; generators of bounded C_0-groups on L_p-spaces and negative generators of positive contraction semigroups on L_p-spaces.*

The following result will be necessary for establishing $\ell_p - \ell_q$ estimates in section 4. It can be found in [8, Proposition 4.10].

Proposition 2.13. *Let $A \in \mathcal{RH}^\infty(X)$ and suppose that $\{h_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{H}^\infty(\Sigma_\theta)$ is uniformly bounded for some $\theta > \theta^{R_\infty}_A$, where Λ is an arbitrary index set. Then the set $\{h_\lambda(A)\}_{\lambda\in\Lambda}$ is R-bounded.*

3. Abstract setting: A characterization of maximal ℓ_p-regularity

Let $\beta \in \mathbb{R}$, $\tau_j \in \mathbb{Z}$, $b \in \ell_1(\mathbb{Z})$ and X be a Banach space. For a given vector-valued sequence $f : \mathbb{Z} \to X$ we consider the abstract discrete equation

$$u(n) = \sum_{j=-\infty}^{n} b(n-j)Au(j) + \sum_{j=1}^{k} \beta_j u(n-\tau_j) + f(n), \qquad n \in \mathbb{Z}, \tag{10}$$

where A is a closed linear operator with domain $D(A)$ defined in a Banach space X. Recall that by $[D(A)]$ we denote the domain of A endowed with the graph norm.

Definition 3.1. Let $1 < p < \infty$ be given. We say that equation (10) has maximal ℓ_p-regularity if for each $f \in \ell_p(\mathbb{Z}; X)$ there exists a unique solution $u \in \ell_p(\mathbb{Z}; [D(A)])$ of (10).

In this section, our purpose is to provide a characterization of maximal ℓ_p-regularity of equation (10). For the sake of simplicity, we will first obtain this characterization for the following equation

$$u(n) = \sum_{j=-\infty}^{n} b(n-j)Au(j) + \beta u(n-\tau) + f(n), \qquad n \in \mathbb{Z}. \tag{11}$$

As a corollary, we will have a full characterization of maximal ℓ_p-regularity for the more general equation (10).

We first introduce the following definition. Observe that, in some sense, it corresponds to the discrete counterpart of the notion of k-regularity introduced in the paper [15]. See also [28].

Definition 3.2. Let $k \in \mathbb{N}_0$ be given. A sequence $b \in \ell_1(\mathbb{Z})$ is called k-regular if there exists a constant $c > 0$ such that $|((1+e^{it})(1-e^{it}))^n[\hat{b}(t)]^{(n)}| \leq c|\hat{b}(t)|$ for all $1 \leq n \leq k$ and all $t \in \mathbb{T}_0$.

Remark 3.3. A simple example of a k-regular sequence is given by $b(n) = \frac{1}{2^n}$, $n \in \mathbb{N}_0$ and 0 otherwise. The 1-regularity follows easily since $\hat{b}(t) = 2(2 - e^{-it})^{-1}$ and $\left|(1+e^{it})(1-e^{it})\frac{[\hat{b}(t)]'}{\hat{b}(t)}\right| = \left|\frac{i(e^{2it}-1)}{(2e^{it}-1)}\right| \leq 2$. The case $k > 1$ follows analogously.

Let $\tau \in \mathbb{Z}$ be given. In what follows we denote by $\delta_\tau : \mathbb{Z} \to \mathbb{R}$ the sequence defined by

$$\delta_\tau(n) = \begin{cases} 1 & n = \tau, \\ \\ 0 & otherwise. \end{cases}$$

We are now ready to prove our main theorem.

Theorem 3.4. *Let X be a UMD space, $1 < p < \infty$, $\beta \in \mathbb{R}$, $b \in \ell_1(\mathbb{Z})$ such that $b(n) = 0$ for all $n \in \mathbb{Z}_-$ and $\tau \in \mathbb{Z}$. Suppose that b is 1-regular, $\hat{b}(t) \neq 0$ for all $t \in \mathbb{T}$ and*

$$\left\{ \frac{(1 - \beta e^{-it\tau})}{\hat{b}(t)} \right\}_{t \in \mathbb{T}} \subset \rho(A),$$

The following assertions are equivalent:

(i) *Equation (11) has maximal ℓ_p-regularity;*
(ii) *$M(t) := (1 - \beta e^{-it\tau} - \hat{b}(t)A)^{-1}$ is an ℓ_p-multiplier from X to $[D(A)]$;*
(iii) *The set $\{M(t) : t \in \mathbb{T}\}$ is R-bounded.*

In addition, if any of the hypothesis holds true, then $u, b * Au \in \ell_p(\mathbb{Z}; X)$ *and there exists a constant* $C > 0$ *(independent of* $f \in \ell_p(\mathbb{Z}; X)$*) such that*

$$\|u\|_{\ell_p(\mathbb{Z};X)} + \|b * Au\|_{\ell_p(\mathbb{Z};X)} \leq C\|f\|_{\ell_p(\mathbb{Z};X)}. \tag{12}$$

