博士論文（要約）

論文題目　An invariant of 3-manifolds via homology cobordisms and knots in lens spaces

（ホモロジーコボルディズムを用いた３次元多様体の不変量とレンズ空間内の結び目）

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My thesis consists of Parts 1 and 2 based on my papers [23] and [21] respectively. We start giving a background for Part 1. In [1], Alexander proved that every connected oriented closed 3-manifold X admits an open book decomposition. In other words, X contains a fibered link. Myers [20] and González-Acuña showed that X admits an open book decomposition with connected binding (see Rolfsen [24, Chapter 10, Section K]). We define the topological invariant $\mathrm{op}(X)$ to be the minimal number g such that X admits an open book decomposition with a page $\Sigma_{g,1}$, where $\Sigma_{g,1}$ denotes an oriented compact surface of genus g with one boundary component. The invariant $\mathrm{op}(X)$ is expected to measure how complicated X is. For instance, $\mathrm{op}(X) = 0$ if and only if X is homeomorphic to the 3-sphere S^3. Since the union of two pages of an open book decomposition is a Heegaard surface, we have the inequality $g(X) \leq 2\,\mathrm{op}(X)$, where $g(X)$ denotes the Heegaard genus of X.

Morimoto [19] started to study genus one fibered knots (GOF-knots) in lens spaces and proved that the lens space $L(p,1)$ contains a GOF-knot, but $L(19,2)$, $L(19,4)$ and $L(19,7)$ do not. In [2], Baker completely determined which lens space $L(p,q)$ contains a GOF-knot, that is, we already know when $\mathrm{op}(L(p,q)) = 1$ holds. Related researches are also found in Baldwin [3] and Sekino [27]. However, the computation of $\mathrm{op}(X)$ is difficult in general. Sakasai [25] introduced a homological analogue $\mathrm{hc}(X)$ of $\mathrm{op}(X)$, which is defined to be the minimal number g such that X is obtained as the closure of a homology cobordism over $\Sigma_{g,1}$. Roughly speaking, a homology cobordism over $\Sigma_{g,1}$ is a 3-manifold which homologically seems to be the product manifold $\Sigma_{g,1} \times [-1,1]$ (see Goussarov [12], Habiro [13] and Garoufalidis-Levine [9]). It is easy to see that $\mathrm{hc}(X) = 0$ if and only if X is an integral homology 3-sphere. Note that $\mathrm{hc}(X)$ is related to $\mathrm{op}(X)$ through the inequality $\mathrm{op}(X) \geq \mathrm{hc}(X)$.

The purpose of Part 1 is to investigate the topological invariant $\mathrm{hc}(X)$. We first refer a result due to Sakasai.

Theorem 1 ([25, Remark 6.10])**.** *The invariant* $\mathrm{hc}(X)$ *depends only on the first Betti number* $b_1(X)$ *and the isomorphism class of the torsion linking form* $\lambda_X \colon TH_1(X) \times TH_1(X) \to \mathbb{Q}/\mathbb{Z}$ *of* X*, where* $TH_1(X)$ *denotes the torsion subgroup of* $H_1(X) := H_1(X;\mathbb{Z})$*.*

Note that $\mathrm{hc}(X)$ is *not* determined only by the isomorphism class of $H_1(X)$. Indeed, for two 3-manifolds $X_q = L(5,1)\sharp L(5,q)$ $(q = 1,2)$, we have $\mathrm{hc}(X_1) = 1 \neq \mathrm{hc}(X_2) = 2$ ([25, Remark 6.10]). In fact, λ_{X_1} and λ_{X_2} are not isomorphic since $\lambda_{X_q} \cong (1/5) \oplus (q/5)$ for $q = 1, 2$. One of the main results of the thesis is the following.

