博士論文

Disposition Strategies and Performance Analysis of Series Configuration Queueing Systems with Blocking Phenomena

(ブロッキング現象を伴うシリーズ構成待ち行列システムの配置戦略と性能評価)

蔡裕立

Acknowledgement

First, I appreciate my Doktorvater, Prof. Dr. Nishinari Katsuhiro. I cannot complete this dissertation without his supervisions and supports. All achievements in the dissertation become possible because of his accurate decisions and executions. I have learned many invaluable experiences from his performances, both in professional and personal sides. This is certainly a remarkable journey for me to complete my Ph.D. dissertation under his supervisions. I would thank Prof. Dr. Daichi Yanagisawa, my senior fellow in Nishinari Lab at the University of Tokyo. He not only served as a role for being a guidance to give me several valuable suggestions about my research, but also provided all necessary kind helps for me all the time. I am very lucky to have such an excellent mentor in my life.

Examination committee members, Prof. Dr. Nishinari Katsuhiro, Prof. Dr. Hiroyuki Morikawa, Prof. Dr. Kazuyuki Aihara, Prof. Dr. Kenji Yamanishi and Prof. Dr. Daichi Yanagisawa are appreciated for reviewing and giving beneficial opinions for the improvement about my dissertation.

Finally, I would like to thank following people for giving me incessant encourages in my life: Ling-Hua Lin and Jui-Kuo Tsai, Yu-Chen Tsai, Ms. Chang Liu, Prof. Dr. Jun Sato, Dr. Juliana Tsai, Prof. Dr. Jyh-Bin Ke, Prof. Dr. Benjamin Fong Chao, Prof. Dr. Lin-Tsang Lee, Prof. Dr. Yint-Zer Shih, Prof. Dr. Hui-Ching Wang, Dr. Kimura Ayako, Dr. Phi Doan, Dr. Hayashi Ryoko, Ms. Charlotte Wiese and Dr. Klaus Kroier, Dr. Chih-Ching Tsai and all my fellows in Nishinari Lab at the University of Tokyo.

蔡裕立

Disposition Strategies and Performance Analysis of Series Configuration Queueing Systems with Blocking Phenomena

(ブロッキング現象を伴うシリーズ構成

待ち行列システムの配置戦略と性能評価)

Contents

CHAPTER 1
INTRODUCTION

1.1 Background

Open queueing networks are very important queueing systems for the optimization of industrial activities in modern economy. Several significant applications of open queueing networks include computer networks, global logistic networks, production line system, supply chain networks, telecommunication system etc. Recently, because of the development of computational facilities, the remarkable trends for applying real data analytics to improve theoretical predictions of specific service queueing systems become more and more important. Therefore, successfully figuring out the theoretical system performance values of a queueing system through exact analysis or numerical analysis is a prerequisite to make the systems work more efficiently by real data analytics. Moreover, the theoretical results can be further validated and improved by real data analysis.

Due to the development of computational power and storage technology of information systems many intractable problems in the past become feasible and applicable in recent years. The big data analytics and internet of things (IoT) are very important topics in the 21th century. Having deep understanding about the characteristic of performances of series configuration queueing systems is beneficial for further designing high efficient automated production systems. In addition, we can imagine that each service station in this kind of system can reflect its statuses of average service rate by applying IoT technology. Furthermore, based on the information of mean arrival rate, mean service rate of each service station and other important system parameters, it is expected to design a controlling center to make the queueing system work more efficiently and reflect real-time situation of operations. Therefore, quantitatively evaluating performance measures of this kind of system is a necessary prerequisite to make the system work smartly.

In this thesis, series configuration queueing systems with perfect service stations and the service stations subject to breakdowns and repairs are studied. This kind of the system is similar with the automated assembly lines in automobile industry. It is expected to apply our theoretical results to real cases in order to save operational costs of each automated assembly line. On the other hand, self-blocking system with infinite waiting capacity before the system is also investigated. Since the growth of population in cities and urbanization become significant recently, the insights of self-blocking queueing system can be applied to taxi cabs in metropolitan areas. We propose disposition strategies for the system work in better operational efficiency by setting system parameters based on the information of numerical simulations.

Traditional studies on queueing systems focused on deriving exact formula of related performance measures and steady-state probabilities. However, when the queuing systems become complex, it is almost impossible analyze problems in this way. Therefore, we adopt numerical methodologies to study the topics. The steady-state analyses of the systems are conducted by matrix-geometric method. We demonstrate that matrix-geometric method is a very powerful tool to study quasi-birth-death process, because we not only obtain the exact formula of stability conditions of the systems, but also evaluated related important performance measures consisted of steady-state probabilities. The information of the performance measures provides theoretical basis for real applications and benefit practitioners working in industries. Major contributions of the thesis are summarized as follows:
we 1) constructed structure generator matrices of the series configuration systems, 2) derived stability conditions in exact forms consisting of system parameters, 3) evaluated numerical performance measures of the systems based on steady-state probabilities, 4) provided transient analysis and investigated dynamic properties for the series configuration systems, 5) proposed application insights of disposition strategies for the system working in high performance way.

1.2 Literature Review

The literature on the series configuration queueing system with blocking phenomena can be traced back to Hunt [1]. He studied four particular cases of service facilities in series including infinite storage space between stages, no storage space between stages, finite storage space between stages, and the case of the unpaced belt-production line. Avi-Itzhak et al. [2] investigated a queueing system consisting of a sequence of two service stations with infinite queue allowable before the first station and no queue allowable between the stations. They obtained the moment generating functions of the steady-state queueing times and the generating functions of the steady numbers of customers in the various parts of the system. Avi-Itzhak et al. [3] studied a queueing system with sequence stations following an ordered service type. They assumed the arrival process is arbitrary and the time to serve each customers at the working stations is regular. Altiok [4] presented an approximate method for the analysis of open networks of queues in tandem and with blocking phenomena. He evaluated the steady-state probabilities of the number of customers at each station based on a specific method of decomposition where the total network is broken down into queues. Langaris et al. [5] provided a method to analyze the waiting time of a two-stage queueing system with blocking phenomena. They further considered the separation of the concepts between effective service time and the blocked time in the first service station. Papadopoulos et al. [6] developed an algorithm to model characteristics of production lines with no intermediate buffers. The marginal probability distribution of the number of units in each machine, the mean queue length and the throughput of the system can be obtained by their method. Avi-Itzhak et al. [7] assumed the just-in-time input for a queueing system with no buffers between servers under the communication and the blocking schemes. They derived explicit expressions for residence times, departure times, equilibrium throughput and other performance measures for the case of equal service requirements at all servers. Akyildiz et al. [8] derived the exact equilibrium state probability distributions for two-station queueing networks with blocking-after-service mechanism. Avi-Itzhaket al. [9] generalized a queueing system under k-stage blocking. They discovered a result that for $k > 1$, the waiting times are not order-insensitive while the G/D/1 equivalence is maintained.

Mathematical analysis and related applications of matrix-geometric method was systematically studied by Neuts [10]. Gomez-Corral [11] applied a general theory on quasi-birth-and-death processes to study a special kind of queueing system with blocking and repeated attempts. Gomez-Corral [12] investigated a two-stage tandem queue with blocking under the presence of a secondary flow of disasters. He determined the stationary distribution at departure epochs by using spectral analysis and calculated the stationary distribution at an arbitrary time. Gomez-Corral [13] studied queueing networks with blocking under the assumption of input units follow Markovian Arrival Process and applied the general theory on Markov renewal processes of M/G/1-type in their analysis. Gomez-Corral et al. [14] considered a two-stage tandem G-queue with blocking, service requirements of phase type and arrivals of units and of signals. They further investigated the influence of several flows of signals on the performance evaluation of the queueing model through various probabilistic descriptors. Gomez-Corral et al. [15] studied the influence of the dependence between units and signals on the performance evaluation of the continuous-time Markov chain describing the state of the network at arbitrary times. Bierbooms et al. [16] developed approximate methods for fluid flow production lines with multi-server workstations and finite buffers. Their method is suitable for the estimations of characteristics of longer production lines. Bierbooms et al. [17] proposed an approximation method to determine the throughput and mean sojourn time of single server tandem queues with general service times and finite buffers by decomposition method. Zhou et al. [18] studied a two-stage tandem queueing network with Markovian arrival process inputs and buffer sharing. They discovered that the buffer sharing policy is more flexible when the inputs have large variant and are correlated. Hillier [19] considered the optimal design of unpaced assembly lines. He analyzed the joint optimization of both the allocation of workload and the allocation of buffer spaces simultaneously when the objective is to maximize the revenue from throughput minus the cost of work-in-process inventory. Sakuma et al. [20] proposed Whitt's approximation to obtain the stationary distribution of an assembly-like queueing system with generally distributed time-constraint. Shin et al. [21] developed an approximation method for throughput in tandem queues with multiple independent reliable servers at each stage and finite buffers between service stations. Hudson et al. [22] gave complete reviews for the topics about unbalanced unpaced serial production lines. Several unanswered questions about the performance of assembly line are described in this work. Sani and Daman [24] studied a M/G/2 queueing system with an exponential server and a general server under a controlled queue discipline. The steady state distribution for the number of customers in the system, mean waiting time, mean queue length and blocking probability for the queueing system are derived. Ramasamy et al. [25] presented the steady state analysis of a heterogeneous server queueing system, Geo/G/2. Services containing discrete in nature can be applied through their analysis in many areas of communication, telecommunications, business and computer systems. Tsai et al. [26-28] discussed series configuration queueing systems with four service stations. They proposed general disposition strategies of the system based on original inductions of this works. Tsai et al. [29] further extended the series configuration queueing system by considering the conditions of system breakdowns and repairs. The disposition strategies of this kind of queueing system are suggested according to their theoretical and numerical investigations. Baumann and Sandmann [30] studied multi-server tandem queues where both stations have a finite buffer and all services times are phase-type distributed. An exact computational analysis of various steady-state performance measures and numerical results are presented. Vinarskiy [31] considered a model of an open exponential queueing network which shares a common buffer of limited capacity. The performance evaluation method could be applied in queueing network design.

We further cite surveys and bibliographies in this important topic by Perros [32-33], Onvural [34], Balsamo [35] and Hall et al. [36], two major monographs by Perros [37] and Balsamo et al. [38], and other special collections from journals [39-40]. Decomposition methods applied to study tandem queueing systems can be referred to Hillier et al. [41] and Perros et al. [42].

The summary of past research on the series configuration queueing systems is shown in Table 1. It is noted that most of the researches focused on studying the system consisting of two service stations by relaxing assumptions of Poisson arrivals or exponential service time, because of the complexity of the problem. Other researchers tried to figure the characteristics of the system which the queue between service stations is allowed. There were no related researches on the disposition strategies for the system with different service rates of the servers.

Table 1. Summary of past researches on series configuration queueing systems

Author (Year)	Stability conditions with different service rate 2, 3, 4 servers	Queue allowed between servers	Queue not allowed between servers	Relax Poisson arrival	Relax exponential service time	Disposition strategies
Hunt (1956)	2 and 3 stations	✓	✓	x	x	x
Avi-Itzhak et al. (1965)	x	x	✓	x	✓	x
Avi-Itzhak et al. (1965)	x	✓	x	✓	✓	x
Altiok (1982)	x	✓	x	x	x	x
Langaris et al. (1984)	Study on 2 stations	✓	x	x	x	x
Papadopoulos et al. (1993)	x	x	✓	x	x	
Avi-Itzhak et al. (1993)	x	x	✓	✓	x	x
Akyildiz et al. (1994)	Study on 2 stations	✓	x	x	x	x
Avi-Itzhak et al. (1995)	x	✓	x	✓	✓	x
Gomez-Corral (2002)	Study on 2 stations	✓	x	✓	✓	x
Gomez-Corral (2002)	Study on 2 stations	x	✓	✓	✓	x
Gomez-Corral (2002)	Study on 2 stations	x	✓	✓	✓	x
Gomez-Corral (2006)	Study on 2 stations	✓	x	✓	✓	x
Gomez-Corral (2009)	Study on 2 stations	✓	x	✓	✓	x
Bierbooms et al. (2012)	x	✓	x	x	x	x
Bierbooms et al. (2013)	x	✓	x	✓	✓	x

Table 1. Summary of past researches on series configuration queueing systems

Author (Year)	Stability conditions with different service rate 2, 3, 4 servers	Queue allowed between servers	Queue not allowed between servers	Relax Poisson arrival	Relax exponential service time	Disposition strategies
Zhou et al. (2013)	Study on 2 stations	✓	x	✓	x	x
Hillier (2013)	x	✓	x	x	✓	x
Sakuma et al. (2014)	x	✓	x	x	✓	x
Shin et al. (2014)	Study on 2 stations	✓	x	✓	x	x
Baumann et al. (2017)	Study on 2 stations	✓	x	✓	✓	x
Vinarskiy (2017)	x	✓	x	✓	x	x
Tsai et al. (2016, 2017)	✓	x	✓	x	x	✓

Abate and Whitt [43] presented an approximation method to investigate the transient behavior of M/M/1 queueing system. The method can help determine whether steady-state descriptions are reasonable or not in the condition that the arrival and service rates are nearly constant over time interval. Abate and Whitt [44] showed how Laplace transform analysis can obtain insights about transient behavior of the M/M/1 queueing system. They further determined the asymptotic behavior of the system through a transform factorization. Bertsimas and Nakazato [45] investigated queueing systems with the class of mixed generalized Erlang distributions. They found simple closed form expressions for the Laplace transforms of the queue length distribution and the waiting time distribution. Abate and Whitt [46] gave the time-dependent moments of the workload process in the M/G/1 queue. They obtained results for the covariance function of the stationary workload process. Various time-dependent characteristics described in terms of the steady-state workload distribution are demonstrated. Choudhury et al. [47] proposed an algorithm for numerically inverting multidimensional transforms. This method can be applied to both continuous variables and discrete variables transformations. They applied the method to invert the two-dimensional transforms of the joint distribution of the duration of a busy period, the number served in the busy period, time-dependent transient queue-length and workload distributions in the M/G/1 queue. Kaczynski et al. [48] derived the exact distribution of the nth customer's mean waiting time in the system in an M/M/s system with k customers initially present. Algorithms for evaluating the covariance between mean waiting time in the system and for an M/M/1 with k customers at beginning of the state of the system was developed. Kim and Whitt [49] showed that the bias of the steady-state Little's law can be estimated and reduced by applying time-varying Little's law. Kim and Whitt [50] advocated a statistical approach to study characteristics of Little's law for queueing systems with non-stationary distributions. They presented their theoretical analysis with data from a call center and simulation experiments. Tsai et al. [51] investigated the transient analysis of series configuration system consisting of two service stations by Runge-Kutta method. Dynamic performance measures are evaluated numerically. Fitting equations describing the dynamic properties of the system are provided.

Neuts and Lucantoni [52] have first considered a queueing system with N servers subject to breakdowns and repairs by means of Markovian chain. They discussed the stationary distributions of various waiting times and presented the effect of utilizing interactive computation in answering questions on the behavior, design and control of certain service systems. Papadopoulos and Heavey [53] reviewed the works about the design and analysis of manufacturing system by queueing networks. Gray, Wang and Scott [54] discussed a queueing model with multiple-vacation and server breakdowns. Thomas, Thornley and Zatschler [55] developed an iterative method to deal with a class of open queueing networks with server breakdowns. Chakka, Ever and Gemikonakli [56] presented a multi-node open network with heterogeneous nodes and finite sized buffers. An approximated model based on an IPP departure process was developed. Chakka, Ever and Gemikonakli [57] subsequently analyzed the modeling of joint-state for open queueing networks with breakdowns, repairs and finite buffers.

Neuts [58] presented a series system with a finite intermediate waiting room. He demonstrated that this system can be studied in terms of an imbedded semi-Markov process. Brandwajn and Jow [59] described an approximation method for a tandem queue with blocking caused by finite buffers between servers. Vuuren et al. [60] conducted performance analysis for multi-server tandem queues with finite buffers and blocking. Ke and Tsai [61] investigated self-blocking system with infinite queue setting in front of the system. They derived exact formula of stability conditions and proposed disposition strategies for the system working in better operations which can be applied to gasoline stations in metropolitan areas. Tsai et al. [62] further extends the works of Ke and Tsai [61] for the self-blocking system consisting of three service stations. The study suggested that the results can be applied to taxi cabs.

1.3 Organization of the Thesis

In this thesis, we first study steady-state performance analysis of series configuration system consisting of two, three and four service stations with blocking phenomena due to there are no waiting space between each service station. Exact stability conditions for the systems with heterogeneous service rates are derived. General disposition strategies for increasing efficiency of the system are proposed through the results of simulations. In Chapter 3, we investigate transient analysis for the system with two service stations. Important performance measurements are estimated by transient state probabilities. The results of simulations show that the disposition strategies for improving operational efficiency of the system are consistent in our proposition for the steady-state analysis of the system. In Chapter 4, we consider the system performance subject to breakdowns and repairs. This kind of problems is very important and applicable in real assembly line. Breakdown rate and repair rate of servers are introduced to evaluate related performance measures. Stability conditions are consistent with numerical results. Disposition strategies for increasing operational efficiency of the system are suggested. In Chapter 5, general disposition strategy of self-blocking queueing system is studied. Exact stability conditions of this kind of system with three service stations are derived. We expect that the suggested disposition strategy can be applied in taxi cabs with large capacity of queue in the system. Chapter 6 presents experimental results for the series configuration system with two service stations. We calculate mean waiting time in the queue through real data collected from the experiments in order to validate the concepts that different disposition strategies will cause different operational efficiency for the series configuration system. The stability conditions of the system are also tested through the experiments. Chapter 7 concludes all works in the thesis and suggests possible research topics in the future.

CHAPTER 2
PERFORMANCE ANALYSIS OF SERIES CONFIGURATION SYSTEM WITH PERFECT SERVICE STATIONS

2.1 Preface

Open queueing networks with no buffers among service stations arranged in series form are popular queueing systems in automobile production line and manufacturing industries. The production costs in this kind of business activities are huge, because of inefficient operational procedures. A manufacturing company always contains many production line systems in their factory. If the efficiency of the automatic production systems could be increased by disposing better production strategies in each production line, the production costs can be saved remarkably.

In this work, we try to propose disposition strategies for this kind of system in order to increase the operational efficiency of the system (i.e. reducing mean waiting time of the components in the system) based on numerical simulations of steady-state analysis. We provide theoretical steady-state analysis for the system consisting of two, three and four service stations. Several important performance measures are investigated for the system with different service rate of the each server. For the simplicity of formulation, the service stations in the system are assumed to be perfect working stations which mean that the server can work continuously without any problem regarding breakdowns. The service discipline of the system requires customers (or components) to enter the system in order. A customer should enter the first server of the system to receive the service, then goes into the second server after complete its service in the first station and so on. We define the complete service as after a customer complete all services in each server, the customer can leave the system from the terminal server.

- **Problem Formulation**

The queueing system consists of independent service stations in series configuration and operates simultaneously. Figure 1 shows the series configuration system with four servers. Poisson arrival process with mean arrival rate λ is assumed. The time to serve a customer in each station follows exponentially distributed with mean service time $\frac{1}{\mu}$. A complete service means that customers enter all of the service stations in order, and finish the services in each service station. There are no queues (buffers) among service stations, so that blocking phenomena would happen in this kind of the system. The phenomenon called blocking after service happens in the case that a customer completes the service in a service station, but another customer in the next station has not finished the service yet. The customer who completed the service is blocked by the customer who is still receiving the service located next station. An infinite capacity queue is allowed in front of the first service station. In addition, only one customer can enter each service station at a time and the service rate is independent of the number of customers. The service of the system obeys the first come first serve (FCFS) discipline.

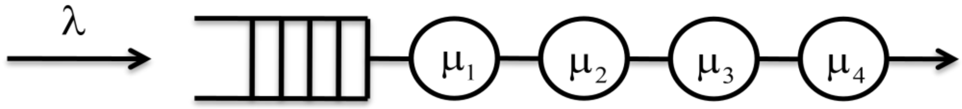


Figure 1. Series configuration queueing system with four service stations.

2.1.1 Contributions and Outline

Major theoretical results in this chapter including 1) developing steady-state structured generator matrix of the system consisting of two, three and four service stations, 2) deriving stability conditions for the system with two, three and four service stations in exact form, 3) evaluating the steady-state probabilities of the system by applying matrix-geometric method and performance measures, such as mean number in the system, mean waiting time in the system and blocking probability, 4) proposing disposition strategies for the system in order to make the system work in high performance ways. Practically, our numerical values of performance measures provide theoretical basis for the practitioners better to grasp the characteristics of the series configuration systems. The insights can also be applied to the real case that design high performance queueing systems.

The rest of the content in this chapter is organized as follows. The notations used in our model and detailed descriptions of matrix-geometric method applied to the system with two, three and four service stations and major performance measures for the system are given in section 2. Numerical results and the proposed disposition strategies for the case studies of the system are presented in section 3. Finally, we conclude with discussions of our works and indicate possible directions for future research in section 4.

2.2 Modeling Framework

2.2.1 Notations

In this section, we introduce notations used in our model framework. Mean arrival rate of Poisson arrivals is denoted as λ. We reserve the notations μ_1, μ_2, μ_3 and μ_4 for the mean service rate of the station-1, the station-2, the station-3, and the station-4 respectively. We use P_{n_1,n_2,n_3,n_4,n_5} to denote the steady-state probability

P_{n_1,n_2,n_3,n_4,n_5} of n_1 customer in the station-4 and n_2 customer in the station-3 and n_3 customer in the station-2 and n_4 customer in the station-1 and n_5 customer in the queue.

For instance, the steady-state probability $P_{0,1,b,b,3}$ means that there is a customer who is blocked in the station-1 and the station-2, since the customer in the station-3 is still receiving the service. There is no customer in the station-4. There are concurrently 3 customers waiting in the queue. Similarly, for the system consisting of three and two service stations, the notation P_{n_1,n_2,n_3,n_4} and P_{n_1,n_2,n_3} are used to mean the steady-state probability and the states of each service stations, respectively.