Proof. We first show (i) implies (ii). Let $f \in \ell_p(\mathbb{Z}; X)$ be given. By hypothesis there exists a unique sequence $u : \mathbb{Z} \to [D(A)]$ such that $u \in \ell_p(\mathbb{Z}; [D(A)])$ satisfies:

$$u(n) = \sum_{j=-\infty}^{n} b(n-j)Au(j) + \beta u(n-\tau) + f(n), \qquad n \in \mathbb{Z}. \tag{13}$$

Let $T_\alpha : \ell_p(\mathbb{Z}; X) \to \ell_p(\mathbb{Z}; [D(A)])$ be defined by $T_\alpha(f) = u$. It can be easily shown using the closed graph theorem that T_α is bounded. Since $b \in \ell_1(\mathbb{Z})$, we obtain the following identities:

$$\begin{aligned}
(b \circ \check{S})(n) &= \sum_{j=0}^{\infty} b(j)\check{S}(j+n) = \sum_{j=0}^{\infty} b(j)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n+j)t}S(t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\Big(\sum_{j=0}^{\infty} e^{ijt}b(j)\Big)S(t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\widehat{b}(-t)S(t)dt =: \check{(\widehat{b}_- \cdot S)}(n), \qquad (14)
\end{aligned}$$

valid for any $S \in C^\infty_{per}(\mathbb{T}, \mathcal{B}(X, Y))$. Therefore, using the hypothesis, the fact that $M \in C^\infty_{per}(\mathbb{T}, \mathcal{B}(X, [D(A)])$, and the identity $I = M(-t) - \beta e^{it\tau}M(-t) - b(-t)AM(-t)$ we get

$$\begin{aligned}
\langle T_\alpha f, \check{\varphi}\rangle = \langle u, \check{\varphi}\rangle &= \sum_{n\in\mathbb{Z}} \check{\varphi}(n)u(n) = \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\varphi(t)u(n)dt \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\varphi(t)(1 - \beta e^{it\tau} - \hat{b}(-t)A)^{-1}u(n)dt \\
&- \beta\sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{it\tau}(1 - \beta e^{it\tau} - \hat{b}(-t)A)^{-1}u(n)e^{int}\varphi(t)dt \\
&- \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} (1 - \beta e^{it\tau} - \hat{b}(-t)A)^{-1}\hat{b}(-t)Au(n)e^{int}\varphi(t)dt \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\varphi(t)M(-t)u(n)dt \\
&- \beta\sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\widehat{\delta}_\tau(t)\varphi(t)M(-t)u(n)dt \\
&- \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\varphi(t)M(-t)\hat{b}(-t)Au(n)dt \\
&= \langle u, \check{(\varphi \cdot M_-)}\rangle - \beta\langle u, \check{(\widehat{\delta_\tau}_- \cdot \varphi \cdot M_-)}\rangle - \langle Au, \check{(\hat{b}_- \cdot \varphi \cdot M_-)}\rangle \\
&= \langle u, \check{(\varphi \cdot M_-)}\rangle - \beta\langle u, \delta_\tau \circ \check{(\varphi \cdot M_-)}\rangle - \langle Au, b \circ \check{(\varphi \cdot M_-)}\rangle,
\end{aligned}$$

for all $\varphi \in C^\infty_{per}(\mathbb{T})$, where $\widehat{\delta_\tau}(t) = e^{-i\tau t}$, and in the last equality we have used (14) with $S = \varphi \cdot M_-$. Therefore using (6) we get

$$\begin{aligned}\langle u, \check{\varphi}\rangle &= \langle u, (\varphi \cdot M_-)\check{}\rangle - \beta\langle \delta_\tau * u, (\varphi \cdot M_-)\check{}\rangle - \langle b * Au, (\varphi \cdot M_-)\check{}\rangle \\ &= \langle u - \beta u_\tau - b * Au, (\varphi \cdot M_-)\check{}\rangle\end{aligned} \tag{15}$$

where $(\delta_\tau * u)(n) = u(n-\tau) := u_\tau(n)$. We conclude that $\langle T_\alpha f, \check{\varphi}\rangle = \langle f, (\varphi \cdot M_-)\check{}\rangle$, for all $\varphi \in C^\infty_{per}(\mathbb{T})$ and $f \in \ell_p(\mathbb{Z}; X)$. This proves the claim and the theorem.