Theorem 2. *The equality* $\mathrm{hc}(X\sharp(\sharp^2 S^1 \times S^2)) = \mathrm{hc}(X) + 1$ *holds for every connected oriented closed* 3*-manifold* X*.*

Theorems 1 and 2 enable us to reduce the computation of $\mathrm{hc}(X)$ to the computation of $\mathrm{hc}(Y)$ or $\mathrm{hc}(Y\sharp S^1 \times S^2)$ for a rational homology 3-sphere Y with $\lambda_X \cong \lambda_Y$. Here we focus on lens spaces which are basic examples of rational homology 3-spheres. The author was informed by Sakasai the following sufficient condition for $\mathrm{hc}(L(p,q)) = 1$ when q is odd: $p(p+4)$ or $p(p-4)$ is a quadratic residue mod q. In fact, we obtain the following.

Theorem 3. *The equality* $\mathrm{hc}(L(p,q)) = 1$ *holds for every lens space* $L(p,q)$*.*

Corollary 4. *The following equality holds:*

$$\mathrm{hc}(L(p,q)\sharp S^1 \times S^2) = \begin{cases} 1 & \textit{if } q \textit{ or } -q \textit{ is a quadratic residue} \bmod p, \\ 2 & \textit{otherwise.} \end{cases}$$

As a consequence of Theorems 1–3 and Corollary 4, if $H_1(X) \cong \mathbb{Z}/p \oplus \mathbb{Z}^r$ with torsion linking form (q/p) for some $p \geq 2$ and q, then

$$\mathrm{hc}(X) = \begin{cases} (r+2)/2 & \text{if } r \text{ is even,} \\ (r+1)/2 & \text{if } r \text{ is odd, and if } q \text{ or } -q \text{ is a quadratic residue mod } p, \\ (r+3)/2 & \text{otherwise.} \end{cases}$$

In order to prove Theorem 3, we find a surface $\Sigma \subset L(p,q)$ of genus one whose complement $L(p,q) \setminus \mathrm{Int}(\Sigma \times [-1,1])$ is a homology cobordism. The following theorem and a well-known fact about binary quadratic forms allow us to construct a desired surface Σ.

Theorem 5. *Let* m *be an integer and* n *a positive integer not equal to five with* m*,* n *coprime. Then there exist infinitely many pairs* $(\varepsilon, l) \in \{1, -1\} \times \{$*odd primes*$\}$ *such that the congruence equation* $nx(x+1) \equiv \varepsilon \mod l$ *has a solution and* $l \equiv m \mod n$ *holds.*

This theorem follows from the Chebotarev density theorem which computes the Dirichlet density of a certain set of primes.

We also discuss the invariant $\mathrm{hc}(X)$ from a knot-theoretic point of view. The main object is a homologically fibered knot (see Goda-Sakasai [10]), in terms of which an alternative definition of $\mathrm{hc}(X)$ is given. For example, Theorem 3 is rephrased as follows: Every lens space contains a homologically fibered knot of genus one. We prove that the homological fiberedness in a rational homology 3-sphere Y is characterized by the Alexander polynomial.

Theorem 6. *Let $K \subset Y$ be a null-homologous knot. Then, K is homologically fibered if and only if the Alexander polynomial $\Delta_K(t)$ is monic (up to sign) and its breadth equals $2g(K)$.*

This theorem allows us to show the following corollary concerning the connected sum of homologically fibered knots, which is an analogue of the result on fibered knots due to Gabai [8].

Corollary 7. *If $K_1 \subset X_1$ and $K_2 \subset X_2$ are homologically fibered knots, then $K_1 \sharp K_2$ is again a homologically fibered knot. Moreover, in the case where X_1 and X_2 are rational homology 3-spheres, the converse is true.*

We next develop an introduction for Part 2. Note that a homologically fibered knot is, by definition, null-homologous. In contrast, we focus on a knot K' whose homology class $[K']$ is a generator of $H_1(L(p,q))$ in Part 2. Then the preimage $\pi^{-1}(K')$ is a knot with free period p, where $\pi\colon S^3 \to L(p,q)$ is the p-fold cyclic cover.