2.2.2 Matrix-Geometric Method

Let denote $\mathbf{P} = [\mathbf{P_0}, \mathbf{P_1}, \mathbf{P_2}, ...]$ as steady-state probability vector corresponding to the transition matrix Q. Note that the steady-state probability vector comprises steady-state probabilities of the quasi-death-birth process. The detailed compositions of the sub-matrices of the transition matrix Q for the system with two and three service and four stations are given in Appendix A. The equilibrium equation of the quasi-birth-death process can be described as $\mathbf{P}\mathrm{Q} = \mathbf{0}$, while the normalization condition of the steady-state probability is $\mathbf{P1} = 1$. Then, the global balance equations of the quasi-birth-death process can be written as

$$\mathbf{P_0}\mathrm{B}_{0,0} + \mathbf{P_1}\mathrm{B}_{1,0} = \mathbf{0}, \tag{1}$$

$$\mathbf{P_0}\mathrm{B}_{0,1} + \mathbf{P_1}\mathrm{A}_1 + \mathbf{P}_2\mathrm{A}_2 = \mathbf{0}, \tag{2}$$

$$\mathbf{P}_\mathrm{i}\mathrm{A}_0 + \mathbf{P}_{\mathrm{i+1}}\mathrm{A}_1 + \mathbf{P}_{\mathrm{i+2}}\mathrm{A}_2 = \mathbf{0}, \qquad \mathrm{i} \geq 1. \tag{3}$$

There exist a rate matrix R, and the following recurrence relation can be constructed

$$\mathbf{P_i} = \mathbf{P_{i-1}}\mathrm{R} = \mathbf{P_1}\mathrm{R}^{\mathrm{i}-1}, \qquad \mathrm{i} > 1. \tag{4}$$

The unknown rate matrix R can be obtained by substituting (4) into (3), and simply to matrix quadratic equation

$$\mathrm{A}_0 + \mathrm{RA}_1 + \mathrm{R}^2\mathrm{A}_2 = 0. \tag{5}$$

The simplified equations of (1) and (2) can be represented as

$$\mathbf{P_0}\mathrm{B}_{0,0} + \mathbf{P_1}\mathrm{B}_{1,0} = \mathbf{0}, \tag{6}$$

$$\mathbf{P_0}\mathrm{B}_{0,1} + \mathbf{P_1}(\mathrm{A}_1 + \mathrm{RA}_2) = \mathbf{0}. \tag{7}$$

According to Bloch et al. [23], the normalization condition equation that only involves $\mathbf{P_0}$ and $\mathbf{P_1}$ is given by

$$\mathbf{P_0}\mathbf{1} + \mathbf{P_1}(\mathrm{I} - \mathrm{R})^{-1}\mathbf{1} = 1, \tag{8}$$

where I is the identity matrix with same size as the rate matrix R.

We apply an iterative method by successive substitution, described in Neuts [10] to solve the rate matrix R from (5). Taking (6), (7) and (8) into account, the steady-state probability vector of $\mathbf{P_0}$ and $\mathbf{P}_1$ can be obtained by solving following matrix equation

$$(\mathbf{P_0}, \mathbf{P_1})\begin{pmatrix} \mathrm{B}_{0,0} & \mathrm{B}_{0,1}^{*} & \mathbf{1} \\ \mathrm{B}_{1,0} & (\mathrm{A}_1 + \mathrm{R}\mathrm{A}_2)^{*} & (\mathrm{I} - \mathrm{R})^{-1}\mathbf{1} \end{pmatrix} = (\mathbf{0}, 1). \tag{9}$$

where $(.)^{*}$ indicates that the last column of the included matrix is removed to avoid linear dependency.

2.2.3 Stability Conditions

Definition 1. Stability of a Queueing System

A queueing system is stable if the number of customers in the system grows with bound.

For the stability of the queueing systems, the stability condition is described by Neuts [10]

$$\mathbf{P}_{\mathrm{A}}\mathrm{A}_0\mathbf{1} < \mathbf{P}_{\mathrm{A}}\mathrm{A}_2\mathbf{1}, \tag{10}$$

where $\mathbf{P}_{\mathrm{A}}$ is the steady-state probability vector corresponding to the conservative stable matrix A.

The conservative stable matrix is defined to be

$$\mathrm{A} = \mathrm{A}_0 + \mathrm{A}_1 + \mathrm{A}_2. \tag{11}$$

Solving the following system equations with normalization condition, we can obtain the steady-state probability $\mathbf{P_A}$.

$$\mathbf{P_A}\mathrm{A} = \mathbf{0}, \tag{12}$$

$$\sum_{i=0}^{n} \mathrm{P}_{\mathrm{A},i} = 1. \tag{13}$$

Substituting the steady-state probability $\mathbf{P_A}$ and the sub-matrix A_0 and A_2 of the system consisting of two, three and four service stations into (10), the exact formulae of stability conditions can be derived.

2.2.4 Exact Form of Stability Conditions and Performance Measures

In this section, we define the performance measures for the series configuration system consisting of two, three and four service stations. Performance measures include mean number in the system, mean number in the queue, mean waiting time in the system, mean waiting time in the queue and blocking probability of the service stations in front of the terminal station. Exact formulae of stability conditions for the system with equivalent and different service rates are also given in the section.

- **Two Service Stations**
- **Stability Conditions**

The stability conditions of the system consisting of two service stations is given by

(1) For $\mu_1 \neq \mu_2$

$$\lambda < \frac{\mu_1\mu_2(\mu_1+\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2}. \tag{14}$$

(2) Special case: $\mu_1 = \mu_2 = \mu$

$$\lambda < \frac{2}{3}\mu. \tag{15}$$

<proof>

The transition matrix of the series configuration queueing system with two service stations can shown as

$$Q = \begin{bmatrix} B_{0,0} & A_0 & 0 & 0 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & 0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & \cdots \\ 0 & 0 & 0 & 0 & A_2 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The following sub-matrices show the composition of the transition matrix corresponding to the quasi-birth-death process for the system with two service stations.

$$B_{0,0} = \begin{bmatrix} -\lambda & 0 & 0 \\ \mu_2 & -(\lambda+\mu_2) & 0 \\ 0 & \mu_2 & -(\lambda+\mu_2) \end{bmatrix}, \quad A_0 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -(\lambda+\mu_1) & 0 & 0 \\ \mu_2 & -(\lambda+\mu_1+\mu_2) & 0 \\ 0 & \mu_2 & -(\lambda+\mu_2) \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & \mu_1 & 0 \\ 0 & 0 & \mu_1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now derive the stability conditions of the system with two service stations.

The stability condition (14) can be derived by (10)

$$\mathbf{P}_A A_0 \mathbf{1} < \mathbf{P}_A A_2 \mathbf{1},$$

We first evaluate the conservative stable matrix A

$$A = A_0 + A_1 + A_2 = \begin{pmatrix} -\mu_1 & \mu_1 & 0 \\ \mu_2 & -(\mu_1+\mu_2) & \mu_1 \\ 0 & \mu_2 & -\mu_2 \end{pmatrix}.$$

Then we obtain the steady-state probability vector $\mathbf{P}_A = [P_{A,0}, P_{A,1}, P_{A,2}]$ by solving following system equations with normalization condition

$$\mathbf{P}_{\mathbf{A}} A = \mathbf{0}, \tag{16}$$

$$\sum_{i=0}^{2} P_{A,i} = 1. \tag{17}$$

The steady-state probability vector is given by

$$\mathbf{P}_A = [\frac{\mu_2^2}{\mu_1^2+\mu_1\mu_2+\mu_2^2}, \frac{\mu_1\mu_2}{\mu_1^2+\mu_1\mu_2+\mu_2^2}, \frac{\mu_1^2}{\mu_1^2+\mu_1\mu_2+\mu_2^2}].$$

We get the result of the left-hand side of (10)

$$\mathbf{P}_A A_0 \mathbf{1} = \frac{\lambda\mu_2^2}{\mu_1^2+\mu_1\mu_2+\mu_2^2} + \frac{\lambda\mu_1\mu_2}{\mu_1^2+\mu_1\mu_2+\mu_2^2} + \frac{\lambda\mu_1^2}{\mu_1^2+\mu_1\mu_2+\mu_2^2} = \lambda,$$

and the right-hand side of (10)

$$\mathbf{P}_A A_2 \mathbf{1} = 0 + \frac{\mu_1\mu_2^2}{\mu_1^2+\mu_1\mu_2+\mu_2^2} + \frac{\mu_1^2\mu_2}{\mu_1^2+\mu_1\mu_2+\mu_2^2} = \frac{\mu_1\mu_2(\mu_1+\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2}.$$

Finally, we obtain the stability condition of the system with two service stations

$$\lambda < \frac{\mu_1\mu_2(\mu_1+\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2}.$$

For the system with same service rate, we set $\mu_1 = \mu_2 = \mu$

$$\lambda < \frac{\mu^2(2\mu)}{3\mu^2} = \frac{2}{3}\mu.$$

- **Performance Measures**

  Performance measures for the system consisting of two service stations are defined by

(1) Mean number of customers in the system

$$L_2 = (P_{1,0,0} + P_{0,1,0} + P_{1,b,0}) + \sum_{n=2}^{\infty}(P_{1,b,n-1} + P_{1,1,n-2} + P_{0,1,n-1}) \cdot n. \tag{18}$$

(2) Mean number of customers in the queue

$$L_{2;q} = \sum_{n=1}^{\infty}(P_{1,b,n} + P_{1,1,n} + P_{0,1,n}) \cdot n. \tag{19}$$

(3) Mean waiting time in the system (Little's Law)

$$W_2 = \frac{L_2}{\lambda}. \tag{20}$$

(4) Mean waiting time in the queue (Little's Law)

$$W_{2;q} = \frac{L_{2;q}}{\lambda}. \tag{21}$$

(5) Blocking probability of the customer in the station-1

$$P_{2;b} = \sum_{n=0}^{\infty} P_{1,b,n}. \tag{22}$$

- **Three Service Stations**
- **Stability Conditions**

Following formulae shows the stability conditions for the system consisting of three service stations.

(1) For $\mu_1 \neq \mu_2 \neq \mu_3$

$$\lambda < \frac{N_3}{D_3}, \tag{23}$$

where

$$N_3 = \mu_1\mu_2\mu_3(\mu_1 + \mu_2)(\mu_2 + \mu_3)(\mu_1^3 + \mu_1^2\mu_2 + 3\mu_1^2\mu_3 + \mu_1\mu_2\mu_3 + 3\mu_1\mu_3^2 + \mu_2\mu_3^2 + \mu_3^3),$$

and

$$D_3 = \mu_1^5(\mu_2^2+\mu_2\mu_3+\mu_3^2)+\mu_1^4(2\mu_2^3+5\mu_2^2\mu_3+5\mu_2\mu_3^2+3\mu_3^3)$$
$$+\mu_1^3(\mu_2^4+5\mu_2^3\mu_3+8\mu_2^2\mu_3^2+7\mu_2\mu_3^3+3\mu_3^4)$$
$$+\mu_1^2(\mu_2^4\mu_3+5\mu_2^3\mu_3^2+8\mu_2^2\mu_3^3+5\mu_2\mu_3^4+\mu_3^5)$$
$$+\mu_1(\mu_2^4\mu_3^2+5\mu_2^3\mu_3^3+5\mu_2^2\mu_3^4+\mu_2\mu_3^5)+(\mu_2^4\mu_3^3+2\mu_2^3\mu_3^4+\mu_2^2\mu_3^5).$$

(2) Special case: $\mu_1=\mu_2=\mu_3=\mu$

$$\lambda < \frac{22}{39}\mu. \tag{24}$$

<proof>

The proof is shown in Appendix A1.

Next, we study the behavior of the system consisting of three service stations. We consider taking the limit of the service rate of each service station to zero, respectively

$$\lim_{\mu_1\to 0}\frac{N_3}{D_3}=0, \tag{25}$$

and

$$\lim_{\mu_2\to 0}\frac{N_3}{D_3}=0, \tag{26}$$

and

$$\lim_{\mu_3\to 0}\frac{N_3}{D_3}=0. \tag{27}$$

The results show that if one of the service rates of the service stations approaches to zero, the mean arrival rate also should be lowered to zero. This means that if one of the service stations is failed, the impact to the whole queueing system is fateful. Since the service rate of any service stations down to zero, the number of customers in the queue would growth rapidly and tend to diverge.

- **Performance Measures**

  Performance measures for the system consisting of three service stations are defined by

  (1) Mean number of customers in the system

$$L_3=(P_{0,0,1,0}+P_{0,1,0,0}+P_{1,0,0,0}+P_{1,b,0,0}+P_{0,1,b,0})+2(P_{0,0,1,1}+P_{0,1,1,0}+P_{1,0,1,0}+P_{1,1,0,0})$$
$$+\sum_{n=3}^{\infty}(P_{0,0,1,n-1}+P_{0,1,1,n-2}+P_{1,0,1,n-2}+P_{1,1,1,n-3})\cdot n+\sum_{n=2}^{\infty}(P_{1,b,1,n-2}+P_{0,1,b,n-1}+P_{1,1,b,n-2})\cdot n$$
$$+\sum_{n=1}^{\infty}(P_{1,b,b,n-1})\cdot n. \tag{28}$$

(2) Mean number of customers in the queue

$$
\begin{aligned}
L_{3;q} &= (P_{0,0,1,1} + P_{0,1,1,1} + P_{1,0,1,1} + P_{0,1,b,1}) + 2(P_{0,0,1,2}) \\
&+ \sum_{n=3}^{\infty} (P_{0,0,1,n}) \cdot n + \sum_{n=2}^{\infty} (P_{0,1,1,n} + P_{1,0,1,n} + P_{0,1,b,n}) \cdot n \\
&+ \sum_{n=1}^{\infty} (P_{1,1,1,n} + P_{1,b,1,n} + P_{1,b,b,n} + P_{1,1,b,n}) \cdot n.
\end{aligned} \tag{29}
$$

(3) Mean waiting time in the system (Little's Law)

$$W_3 = \frac{L_3}{\lambda}. \tag{30}$$

(4) Mean waiting time in the queue (Little's Law)

$$W_{3;q} = \frac{L_{3;q}}{\lambda}. \tag{31}$$

(5) Blocking probability of the customer in the station-1

$$P_{3;b,1} = \sum_{n=0}^{\infty} P_{1,b,b,n} + P_{0,1,}. \tag{32}$$

(6) Blocking probability of the customer in the station-2

$$P_{3;b,2} = \sum_{n=0}^{\infty} P_{1,b,b,n} + P_{1,b,0,n}. \tag{33}$$

(7) Blocking probability of the customer in the station-1 and the station-2

$$P_{3;b,12} = \sum_{n=0}^{\infty} P_{1,b,b,n}. \tag{34}$$

- **Four Service Stations**
- **Stability Conditions**

Following formulae shows the stability conditions for the system consisting of three service stations.

(1) For $\mu_1 \neq \mu_2 \neq \mu_3 \neq \mu_4$

$$\lambda < \frac{N_4}{D_4}, \tag{35}$$

where the exact results of the N and the D are shown in the supplementary document I.

(2) Special case: $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu$

$$\lambda < \frac{4024}{7817}\mu. \tag{36}$$

<*proof*>
The proof is shown in Appendix A2.

- **Performance Measures**

  Performance measures for the system consisting of three service stations are defined by

  (1) Mean number of customers in the system

$$
\begin{aligned}
L_4 &= (P_{0,0,0,1,0} + P_{0,0,1,0,0} + P_{0,1,0,0,0} + P_{1,0,0,0,0} + P_{1,b,0,0,0} + P_{0,0,1,b,0} + P_{0,1,b,0,0} + P_{0,1,b,b,0} + P_{1,b,b,0,0}) \\
&+2(P_{0,0,0,1,1} + P_{0,0,1,1,0} + P_{0,1,0,1,0} + P_{1,0,0,1,0} + P_{1,0,1,0,0} + P_{1,1,0,0,0} + P_{0,1,1,0,0} \\
&+P_{1,b,0,1,0} + P_{0,0,1,b,1} + P_{0,1,1,b,0} + P_{1,1,b,0,0} + P_{0,1,b,1,0} + P_{1,0,1,b,0} + P_{1,b,1,0,0}) \\
&+3(P_{0,0,0,1,2} + P_{0,0,1,1,1} + P_{0,1,0,1,1} + P_{1,0,0,1,1} + P_{1,0,1,1,0} + P_{1,1,0,1,0} + P_{0,1,1,1,0} + P_{1,1,1,0,0}) \\
&+\sum_{n=4}^{\infty}(P_{0,0,0,1,n-1} + P_{0,0,1,1,n-2} + P_{0,1,0,1,n-2} + P_{1,0,0,1,n-2} + P_{1,0,1,1,n-3} + P_{1,1,0,1,n-3} + P_{0,1,1,1,n-3} + P_{1,1,1,1,n-4}) \cdot n \\
&+\sum_{n=3}^{\infty}(P_{1,b,0,1,n-2} + P_{0,0,1,b,n-1} + P_{0,1,1,b,n-2} + P_{1,1,b,1,n-3} + P_{0,1,b,1,n-2} + P_{1,0,1,b,n-2} + P_{1,b,1,1,n-3} + P_{1,1,1,b,n-3}) \cdot n \\
&+\sum_{n=2}^{\infty}(P_{0,1,b,b,n-1} + P_{1,b,b,1,n-2} + P_{1,1,b,b,n-2} + P_{1,b,1,b,n-2}) \cdot n \\
&+\sum_{n=1}^{\infty}(P_{1,b,b,b,n-1}) \cdot n.
\end{aligned}
\tag{37}
$$

  (2) Mean number of customers in the queue

$$
\begin{aligned}
L_{4;q} &= (P_{0,0,0,1,1} + P_{0,0,1,b,1} + P_{0,0,1,1,1} + P_{0,1,0,1,1} + P_{1,0,0,1,1} + P_{1,0,1,1,1} + P_{1,1,0,1,1} \\
&+P_{1,b,0,1,1} + P_{0,1,1,1,1} + P_{0,1,1,b,1} + P_{0,1,b,b,1} + P_{0,1,b,1,1} + P_{1,0,1,b,1}) \\
&+2(P_{0,0,0,1,2} + P_{0,0,1,1,2} + P_{0,1,0,1,2} + P_{1,0,0,1,2} + P_{0,0,1,b,2}) \\
&+3(P_{0,0,0,1,3}) + \sum_{n=4}^{\infty}(P_{0,0,0,1,n}) \cdot n + \sum_{n=3}^{\infty}(P_{0,0,1,1,n} + P_{0,1,0,1,n} + P_{1,0,0,1,n} + P_{0,0,1,b,n}) \cdot n \\
&+\sum_{n=2}^{\infty}(P_{1,0,1,1,n} + P_{1,1,0,1,n} + P_{1,b,0,1,n} + P_{0,1,1,1,n} + P_{0,1,1,b,n} + P_{0,1,b,b,n} + P_{0,1,b,1,n} + P_{1,0,1,b,n}) \cdot n \\
&+\sum_{n=1}^{\infty}(P_{1,1,1,1,n} + P_{1,1,b,1,n} + P_{1,b,b,1,n} + P_{1,b,1,1,n} + P_{1,1,1,b,n} + P_{1,b,b,b,n} + P_{1,1,b,b,n} + P_{1,b,1,b,n}) \cdot n.
\end{aligned}
\tag{38}
$$

  (3) Mean waiting time in the system (Little's Law)

$$
W_4 = \frac{L_4}{\lambda}. \tag{39}
$$

(4) Mean waiting time in the queue (Little's Law)

$$W_{4;q} = \frac{L_{4;q}}{\lambda}. \tag{40}$$

(5) Blocking probability of the customer in the station-1

$$P_{4;b,1} = \sum_{n=0}^{\infty} P_{0,0,1,b,n} + P_{0,1,1,b,n} + P_{0,1,b,b,n} + P_{1,0,1,b,n} + P_{1,1,1,b,n} + P_{1,b,b,b,n} + P_{1,1,b,b,n} + P_{1,b,1,b,n}. \tag{41}$$

(6) Blocking probability of the customer in the station-2

$$P_{4;b,2} = \sum_{n=0}^{\infty} P_{1,1,b,1,n} + P_{0,1,b,1,n} + P_{0,1,b,b,n} + P_{1,b,b,1,n} + P_{1,b,b,b,n} + P_{1,1,b,b,n}. \tag{42}$$

(7) Blocking probability of the customer in the station-3

$$P_{4;b,3} = \sum_{n=0}^{\infty} P_{1,b,0,1,n} + P_{1,b,1,1,n} + P_{1,b,b,1,n} + P_{1,b,b,b,n} + P_{1,b,1,b,n}. \tag{43}$$

Proposition 2.2.1. Disposition strategies for the series configuration queueing system consisting of the arbitrary number of service stations with different service rates are different.

(1) Series configuration queueing system with **the odd number** of service stations

It is better to arrange lower service rate for the first service station compared with other service stations in the system in order to obtain the best operational efficiency for the system with the odd number of service stations.

(2) Series configuration queueing system with **the even number** of service stations

We suggest setting higher service rates for the service stations in front of the terminal station as possibly as we can. In this way, the mean waiting time in the system would be the shortest compared with other disposition strategies.

2.3 Numerical Results

Numerical experiments for the queueing system consisting of two, three and four service stations are provided in this section. Performance metrics of the system with equivalent service rates (i.e. $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu$) and with different service rates are presented. Performance measures include mean number in the system, blocking probabilities of the customers before the terminal service station, mean waiting time in the system and mean waiting time in the queue.

Based on the results of simulation, we will suggest better disposition strategies to increase operation efficiency for the system.

2.3.1 Two Service Stations

(1) Each Service Station with Same Service Rate

We first present how mean number in the system and blocking probabilities of the station-1 change as the mean arrival rate varies λ from 0.01 to 0.6. We observe that mean number in the system increases as λ increases as shown in Figure 2. It is noted that the mean number of customer in the system increases rapidly as λ approaches to 0.66. This numerical result verifies the stability condition derived in the Section 2.2.4. The trends of blocking probabilities also increase and finally approach to about 0.33. as shown in Figure 3.