We now show that $(iii) \implies (ii)$ We claim that, by hypothesis, the set $\{(e^{it} - 1)(e^{it} + 1)M'(t)\}_{t\in\mathbb{T}}$ is R-bounded. Indeed, given $t \in \mathbb{T}$ and observing that $M(t) = \frac{1}{\hat{b}(t)}[\frac{(1-\beta e^{-it\tau})}{\hat{b}(t)} - A]^{-1}$ we obtain that

$$M'(t) = -M(t)\frac{[\hat{b}(t)]'}{\hat{b}(t)} - M(t)^2 i\tau\beta e^{-it\tau} + M(t)^2\frac{[\hat{b}(t)]'}{\hat{b}(t)}(1 - \beta e^{-it\tau}) \quad t \in \mathbb{T}. \tag{16}$$

Therefore,

$$\begin{aligned}(1 - e^{it})(1 + e^{it})M'(t) = &-(1 - e^{it})(1 + e^{it})M(t)\frac{[\hat{b}(t)]'}{\hat{b}(t)} - (1 - e^{it})(1 + e^{it})M(t)^2 i\tau\beta e^{-it\tau} \\ &+ (1 - e^{it})(1 + e^{it})M(t)^2\frac{[\hat{b}(t)]'}{\hat{b}(t)}(1 - \beta e^{-it\tau}), \quad t \in \mathbb{T}.\end{aligned}$$

From [1, Proposition 2.2.5] and the 1-regularity of b we conclude that the set $\{(1 - e^{it})(1 + e^{it})M'(t) : t \in \mathbb{T}\}$ is R- bounded and the claim is proven. Finally from Theorem 2.7 we obtain (ii).

It is clear that (ii) implies (iii) follows directly from Theorem 2.8.

It only remains to show that (ii) implies (i). We first claim that $N(t) := \hat{b}(t)(1 - \beta e^{-it\tau} - \hat{b}(t)A)^{-1}$ and $S(t) := e^{-it\tau}(1 - \beta e^{-it\tau} - \hat{b}(t)A)^{-1}$ are ℓ_p-multipliers. Indeed, since $b \in \ell_1(\mathbb{Z})$ we have that $N(t) = \hat{b}(t)M(t)$ and $S(t) = e^{-it\tau}M(t)$ are R-bounded sets. On the other hand, the R-boundedness of the sets $\{N(t)\}_{t\in\mathbb{T}}$ and $\{S(t)\}_{t\in\mathbb{T}}$, the boundedness of $\hat{b}(t)$ (which follows from the fact that $b \in \ell_1(\mathbb{Z})$), the 1-regularity of b and the identities:

$$\begin{aligned}(1 - e^{it})(1 + e^{it})N'(t) &= (1 - e^{it})(1 + e^{it})\frac{[\hat{b}(t)]'}{\hat{b}(t)}\hat{b}(t)M(t) + (1 - e^{it})(1 + e^{it})M'(t)\hat{b}(t), \\ (1 - e^{it})(1 + e^{it})S'(t) &= -i\tau e^{-it\tau}M(t)(1 - e^{it})(1 + e^{it}) + e^{-it\tau}M'(t)(1 + e^{it})(1 - e^{it}).\end{aligned}$$

show that the sets $\{(1-e^{it})(1+e^{it})N'(t) : t \in \mathbb{T}\}$ and $\{(1-e^{it})(1+e^{it})S'(t) : t \in \mathbb{T}\}$ are also R-bounded and then the claim holds by Theorem 2.7. Let $f \in \ell_p(\mathbb{Z}; X)$ be given. By hypothesis, there exists $u \in \ell_p(\mathbb{Z}; [D(A)])$ such that

$$\sum_{n\in\mathbb{Z}} u(n)\check{\varphi}(n) = \sum_{n\in\mathbb{Z}} (\varphi \cdot M_-)\check{}(n)f(n), \tag{17}$$

for all $\varphi \in C^\infty_{per}(\mathbb{T})$. Define $N(t) = \hat{b}(t)M(t)$ and $S(t) = \widehat{\delta_\tau}(t)M(t)$. Moreover, there exist $v, w \in \ell_p(\mathbb{Z}; [D(A)])$ such that

$$\sum_{n\in\mathbb{Z}} v(n)\check{\psi}(n) = \sum_{n\in\mathbb{Z}} (\psi \cdot N_-\check{)}(n) f(n), \tag{18}$$

and

$$\sum_{n\in\mathbb{Z}} w(n)\check{\eta}(n) = \sum_{n\in\mathbb{Z}} (\eta \cdot S_-\check{)}(n) f(n), \tag{19}$$

for all $\psi, \eta \in C^\infty_{per}(\mathbb{T})$ where

$$(\psi \cdot N_-\check{)}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\psi(t)\widehat{b}(-t)M(-t)dt, \tag{20}$$

and

$$(\eta \cdot S_-\check{)}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\eta(t)\widehat{\delta}_\tau(-t)M(-t)dt. \tag{21}$$

Observe that by hypothesis $\varphi(t) = \psi(t)\widehat{b}(-t) \in C^\infty_{per}(\mathbb{T})$. Setting φ in (17), from (18) and (20) we get

$$\langle u, (\psi \cdot \widehat{b}_-\check{)}\rangle = \langle v, \check{\psi}\rangle.$$