Let K be a torus knot and $p \geq 2$ an integer. Hartley [14] created a decision argument to know whether K can be represented as the preimage of a knot in the lens space $L(p,q)$ for some q. For example, the trefoil $T_{3,2}$ does not appear as the preimage of any knot in $L(2,1) \cong \mathbb{R}P^3$. The Alexander polynomial of torus knots was used in his proof.

Let K be a knot such that the outer automorphism group $\mathrm{Out}(\pi_1(S^3 \setminus K))$ is trivial. For instance, 9_{32}, 9_{33} and 24 prime knots with 10 crossings (and their mirror images) satisfy this condition (see Kawauchi [16, Appendix F.2], Kodama-Sakuma [17, Table 3.1]). Then it follows from Borel's theorem (Conner-Raymond [7, Theorem 3.2]) that K cannot be represented as the preimage of any knot in any lens space (see [16, Theorems 10.6.2 and 10.6.6(1)]).

The purpose of Part 2 is to deduce the above facts from a single result. Let G be a group and $p \geq 2$ an integer which is not necessarily prime. We introduce the subgroup $\mathrm{C}^p(G)$ generated by the commutators and the pth power of each element of G. The case $p = 2$ was studied, for example, in Sun [29] and Haugh-MacHale [15], from a purely algebraic point of view. Moreover, the case where p is prime can be found in Stallings [28] and Cochran-Harvey [6]. We apply C^p to the fundamental group of the complement of a knot in a 3-manifold. Let Σ be an integral homology 3-sphere and let $\pi\colon \Sigma \to \Sigma'$ be a p-fold cyclic cover. The main result of Part 2 is the following.

Theorem 8. *Let K' be a knot in Σ' with connected preimage $K = \pi^{-1}(K')$. Then the image of the induced map $\pi_*\colon \pi_1(\Sigma \setminus K) \rightarrowtail \pi_1(\Sigma' \setminus K')$ coincides with the subgroup $\mathrm{C}^p(\pi_1(\Sigma' \setminus K'))$.*

Considering the case of $\Sigma = S^3$ and $\Sigma' = L(p,q)$, we obtain the corollaries below as mentioned before.

Corollary 9. *Let $m, n, p \geq 2$ be integers with $\gcd(m,n) = 1$. Then there exist an integer q and a knot K' in $L(p,q)$ such that $\pi^{-1}(K')$ is isotopic to the torus knot $T_{m,n}$ or its mirror image if and only if $\gcd(mn,p) = 1$.*

Moreover, Chbili [5] determined the above q, and we also observe it via Corollary 9.

Corollary 10. *A knot K in S^3 with $\mathrm{Out}(\pi_1(S^3 \setminus K)) = 1$ is not represented as the preimage of any knot in any lens space.*

When a knot K is represented as a preimage, there is still the question of whether K' is uniquely determined. Sakuma [26], Boileau and Flapan [4] independently proved that if the preimages of oriented knots K'_1, K'_2 are the same prime knot, then there exists a diffeomorphism $(L(p,q), K'_1) \to (L(p,q), K'_2)$ preserving the orientations both of $L(p,q)$ and K'_i's. Note that K'_1, K'_2 are not necessarily ambient isotopic to each other. They also showed that the same is true for a composite knot under a condition regarding "slopes". Furthermore, Manfredi [18] recently constructed two knots in a lens space such that they are not ambient isotopic to each other but their preimages are the unknot.

Our work is closely related to periodic diffeomorphisms of (S^3, K) without fixed points. Indeed, such a diffeomorphism of period p induces a cyclic cover $(S^3, K) \to (L(p,q), K')$ for some q, and vice versa. Hartley [14] gave a list of possible free periods of prime knots up to 10 crossings. Moreover, González-Acuña and Whitten [11, Chapter 4] proved that any non-torus two-bridge knot has no free period.

In the thesis, we shall discuss whether the braid group B_n is a C^p-group for each n, p and study the series of subgroups obtained from a group G by taking $\mathrm{C}^p(-)$ repeatedly. Finally, note that a part of Part 2 is also found in author's article [22] in the proceedings.

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