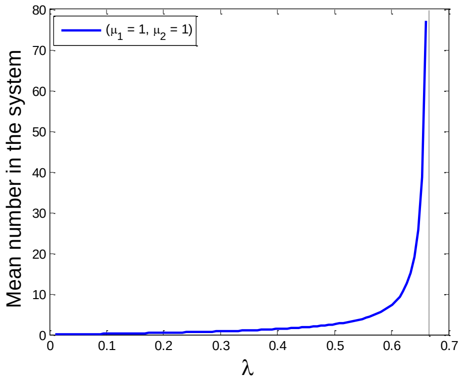


Figure 2. Mean number in the system (two service stations)

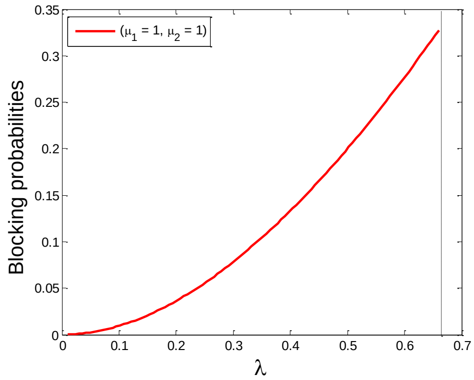
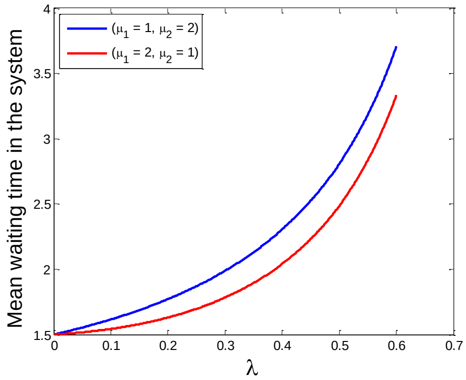


Figure 3. Blocking probability (two service stations)

(2) Each Service Station with Different Service Rates

Next, the impact of different rates causing distinct performances for the series configuration system is investigated. We set $\mu_1 = 2, \mu_2 = 1$ and $\mu_1 = 1, \mu_2 = 2$, then vary the mean arrival rate λ from 0.01 to 0.60. It is noted that if we dispose higher service rate for the station-1, the mean waiting time in the system is less than that of setting higher service rate for the station-2. Since setting higher service rate for the station-1 would advance the opportunity for the customers waiting in the queue to enter the first service station, it can reduce mean waiting time in the queue for customers, as show in Figure 4 and Figure 5, respectively. It is suggested that setting higher service rate for the station-1 of the system with two service stations in order to maintain higher operational efficiency when the service rate of each service station is different.



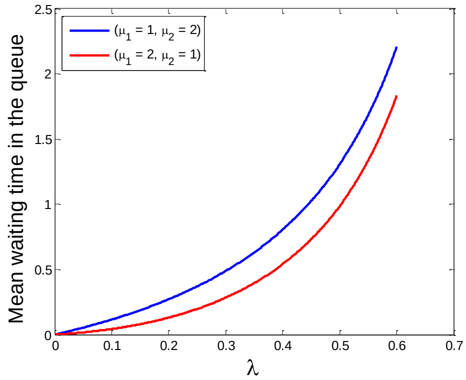


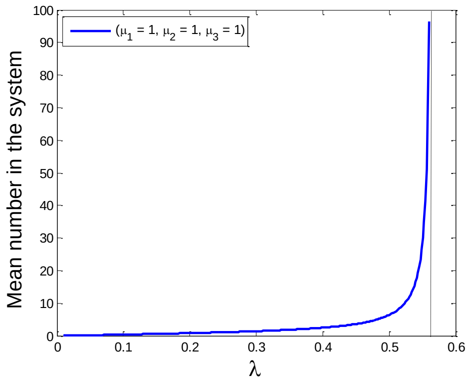
Figure 4. Mean waiting time in the system (two service stations)

Figure 5. Mean waiting time in the queue (two service stations)

2.3.2 Three Service Stations

(1) Each Service Station with Same Service Rate

The mean number in the system and blocking probability of the station-1 and the station-2 as a function of mean arrival rate of the system consisting of three service stations are shown in Figure 6 and Figure 7, respectively. The numerical results of mean number in the system are consistent with the exact results of stability conditions we derived in the section 2.2.4 which shows the upper bound of the stability condition approach to $\frac{22}{39}(\approx 0.564)$. In addition, it is noted that the blocking probability of the station-1 is higher than that of the station-2. The blocking probability of the station-1 and the station-2 happening simultaneously is relatively low in this case.



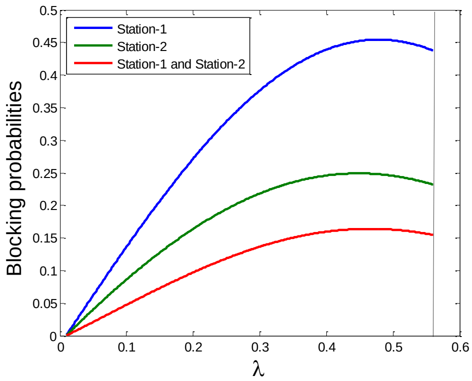


Figure 6. Mean number in the system (three service stations)

Figure 7. Blocking probability (three service stations)

(2) Disposing the Service Rates of Two Service Stations

In the case of different service rates, we assume that we are able to control the service rates of two service stations and the service rates of one service station at one time.

Intuitively, it is better to set higher service rates for the service stations before the terminal service station (i.e. the last service station of the series configuration queueing system) according to the results of the system with two service stations. In the case of controlling service rates of two service stations, we set $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1$ and $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2$ and $\mu_1 = 2, \mu_2 = 1, \mu_3 = 2$, then vary the mean arrival rate λ from 0.01 to 0.75. The numerical results suggest that setting higher service rates for the station-2 and the station-3 result in best operational efficiency, as shown in Figure 8. Since setting lower service rate for the station-1 would cause less waiting in the queue, it makes easier for the customer to enter service stations to receive their services.

It is observed that the mean waiting time in the system of the case $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1$ is higher than that of the case $\mu_1 = 2, \mu_2 = 1, \mu_3 = 2$ when the mean arrival rate is lower than 0.56, as show in Figure 9. This result reveals the fact that when the mean arrival rate is lower than 0.56, it takes longer time to complete services in the service stations for the case $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1$, since the mean waiting time in the queue is almost the same for both cases. When the mean arrival is greater than 0.56, the mean waiting time in the queue in the case of $\mu_1 = 2, \mu_2 = 1, \mu_3 = 2$ becomes relatively longer than that of the case $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1$. We finally observed that the mean waiting time in the system in the case of $\mu_1 = 2, \mu_2 = 1, \mu_3 = 2$ is larger than that of the case $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1$.

We suggest $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2$ as the best disposition strategy, when we are able to control service rates of two service stations for the system.

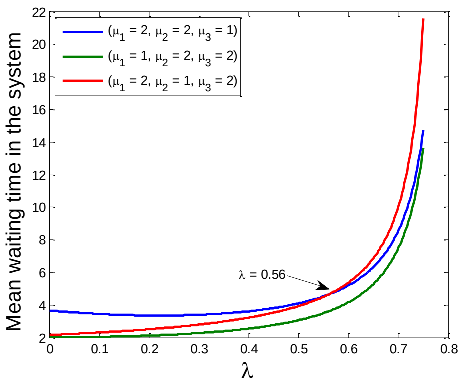


Figure 8. Mean waiting time in the system (three stations)

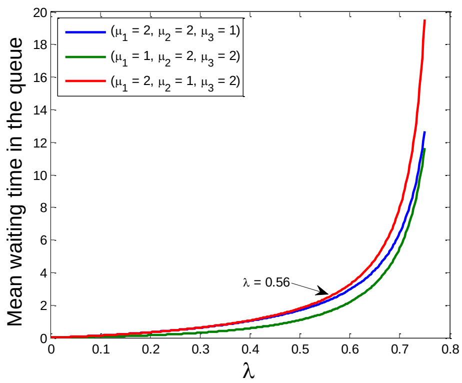


Figure 9. Blocking probability (three stations)

(2) Disposing the Service Sate of a Service Station

Next, we study the case of controlling service rate of one service station, we set $\mu_1 = 2, \mu_2 = 1, \mu_3 = 1$ and $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1$ and $\mu_1 = 1, \mu_2 = 1, \mu_3 = 2$, then vary the mean arrival rate λ from 0.01 to 0.6. The plots are presented in Figure 10, which shows that in the case of $\mu_1 = 2, \mu_2 = 1, \mu_3 = 1$, the mean waiting time is the greatest compared with other two cases. In this disposition strategy, the customers in the queue are difficult to enter the service stations, because of the high blocking probability of the station-1.

It is noted that the mean waiting time in the system of the case $\mu_1 = 1, \mu_2 = 1, \mu_3 = 2$ is lower than that of the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1$ when the mean arrival rate is lower than 0.46. Since the mean waiting time in the queue is almost the same for both cases, we discover that the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1$ would result in longer time to complete the services in the service stations, as shown in Figure 11. The waiting time in the queue in the case of $\mu_1 = 1, \mu_2 = 1, \mu_3 = 2$ becomes relatively longer than that of the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1$, when the mean arrival is greater than 0.46. It is observed that the mean waiting time in the system in the case of $\mu_1 = 1, \mu_2 = 1, \mu_3 = 2$ is larger than that of the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1$. The setting of lower service rates in the station-1 and the station-2 makes customers take longer waiting time in the queue.

It is suggested that set $\mu_1 = 1, \mu_2 = 1, \mu_3 = 2$ as the best disposition strategy when the mean arrival rate is lower than 0.46. Conversely, the case of $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1$ is a relatively better disposition strategy when the mean arrival rate becomes larger than 0.46 in the case that we can control only one of the service rates for the system consisting of three service stations.

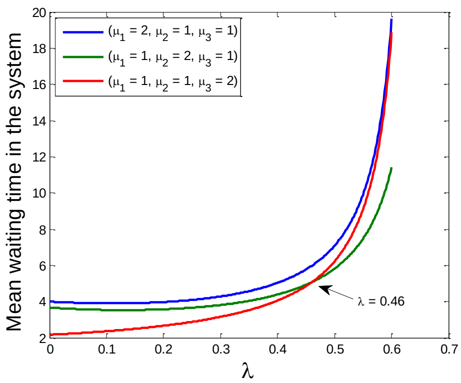


Figure 10. Mean waiting time in the system (three service stations)

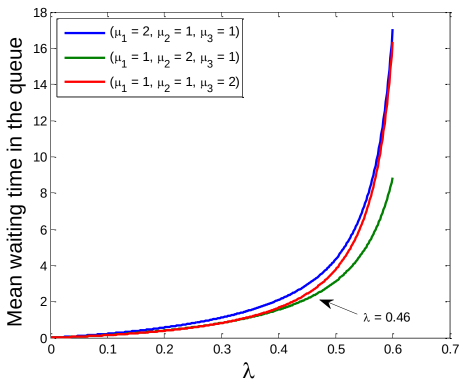


Figure 11. Blocking probability (three service stations)

2.3.3 Four Service Stations

(1) Each Service Station with Same Service Rate

The increasing trends of mean number in the system and blocking probabilities as a function of mean arrival rate λ is studied. Figure 12. presents the mean number in the system for the system consisting of four service stations. It is observed that the upper bound of the stability condition of the mean number in the system approaches to $\frac{4024}{7817}(\approx 0.514)$, which proves the consistence of the exact results derived in the section 2.2.4. Blocking probability of the station-1, the station-2 and the station-3 as a function of mean arrival rate of the system consisting of four service stations is shown in Figure 13. Furthermore, it is investigated that the blocking probability of the station-1 is higher than that of the station-2 and of the station-3 in this case.

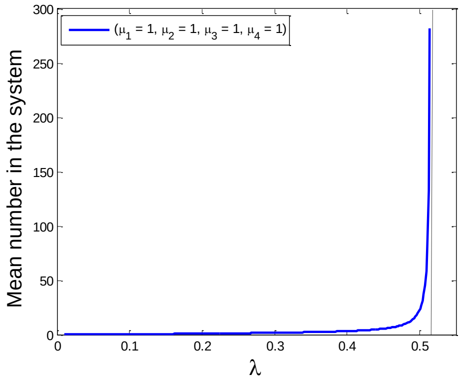


Figure 12. Mean waiting time in the system (four service stations)

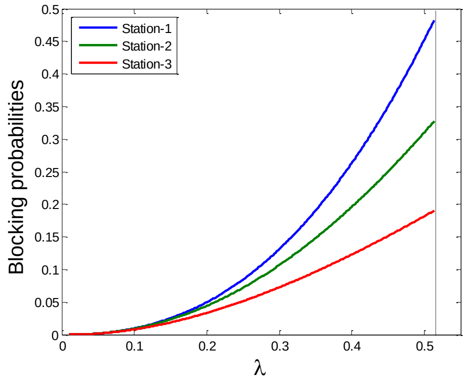


Figure 13. Blocking probability (four service stations)

(2) Disposing the Service Rates of Three Service Stations

In the case of different service rates, we study the conditions that we can concurrently control the service rates of three service stations and the service rate of only one service station for the system consisting of four service stations.

First, we investigate the cases that we are able to control three service rates of the service stations in this system. We set $\mu_1 = 2, \mu_2 = 2, \mu_3 = 2, \mu_4 = 1$ and $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1, \mu_4 = 2$ and $\mu_1 = 2, \mu_2 = 1, \mu_3 = 2, \mu_4 = 2$ and $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2, \mu_4 = 2$, then vary the mean arrival rate λ from 0.01 to 0.7. It is suggested to set higher service rates for the station-1, the station-2 and the station-3 in order to obtain the best operational efficiency for the system, as shown in Figure 14. This best disposition strategy for the system consisting of four service stations is accordant with the result of the system comprising two service stations indicated in previous numerical simulations.

Since the mean waiting time in the system of the case $\mu_1 = 2, \mu_2 = 1, \mu_3 = 2, \mu_4 = 2$ is always higher than that of the case $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1, \mu_4 = 2$ for all mean arrival rates. On the other hand, we compare the cases between $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2, \mu_4 = 2$ and $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1, \mu_4 = 2$. It is investigated that the mean waiting time of the system of the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2, \mu_4 = 2$ is higher than that of the case $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1, \mu_4 = 2$ when mean arrival rate is lower than 0.65. This result shows that when the mean arrival rate is lower than 0.65, the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2, \mu_4 = 2$ causes longer time for the customers waiting in the queue as show in Figure 15. The mean waiting time in the queue of the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2, \mu_4 = 2$ becomes shorter than that of the case $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1, \mu_4 = 2$ when the mean arrival rate is greater than 0.65.

We suggest the case $\mu_1 = 2, \mu_2 = 2, \mu_3 = 2, \mu_4 = 1$ as the best disposition strategy, when we are able to control service rates of three service stations for the system.

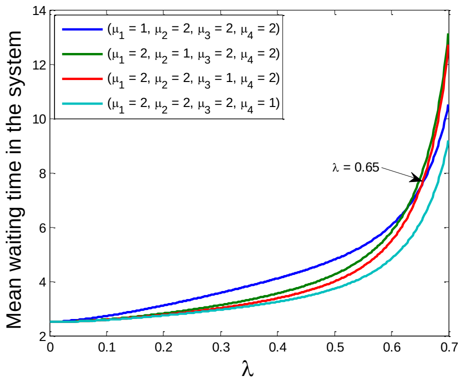


Figure 14. Mean waiting time in the system (four service stations)

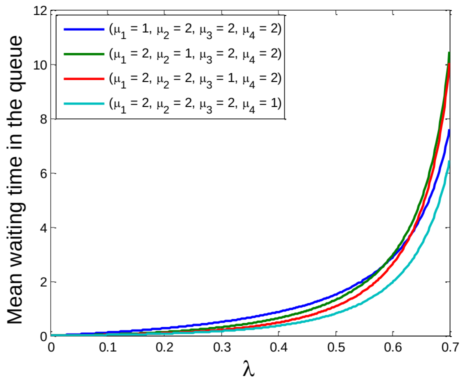


Figure 15. Mean waiting time in the queue (four service stations)

(3) Disposing the Service Rate of a Service Station

Next, we study the case of controlling service rate of one service station, we set $\mu_1 = 2, \mu_2 = 1, \mu_3 = 1, \mu_4 = 1$ and $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1, \mu_4 = 1$ and $\mu_1 = 1, \mu_2 = 1, \mu_3 = 2, \mu_4 = 1$ and $\mu_1 = 1, \mu_2 = 1, \mu_3 = 1, \mu_4 = 2$ then vary the mean arrival rate λ from 0.01 to 0.5. It is investigated that the mean waiting time is the greatest in the case of $\mu_1 = 1, \mu_2 = 1, \mu_3 = 1, \mu_4 = 2$ compared with other three cases as shown in Figure 16. This disposition strategy makes the customers in the queue difficult to enter the service stations, since the mean waiting time in the queue is higher than other three cases.

Similar to the case studies of controlling three service stations in previous section, we note that the mean waiting time in the system in the case of $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1, \mu_4 = 1$ is always lower than that of the case

$\mu_1 = 1, \mu_2 = 1, \mu_3 = 2, \mu_4 = 1$. We compare the case $\mu_1 = 2, \mu_2 = 1, \mu_3 = 1, \mu_4 = 1$ with the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1, \mu_4 = 1$ for discussing the disposition strategies. We consider that the mean waiting time in the system of the case $\mu_1 = 2, \mu_2 = 1, \mu_3 = 1, \mu_4 = 1$ is lower than that of the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1, \mu_4 = 1$ when the mean arrival rate is lower than 0.42. It is noted the mean waiting time in the queue is almost the same for both cases, so setting higher service rate for the station-1 is better to make customer to enter the service stations when mean arrival rate is lower than 0.42, as shown in Figure 17. When the mean arrival is greater than 0.42, it is observed that the mean waiting time in the system in the case of $\mu_1 = 2, \mu_2 = 1, \mu_3 = 1, \mu_4 = 1$ is larger than that of the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1, \mu_4 = 1$. While the increasing of the mean arrival rate, the setting of lower service rates in the station-1 and the station-2 and the station-3 makes customers take longer waiting time in the queue.

We suggest that setting $\mu_1 = 2, \mu_2 = 1, \mu_3 = 1, \mu_3 = 1$ as the best disposition strategy when the mean arrival rate is lower than 0.42. On the other hand, for the case that we can control only one of the service rates for the system consisting of four service stations, we observe that case of $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1, \mu_4 = 1$ is a relatively better disposition strategy compared with the case $\mu_1 = 2, \mu_2 = 1, \mu_3 = 1, \mu_3 = 1$ when the mean arrival rate becomes larger than 0.42.

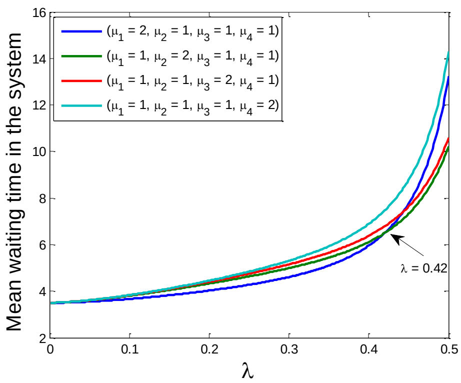


Figure 16. Mean waiting time in the system (four service stations)

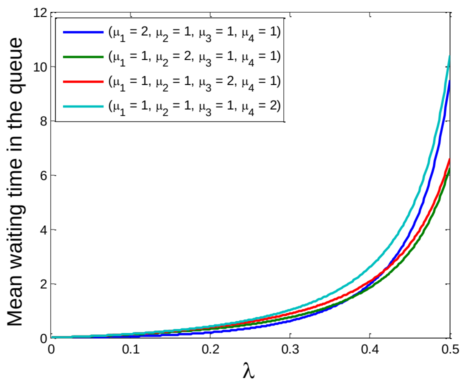


Figure 17. Mean waiting time in the queue (four service stations)

2.4 Summary and Discusstions

In this chapter, the steady-state probabilities of the series configuration queueing system consisting of two, three and four service stations with different service rates are successfully evaluated by matrix-geometric method. Therefore, matrix-geometric method is a powerful tool to study queueing systems with complex quasi-birth-death process. We further derived the exact form of stability conditions which is important for studying steady-state analysis for the series configuration systems. Numerical results of performance measures based on the evaluated steady-state probabilities are provided in order to investigate more characteristics of the systems. Disposition strategies to increase the operational efficiency of the series configuration queueing system are proposed through series numerical simulations.

Intuitively, arranging higher service rate for the service stations in front of the terminal station can be applied to the series configuration queueing system consisting of more than 2 service stations. Surprisingly, the simulations presented the opposite results on the intuitions. According to our simulations results, we suggest that a better disposition strategy for the system consisting of the even number of service stations would be increasing the service rates for the first service station and those stations near the terminal station in order to make the series configuration system work more efficiently. On the other hand, setting lower service rate for the first service station compared with other service stations in the system in order to obtain the best operational efficiency for the system with the odd number of service stations.

CHAPTER 3
TRANSIENT ANALYSIS OF A SERIES CONFIGURATION QUEUEING SYSTEM WITH BLOCKING PHENOMENA

3.1 Preface

Traditionally, most of the research on queueing systems focused on steady-state analysis, because it is relatively easier to evaluate steady-state probabilities and derive its related stability conditions based on the information of structure generator matrix. Although steady-state analysis can demonstrate important properties for the queueing system, it still lack the information of the system evolving from its initial conditions In addition, it is always need long time for a dynamic system turning into steady-state. Therefore, we try to conduct the transient analysis and study the dynamic behaviors for the series configuration queueing system consisting of two service stations in this chapter.

The series configurations system consisting of two service stations is shown in Figure 1. The quasi-birth-death process of the system with two service stations is shown in Figure 2. Transient state probabilities are estimated by Runge-Kutta method to solve the master equation of the system. Dynamic performance measures evolving with time including mean number in the system, mean waiting time in the system, blocking probability of the station-1 and rejecting probability of the system are evaluated through transient probabilities. For the system with large capacity, we discover that some of the performance measures are convergent to the results of our previous studies on steady-state analysis of the system. Simulation results reveal that there is still difference of operational efficiency of the system by setting different service rates for the service stations. We propose better disposition strategies for the system working in higher performance by numerical results.

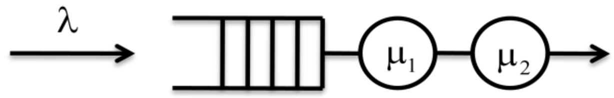


Figure 1. Series configuration queueing system with two service stations

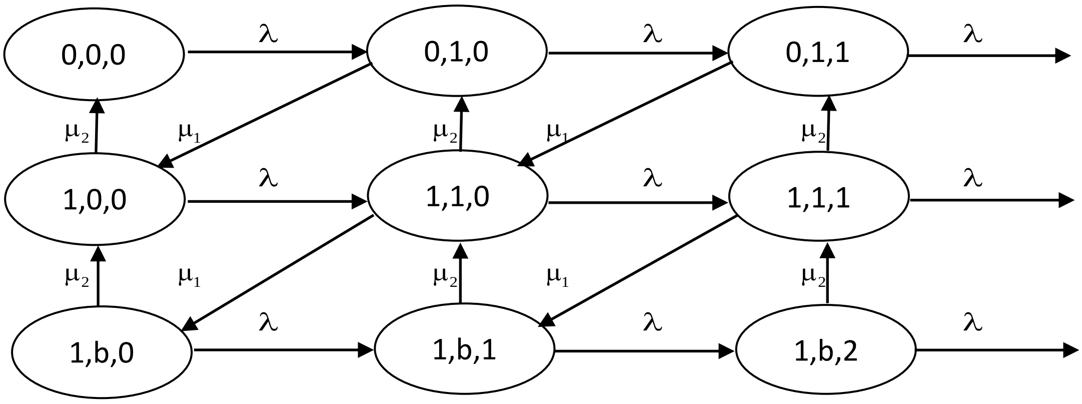


Figure 2. Quasi-birth-death process of the system with two service stations

The rest of the content in this chapter is organized as follows. The summary of notations used in our model and problem description are introduced in the beginning of next section. Formulation of dynamic behavior of the system is also described in the Section 3.2. Numerical results of the transient analysis of the system and disposition strategies are presented in Section 3.3. Finally, we conclude with discussions of our works and indicate possible directions for future research in section 4.