From Lemma 2.4 we conclude from the above identity that

$$v(n) = (b * u)(n), \quad n \in \mathbb{Z}. \tag{22}$$

Now, considering $\varphi(t) = \eta(t)\widehat{\delta}_\tau(-t)$ in (19) we obtain from (19) and (21) the identity $\langle w, \check{\eta}\rangle = \langle u, (\eta \cdot \widehat{\delta_\tau}_-)\check{\rangle}$. Observe that $\varphi \in C^\infty_{per}(\mathbb{T})$ because $\tau \in \mathbb{Z}$. Again, making use of Lemma 2.4, we conclude from the above identity that

$$w(n) = (\delta_\tau * u)(n) = u(n-\tau), \quad n \in \mathbb{Z}. \tag{23}$$

Since $AN(t) + \beta S(t) = M(t) - I$, after multiplication by $e^{int}\varphi(t)$ and integration over the interval $(-\pi, \pi)$, we have

$$A(\varphi \cdot N_-\check{)}(n) + \beta(\varphi \cdot S_-\check{)}(n) = (\varphi \cdot M_-\check{)}(n) - \check{\varphi}I,$$

for all $\varphi \in C^\infty_{per}(\mathbb{T})$. Then we obtain

$$\langle f, A(\varphi \cdot N_-\check{)}\rangle + \beta\langle f, (\varphi \cdot S_-\check{)}\rangle = \langle f, (\varphi \cdot M_-\check{)}\rangle - \langle f, \check{\varphi}\rangle,$$

and by replacing (17), (18) and (19) in the above identity we obtain

$$\sum_{n\in\mathbb{Z}} Av(n)\check{\varphi}(n) + \beta\sum_{n\in\mathbb{Z}} w(n)\check{\varphi}(n) = \sum_{n\in\mathbb{Z}} u(n)\check{\varphi}(n) - \sum_{n\in\mathbb{Z}} \check{\varphi}(n)f(n),$$

for all $\varphi \in C^\infty_{per}(\mathbb{T})$. Considering (22) and (23) and taking into account that A is a closed linear operator we conclude that u satisfies the equation (11). We have proven the existence of a solution. It remains to prove the uniqueness.
Let $u : \mathbb{Z} \to X$ be a solution of (11) with $f \equiv 0$. For all $\varphi \in C^\infty_{per}(\mathbb{T})$ and (15), we obtain

$$\langle u, \check{\varphi}\rangle = \langle u - b * Au - \beta u_\tau, (\varphi \cdot M_-\check{)}\rangle = 0.$$

Taking $\varphi_k(t) := e^{-ikt}$, $k \in \mathbb{Z}$ we obtain $u \equiv 0$ and then the claim is proven. Finally, the last assertion is a direct consequence of the closed graph theorem.

□

Remark 3.5. Observe that only (ii) implies (i) requires that $b(n) = 0$ for all $n \in \mathbb{Z}_-$ meanwhile the other ones remain true for any $b \in \ell_1(\mathbb{Z})$.

As a direct consequence of the fact that R-boundedness is equivalent to boundedness in Hilbert spaces we obtain the following corollary.

Corollary 3.6. *If the space X in Theorem 3.4 is a Hilbert space condition (iii) can be replaced by*

$$\sup_{t\in\mathbb{T}} \left\| (1 - \beta e^{-it\tau} - \hat{b}(t)A)^{-1} \right\| < \infty.$$

As a corollary of Theorem 3.4, we immediately obtain the following result that shows maximal ℓ_p-regularity when equation (11) has a finite number of delays.

Theorem 3.7. *Let X be a UMD space, $1 < p < \infty$, $\beta \in \mathbb{R}$, $b \in \ell_1(\mathbb{Z})$ such that $b(n) = 0$ for all $n \in \mathbb{Z}_-$ and $\tau_j \in \mathbb{Z}$. Suppose that b is 1-regular, $\hat{b}(t) \neq 0$ for all $t \in \mathbb{T}$ and*

$$\left\{ \frac{1 - \sum_{j=1}^{k} \beta_j e^{-it\tau_j}}{\hat{b}(t)} \right\}_{t\in\mathbb{T}} \subset \rho(A).$$

The following assertions are equivalent:

(i) *Equation*

$$u(n) = \sum_{j=-\infty}^{n} b(n-j)Au(j) + \sum_{j=1}^{k} \beta_j u(n-\tau_j) + f(n), \qquad n \in \mathbb{Z},$$

has maximal ℓ_p-regularity;

(ii) $M(t) := (1 - \sum_{j=1}^{k} \beta_j e^{-it\tau_j} - \hat{b}(t)A)^{-1}$ *is an ℓ_p-multiplier from X to $[D(A)]$;*