3.2 Modeling Framework

3.2.1 Problem Formulation and Notations

In our analysis, a series configuration queueing system consisting of two service stations operates independently and simultaneously. We assume Poisson arrival process with mean arrival rate λ. The time to serve a customer in each service station obeys exponential distribution with mean service time $\frac{1}{\mu}$. Customers should enter each service station to receive services in order. After complete the service in each server, the customer can leave the system. The existence of blocking phenomena after the service in the station-1 is because there is no queue between service stations. This phenomenon happen in the condition that a customer has completed the service in the station-1, but another customer is still receiving service in the station-2. In front of the first service station, there is a queue with finite capacity. Each service station can serve a customer at a time and the service time is independent of the number of customers. This system obeys the first come first serve (FCFS) discipline.

The notations μ_1 and μ_2 denote the service rate of the station-1 and the station-2, respectively. Moreover, we use $P_{n_1,n_2,n_3}(t)$ to denote the transient probability $P_{n_1,n_2,n_3}(t)$ of n_1 customer in the station-2 and n_2 customer in the station-1 and n_3 customer in the queue. For instance, the steady-state probability $P_{1,b,5}(t)$ means that there is a customer who is blocked in the station-1, since the customer in the station-2 is still receiving the service. There are five customers waiting in the queue.

3.2.2 Master Equations

The series configuration queueing system consisting of two service stations is modeled by continuous-time Markov process. The capacity of the system is supposed to be equal or larger than 5 (i.e. $K \geq 5$). According to the quasi-birth-and-death process, the dynamic behavior of the system can be described as following system of differential equations:

$$\frac{dP_{0,0,0}(t)}{dt} = -\lambda P_{0,0,0}(t) + \mu_2 P_{1,0,0}(t), \tag{1}$$

$$\frac{dP_{1,0,0}(t)}{dt} = -(\lambda + \mu_2)P_{1,0,0}(t) + \mu_2 P_{1,b,0}(t) + \mu_1 P_{0,1,0}(t), \tag{2}$$

$$\frac{dP_{1,b,0}(t)}{dt} = -(\lambda + \mu_2)P_{1,b,0}(t) + \mu_1 P_{1,1,0}(t), \tag{3}$$

$$\frac{dP_{0,1,i}(t)}{dt} = -(\lambda + \mu_1)P_{0,1,i}(t) + \lambda P_{0,0,i}(t) + \mu_2 P_{1,1,i}(t), \quad i = 0,1,2,\ldots,K-4. \tag{4}$$

$$\frac{dP_{1,1,i}(t)}{dt} = -(\lambda + \mu_1 + \mu_2)P_{1,1,i}(t) + \lambda P_{1,0,i}(t) + \mu_2 P_{1,b,i+1}(t) + \mu_1 P_{0,1,i+1}(t),$$
$$i = 0,1,2,\ldots,K-4. \tag{5}$$

$$\frac{dP_{1,b,i+1}(t)}{dt} = -(\lambda + \mu_2)P_{1,b,i+1}(t) + \lambda P_{1,b,i}(t) + \mu_1 P_{1,1,i+1}(t), \quad i = 0,1,2,\ldots,K-3. \tag{6}$$

$$\frac{dP_{0,1,K-3}(t)}{dt} = -(\lambda + \mu_1)P_{0,1,K-3}(t) + \lambda P_{0,1,K-4}(t) + \mu_2 P_{1,1,K-3}(t), \tag{7}$$

$$\frac{dP_{1,1,K-3}(t)}{dt} = -(\lambda + \mu_1 + \mu_2)P_{1,1,K-3}(t) + \lambda P_{1,1,K-4}(t) + \mu_2 P_{1,b,K-2}(t) + \mu_1 P_{0,1,K-2}(t), \tag{8}$$

$$\frac{dP_{1,b,K-2}(t)}{dt} = -\mu_2 P_{1,b,K-2}(t) + \lambda P_{1,b,K-3}(t) + \mu_1 P_{1,1,K-2}(t), \tag{9}$$

$$\frac{dP_{0,1,K-2}(t)}{dt} = -(\lambda + \mu_1)P_{0,1,K-2}(t) + \lambda P_{0,1,K-3}(t) + \mu_2 P_{1,1,K-2}(t), \tag{10}$$

$$\frac{dP_{1,1,K-2}(t)}{dt} = -(\mu_1 + \mu_2)P_{1,1,K-2}(t) + \lambda P_{1,1,K-3}(t) + \mu_1 P_{0,1,K-1}(t), \tag{11}$$

$$\frac{dP_{1,1,K-1}(t)}{dt} = -\mu_1 P_{1,1,K-1}(t) + \lambda P_{0,1,K-2}(t). \tag{12}$$

The normalization condition of the system at each time step t is

$$P_{0,0,0}(t)+P_{1,0,0}(t)+\sum_{i=0}^{K-1}P_{0,1,i}(t)+\sum_{i=0}^{K-2}P_{1,1,i}(t)+\sum_{i=0}^{K-2}P_{1,b,i}(t)=1. \tag{13}$$

Runge-Kutta numerical method is applied to evaluate transient probabilities of the system. Meanwhile, we assume that the system is empty at the beginning of the system state. This means that the initial conditions of the system are given by

$$P_{0,0,0}(0)=1;\ P_{1,0,0}(0)+\sum_{i=0}^{K-1}P_{0,1,i}(0)+\sum_{i=0}^{K-2}P_{1,1,i}(0)+\sum_{i=0}^{K-2}P_{1,b,i}(0)=0. \tag{14}$$

Although there are no concerns about the stability conditions of the system in transient analysis, we still cite the stability conditions from our previous works Tsai [26-28] here as indications for the cases of numerical results in Section 3.3.

Theorem 1. The stability conditions of the series configuration queueing system consisting of two service stations

(1) For $\mu_1 \neq \mu_2$

$$\lambda < \frac{\mu_1(\mu_1\mu_2+\mu_2^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2}.$$

(2) Special case: $\mu_1=\mu_2=\mu$

$$\lambda < \frac{2}{3}\mu.$$

3.2.3 Performance Measures

In this section, we define performance measures for the series configuration system consisting of two service stations. Mean number in the system, mean number in the queue, mean waiting time in the system, mean waiting time in the queue, blocking probability of the station-1 and rejecting probability of the system are described by transient probabilities for further studies of dynamic behavior of the system through numerical analysis.

- **Performance measures**

  Performance measures for the system consisting of two service stations are defined by

(1) Mean number of customers in the system

$$\begin{aligned}L(t)&=[P_{1,0,0}(t)+P_{0,1,0}(t)+P_{1,b,0}(t)]\\&+\sum_{n=2}^{K-1}[P_{1,b,n-1}(t)+P_{1,1,n-2}(t)+P_{0,1,n-1}(t)]\cdot n+P_{1,1,K-2}(t)\cdot K.\end{aligned} \tag{15}$$

(2) Mean number of customers in the queue

$$L_q(t) = \sum_{n=1}^{K-2} [P_{1,b,n}(t) + P_{1,1,n}(t) + P_{0,1,n}(t)] \cdot n. \tag{16}$$

(3) Blocking probability of the customer in the station-1

$$P_b(t) = \sum_{n=0}^{\infty} P_{1,b,n}(t). \tag{17}$$

(4) Rejecting probability of the system

$$P_r(t) = P_{1,b,K-2}(t) + P_{1,1,K-2}(t) + P_{0,1,K-1}(t). \tag{18}$$

(5) Mean waiting time in the system (Little's Law)

$$W(t) = \frac{L(t)}{\lambda_{eff}(t)}. \tag{19}$$

where $\lambda_{eff}(t) \equiv \lambda\,[1 - P_r(t)]$ is the effective mean arrival rate.

(6) Mean waiting time in the queue (Little's Law)

$$W_q(t) = \frac{L_q(t)}{\lambda_{eff}(t)}. \tag{20}$$

3.3 Numerical Results

In this section, numerical experiments for the queueing system consisting of two service stations are performed. Both performance metrics of the system with equivalent service rates (i.e. $\mu_1 = \mu_2 = \mu$) and different service rates are presented to study the dynamic behavior of the system. We proposed the disposition strategies based on the numerical results to make the system work in a better operational efficient way. The relaxation time and its fitting equations for the meaning waiting time in the system is provided to investigate the dynamic properties of the system.

3.3.1 Each Service Station with Same Service Rate

We first study the effects of the capacity of the queue on decay rate of different performance measures. We fix $\mu_1 = \mu_2 = 1$, $\lambda = 0.666$ and set $K = 200$ and $K = 50$. It is investigated how mean number in the system, mean waiting time in the system, blocking probability and rejecting probability of the system evolve with time when the finite capacity of the queue is 200, as shown in Figure 3 ~ Figure 6. It is observed that the decay speed of each performance converges to the steady-states is pretty slow, because it almost takes 50000 (time steps) from the transition states to become steady states. On the other hand, we discover that the speed of convergence is faster in the cases that we set lower capacity (K = 50) for the system, as shown in Figure 7 ~ Figure 10. It is noted that the steady-state of blocking probability in both cases approach to 0.33. This result is consistent with our simulations in previous work Tsai [26-28]. Furthermore, it is noted that the difference of mean waiting time in the system is not large for each case with different mean arrival rate in the early stage of transient states, as shown in Figure 12. The blocking probabilities still approach 0.33 in all cases when the transient states become steady, as shown in Figure 13.

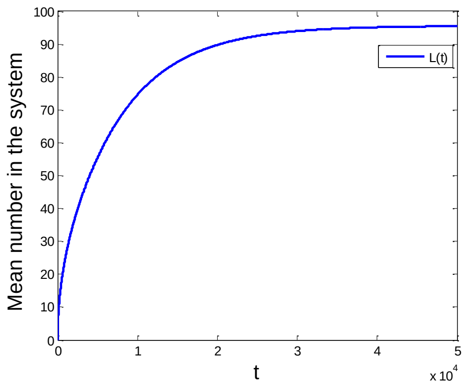


Figure 3. Mean number in the system (K = 200)

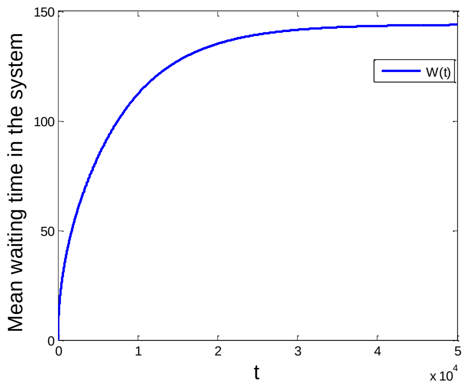


Figure 4. Mean waiting time in the system (K = 200)

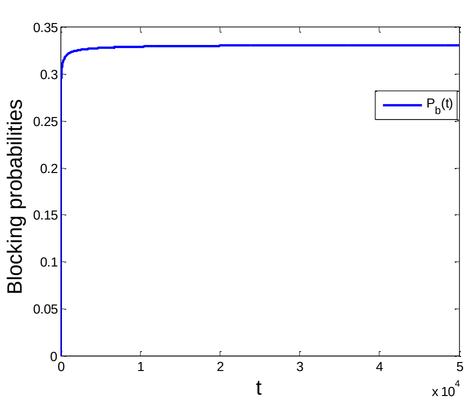


Figure 5. Blocking probability (K = 200)

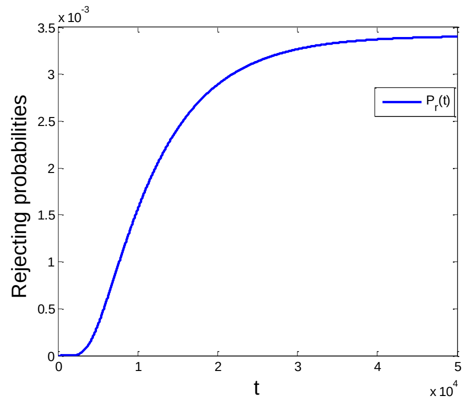


Figure 6. Rejecting probability (K = 200)

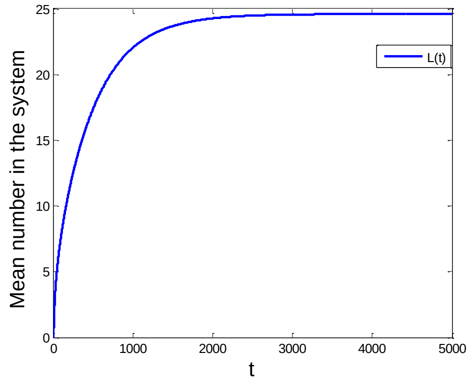


Figure 7. Mean number in the system (K = 50)

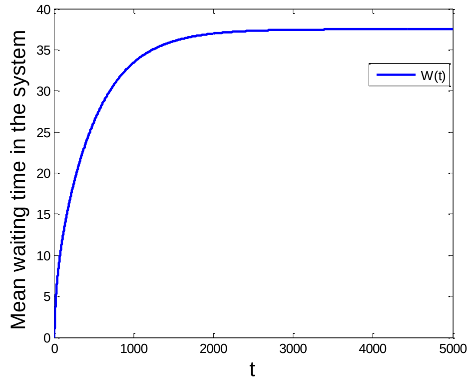


Figure 8. Mean waiting time in the system (K = 50)

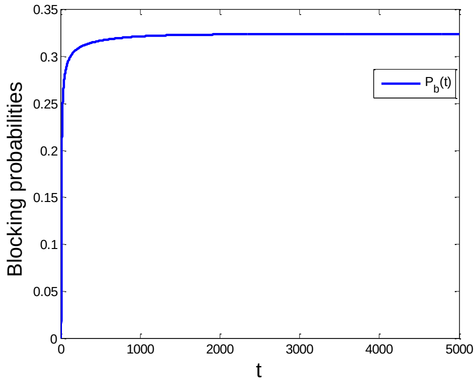


Figure 9. Blocking probability (K = 50)

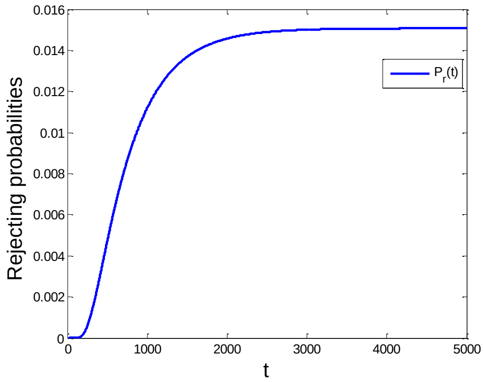


Figure 10. Rejecting probability (K = 50)

We continue to study the effects of various mean arrival rates on performance measures of the system. We set $\mu_1 = \mu_2 = 1$, $K = 50$ and varies mean arrival rate from 1~5. It can be easily observed that the convergent speed of each performance measure increases as mean arrival rate increases, as shown in Figure 11 ~ Figure 14., respectively.

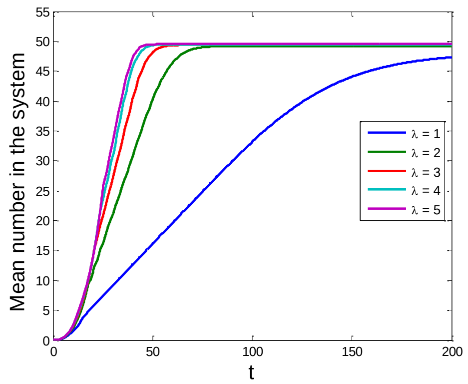


Figure 11. Mean number in the system (K = 50)

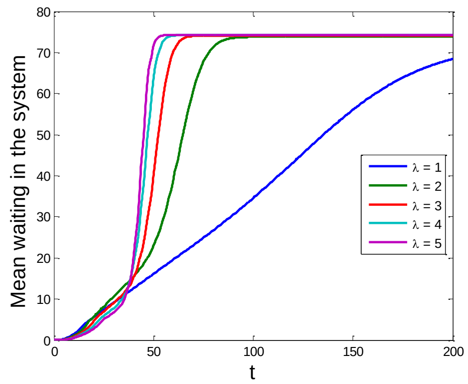


Figure 12. Mean waiting time in the system (K = 50)

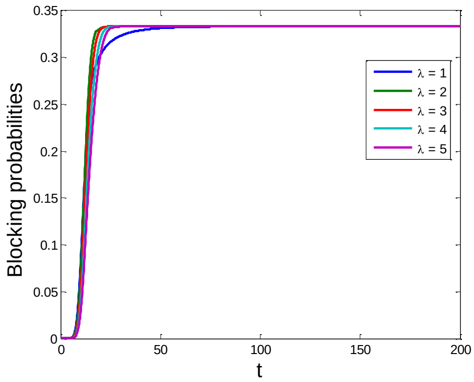


Figure 13. Blocking probability (K = 50)

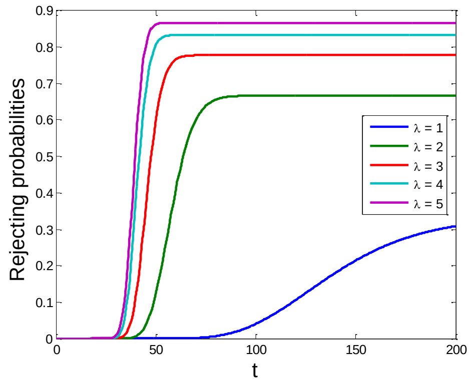


Figure 14. Rejecting probability (K = 50)

3.3.2 Each Service Station with Different Service Rates

Next, we study the impact of setting different service rates for the service stations on operational efficiency of the system (i.e. mean waiting time in the system), and suggest disposition strategy for the system operating more efficiently. The capacity setting in the following cases is $K = 15$.

The following cases are considered:

Case 1

We choose $\mu_1 = 1,\ \mu_2 = 2$ and $\mu_1 = 2,\ \mu_2 = 1$ and set $\lambda = 0.2$.

Case 2

We choose $\mu_1 = 1,\ \mu_2 = 2$ and $\mu_1 = 2,\ \mu_2 = 1$ and set $\lambda = 0.4$.

Case 3

We choose $\mu_1 = 1,\ \mu_2 = 2$ and $\mu_1 = 2,\ \mu_2 = 1$ and set $\lambda = 0.66$.

Case 4

We choose $\mu_1 = 1,\ \mu_2 = 2$ and $\mu_1 = 2,\ \mu_2 = 1$ and set $\lambda = 0.7$.

Case 5

We choose $\mu_1 = 1,\ \mu_2 = 2$ and $\mu_1 = 2,\ \mu_2 = 1$ and set $\lambda = 0.74$.

Case 6

We choose $\mu_1 = 1,\ \mu_2 = 2$ and $\mu_1 = 2,\ \mu_2 = 1$ and set $\lambda = 0.78$.

As time goes by, we discover that disposing different service rate for the service stations causes different operational efficiency of the system, as shown in Figure 15. ~ Figure 20., respectively. It can also be observed that the speed of decay to the steady-state of the mean waiting time in the system decreases as mean arrival rate increases. We finally suggest that disposing higher service rate for the station-1 in order to keep the high performance operations of the series configuration queueing system consisting of two service stations. Transient analysis shows the same pattern of the disposition strategy as our previous works on steady-state analysis Tsai [26-28].

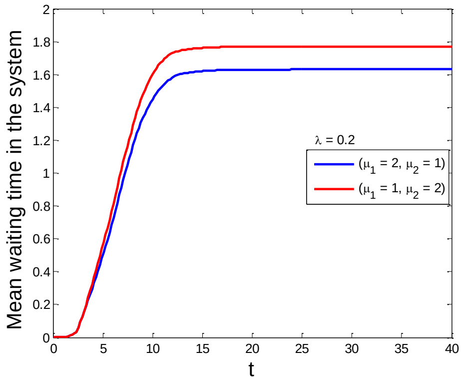


Figure 15. Mean waiting time in the system (K = 15)

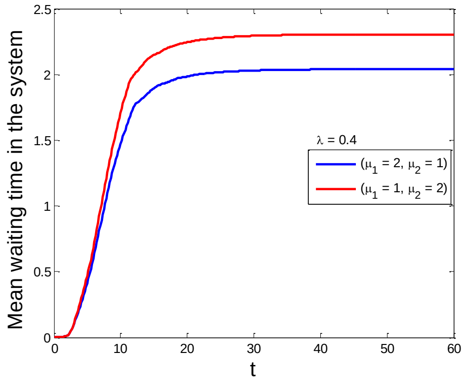


Figure 16. Mean waiting time in the system (K = 15)

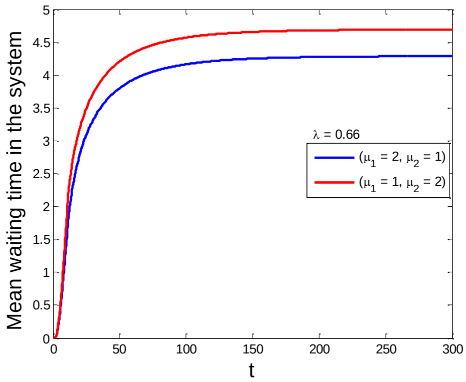


Figure 17. Mean waiting time in the system (K = 15)

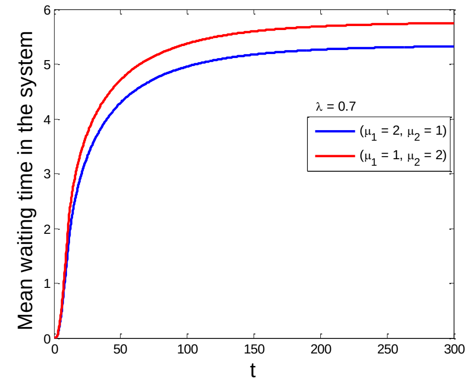


Figure 18. Mean waiting time in the system (K = 15)

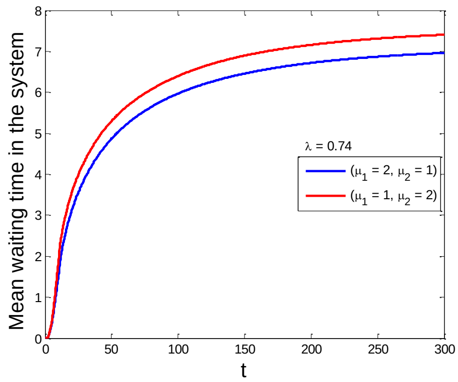


Figure 19. Mean waiting time in the system (K = 15)

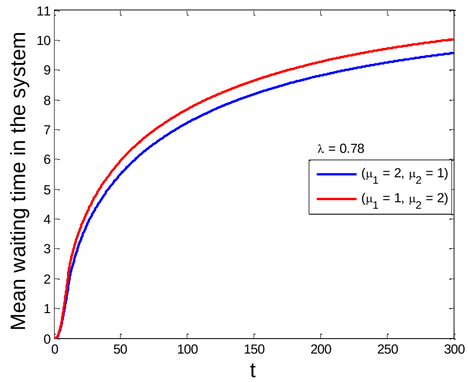


Figure 20. Mean waiting time in the system (K = 15)

3.3.3 Analysis of Relaxation Time

In this section, we analyze the relaxation time of transient states of the system. Here, the relaxation time is denoted as R. It is noted that the relaxation time of the system in disposition strategy $\mu_1 = 2, \mu_2 = 1$ is longer than that of the $\mu_1 = 1, \mu_2 = 2$ as show in Figure 21~ Figure 28. The relaxation time is increasing significantly as mean arrival rate λ increases. The dynamic properties of the system with different combination of parameters can be described by fitting equations in each case.