(iii) *The set $\{M(t) : t \in \mathbb{T}\}$ is R-bounded.*

In addition, if any of the hypothesis holds true, then $u, b * Au \in \ell_p(\mathbb{Z}; X)$ and there exists a constant $C > 0$ (independent of $f \in \ell_p(\mathbb{Z}; X)$) such that

$$\|u\|_{\ell_p(\mathbb{Z};X)} + \|b * Au\|_{\ell_p(\mathbb{Z};X)} \leq C\|f\|_{\ell_p(\mathbb{Z};X)}. \tag{24}$$

4. Applications

In order to analyze maximal ℓ_p-regularity of some concrete models such as the discrete wave and Kuznetsov equations we first state the following abstract result concerning $\ell_p - \ell_q$ estimates. It is enough to take $X = \ell_q(\mathbb{Z}^N)$ in Theorem 3.4; then, for any closed linear operator $A_q : D(A_q) \subset \ell_q(\mathbb{Z}^N) \to \ell_q(\mathbb{Z}^N)$ we obtain the following theorem.

Theorem 4.1. *Let $1 < p, q < \infty$, $\beta \in \mathbb{R}$, $b \in \ell_1(\mathbb{Z})$ such that $b(n) = 0$ for all $n \in \mathbb{Z}_-$ and $\tau \in \mathbb{Z}$. Suppose that b is 1-regular, $\hat{b}(t) \neq 0$ for all $t \in \mathbb{T}$,*

$$\left\{ \frac{(1 - \beta e^{-it\tau})}{\hat{b}(t)} \right\}_{t\in\mathbb{T}} \subset \rho(A_q),$$

and that the set $\{(1-\beta e^{-it\tau}-\hat{b}(t)A_q)^{-1} \ : \ t\in\mathbb{T}\}$ *is R-bounded. Then, for any* $f\in\ell_p(\mathbb{Z},\ell_q(\mathbb{Z}^N))$ *there exists a unique solution* $u\in\ell_p(\mathbb{Z};[D(A_q)])$ *of the equation*

(25)
$$u(n,\mathbf{m})=\sum_{j=-\infty}^{n} b(n-j)A_q u(j,\mathbf{m})+\beta u(n-\tau,\mathbf{m})+f(n,\mathbf{m}),\qquad n\in\mathbb{Z},\quad \mathbf{m}\in\mathbb{Z}^N,$$

such that u, $b*A_qu\in\ell_p(\mathbb{Z};\ell_q(\mathbb{Z}^N))$ *and the following* $\ell_p-\ell_q$ *estimate holds:*

$$\Big(\sum_{n\in\mathbb{Z}}\|u(n)\|^p_{\ell_q(\mathbb{Z}^N)}\Big)^{1/p}+\Big(\sum_{n\in\mathbb{Z}}\|(b*A_qu)(n)\|^p_{\ell_q(\mathbb{Z}^N)}\Big)^{1/p}\leq C\Big(\sum_{n\in\mathbb{Z}}\|f(n)\|^p_{\ell_q(\mathbb{Z}^N)}\Big)^{1/p},\tag{26}$$

where $C>0$ *is a constant independent of* $f\in\ell_p(\mathbb{Z},\ell_q(\mathbb{Z}^N))$.

4.1. **The discrete wave equation.** We first introduce the following notation. Let $\mathbf{n}=(n_1,\dots,n_N)\in\mathbb{Z}^N$. We are going to consider the *multidimensional* discrete Laplacian $\Delta_{d,N}$, defined as

$$\Delta_{d,N}\varphi(\mathbf{n})=\sum_{j=1}^{N}\big(\varphi(\mathbf{n}+e_j)-2\varphi(\mathbf{n})+\varphi(\mathbf{n}-e_j)\big),\tag{27}$$

where e_j denotes the unit vector in the positive direction of the j-th coordinate. The operator $\Delta_{d,N}$ maps $\ell_q(\mathbb{Z}^N)$ into itself boundedly for all $1\leq q\leq\infty$ and

$$\sigma(\Delta_{d,N})=\Big\{-\sum_{j=1}^{N}4\sin^2(\theta_j/2)\Big\}_{\boldsymbol{\theta}\in(-\pi,\pi]^N},$$

see [23] for more information.

Let us consider the following time discrete version of the wave equation with the discrete Laplacian

$$\boldsymbol{\Delta}^{\mathbf{2}}_{\mathbf{r}}u(n,\mathbf{m})=\Delta_{d,N}u(n,\mathbf{m})+f(n,\mathbf{m}),\quad n\in\mathbb{Z},\quad \mathbf{m}\in\mathbb{Z}^N,\quad \rho>0,\tag{28}$$

where $r>0$ and $\boldsymbol{\Delta}_{\mathbf{r}}$ is two-times the forward r-difference operator acting on the first variable defined by $\boldsymbol{\Delta}_{\mathbf{r}}f(n):=f(n+1)-rf(n)$. In order words, it corresponds to the discretization in time of the second order differential operator ∂_{tt}. Rewriting (28) as

$$u(n+2,\mathbf{m})-2ru(n+1,\mathbf{m})+r^2u(n,\mathbf{m})=\Delta_{d,N}u(n,\mathbf{m})+f(n,\mathbf{m})$$

we observe that this equation fits in the abstract model (25) choosing $A_q=-\Delta_{d,N}$, $\tau_1=-1,\tau_2=-2,\beta_1=2r,\beta_2=-1$ and $b(n)=-\delta_0(n)$. Moreover, if we assume the condition

$$r>1+\sqrt{2}\tag{29}$$

then we obtain

$$\big\{r^2-2re^{it}+e^{2it}\big\}_{t\in\mathbb{T}}\subset\rho(\Delta_{d,N}).$$

In fact, note that $r>1+\sqrt{2}>1-\sqrt{2}$ and $\sigma(\Delta_{d,N})=[-4,0]$ implies $\Re(r^2-2re^{it}+e^{2it})=r^2-2r\cos t+\cos 2t>r^2-2r-1=(r-(1-\sqrt{2})(r-(1+\sqrt{2}))>0$, proving the claim.

We define the function

$$h_t(z):=(r^2-2re^{it}+e^{2it}+z)^{-1},\quad t\in(-\pi,\pi),\quad z\in\Sigma_{\pi/2},$$

then for any $z \in \Sigma_{\pi/2}$ and $t \in (-\pi, \pi)$ we have

$$|h_t(z)| = |\int_0^\infty e^{-(r^2-2re^{it}+e^{2it}+z)s} ds| \leq \frac{1}{r^2 - 2r\cos t + \cos 2t + \Re(z)} < \frac{1}{r^2-2r-1}.$$

Then, Proposition 2.13 shows that the set $\{(r^2-2re^{it}+e^{2it}-\Delta_{d,N})^{-1}\}_{t\in\mathbb{T}}$ is R-bounded. We conclude from Theorem 3.7 the following result.

Theorem 4.2. *Let $f \in \ell_p(\mathbb{Z}; \ell_q(\mathbb{Z}^N))$, $1 < p < \infty$ be given and suppose that*

$$r > 1 + \sqrt{2}.$$

Then the numerical solution $(u(n, \mathbf{m}))_{n\in\mathbb{Z},\, \mathbf{m}\in\mathbb{Z}^N}$ of (28), obtained by the forward Euler r-method exists, belongs to $u \in \ell_p(\mathbb{Z}; \ell_q(\mathbb{Z}^N))$ and satisfies the discrete maximal $\ell_p - \ell_q$ regularity estimate

$$\Big(\sum_{n\in\mathbb{Z}} \|u(n)\|^p_{\ell_q(\mathbb{Z}^N)}\Big)^{1/p} + \Big(\sum_{n\in\mathbb{Z}} \|(\Delta_{d,N}u)(n)\|^p_{\ell_q(\mathbb{Z}^N)}\Big)^{1/p} \leq C\Big(\sum_{n\in\mathbb{Z}} \|f(n)\|^p_{\ell_q(\mathbb{Z}^N)}\Big)^{1/p}, \tag{30}$$

where the constant $C > 0$ is independent of f.

4.2. **The discrete Kuznetsov equation.** We consider the linearized Kuznetsov equation:

$$u_{tt} - c^2\Delta u - \nu\epsilon\Delta u_t = f(t), \qquad t \in \mathbb{R},$$

where Δ is the Laplacian operator in $\mathbb{R}^N$. The discrete version reads as

$$\nabla^{\mathbf{2}}_{\mathbf{r}} u(n, \mathbf{m}) = c^2\Delta_{d,N}u(n, \mathbf{m}) + \nu\epsilon\Delta_{d,N}\nabla_{\mathbf{r}}u(n, \mathbf{m}) + f(n, \mathbf{m}), \quad n \in \mathbb{Z}, \quad \mathbf{m} \in \mathbb{Z}^N, \tag{31}$$

where $r > 0$ and $\nabla^{\mathbf{2}}_{\mathbf{r}}$ is two-times the backward r-difference operator acting on the first variable defined by $\nabla_{\mathbf{r}} f(n) := f(n) - rf(n-1)$. Simplifying, we arrive at the following system

$$\begin{aligned} &u(n, \mathbf{m}) - 2ru(n-1, \mathbf{m}) + r^2u(n-2, \mathbf{m}) \\ &= c^2\Delta_{d,N}u(n, \mathbf{m}) + \nu\epsilon\Delta_{d,N}u(n, \mathbf{m}) - \nu\epsilon r\Delta_{d,N}u(n-1, \mathbf{m}) + f(n, \mathbf{m}) \\ &= (c^2 + \nu\epsilon)\Delta_{d,N}u(n, \mathbf{m}) - \nu\epsilon r\Delta_{d,N}u(n-1, \mathbf{m}) + f(n, \mathbf{m}) \end{aligned}$$