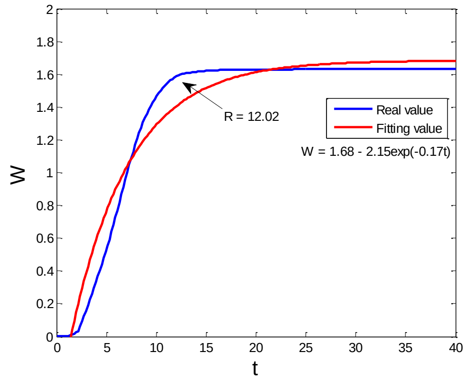


Figure 21. Mean waiting time in the system $(\lambda = 0.2, \mu_1 = 2, \mu_2 = 1)$

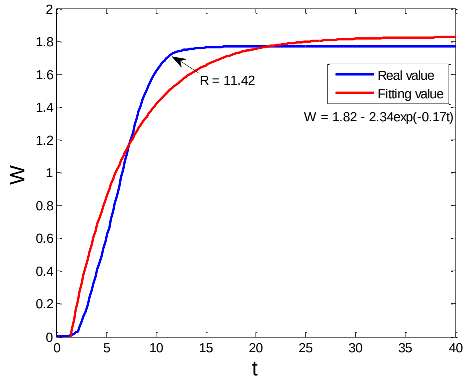


Figure 22. Mean waiting time in the system $(\lambda = 0.2, \mu_1 = 1, \mu_2 = 2)$

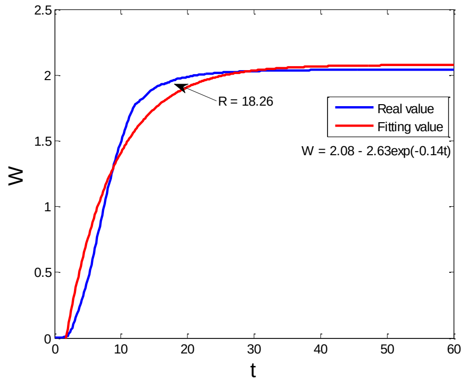


Figure 23. Mean waiting time in the system $(\lambda = 0.4, \mu_1 = 2, \mu_2 = 1)$

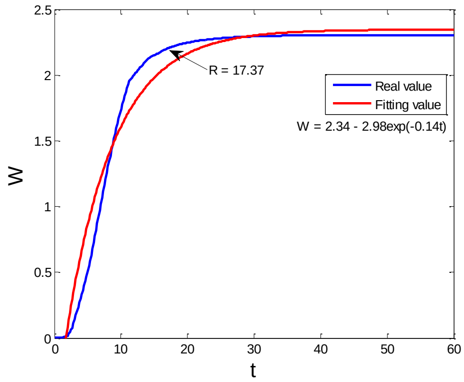


Figure 24. Mean waiting time in the system $(\lambda = 0.4, \mu_1 = 1, \mu_2 = 2)$

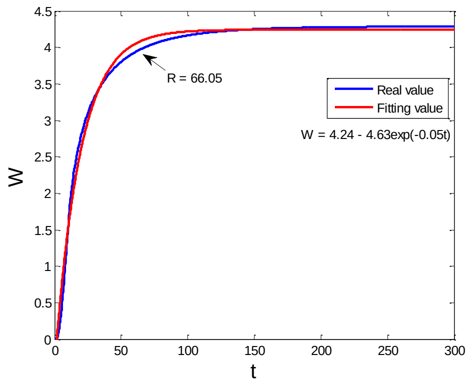


Figure 25. Mean waiting time in the system $(\lambda = 0.66, \mu_1 = 2, \mu_2 = 1)$

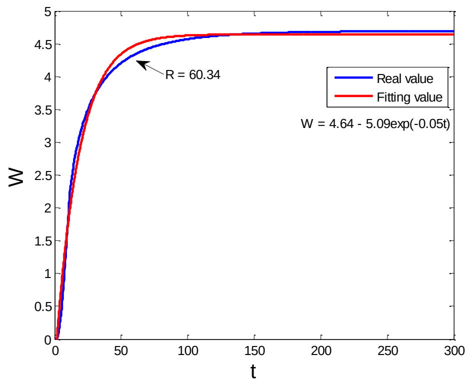


Figure 26. Mean waiting time in the system $(\lambda = 0.66, \mu_1 = 1, \mu_2 = 2)$

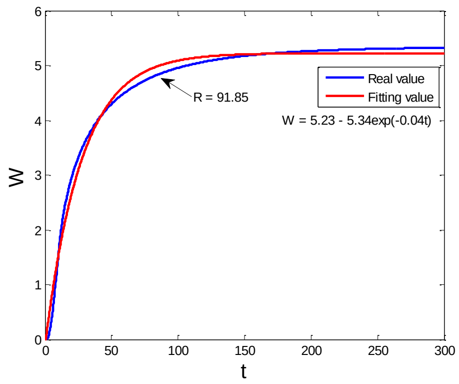


Figure 27. Mean waiting time in the system $(\lambda = 0.7,\ \mu_1 = 2,\ \mu_2 = 1)$

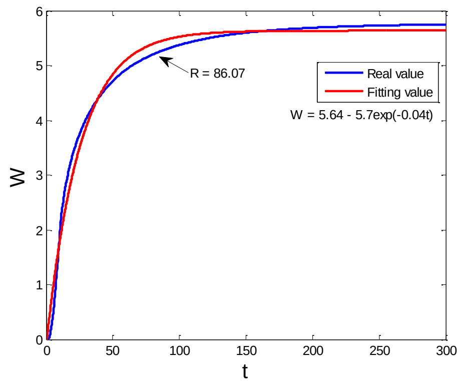


Figure 28. Mean waiting time in the system $(\lambda = 0.7,\ \mu_1 = 1,\ \mu_2 = 2)$

3.4 Summary and Discusstions

We successfully solved the transient probabilities of the series configuration queueing system consisting of two service stations by Runge-Kutta method. We also calculated important transient performance measures, such as mean number in the system, mean waiting time in the system, blocking probability and rejecting probability of the system to investigate the dynamic behavior of the system. Moreover, the convergent speed of each performance measure of the system depending on the capacity of the system and the mean arrival rate is investigated. Relaxation time and the fitting equation are provided to study the dynamic properties of the system.

We suggest setting higher service rate for the station-1 of the series configuration queueing system with two service stations in order to keep better operational efficiency for the system. This consideration is consistent with the steady-state analysis for the same system in Chapter 2. We have also shown the same results of disposition strategies for the transient analysis of the system in this Chapter.

CHAPTER4
PERFORMANCE ANALYSIS OF SERIES CONFIGURATION SYSTEM SUBJECT TO BREAKDOWNS AND REPAIRS

4.1 Preface

The breakdowns of machines happen frequently in real applications. Recently, the big data analytics has been applied to detect the failures of machines in factory due to development of storage technology and computational platforms. It is possible to detect related failure information before the failure of machines by applying machine learning techniques. In this way, we can maintain the machines more effectively and save other unnecessary costs. In this Chapter, we try to analyze series configuration subject to breakdowns and repairs. The theoretical results provide insights to know the characteristics of the system, so that the operating staffs can maintain the system suitably by means of controlling specific system parameters of the system, such as mean service rate, breakdown rate and repair rate.

The quasi-birth-death process of the queueing system subject to breakdowns and repairs is shown in Figure 1. It is noted that the system works in three possible major working states which include working states of servers without breakdowns, working states of servers with breakdowns of station-1 and working states of servers with breakdowns of station-2. By matrix-geometric method, we can conduct the steady-state analysis for the system and evaluate the failure probability of the station-1, station-2 and reliable probability of the system corresponding to different combinations of system parameters. Sensitivity analysis for the system with various service rates, breakdown rates and repair rates are also demonstrated. We further propose disposition strategies by disposing system parameters to make the system work more efficiently and more reliably based on the information of simulations.

- **Problem Formulation**

In our analysis, the queueing system consists of two independent service stations placed in series configuration and operates simultaneously with server breakdowns. For the simplicity of modeling work, we assume that the service stations cannot become breakdown simultaneously in this study. Poisson arrival process with mean arrival rate λ is assumed. In each station, the average time to serve a customer follows exponential distribution with mean $\frac{1}{\mu}$. The service stations can break down and the breakdown times are exponentially distributed with breakdown rate α. Concurrently, the repair time is assumed to be exponential with mean repair time $\frac{1}{\beta}$. A complete service is defined as customers passing through all of the service stations in order and finishes the final service in the terminal station. There is no queue between service stations. An infinite capacity queue in front of the first station is allowed. Each service station can only serve a customer at a time while the service rate is independent of the number of customers. The service of the system obeys the first come first serve (FCFS) discipline.

4.1.1 Contributions and Outline

Major theoretical results in this chapter including 1) constructing steady-state structure generator matrix equations of the queueing system with two service stations, 2) deriving stability conditions consist of system parameters in exact form, 3) evaluating the steady-state probabilities with different conditions of system parameters, 4) presenting insights through numerical simulations to propose disposition strategies for the system working more smartly and efficiently. Practically, the practitioners can benefit from our theoretical results by disposing finite resources and adjusting related parameters based on the information to increase the operational efficiency of the system in applications. Furthermore performance measures of the system can be quantitatively described by our model, such as mean number in the system, mean waiting time in the system, blocking probability of the station-1, reliable probability of the system, failure probability of each service station.

The rest of the content in this Chapter is organized as follows. First, we summarize notations used in our model in the beginning of next section. Section 2. includes details of matrix-geometric method applied to evaluate steady-state probabilities of the system. Furthermore, the stability conditions and major performance metrics are also included in this section. In section 3., we perform numerical experiments and propose disposition strategies for the system through case studies. Finally, conclusions and discussions of our works and indications of possible directions for future research are included in Section 4.

4.2. Modeling Framework

4.2.1 Notations

In this section, the notations used in our model framework are introduced. In steady-state, the following notations are used.

λ, mean arrival rate of the customers

μ_1, mean service rate of the station-1

μ_2, mean service rate of the station-2

α_1, mean failure rate of the station-1

α_2, mean failure rate of the station-2

β_1, mean repair rate of the station-1

β_2, mean repair rate of the station-2

$P_{0;n_1,n_2,n_3}$, steady-state probability in working states (both service stations work concurrently). $P_{1;n_1,n_2,n_3}$, steady-state probability in failure states of the station-1 (only station-2 works), $P_{2;n_1,n_2,n_3}$, steady-state probability in failure states of the station-2 (only station-1 works), Note that the notation $P_{0;n_1,n_2,n_3}$ is used to denote the steady-state probability in working states $P_{0;n_1,n_2,n_3}$ of n_1 customer in the station-2 and n_2 customer in the station-1 and n_3 customer in the queue. For instance, the notation $P_{1;1,b,7}$ of steady-state probability means that in failure states of the station-1, there is a customer receiving the service in the station-2 and a customer who is blocked in the station-1 by the customer in the station-2 and seven customers waiting to secure services in the queue.

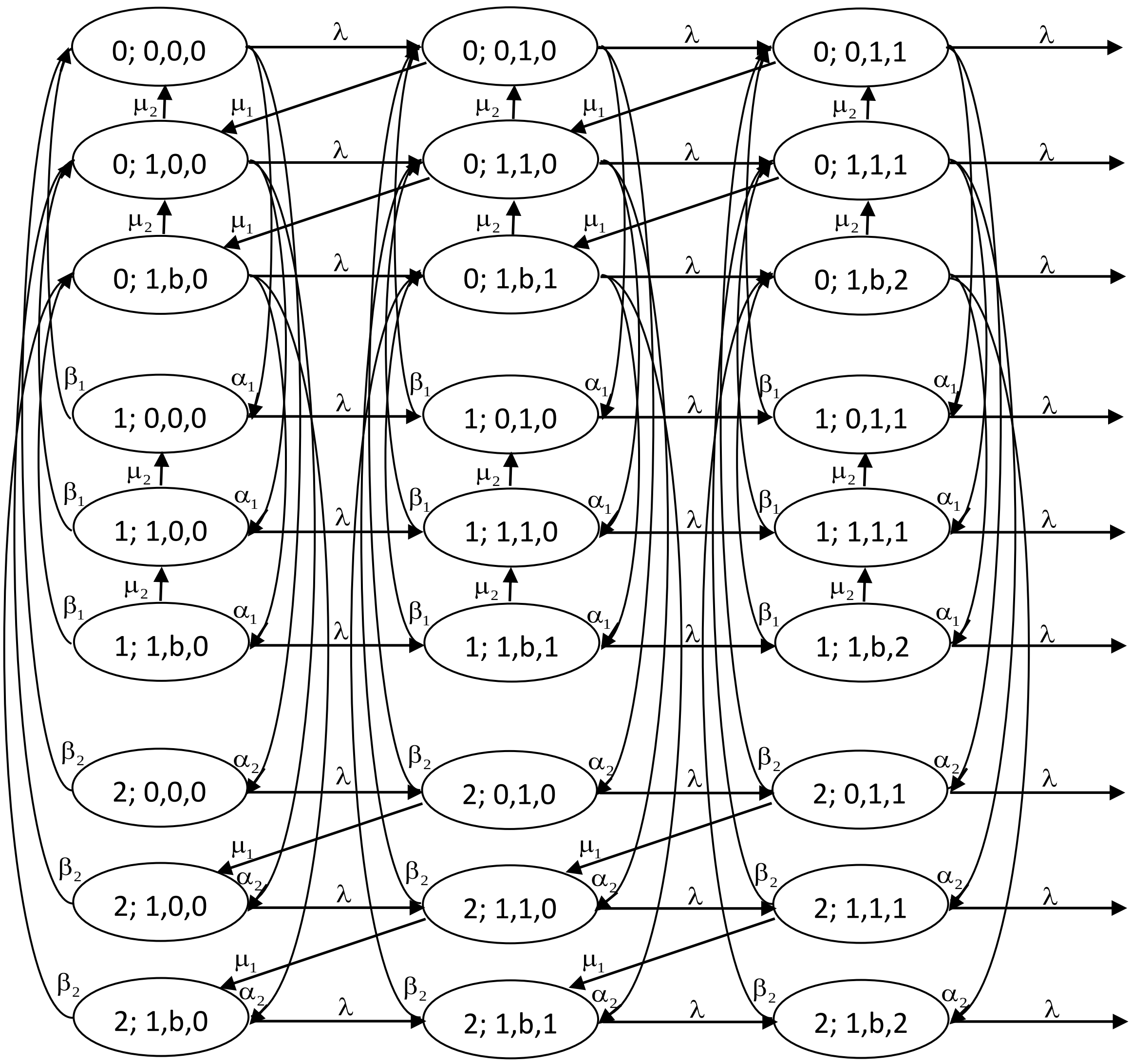


Figure 1. State transition diagram of the queueing system subject to breakdowns and repairs

4.2.2 Matrix-Geometric Method

Let $\mathbf{P} = [\mathbf{P_0}, \mathbf{P_1}, \mathbf{P}_2, ...]$ denote as steady-state probability vector corresponding to the transition matrix Q. The steady-state probability vector contains steady-state probabilities of the quasi-death-birth process in working states, failure states of the station-1 and the station-2. The compositions of the sub-matrices of the transition matrix Q for the system are shown in Appendix A. The steady-state equations of the quasi-birth-death process in vector form with the transition matrix can be written as $\mathbf{P}\mathrm{Q} = \mathbf{0}$, while $\mathbf{P1} = 1$ is the normalization condition of the steady-state probability. Then, the global steady-state equations of the quasi-birth-death process can be described as

$$\mathbf{P_0}\mathrm{B}_{0,0} + \mathbf{P_1}\mathrm{B}_{1,0} = \mathbf{0}, \tag{1}$$

$$\mathbf{P_0}\mathrm{B}_{0,1} + \mathbf{P_1}\mathrm{A}_1 + \mathbf{P}_2\mathrm{A}_2 = \mathbf{0}, \tag{2}$$

$$\mathbf{P}_\mathrm{i}\mathrm{A}_0 + \mathbf{P}_{\mathrm{i+1}}\mathrm{A}_1 + \mathbf{P}_{\mathrm{i+2}}\mathrm{A}_2 = \mathbf{0}, \qquad \mathrm{i} \geq 1. \tag{3}$$

There exists a rate matrix R, and follows the recurrence relations

$$\mathbf{P_i} = \mathbf{P_{i\text{-}1}}\mathrm{R} = \mathbf{P_1}\mathrm{R}^{\mathrm{i}-1}, \qquad \mathrm{i} > 1. \tag{4}$$

We substitute (4) into (3), and simplify to quadratic matrix equation in order to solve the rate matrix R

$$\mathrm{A}_0 + \mathrm{RA}_1 + \mathrm{R}^2\mathrm{A}_2 = 0. \tag{5}$$

The simplified equations of (1) and (2) can be represented as

$$\mathbf{P_0}\mathrm{B}_{0,0} + \mathbf{P_1}\mathrm{B}_{1,0} = \mathbf{0}, \tag{6}$$

$$\mathbf{P_0}\mathrm{B}_{0,1} + \mathbf{P_1}(\mathrm{A}_1 + \mathrm{RA}_2) = \mathbf{0}. \tag{7}$$

The normalization condition equation that only involves $\mathbf{P_0}$ and $\mathbf{P_1}$ can be referred in Bloch et al. [23]

$$\mathbf{P_0}\mathbf{1} + \mathbf{P_1}(\mathrm{I} - \mathrm{R})^{-1}\mathbf{1} = 1, \tag{8}$$

where I is the identity matrix with same size as the rate matrix R.

The rate matrix R is solved by logarithmic reduction method, described in Bloch et al. [23] from (5). Then, taking (6), (7) and (8) into together, the steady-state probability vector of $\mathbf{P_0}$ and $\mathbf{P}_1$ can be obtained by solving following matrix equation

$$(\mathbf{P_0}, \mathbf{P_1})\begin{pmatrix} \mathrm{B}_{0,0} & \mathrm{B}_{0,1}^{*} & \mathbf{1} \\ \mathrm{B}_{1,0} & (\mathrm{A}_1 + \mathrm{RA}_2)^{*} & (\mathrm{I} - \mathrm{R})^{-1}\mathbf{1} \end{pmatrix} = (\mathbf{0}, 1). \tag{9}$$

where $(.)^*$ indicates that the last column of the included matrix is removed to avoid linear dependency.

4.2.3 Stability Conditions

According to Neuts [10], the stability conditions of the system can be derived from following inequality:

$$\mathbf{P}_A A_0 \mathbf{1} < \mathbf{P}_A A_2 \mathbf{1}, \tag{10}$$

where $\mathbf{P}_A$ is the steady-state probability vector corresponding to the conservative stable matrix A.

The conservative stable matrix is defined to be

$$A = A_0 + A_1 + A_2. \tag{11}$$

We can obtain the steady-state probability $\mathbf{P_A}$ by solving the following system equations with normalization condition

$$\mathbf{P_A} A = \mathbf{0}, \tag{12}$$

$$\sum_{i=0}^{8} P_{A,i} = 1. \tag{13}$$

4.2.4 Performance Measures and Exact Results

In this section, we define the performance measures for the series configuration system consisting of two service stations subject to breakdowns and repairs. Performance measures include mean number in the system, mean number in the queue, mean waiting time in the system, mean waiting time in the queue and blocking probability of the station-1, failure probability of each service station and reliable probability of the system. Exact formulae of stability conditions for the system with special and general case are also given in the section.

- **Stability Conditions**

Theorem 1. The following inequalities are necessary and sufficient conditions for the system to be stable.