Choosing $A_q = -\Delta_{d,N}$, $b(n) = -(c^2 + \nu\epsilon)\delta_0(n) + \nu\epsilon r\delta_0(n-1)$, $\beta_1 = 2r, \tau_1 = 1$ and $\beta_2 = -r^2, \tau_2 = 2$ we have $\hat{b}(t) = -(c^2 + \nu\epsilon) + \nu\epsilon re^{-it}$ and it easily follows that b is 1-regular. Assume

$$r < \frac{\nu\epsilon - c^2}{\nu\epsilon} \qquad \text{and} \qquad c^2 < \nu\epsilon. \tag{32}$$

Then

$$\left\{\frac{1 - 2re^{it} + r^2e^{2it}}{(c^2 + \nu\epsilon) + \nu\epsilon re^{-it}}\right\}_{t\in\mathbb{T}} \subset \rho(\Delta_{d,N}).$$

Indeed, we observe that

$$\begin{aligned} \Re\left(\frac{1 - 2re^{it} + r^2e^{2it}}{(c^2 + \nu\epsilon) + \nu\epsilon re^{-it}}\right) &= c^2 + \nu\epsilon - r(2c^2 + \nu\epsilon)\cos t + r^2(c^2 - \nu\epsilon)\cos 2t + \nu\epsilon r^3\cos 3t \\ &\geq [c^2 + \nu\epsilon - r(2c^2 + \nu\epsilon)] - r^2[(c^2 - \nu\epsilon) + \nu\epsilon r] > 0. \end{aligned}$$

We define the function

$$h_t(z) := \frac{1}{(c^2+\nu\epsilon)+\nu\epsilon re^{-it}}\left(\frac{1-2re^{it}+r^2e^{2it}}{(c^2+\nu\epsilon)+\nu\epsilon re^{-it}}+z\right)^{-1}, \quad t\in(-\pi,\pi), \quad z\in\Sigma_{\pi/2}.$$

Using the same arguments than in the subsection 4.1, and using conditions (32) we get

$$\begin{aligned}|h_t(z)| &\leq \frac{1}{|(c^2+\nu\epsilon)+\nu\epsilon re^{-it}|}\frac{1}{[c^2+\nu\epsilon-r(2c^2+\nu\epsilon)]-r^2[(c^2-\nu\epsilon)+\nu\epsilon r]}\\ &\leq \frac{1}{|(c^2+\nu\epsilon)-\nu\epsilon r|}\frac{1}{[c^2+\nu\epsilon-r(2c^2+\nu\epsilon)]-r^2[(c^2-\nu\epsilon)+\nu\epsilon r]}.\end{aligned}$$

Therefore $\sup_{t\in\mathbb{T},z\in\Sigma_{\pi/2}}|h_t(z)|<\infty$. As before, we conclude by using Proposition 2.13, and Theorem 4.1 that under the conditions (32) the equation (31) admits a unique solution satisfying appropriate $\ell_p-\ell_q$-estimates. More precisely, we have the following theorem.

Theorem 4.3. *Let $f\in\ell_p(\mathbb{Z};\ell_q(\mathbb{Z}^N))$, $1<p<\infty$ be given and suppose that*

$$r<\frac{\nu\epsilon-c^2}{\nu\epsilon} \qquad \text{and} \qquad c^2<\nu\epsilon. \tag{33}$$

Then the numerical solution $(u(n,\mathbf{m}))_{n\in\mathbb{Z},\,\mathbf{m}\in\mathbb{Z}^N}$ of (31), obtained by the backward Euler r-method exists, belongs to $u\in\ell_p(\mathbb{Z};\ell_q(\mathbb{Z}^N))$ and is bounded by

$$\begin{aligned}\Big(\sum_{n\in\mathbb{Z}}\|u(n)\|^p_{\ell_q(\mathbb{Z}^N)}\Big)^{1/p}+\Big(\sum_{n\in\mathbb{Z}}\|c^2\Delta_{d,N}u(n)+\nu\epsilon\Delta_{d,N}\nabla_{\mathbf{r}}u(n)\|^p_{\ell_q(\mathbb{Z}^N)}\Big)^{1/p}\\ \leq C\Big(\sum_{n\in\mathbb{Z}}\|f(n)\|^p_{\ell_q(\mathbb{Z}^N)}\Big)^{1/p},\end{aligned}$$

where the constant $C>0$ is independent of f.