(1) Special case: $\mu_1 = \mu_2 = \mu$, $\alpha_1 = \alpha_2 = \alpha$, and $\beta_1 = \beta_2 = \beta$,

$$\lambda < \frac{2\beta\mu(\alpha+\beta+\mu)^2}{(2\alpha+\beta)(3\alpha\beta+3\beta^2+2\alpha\mu+6\beta\mu+3\mu^2)}. \tag{14}$$

(2) General case: $\mu_1 \neq \mu_2$, $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$,

$$\lambda < \frac{N}{D}, \tag{15}$$

where

$$
\begin{aligned}
N = {} & \beta_1\beta_2\mu_1\mu_2(\alpha_1\alpha_2^2\beta_1\mu_1 + \alpha_2^2\beta_1^2\mu_1 + 2\alpha_1\alpha_2\beta_1\beta_2\mu_1 + 2\alpha_2\beta_1^2\beta_2\mu_1 + \alpha_1\beta_1\beta_2^2\mu_1 + \beta_1^2\beta_2^2\mu_1 \\
& + 2\alpha_1\alpha_2\beta_1\mu_1^2 + 2\alpha_2\beta_1^2\mu_1^2 + 2\alpha_1\beta_1\beta_2\mu_1^2 + 2\beta_1^2\beta_2\mu_1^2 + \alpha_1\beta_1\mu_1^3 + \beta_1^2\mu_1^3 + \alpha_1^2\alpha_2\beta_2\mu_2 \\
& + 2\alpha_1\alpha_2\beta_1\beta_2\mu_2 + \alpha_2\beta_1^2\beta_2\mu_2 + \alpha_1^2\beta_2^2\mu_2 + 2\alpha_1\beta_1\beta_2^2\mu_2 + \beta_1^2\beta_2^2\mu_2 + 2\alpha_1^2\alpha_2\mu_1\mu_2 + 2\alpha_1\alpha_2^2\mu_1\mu_2 \\
& + 4\alpha_1\alpha_2\beta_1\mu_1\mu_2 + 2\alpha_2^2\beta_1\mu_1\mu_2 + 2\alpha_2\beta_1^2\mu_1\mu_2 + 2\alpha_1^2\beta_2\mu_1\mu_2 + 4\alpha_1\alpha_2\beta_2\mu_1\mu_2 + 4\alpha_1\beta_1\beta_2\mu_1\mu_2 \\
& + 4\alpha_2\beta_1\beta_2\mu_1\mu_2 + 2\beta_1^2\beta_2\mu_1\mu_2 + 2\alpha_1\beta_2^2\mu_1\mu_2 + 2\beta_1\beta_2^2\mu_1\mu_2 + \alpha_1^2\mu_1^2\mu_2 + 4\alpha_1\alpha_2\mu_1^2\mu_2 + 2\alpha_1\beta_1\mu_1^2\mu_2 \\
& + 4\alpha_2\beta_1\mu_1^2\mu_2 + \beta_1^2\mu_1^2\mu_2 + 4\alpha_1\beta_2\mu_1^2\mu_2 + 4\beta_1\beta_2\mu_1^2\mu_2 + 2\alpha_1\mu_1^3\mu_2 + 2\beta_1\mu_1^3\mu_2 + 2\alpha_1\alpha_2\beta_2\mu_2^2 \\
& + 2\alpha_2\beta_1\beta_2\mu_2^2 + 2\alpha_1\beta_2^2\mu_2^2 + 2\beta_1\beta_2^2\mu_2^2 + 4\alpha_1\alpha_2\mu_1\mu_2^2 + \alpha_2^2\mu_1\mu_2^2 + 4\alpha_2\beta_1\mu_1\mu_2^2 + 4\alpha_1\beta_2\mu_1\mu_2^2 \\
& + 2\alpha_2\beta_2\mu_1\mu_2^2 + 4\beta_1\beta_2\mu_1\mu_2^2 + \beta_2^2\mu_1\mu_2^2 + 2\alpha_1\mu_1^2\mu_2^2 + 2\alpha_2\mu_1^2\mu_2^2 + 2\beta_1\mu_1^2\mu_2^2 + 2\beta_2\mu_1^2\mu_2^2 \\
& + \mu_1^3\mu_2^2 + \alpha_2\beta_2\mu_2^3 + \beta_2^2\mu_2^3 + 2\alpha_2\mu_1\mu_2^3 + 2\beta_2\mu_1\mu_2^3 + \mu_1^2\mu_2^3)
\end{aligned}
$$

and

$$
\begin{aligned}
D = {} & (\alpha_2\beta_1 + \alpha_1\beta_2 + \beta_1\beta_2)(\alpha_2^2\beta_1^2\mu_1^2 + 2\alpha_2\beta_1^2\beta_2\mu_1^2 + \beta_1^2\beta_2^2\mu_1^2 + 2\alpha_2\beta_1^2\mu_1^3 + 2\beta_1^2\beta_2\mu_1^3 + \beta_1^2\mu_1^4 \\
& + \alpha_1\alpha_2\beta_1\beta_2\mu_1\mu_2 + \alpha_2\beta_1^2\beta_2\mu_1\mu_2 + \alpha_1\beta_1\beta_2^2\mu_1\mu_2 + \beta_1^2\beta_2^2\mu_1\mu_2 + 2\alpha_1\alpha_2\beta_1\mu_1^2\mu_2 + 2\alpha_2^2\beta_1\mu_1^2\mu_2 \\
& + 2\alpha_2\beta_1^2\mu_1^2\mu_2 + 2\alpha_1\beta_1\beta_2\mu_1^2\mu_2 + 4\alpha_2\beta_1\beta_2\mu_1^2\mu_2 + 2\beta_1^2\beta_2\mu_1^2\mu_2 + 2\beta_1\beta_2^2\mu_1^2\mu_2 + \alpha_1\beta_1\mu_1^3\mu_2 \\
& + 4\alpha_2\beta_1\mu_1^3\mu_2 + \beta_1^2\mu_1^3\mu_2 + 4\beta_1\beta_2\mu_1^3\mu_2 + 2\beta_1\mu_1^4\mu_2 + \alpha_1^2\beta_2^2\mu_2^2 + 2\alpha_1\beta_1\beta_2^2\mu_2^2 + \beta_1^2\beta_2^2\mu_2^2 \\
& + 2\alpha_1^2\beta_2\mu_1\mu_2^2 + 2\alpha_1\alpha_2\beta_2\mu_1\mu_2^2 + 4\alpha_1\beta_1\beta_2\mu_1\mu_2^2 + 2\alpha_2\beta_1\beta_2\mu_1\mu_2^2 + 2\beta_1^2\beta_2\mu_1\mu_2^2 + 2\alpha_1\beta_2^2\mu_1\mu_2^2 \\
& + 2\beta_1\beta_2^2\mu_1\mu_2^2 + \alpha_1^2\mu_1^2\mu_2^2 + 2\alpha_1\alpha_2\mu_1^2\mu_2^2 + \alpha_2^2\mu_1^2\mu_2^2 + 2\alpha_1\beta_1\mu_1^2\mu_2^2 + 4\alpha_2\beta_1\mu_1^2\mu_2^2 + \beta_1^2\mu_1^2\mu_2^2 \\
& + 4\alpha_1\beta_2\mu_1^2\mu_2^2 + 2\alpha_2\beta_2\mu_1^2\mu_2^2 + 4\beta_1\beta_2\mu_1^2\mu_2^2 + \beta_2^2\mu_1^2\mu_2^2 + 2\alpha_1\mu_1^3\mu_2^2 + 2\alpha_2\mu_1^3\mu_2^2 + 2\beta_1\mu_1^3\mu_2^2 \\
& + 2\beta_2\mu_1^3\mu_2^2 + \mu_1^4\mu_2^2 + 2\alpha_1\beta_2^2\mu_2^3 + 2\beta_1\beta_2^2\mu_2^3 + 4\alpha_1\beta_2\mu_1\mu_2^3 + \alpha_2\beta_2\mu_1\mu_2^3 + 4\beta_1\beta_2\mu_1\mu_2^3 + \beta_2^2\mu_1\mu_2^3 \\
& + 2\alpha_1\mu_1^2\mu_2^3 + 2\alpha_2\mu_1^2\mu_2^3 + 2\beta_1\mu_1^2\mu_2^3 + 2\beta_2\mu_1^2\mu_2^3 + \mu_1^3\mu_2^3 + \beta_2^2\mu_2^4 + 2\beta_2\mu_1\mu_2^4 + \mu_1^2\mu_2^4)
\end{aligned}
$$

- **Performance Measures**

  Performance measures for the system consisting of two service stations subject to breakdowns and repairs are defined by

  (1) Mean number of customers in the system

$$
\begin{aligned}
L = {} & (P_{0;1,0,0} + P_{0;0,1,0} + P_{0;1,b,0}) + \sum_{n=2}^{\infty}(P_{0;1,b,n-1} + P_{0;1,1,n-2} + P_{0;0,1,n-1}) \cdot n \\
& + (P_{1;1,0,0} + P_{1;0,1,0} + P_{1;1,b,0}) + \sum_{n=2}^{\infty}(P_{1;1,b,n-1} + P_{1;1,1,n-2} + P_{1;0,1,n-1}) \cdot n \\
& + (P_{2;1,0,0} + P_{2;0,1,0} + P_{2;1,b,0}) + \sum_{n=2}^{\infty}(P_{2;1,b,n-1} + P_{2;1,1,n-2} + P_{2;0,1,n-1}) \cdot n.
\end{aligned} \tag{16}
$$

  (2) Mean number of customers in the queue

$$
\begin{aligned}
L_q = {} & \sum_{n=1}^{\infty}(P_{0;1,b,n} + P_{0;1,1,n} + P_{0;0,1,n}) \cdot n + \sum_{n=1}^{\infty}(P_{1;1,b,n} + P_{1;1,1,n} + P_{1;0,1,n}) \cdot n \\
& + \sum_{n=1}^{\infty}(P_{2;1,b,n} + P_{2;1,1,n} + P_{2;0,1,n}) \cdot n.
\end{aligned} \tag{17}
$$

(3) Mean waiting time in the system (Little's Law)

$$W = \frac{L}{\lambda}. \tag{18}$$

(4) Mean waiting time in the queue (Little's Law)

$$W_q = \frac{L_q}{\lambda}. \tag{19}$$

(5) Blocking probability of the customer in the station-1

$$P_b = \sum_{n=0}^{\infty} P_{0;1,b,n} + \sum_{n=0}^{\infty} P_{1;1,b,n} + \sum_{n=0}^{\infty} P_{2;1,b,n}. \tag{20}$$

(6) Failure probability of the station-1

$$P_{f,1} = (P_{1;0,0,0} + P_{1;1,0,0} + P_{1;1,b,0}) + \sum_{n=1}^{\infty} P_{1;0,1,n-1} + P_{1;1,1,n-1} + P_{1;1,b,n}. \tag{21}$$

(7) Failure probability of the station-2

$$P_{f,2} = (P_{2;0,0,0} + P_{2;1,0,0} + P_{2;1,b,0}) + \sum_{n=1}^{\infty} P_{2;0,1,n-1} + P_{2;1,1,n-1} + P_{2;1,b,n}. \tag{22}$$

(8) Reliable probability of the system

$$\begin{aligned} P_r &= (P_{0;0,0,0} + P_{0;1,0,0} + P_{0;1,b,0}) + \sum_{n=1}^{\infty} P_{0;0,1,n-1} + P_{0;1,1,n-1} + P_{0;1,b,n} \\ &= 1 - (P_{f,1} + P_{f,2}). \end{aligned} \tag{23}$$

4.3. Numerical Results

In this section, we perform numerical experiments for the queueing system consisting of two service stations subject to breakdowns and repairs to study the effects of various parameters on mean waiting time in the system, blocking probability, failure probability and reliable probability. In addition, sensitivity analysis clearly shows how performance measures vary with specific parameters including service rates, breakdown rates and repair rates. Disposition strategies and methods of controlling parameters to increase operational efficiency for the system according to the results of numerical simulation are proposed.

4.3.1 Validation of Stability Conditions by Numerical Results

First, the consistency of stability conditions is examined by numerical results. We fix $\mu_1 = \mu_2 = 1$, $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 1$, and present the trends of mean number in the system as the mean arrival rate λ varies from 0.01 to 0.352, as shown in Figure 2. It is observed that mean number in the system increases as λ increases. The bound of the stability condition in this case ($\frac{6}{17} \approx 0.352$) validates the exact results derived in section 4.2.4. It is noted that the blocking probability of the station-1 increases as the values of λ increases and the maximum value of the blocking probability of the station-1 is near 0.37, as shown in Figure 3.

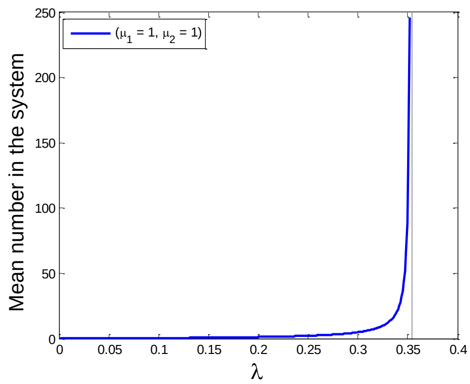


Figure 2. Mean number in the system

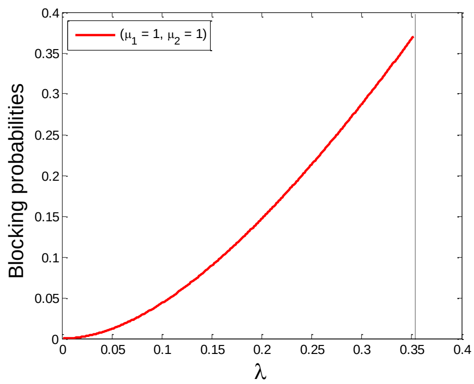
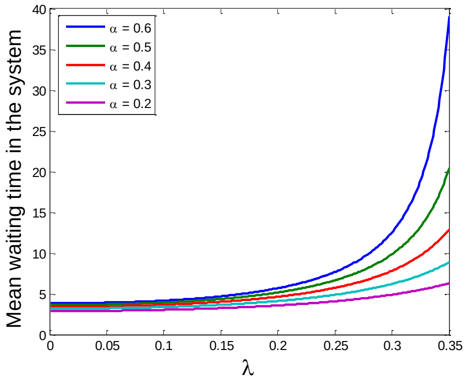


Figure 3. Blocking probability of the station-1

4.3.2 Sensitivity Analysis of Performance Measures

In this section, the impact of mean failure rate of service stations in working states α on the mean waiting time in the system and blocking probability is studied. We set $\mu_1 = \mu_2 = 1$, $\beta_1 = \beta_2 = 0.8$ with various numbers of mean failure rate which increases from 0.2 to 0.6 and vary the mean arrival rate λ from 0.01 to 0.35. It is discovered that both mean waiting time in the system and blocking probability increases as the values of α increases, as shown in Figure 4 and Figure 5, respectively.



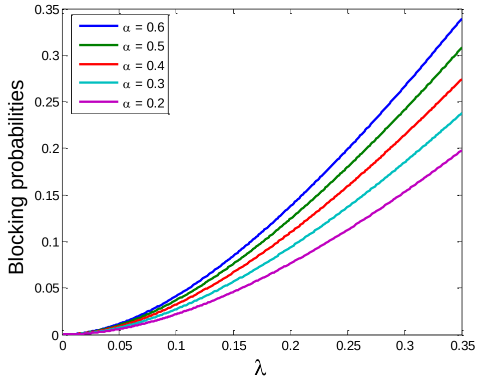
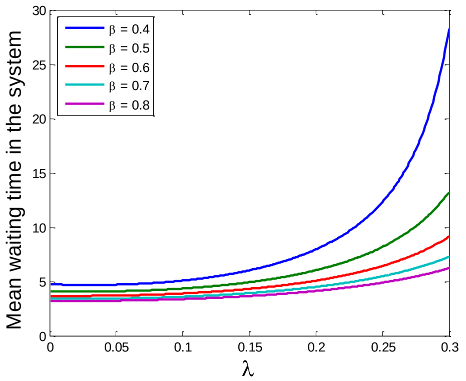


Figure 4. Mean waiting time in the system with respect to failure rate

Figure 5. Blocking probability with respect to failure rate

On the other hand, we investigate the effect of mean repair rate of service stations either in failure states of the station-1 or of the station-2 on mean waiting time in the system and blocking probability. We fix $\mu_1 = \mu_2 = 1$, $\alpha_1 = \alpha_2 = 1$ with different numbers of mean repair rate which increase from 0.4 to 0.8 and vary the mean arrival rate λ from 0.01 to 0.35. It is investigated that the mean waiting time in the system and the blocking probability decreases as β increases, as shown in Figure 6 and Figure 7, respectively.



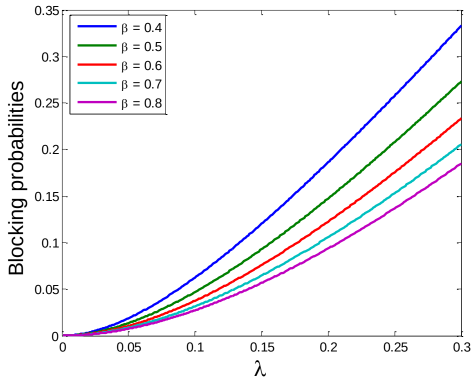


Figure 6. Mean waiting time in the system with respect to repair rate

Figure 7. Blocking probability with respect to repair rate

Next, we study the sensitivity performance of failure probability of the station-1 and the reliable probability of the system with respect to mean failure rate of the station-2 α_2 and vary the mean failure rate of the station-1 α_1 from 0.01 to 1. It is clear that the failure probability of the station-1 and the reliable probability of the system decreases as α_2 increases, as shown in figure 8 and figure 9, respectively. These results mean that in practice, we can control the failure probabilities of the station-1 and the station-2 and reliable probabilities of the system by adjusting the ratio of the mean failure rate of the station-2 α_2 to the mean failure rates of the station-1 α_1 and vice versa.

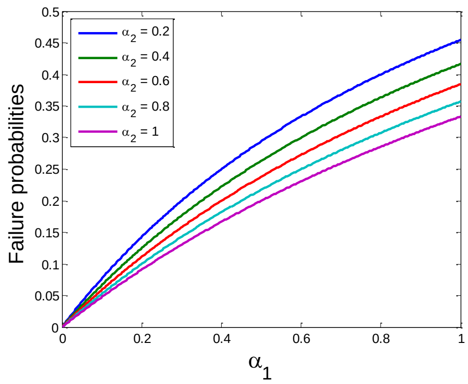


Figure 8. Failure probability of the station-1 with respect to failure rate

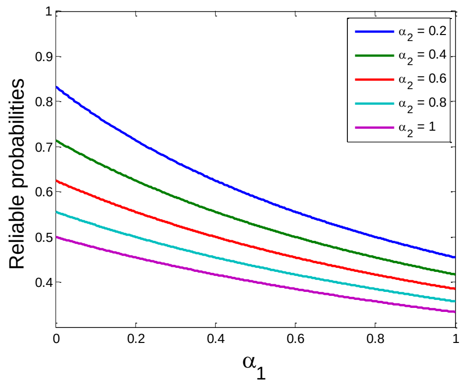


Figure 9. Reliable probability of the system with respect to failure rate

4.3.3 Disposition Strategies

Finally, we consider the influence of disposition strategies of mean service rate, mean failure rate and mean repair rate on the performance of mean waiting time in the system. We set $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 0.8$, and study the cases with different service rates $\mu_1 = 1, \mu_2 = 2$, and $\mu_1 = 2, \mu_2 = 1$, then vary the mean arrival rate λ from 0.01 to 0.35. It is observed that setting higher service rate for the station-1 increases the operational efficiency for the system subject to breakdowns and repairs, as shown in Figure 10. Next, we fix the parameters $\mu_1 = 1, \mu_2 = 1$, $\beta_1 = \beta_2 = 0.8$ and investigate the cases with different failure rates of service stations in working states $\alpha_1 = 0.6, \alpha_2 = 0.3$, and $\alpha_1 = 0.3, \alpha_2 = 0.6$, then vary the mean arrival rate λ from 0.01 to 0.25. It can be seen that if the system with higher failure rate of the station-1, the mean waiting time of the system is higher than that of the case with lower failure rate of the station-1, as shown in Figure 11. Finally, the system parameters with different repair rates are fixed at $\alpha_1 = \alpha_2 = 0.3$, $\mu_1 = \mu_2 = 1$. We present the cases $\beta_1 = 0.4, \beta_2 = 0.8$, and $\beta_1 = 0.8, \beta_2 = 0.4$, and vary the mean arrival rate λ from 0.01 to 0.25. It is noted that disposing higher repair rate for the station-1 can make the system work in a higher performance, as shown in Figure 12. These results show the fact that the bottleneck of the working stations for the series configuration queueing system with two service stations is the station-1 which affects the whole operational performances of the system with perfect working states, referred as Tsai et al. [26-28], and the system subject to breakdowns and repairs. We suggest dispose higher service rate and repair rate and keep lower failure rate for the station-1 of the system in order to increase operational efficiency of the system.

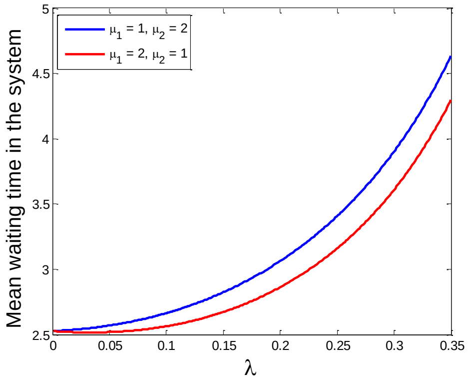


Figure 10. Mean waiting time in the system with different service rate

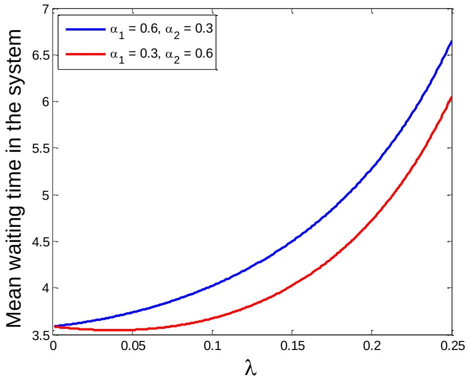


Figure 11. Mean waiting time in the system with different failure rate

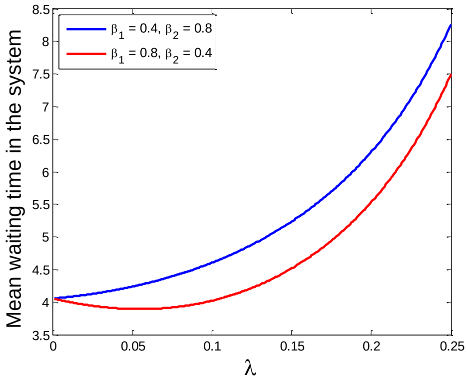


Figure 12. Mean waiting time in the system with different repair rate

4.4 Summary and Discussions

We study a series configuration queueing system consisting of two service stations subject to breakdowns and repairs. The service stations in this kind of system might fail and contain repair procedures for the failure service stations We successfully demonstrate matrix-geometric method is still a powerful tool for further investigating and understanding characteristics of series configurations queueing systems as described in our previous works, Tsai [26-28], even for more complex extensions of the mathematical models and conditions close to the real applications in industries.

Numerical results validate the correctness of the exact formulae of stability conditions. We also discover the fact that the bottleneck of the working stations for the series configuration queueing system with two service stations is the station-1 which significantly influences the whole operational performances of the system with perfect working states and the system subject to breakdowns and repairs. We suggest that setting higher service rates, repair rate and sustain lower failure rate for the station-1 of the system to make the system work more efficiently according to the numerical experiments.

Future research will focus on conducting statistical analysis for real industrial application of manufacturing systems and validate the propositions of the analysis with our theoretical results developed in this research. Transient analysis and reliability analysis of the system would be considered further.

CHAPTER 5
GENERAL DISPOSITION STRATEGIES OF SELF-BLOCKING QUEUEING SYSTEM WITH PERFECT SERVICE STATIONS

5.1 Preface

The research topics in previous Chapters focused on the queueing systems with service discipline that customers should enter each service station orderly to complete the services in the system. That kind of queueing system is popular in manufacturing industries. In this Chapter, we study another kind of series configuration queueing system with different service rules, called self-blocking queueing system. The complete service in this queueing system is defined as a customer should receive a service in either service station. However, since there is no finite buffer among service stations, the blocking phenomena still happen in the case that a customer has completed its service in a service station, but another customer in the next stations is still receiving the service. The first study on the performance analysis of self-blocking queueing system consisting of two service stations was conducted by Ke and Tsai [61]. They notice that the system with two servers can be applied to analyze the operational situations of gasoline stations in urban areas. Now, we extend their studies to the system consisting of three service stations and investigate related important performance measures, such as mean number in the system, mean number in the queue, mean waiting time in the system, mean waiting time in the queue, blocking probabilities of servers located before the terminal stations and mean throughput of each service station. The self-blocking with many service stations can be applied to the real cases in taxi cabs. We finally propose a general disposition strategy for the system to work in higher operational performance based on the numerical results.

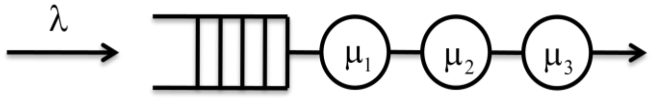


Figure 1. Scheme of self-blocking queueing system

5.1.2 Contributions and Outline

Major theoretical results in this chapter including 1) developing steady-state structure generator matrix equations of the self-blocking queueing system with three service stations, 2) deriving stability conditions of the system in exact form, 3) evaluating the steady-state probabilities with different service rates of the service stations, 4) presenting sensitivity analysis through numerical simulations to propose disposition strategies for the system working more efficiently. In practice, the operational staffs can benefit from our theoretical results by applying disposition strategy for the self-blocking system in a taxi can in order to reduce the mean waiting time for the customers in the queueing system.

The rest of the content in this Chapter is organized as follows. In Section 2, we formulate the problem and provide notation used throughout the article. In Section 3, we construct the steady-state equations corresponding to the quasi-birth-death process of the system and evaluate the steady-state probabilities by matrix-geometric method. We derived the stability conditions of the system in explicit mathematical expressions with general case and special case. In section 4, numerical results are presented to demonstrate the properties of the system and to illustrate different performance measures of the system with different service rates. Finally, we conclude the research topic with discussions and suggest possible future research topics in the last Section.

5.2 Modeling Framework

5.2.1 Problem Formulation and Notations

Three independent service stations placed in series configuration operate simultaneously in the queueing system. We assumed Poisson arrival process with mean arrival rate λ and the time to serve a customer in each service station is exponentially distributed with mean service time $\frac{1}{\mu}$. When the service stations are all in the idle situation, the customer must enter the terminal station directly to receive the service. There are no queues between each service station. A customer can finish the service at any stations then leave the system directly if the self-blocking phenomenon does not happen. The self-blocking phenomenon means that when a customer completes the service in a service station, but the another customer in the next station has not finished the service yet. The customer who is receiving the service blocks the customer who has completed the service in the previous station. The blocking phenomenon happens in the station-1, and the station-2 in this system. We assume an infinite queue in front of the first service station. In addition, the service station can only serve a customer at a time and the service rate is independent of the number of customers. The service of the system obeys the first come first serve (FCFS) discipline.

The following notations are used for modeling the system:

λ Mean arrival rate of Poisson arrivals

μ_1 Mean service rate of the station-1

μ_2 Mean service rate of the station-2

μ_3 Mean service rate of the station-3

The notation P_{n_1,n_2,n_3,n_4} is used to denote the steady-state probability P_{n_1,n_2,n_3,n_4} of n_1 customer in the station-3 and n_2 customer in the station-2 and n_3 customer in the station-1 and n_4 customer in the queue.

5.2.2 Matrix-Geometric Equations

The steady-state probability vector corresponding to the structured generator matrix Q is denoted as $\mathbf{P}=[\mathbf{P_0},\mathbf{P_1},\mathbf{P_2},...]$. The steady-state probability vector can be obtained by solving the system of equations $\mathbf{P}Q=\mathbf{0}$, while obeying the normalization condition $\mathbf{P1}=1$. The global balance equations of the system can be written as

$$\mathbf{P_0}B_{0,0}+\mathbf{P_1}B_{1,0}+\mathbf{P}_2B_{2,0}+\mathbf{P}_3B_{3,0}=\mathbf{0}, \tag{1}$$

$$\mathbf{P_0}B_{0,1}+\mathbf{P_1}A_1+\mathbf{P}_4A_4=\mathbf{0}, \tag{2}$$

$$\mathbf{P}_iA_0+\mathbf{P}_{i+1}A_1+\mathbf{P}_{i+4}A_4=\mathbf{0}, \qquad i\geq 1. \tag{3}$$

A rate matrix R is introduced to construct the following recurrence relations

$$\mathbf{P_i}=\mathbf{P_{i-1}}R=\mathbf{P_1}R^{i-1}, \qquad i>1. \tag{4}$$

Substituting (4) into (3), we can obtain the following characteristic equation of the recurrence relation

$$A_0+RA_1+R^4A_4=0. \tag{5}$$

Therefore, we solve (5) by iteration method for the rate matrix R.
The matrix equations of (1) and (2) can be further simplified as

$$\mathbf{P_0}B_{0,0}+\mathbf{P_1}(B_{1,0}+RB_{2,0}+R^2B_{3,0})=\mathbf{0}, \tag{6}$$

$$\mathbf{P_0}B_{0,1}+\mathbf{P_1}(A_1+R^3A_4)=\mathbf{0}. \tag{7}$$

The normalization condition equation that involves $\mathbf{P_0}$ and $\mathbf{P_1}$ is given by

$$\mathbf{P_0}\mathbf{1}+\mathbf{P_1}(I-R)^{-1}\mathbf{1}=1, \tag{8}$$

where I is the identity matrix with same size as the rate matrix R.
Taking (6), (7) and the normalization condition (8) into account, the steady-state probability vector of $\mathbf{P_0}$ and $\mathbf{P}_1$ can be obtained by solving following matrix equation

$$(\mathbf{P_0},\mathbf{P_1})\begin{pmatrix} B_{0,0} & B^*_{0,1} & \mathbf{1} \\ B_{1,0}+RB_{2,0}+R^2B_{3,0} & (A_1+R^3A_4)^* & (I-R)^{-1}\mathbf{1} \end{pmatrix}=(\mathbf{0},1). \tag{9}$$

where $(.)^*$ indicates that the last column of the included matrix is removed to avoid linear dependency.

5.2.3 Stability Conditions

In order to confirm whether the quasi-birth-death process is tractable by the matrix-geometric method, the stability of the queueing system must be verified by checking the stability condition of Neuts [10]:

$$\mathbf{P}_A A_0 \mathbf{1} < \mathbf{P}_A A_2 \mathbf{1} + 2(\mathbf{P}_A A_3 \mathbf{1}) + 3(\mathbf{P}_A A_4 \mathbf{1}), \tag{10}$$

where $\mathbf{P_A}$ is the steady-state probability vector corresponding to the generator matrix A.

We first write down the generator matrix A:

$$\mathbf{A} = \sum_{i=0}^{4} \mathbf{A}_i \tag{11}$$

Using A, the steady-state probability vector $\mathbf{P_A} = [P_{0A}, P_{1A}, P_{2A}, P_{3A}, P_{4A}, P_{5A}, P_{6A}]$ can be obtained:

$$\mathbf{P_A A} = \mathbf{0}, \tag{12}$$

$$\sum_{i=0}^{6} P_{A,i} = 1, \tag{13}$$

We can derive exact formulae of stability conditions by substituting the steady-state probability into (10) and employing the content of the matrix A_0 and A_2.

5.2.4 Performance Measures and Exact Results

- **Stability Conditions**

Theorem 5.2.1 The stability conditions of the system consisting of three service stations can be concluded as the following two inequalities:

(1) For $\mu_1 \neq \mu_2 \neq \mu_3$

$$\lambda < \frac{N}{D}, \tag{14}$$

where

$$N = 3\mu_1\mu_2\mu_3(\mu_1+\mu_2)(\mu_1+\mu_3)(\mu_2+\mu_3)(\mu_1+\mu_2+\mu_3),$$

and

$$D = \mu_1^4(\mu_2^2+\mu_2\mu_3+\mu_3^2) + \mu_1^3(2\mu_2^3+4\mu_2^2\mu_3+4\mu_2\mu_3^2+2\mu_3^3) + \mu_1^2(\mu_2^4+4\mu_2^3\mu_3+5\mu_2^2\mu_3^2+4\mu_2\mu_3^3+\mu_3^4)$$
$$+\mu_1(\mu_2^4\mu_3+4\mu_2^3\mu_3^2+4\mu_2^2\mu_3^3+\mu_2\mu_3^4) + \mu_2^2(\mu_2^2\mu_3^2+2\mu_2\mu_3^3+\mu_3^4).$$

(2) Special case: $\mu_1 = \mu_2 = \mu_3 = \mu$

$$\lambda < \frac{18}{11}\mu. \tag{15}$$

- **Performance Measures**

Performance measures of the self-blocking queueing system with infinite capacity are defined as:

(1) Mean number of customers in the system

$$L = (P_{0,0,1,0} + P_{0,1,0,0} + P_{1,0,0,0} + P_{1,b,0,0} + P_{0,1,b,0}) + 2(P_{0,0,1,1} + P_{0,1,1,0} + P_{1,0,1,0} + P_{1,1,0,0})$$
$$+\sum_{n=3}^{\infty}(P_{0,0,1,n-1} + P_{0,1,1,n-2} + P_{1,0,1,n-2} + P_{1,1,1,n-3})\cdot n + \sum_{n=2}^{\infty}(P_{1,b,1,n-2} + P_{0,1,b,n-1} + P_{1,1,b,n-2})\cdot n$$
$$+\sum_{n=1}^{\infty}(P_{1,b,b,n-1})\cdot n\,. \tag{16}$$

(2) Mean number of customers in the queue

$$L_q = (P_{0,0,1,1} + P_{0,1,1,1} + P_{1,0,1,1} + P_{0,1,b,1}) + 2(P_{0,0,1,2})$$
$$+\sum_{n=3}^{\infty}(P_{0,0,1,n})\cdot n + \sum_{n=2}^{\infty}(P_{0,1,1,n} + P_{1,0,1,n} + P_{0,1,b,n})\cdot n + \sum_{n=1}^{\infty}(P_{1,1,1,n} + P_{1,b,1,n} + P_{1,b,b,n} + P_{1,1,b,n})\cdot n$$
$$\overset{\text{if } \mu_1=\mu_2=\mu_3}{=} \mathbf{P}_1[R(I-R)^{-1}(I-R)^{-1}]\cdot\mathbf{1} \tag{17}$$

(3) Mean waiting time in the system (Little’s Law)

$$W = \frac{L}{\lambda}. \tag{18}$$

(4) Mean waiting time in the queue (Little’s Law)

$$W_q = \frac{L_q}{\lambda}. \tag{19}$$

(5) Blocking probability of the customer in the station-1

$$P_{b,1} = \sum_{n=0}^{\infty} P_{1,b,b,n} + P_{0,1,b,n} + P_{1,1,b,n}\,. \tag{20}$$

(6) Blocking probability of the customer in the station-2

$$P_{b,2} = \sum_{n=0}^{\infty} P_{1,b,b,n} + P_{1,b,0,n}\,. \tag{21}$$

(7) Mean throughput of the system

$$T = \mu_1[\sum_{n=0}^{\infty} P_{0,0,1,n} + P_{0,1,1,n} + P_{1,1,1,n} + P_{1,b,1,n}]$$
$$+\mu_2[P_{0,1,0,0} + P_{1,1,0,0} + \sum_{n=0}^{\infty} P_{0,1,b,n} + P_{0,1,1,n} + P_{1,1,1,n} + P_{1,1,b,n}] \tag{22}$$
$$+\mu_3[P_{1,0,0,0} + P_{1,1,0,0} + P_{1,b,0,0} + \sum_{n=0}^{\infty} P_{1,1,1,n} + P_{1,b,1,n} + P_{1,b,b,n} + P_{1,1,b,n}]\,.$$

Proposition 5.2.1 Disposition strategies for the self-blocking queueing system consisting of the arbitrary number of service stations with different service rates are same.

We propose disposition strategies for the system based on previous research conducted by Ke and Tsai [61] and this work in order to increase the operational efficiency of the system.

(1) Self-blocking queueing system with **the arbitrary number** of service stations

It is better to arrange higher service rate for the terminal service station and the service stations near the terminal station compared with other service stations in the system in order to obtain better operational efficiency for the system.

5.3 Numerical Results

In this section, numerical experiments are performed to study characteristics of the self-blocking system consisting of three service stations. Numerical validation of the exact results of stability conditions is presented. Moreover, disposition strategies are suggested to make the system work more efficiently in accordance with simulations.

5.3.1 Each Service Station with Same Service Rate

First, we investigate the trends of mean number in the system and mean throughput of each service station as a function of mean arrival rate λ. Mean number in the system is presented in Figure 2. The upper bound of the stability condition of the mean number in the system approaches to $\frac{18}{11}(\approx 1.636)$. This numerical result is consistent with the exact formula given in the Section 5.2.4. Mean throughput of each service stations as a function of mean arrival rate is shown in Figure 3. It is investigated that the mean throughput of the station-1 is higher than that of the station-2 and of the station-3 in the condition that all of the service stations are set in the same service rate.

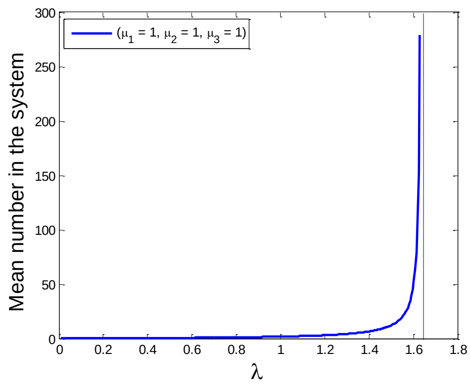


Figure 2. Mean number in the system

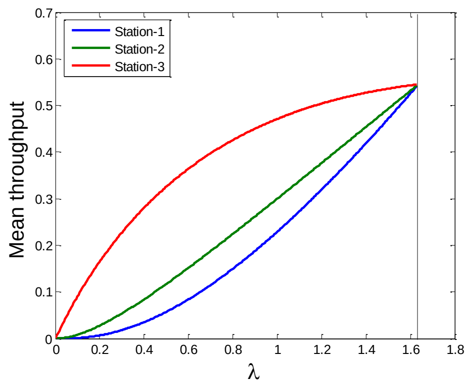


Figure 3. Mean throughput of each station

5.3.2 Each Service Station with Different Service Sates

- **Disposing the Service Rates of Two Service Stations**

We study the disposition conditions that we can concurrently control the service rates of two service stations and the service rate of only one service station for the system consisting of three service stations.

First, in the cases that we are able to control two service rates of the service stations in this system. We set $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1$ and $\mu_1 = 2, \mu_2 = 1, \mu_3 = 2$ and $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2$ and then vary the mean arrival rate λ from 0.01 to 2. It is observed that setting higher service rates for the station-2, and the station-3 is a better disposition strategy than that of other two cases, as shown in Figure 4. We suggest the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2$ as the best disposition strategy, when we are able to control service rates of two service stations for the system.

- **Disposing the Service Rate of a Service Station**

Next, the cases of controlling service rate of one service station are presented. We set $\mu_1 = 2, \mu_2 = 1, \mu_3 = 1$ and $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1$ and $\mu_1 = 1, \mu_2 = 1, \mu_3 = 2$, then vary the mean arrival rate λ from 0.01 to 1.6. It is investigated that the mean waiting time is the lowest in the case of $\mu_1 = 1, \mu_2 = 1, \mu_3 = 2$ compared with other two cases as shown in Figure 5. Therefore, the case $\mu_1 = 1, \mu_2 = 1, \mu_3 = 2$ is suggested as the best disposition strategy, when we can control service rates of only one service station for the system.

Note that, in both case studies, numerical computations of all cases should obey the stability conditions derived in the section 5.2.4. in order to satisfy the ergodicity condition of the steady-state probabilities.

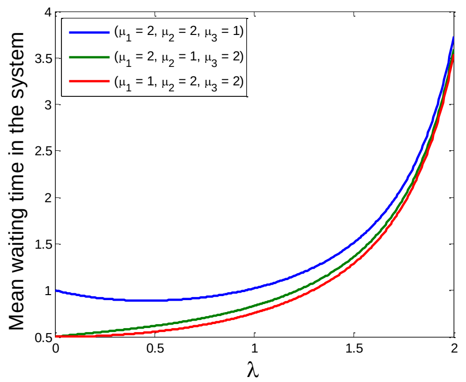


Figure 4. Mean number in the system (Disposing two service stations)

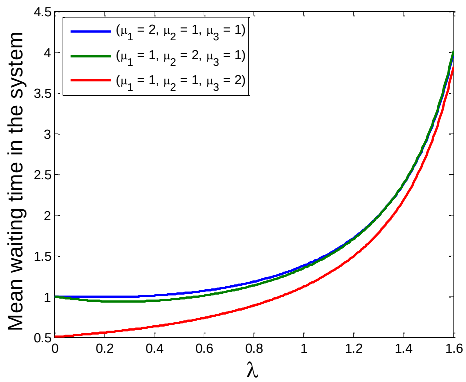


Figure 5. Mean waiting time in the system (Disposing a service station)

5.4 Summary and Discussions

In this Chapter, performance analysis of self-blocking queueing system consisting of three service stations is studied. The steady-state probabilities of the system with infinite capacity are evaluated by matrix-geometric method. We have also derived stability conditions of the queueing system for both service stations setting same service rates and each service station arranging different service rates. Numerical simulation on mean number of customers in the system of the self-blocking system confirms the correctness of exact formulae of stability conditions.

Numerical results of performance measures of the self-blocking queueing system indicate that it is always better to dispose higher service rate for the terminal service station in order to effectively reduce the mean waiting time in the system. In this way of disposition, the system can operate in a more efficient way. General disposition strategies for the system consisting of the arbitrary number of service stations are proposed to increase operational efficiency of the system.

Transient analysis and the general service time distributions will be considered for future research.

CHAPTER 6
EXPERIMENTS AND STATISTICAL ANALYSIS

6.1 Preface

The major purpose in this chapter is to design a plan to validate the theoretical results of disposition strategies for queueing system consisting of two service stations with blocking phenomena. According to Tsai et al. [26-28], if we dispose different service rate for the system with blocking phenomena in steady-state, it shows that the operational performance of the system would be different, as shown in Figure 1.

On the other hand, we investigate that the effect of the disposition strategies is not significant before the system turn into the steady state based on the theoretical transient results. However, the disposition strategies are still effective when the system reach to the turning point between transient state and steady state, as shown in Figure 2. We design the experiment based on pedestrian dynamics. Although our theoretical results majorly focus on automatic automobile assembly line, the results of the experiment can be treated as a basic approximation for more refined experiments in real applications in the industries.

We measure waiting time in the queue for those customers certainly wait in the queue to receive the service from working stations. Four cases are designed to study the significant effects of disposition strategies that affect the operational performance of the system consisting of two service stations. The first and the second case demonstrate the mean arrival rate following steady-state situation. The transient state of the system with different disposition strategies is presented in the third and the fourth case, respectively. The mean waiting time in the queue is calculated to investigate the effect of different disposition strategies for the performance of the system with two service stations.

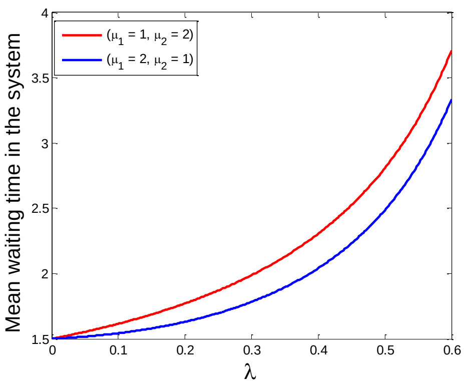


Figure 1. Mean waiting time in the system (steady state)

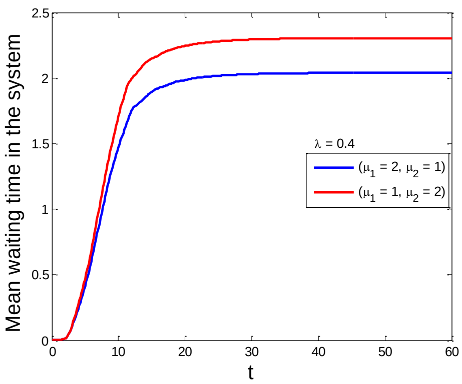


Figure 2. Mean waiting time in the system (transient state)

6.2 Experimental Process

- **Processes of the Experiment**

1. Generate service time from exponential distribution based on a specific average service rate.
2. Generate mean arrival time from geometric distribution based on a specific average arrival rate.
3. Investigate whether there are different operational efficiencies between disposition strategies suggested in our theoretical results depending on different arrival rates of the customers.
4. Observe the oscillation phenomena for the system with different disposition strategies when the system is still in transient state.

- **Processes for the Pedestrian**

Since the mean waiting time in the system is equal to the sum of the mean waiting time in the queue and the mean service time of the service stations. Our results show that different disposition strategies will majorly cause the different mean waiting time in the queue for customers.

Thus, we define the following quantity to calculate mean waiting time in the queue according to the number of customers arriving to the system.

$$W_q = \frac{\sum_{i}^{n} x_{q,i}}{n}$$

where, n is the total number of customers entering the system, $x_{q,i}$ is the waiting time for the ith customer in the queue. We still record the service time for the ith customer, denoted as $x_{s,i}$.

We assign electronic card to each customer and record both their waiting time in the queue and the time to complete their service when they are in the service stations.