As a dual example, we examine the following discrete version of the Kuznetsov equation:

$$\mathbf{\Delta_r^2}u(n,\mathbf{m})=c^2\Delta_{d,N}u(n,\mathbf{m})+\nu\epsilon\Delta_{d,N}\mathbf{\Delta_r}u(n,\mathbf{m})+f(n,\mathbf{m}),\ n\in\mathbb{Z},\ \mathbf{m}\in\mathbb{Z}^N \tag{34}$$

where $\mathbf{\Delta_r}$ is the forward r-difference operator. Equivalently:

$$\begin{aligned}&u(n+2,\mathbf{m})-2ru(n+1,\mathbf{m})+r^2u(n,\mathbf{m})\\ &=(c^2-\nu\epsilon r)\Delta_{d,N}u(n,\mathbf{m})+\nu\epsilon\Delta_{d,N}u(n+1,\mathbf{m})+f(n,\mathbf{m}).\end{aligned}$$

Suppose that the following conditions between the coefficients of the equation hold

$$\frac{c^2+\nu\epsilon}{2c^2-\epsilon\nu}<r<\frac{c^2-3\nu\epsilon}{\nu\epsilon}, \qquad c^2>3\nu\epsilon. \tag{35}$$

Then we have

$$\left\{\frac{r^2-2re^{it}+e^{2it}}{(c^2-\nu\epsilon r)+\nu\epsilon re^{it}}\right\}_{t\in\mathbb{T}}\subset\rho(\Delta_{d,N}).$$

In fact, a simple computation shows that $\Re(\frac{r^2-2re^{it}+e^{2it}}{(c^2-\nu\epsilon r)+\nu\epsilon re^{it}}) > 0$ if and only if $P(t) := \Re((r^2-2re^{it}+e^{2it})((c^2-\nu\epsilon r)+\nu\epsilon re^{-it})) > 0$. Now, we observe that, in view of the hypothesis (35) we have

$$\begin{aligned} P(t) &= (c^2-\nu\epsilon r)\cos 2t + (\nu\epsilon - 2rc^2 + 2\epsilon\nu r^2 + r^2\nu\epsilon)\cos t + (r^2c^2 - \nu\epsilon r^3 - 2r\nu\epsilon) \\ &\geq -(c^2-\nu\epsilon r) - (\nu\epsilon - 2rc^2 + 2\epsilon\nu r^2 + r^2\nu\epsilon) + (r^2c^2 - \nu\epsilon r^3 - 2r\nu\epsilon) \\ &= [(2c^2-\epsilon\nu)r - (c^2+\nu\epsilon)] + r^2[(c^2-3\nu\epsilon) - \nu\epsilon r] > 0. \end{aligned}$$

This proves the claim. We now define the following complex valued function

$$h_t(z) := \left(\frac{r^2-2re^{it}+e^{2it}}{(c^2-\nu\epsilon r)+\nu\epsilon re^{it}} + z\right)^{-1}.$$

Then, as a consequence of the above computation, we obtain the following estimate

$$|h_t(z)| \leq \frac{1}{[(2c^2-\epsilon\nu)r - (c^2+\nu\epsilon)] + r^2[(c^2-3\nu\epsilon) - \nu\epsilon r]}$$

proving that the set $\{h_t\}_{t\in\mathbb{T}} \subset \mathcal{H}^\infty(\Sigma_{\pi/2})$ is uniformly bounded and then the set

$$\{h_\lambda(\Delta_{d,N})\}_{t\in\mathbb{T}}$$

is R-bounded and the conclusion follows as before. We arrive at the following result.

Theorem 4.4. *Let $f \in \ell_p(\mathbb{Z};\ell_q(\mathbb{Z}^N))$, $1<p<\infty$ be given and suppose that*

$$\frac{c^2+\nu\epsilon}{2c^2-\epsilon\nu} < r < \frac{c^2-3\nu\epsilon}{\nu\epsilon}, \qquad c^2 > 3\nu\epsilon. \tag{36}$$

Then the numerical solution $(u(n,\mathbf{m}))_{n\in\mathbb{Z},\,\mathbf{m}\in\mathbb{Z}^N}$ of (34), obtained by the forward Euler r-method exists, belongs to $u \in \ell_p(\mathbb{Z};\ell_q(\mathbb{Z}^N))$ and satisfies the discrete maximal $\ell_p-\ell_q$ regularity estimate

$$\begin{aligned} \Big(\sum_{n\in\mathbb{Z}} \|u(n)\|^p_{\ell_q(\mathbb{Z}^N)}\Big)^{1/p} + \Big(\sum_{n\in\mathbb{Z}} \|c^2\Delta_{d,N}u(n) + \nu\epsilon\Delta_{d,N}\mathbf{\Delta_r}u(n)\|^p_{\ell_q(\mathbb{Z}^N)}\Big)^{1/p} \\ \leq C\Big(\sum_{n\in\mathbb{Z}} \|f(n)\|^p_{\ell_q(\mathbb{Z}^N)}\Big)^{1/p}, \end{aligned}$$

where the constant $C>0$ is independent of f.

Remark 4.5. It is interesting to observe that the case $r=1$ can be reached under the hypothesis:

$$c^2 > 4\nu\epsilon$$

which shows that insofar as the damping term in (34) is not too small, the possibility of discretizing the temporal derivative by means of the usual backward difference operator increases. This reveals that in order to have $\ell_p-\ell_q$ estimates for the model (25), the difference operator that will be used in the temporal discretization of the equation will depend on the structure of the equation, i.e. on the parameters β and b.

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