- **Design of the Experiment**

The experiment process is shown in Figure 3. We assign five staffs to work in the experiment with different responsibilities. The staff 1 and the staff 2 work in the station-1 and the station-2, respectively. They are responsible to confirm the customers entering the service station complete the service in each station. The staff 3 is responsible to record the waiting time in the queue when there are some customers waiting in the queue for the service. The staff 4 tells the customers that they can enter the queueing system based on the pseudo-random number generators of the specific mean arrival rate of geometric distributions. The staff 5 gives instructions for the attendants regarding specific problems in the experiment.

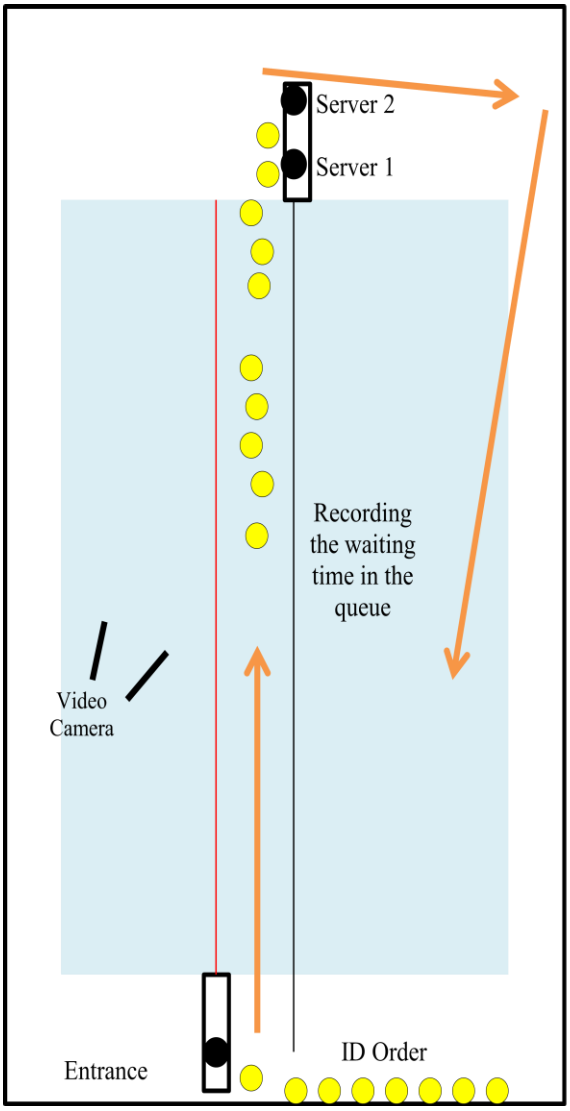


Figure 3. The experiment process

6.3 Experimental Results

In this section, we present the data collected from experiments and calculate the mean waiting time in the queue for each case.

We consider the following cases:

Case 1

We choose $\frac{1}{\mu_1} = 4$, $\frac{1}{\mu_2} = 8$ and set $\frac{1}{\lambda} = 16$.

Case 2

We choose $\frac{1}{\mu_1} = 8$, $\frac{1}{\mu_2} = 4$ and set $\frac{1}{\lambda} = 16$.

Case 3

We choose $\frac{1}{\mu_1}=4$, $\frac{1}{\mu_2}=8$ and set $\frac{1}{\lambda}=8$.

Case 4

We choose $\frac{1}{\mu_1}=8$, $\frac{1}{\mu_2}=4$ and set $\frac{1}{\lambda}=8$.

The major purposes for conducing the experiments are to validate two theoretical results. First, we should validate whether the operational efficiency of the system is different by disposing different service rates for each service station. Second, the stability conditions derived in steady-state analysis would be test to observe whether the mean waiting time in the queue growths significantly or not. Table 1. ~ Table 4. show that the waiting time in the queue and their corresponding ID number of customers entering to the system. There are total 30 customers joining in the experiments.

Table 1. Waiting time in the queue for the case 1

ID No.	Waiting time in the queue (MSEC)
012E30D2008A4285	12476
012E30D2008A4492	21053
012E30D2008A4743	9174
012E30D2008A4282	9709
012E30D2008A424F	9321
012E30D2008A4634	10646
012E30D2008A4667	15151
012E30D2008A4876	19734

(MSEC: Millisecond)

Table 2. Waiting time in the queue for the case 2

ID No.	Waiting time in the queue (MSEC)
012E30D2008A424D	8493
012E30D2008A477F	12144
012E30D2008A4493	15755
012E30D2008A4285	14829
012E30D2008A4492	16128
012E30D2008A4743	24052
012E30D2008A4282	13093
012E30D2008A424F	12729

012E30D2008A4634	23348
012E30D2008A4667	12084
012E30D2008A4876	13063
012E30D2008A4482	17044

Table 3. Waiting time in the queue for the case 3

ID No.	Waiting time in the queue (MSEC)
012E30D2008A4493	12214
012E30D2008A4285	24458
012E30D2008A4492	38977
012E30D2008A444A	32584
012E30D2008A445A	38715
012E30D2008A427C	31362
012E30D2008A4743	31162
012E30D2008A4282	40616
012E30D2008A424F	32726
012E30D2008A4634	33657
012E30D2008A4683	43127
012E30D2008A4862	45174
012E30D2008A488D	51596
012E30D2008A463D	46346
012E30D2008A4667	51309
012E30D2008A4876	56952
012E30D2008A4883	55570
012E30D2008A484E	62524
012E30D2008A469C	58816
012E30D2008A4864	59516
012E30D2008A468E	68619
012E30D2008A4856	71125
012E30D2008A483B	49256
012E30D2008A4482	35900

Table 4. Waiting time in the queue for the case 4

ID No.	Waiting time in the queue (MSEC)
012E30D2008A4452	9399
012E30D2008A477F	19633
012E30D2008A4493	28391
012E30D2008A4285	39306
012E30D2008A4492	40094
012E30D2008A444A	45074
012E30D2008A445A	53037
012E30D2008A427C	40952
012E30D2008A4743	54474
012E30D2008A4282	52966
012E30D2008A424F	45834
012E30D2008A4634	53070
012E30D2008A4683	53509
012E30D2008A4862	56527
012E30D2008A488D	58913
012E30D2008A4667	41920
012E30D2008A463D	51913
012E30D2008A4876	54245
012E30D2008A4883	59513
012E30D2008A484E	58434
012E30D2008A469C	53059
012E30D2008A4864	62590
012E30D2008A468E	63208
012E30D2008A4856	64292
012E30D2008A483B	42158
012E30D2008A4482	51325

- **Validations of Disposition Strategies**

Based on the theoretical results for the series configuration queueing system consisting of two service stations, we propose that it is better to set higher service rate for the server-1 in order to make the system work in better operational efficiency. Table 5. shows the summary of experiments. Experiment 1 and experiment 2 is compared with different service rate of the system with longer inter-arrival time. On the other hand, the system with shorter inter-arrival time with different service is designed between experiment 3 and experiment 4.

The results of statistical test for the disposition strategies are shown in Table 6. Although the mean waiting time in the queue is consistent with our theoretical predictions for both cases, the statistical test shows that the experiment result is not significant due to large variance of the observational data. The major reason causing the large variance of the experimental data is that we just assign 30 customers in the experiment. Since it is noted that the waiting time in the queue for the customers entering the system earlier is almost zero, the variance of the observational data becomes large. However, if we increase the number of customers in the experiments, it is expected that the results of statistical test would become significant, because there are more and more records for the customers waiting in the queue.

Table 5. Summary of the experiments

No. of Experiments	Entrance $\frac{1}{\lambda}$	Server 1 $\frac{1}{\mu_1}$	Server 2 $\frac{1}{\mu_1}$	Theoretical Predictions
Exp01	16	4	8	**Better Strategy**
Exp02	16	8	4	
Exp03	8	4	8	**Better Strategy**
Exp04	8	8	4	

Table 6. Statistical test for the significance of mean waiting in the queue with different service rates

No. of Experiments	Mean waiting time in the queue W_q	Statistical Test $U = \frac{\overline{X} - \overline{Y}}{\sqrt{\frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}}}$
Exp01	3.58 ± 6.36	1.35 (**Not Significant**)
Exp02	6.09 ± 7.96	
Exp03	35.74 ± 21.89	1.11 (**Not Significant**)
Exp04	41.79 ± 20.32	

- **Validations of Stability Conditions**

Next, we do the statistical test to compare the significance for the system with different inter-arrival time. It is discovered that the mean waiting time in the queue for the system with different inter-arrival shows statistical significance in both cases as shown in Table 7. This means that the stability conditions of the steady-state analysis may be applied to real cases. Therefore, the assumption of our theoretical models for the Poisson arrival is not so strong.

Table 7. Statistical test for the significance of mean waiting in the queue with different arrival rates

No. of Experiments	Mean waiting time in the queue W_q	Statistical Test $U = \frac{\overline{X} - \overline{Y}}{\sqrt{\frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}}}$
Exp01	3.58 ± 6.36	7.73 (**Significant**)
Exp03	35.74 ± 21.89	
Exp02	6.09 ± 7.96	8.96 (**Significant**)
Exp04	41.79 ± 20.32	

- **Theoretical Analysis of Variance of Mean Waiting Time in the Queue**

We observe that the variance of the experimental data is quite large. Now, we try to explain this phenomenon from theoretical analysis. Let W and L denote the mean waiting time in the queue and mean number in the queue, respectively. We can evaluate the variance of mean waiting time in the queue by Little's law,

$$\mathrm{Var}[W] = \mathrm{Var}[\frac{L}{\lambda}] = \frac{1}{\lambda^2}\mathrm{Var}[L] = \frac{1}{\lambda^2}\left(E[L^2] - (E[L])^2\right).$$

The theoretical values of errors reflect that the variance is large for both cases as shown in Figure 4 and Figure 5. Since we assume the time to serve a customer follows exponential distribution, the numerical value also show large variances. However, it is possible to reduce the variance of observational values by increasing the number of customer in experiments. Therefore, the different disposition strategies would become significant for the system consisting of two service stations.

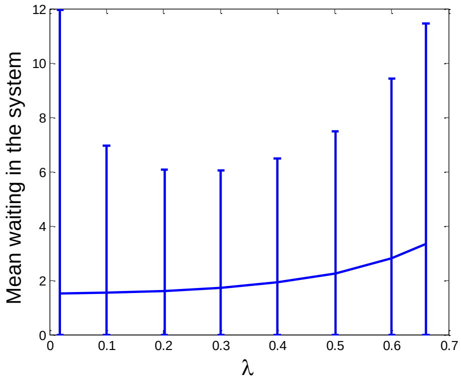


Figure 4. Mean waiting time in the system $(\mu_1 = 2,\ \mu_2 = 1)$

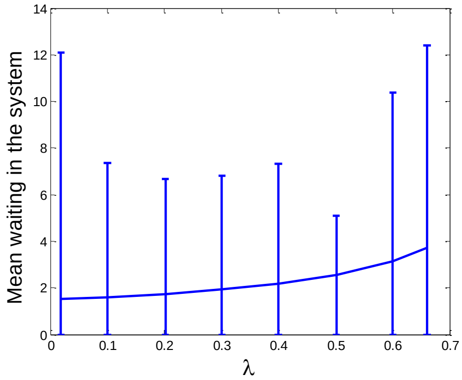


Figure 5. Mean waiting time in the system $(\mu_1 = 1,\ \mu_2 = 2)$

6.4 Summary and Discussions

We design a simple experiment to validate our theoretical results of stability conditions and disposition strategies for the series configuration queueing system consisting of two service stations. We set the mean arrival rate to satisfy the steady-state conditions and the transient conditions with different disposition strategies for the service stations.

The statistical test of the difference of mean waiting time in the queue with different disposition strategies is insignificant, because the number of customer assigned in the experiment is not enough. Theoretical analysis on the mean waiting time in the queue also shows that the variance of mean waiting time in the queue is large. In the future, we can increase the number of customers to collect more experimental data in order to reduce the variance of the data. On the other hand, The statistical test of the difference of mean waiting time in the queue with different inter-arrival time is highly significant. This means that the assumption of Poisson arrival in the theoretical model is not so strong, because we can apply theoretical results of steady-state analysis to real cases.

It is worth to design other experiments to further investigate the characteristics of the system consisting of more than two service stations. In the future, we intend to propose our results of the research to the industries for real applications.

CHAPTER 7
CONCLUSIONS

In this thesis, we study the performance analysis of series configuration queueing systems which are very important in traditional manufacturing industry, assembly line of automobile and semiconductor industry etc. We provide the theoretical steady-state analysis for the systems. The information of performance analysis reveals important characteristics of the systems. We expect that the information will be useful for the design of smart factory in the ear of Industry 4.0 and Internet of Things (IoT). Theoretical results of stability conditions are shown to be statistical significant by pedestrian experiments with different arrival rates.

In Chapter 2, series configuration queueing systems consisting of two, three and four service stations are investigated. This kind of system is very popular for the applications of assembly line in manufacturing industry. We construct the structure generator matrix and evaluate the steady-state probabilities by matrix-geometric method. The performance measures includes mean number in the system, mean number in the queue, mean waiting time in the system, mean waiting time in the queue, blocking probability of the service stations before the terminal station. Stability conditions are derived in exact form. Disposition strategies based on the numerical simulations are suggested to make the systems work efficiently.

Transient analysis for the series configuration system consisting of two servers is discussed in Chapter 3. A queue with finite capacity is allowed in front of the system. Master equation is constructed by the quasi-birth-death process of the system. We apply Runge-Kutta method to evaluate dynamic state probabilities. Dynamic performance measures including mean number in the system, mean number in the queue, mean waiting time in the system, mean waiting time in the queue, blocking probability of the station-1 and rejecting probability are defined. Numerical results show that there is different operational efficiency of the system if the service rates of servers are different. The disposition strategies are consistent with the results of steady-state analysis.

Chapter 4 is the topic about series configuration system subject to breakdowns and repairs. In practice, it is possible that the servers may go breakdowns due to some operational problems of machines. We extend the model formulation by introducing breakdown phenomenon with average rates follows exponential distribution and repair mechanism for the system. The steady-state probabilities in working states, failure states of the station-1 and failure states of the station-2 are evaluated by matrix-geometric method. Furthermore, we numerically estimate the performance measures including mean number in the system, mean number in the queue, mean waiting time in the system, mean waiting time in the queue, blocking probability failure probability of the station-1, failure probability of the station-2, reliable probability of the system. Exact formula of the stability conditions is successfully derived. Sensitivity analyses with respect important system parameters are investigated to study the properties of the system. Numerical results also show that it is better to set higher service rate and repair rate for the station-1 in order to keep high operational performance for the system. On the other hand, maintaining lower breakdown rate for the station-1 compared with the station-2 make the system work more efficiently.

The general disposition strategy for self-blocking queueing system is proposed in Chapter 5. We demonstrate that the matrix-geometric method can still be applied to evaluate steady-state probability of self-blocking queueing system. Performance measures including mean number in the system, mean number in the queue, mean waiting time in the system, mean waiting time in the queue, blocking probability and mean throughput of each service station are defined. Stability conditions are derived in exact form. It is suggested that disposing higher service rate for the terminal station and servers near the terminal station in order to make the system work in higher operational efficiency. The disposition strategy is expected to be applied to taxi cab queueing systems.

In Chapter 6, we design four experiments to validate our propositions of disposition strategies and stability conditions for the series configuration queueing system consisting of two service stations. Although the mean waiting time in the queue for different disposition strategies is not statistical significant, the stability conditions for different mean arrival rates are statistically significant. The statistical results show that the assumption of Poisson arrival in our model is not so strong, because it may be applied to real cases. It is presented that the variance of mean waiting time in the queue with different disposition strategies is large by theoretical analysis. Regarding the statistically insignificant disposition strategies, we will increase the number of samples for the design of experiments in the future. It is expected that the effects by setting different service rates for servers would become significant.

Series configuration queueing system subject to working breakdowns and repairs, and the system subject to simultaneous breakdowns are first considered as research topics next step. The extension of stability conditions is expected to be obtained by matrix-geometric method. We will concurrently study self-blocking systems in the cases of breakdowns and repairs. Series configuration systems and self-blocking systems with general service distributions are worth to be important research in the future.

Bibliography

[1] G.C. Hunt, "Sequential arrays of waiting lines," Operations Research, vol. 4, no. 6, pp. 674–683, 1956.

[2] B. Avi-Itzhak and M. Yadin, "A sequence of two servers with no intermediate queue," Management Science, vol. 11, no. 5, pp. 553-564, 1965.

[3] B. Avi-Itzhak and M. Yadin, "A sequence of service stations with arbitrary input and regular service times," Management Science, vol. 11, no. 5, pp. 565-571, 1965.

[4] T. Altiok, "Approximate analysis of exponential tandem queues with blocking," European Journal of Operational Research, vol. 11, no. 4, pp. 390-398, 1982.

[5] C. Langaris and B. Conolly, "On the waiting time of a two stage queueing system with blocking," Journal of Applied Probability, vol. 21, no. 3, pp. 628-638, 1984.

[6] H.T. Papadopoulos and M.E.J. O'Kelly, "Exact analysis of production lines with no intermediate buffers," European Journal of Operational Research, vol. 65, no. 1, pp. 118-137, 1993.

[7] B. Avi-Itzhak and S. Halfin, "Servers in tandem with communication and manufacturing blocking," Journal of Applied Probability, vol. 30, no. 2, pp. 429-437, 1993.

[8] I.F. Akyildiz and Horst von Brand, "Exact solutions for networks of queues with blocking-after-service," Theoretical Computer Science, vol. 125, no. 1, pp. 390-398, 1994.

[9] B. Avi-Itzhak and H. Levy, "A sequence of servers with arbitrary input and regular service times revisited," Management Science, vol. 41, no. 6, pp. 1039-1047, 1995.

[10] M.F. Neuts, Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach. New York: Dover Publications, 1995, ch. 1.

[11] A. Gomez-Corral, "A matrix-geometric approximation for tandem queues with blocking and repeated attempts," Operations Research Letters, vol. 30, no. 6, pp. 360-374, 2002.

[12] A. Gomez-Corral, "On a tandem G-network with blocking," Advances in Applied Probability, vol. 34, no. 3, pp. 626-661, 2002.

[13] A. Gomez-Corral, "A tandem queue with blocking and Markovian arrival process," Queueing Systems, vol. 41, no. 4, pp. 343-370, 2002.

[14] A. Gomez-Corral and M.E. Martos "Performance of two-stage tandem queues with blocking: The impact of several flows of signals," Performance Evaluation, vol. 63, no. 9-10, pp. 910-938, 2006.

[15] A. Gomez-Corral and M.E. Martos, "Marked Markovian arrivals in a tandem G-network with blocking," Methodology and Computing in Applied Probability, vol. 11, no. 4, pp. 621-649, 2009.

[16] R. Bierbooms, I.J.B.F. Adan, and M. van Vuuren, "Performance analysis of exponential production lines with fluid flow and finite buffers," IIE Transactions, vol. 44, no. 12, pp. 1132-1144, 2012.
[17] R. Bierbooms, I.J.B.F. Adan, and M. van Vuuren, "Approximate analysis of single-server tandem queues with finite buffers," Annals of Operations Research, vol. 209, no. 1, pp. 67-84, 2013.
[18] W. Zhou, Z. Lian, W. Xu, and W. Huang, "A two-stage queueing network with MAP inputs and buffer sharing," Applied Mathematical Modelling, vol. 37, no. 6, pp. 3736-3747, 2013.
[19] M. Hiller, "Designing unpaced production lines to optimize throughput and work-in-process inventory," IIE Transactions, vol. 45, no. 5, pp. 516-527, 2013.
[20] Y. Sakuma and A. Inoie, "An approximation analysis for an assembly-like queueing system with time-constraint items," Applied Mathematical Modelling vol. 38, no. 24, pp. 5870-5882, 2014.
[21] Y. W. Shin and D.H. Moon, "Approximation of throughput in tandem queues with multiple servers and blocking," Applied Mathematical Modelling, vol. 38, no. 24, pp. 6122-6132, 2014. W. Shin and D.H. Moon, "Approximation of throughput in tandem queues with multiple servers and blocking," Applied Mathematical Modelling, vol. 38, no. 24, pp. 6122-6132, 2014.
[22] S. Hudson, T. McNamara, and S. Shaaban, "Unbalanced lines: where are we now," International Journal of Production Research, vol. 53, pp. 1895-1911, 2015.
[23] G. Bolch and S. Greiner, Queueing Networks and Markov Chains: Modeling and Performance Evaluation with Computer Science Applications. New Jersey: Wiley-Interscience, 2nd edition, 2006, ch. 3.
[24] S. Sani and O.A. Daman, "The M/G/2 queue with heterogeneous servers under a controlled service discipline: stationary performance analysis," IAENG International Journal of Applied Mathematics, vol. 45, no. 1, pp. 31-40, 2015.
[25] S. Ramasamy, O.A. Daman, and S. Sani, "Discrete-time Geo/G/2 queue under a serial and parallel queue disciplines," IAENG International Journal of Applied Mathematics, vol. 45, no. 4, pp. 354-363, 2015.
[26] Y.L. Tsai, D. Yanagisawa, and K. Nishinari "Performance analysis of series configuration queueing system with four service stations," Lecture Notes in Engineering and Computer Science: Proceedings of The International MultiConference of Engineers and Computer Scientists 2016, IMECS 2016, 16-18 March, 2016, Hong Kong, pp. 931–935.
[27] Y.L. Tsai, D. Yanagisawa and K. Nishinari, "Disposition strategies for open queueing networks with different service rates," Engineering Letters, vol. 24, no.4, pp. 418-428, 2016.

[28] Y.L. Tsai, D. Yanagisawa and K. Nishinari, “General disposition strategies of series configuration queueing systems,” IAENG International Journal of Applied Mathematics, vol. 46, no. 3, pp. 317-323, 2016.
[29] Y.L. Tsai, D. Yanagisawa, and K. Nishinari, “Performance analysis of open queueing networks subject to breakdowns and repairs,” Engineering Letters, vol. 24, no. 2, pp. 207-214, 2016.
[30] H. Baumann, W. Sandmann, “Multi-sever tandem queue with Markovian arrival process, phase-type service times, and finite buffers,” European Journal of Operational Research, vol. 256, pp. 187-195, 2017.
[31] M. Vinarskiy, “A method of approximate analysis of an open exponential queuing network with losses due to finite shared buffers in multi-queue nodes,” European Journal of Operational Research, vol. 258, pp. 207-215, 2017.
[32] H.G. Perros, “A bibliography of papers on queueing networks with finite capacity queues,” Performance Evaluation, vol. 10, no. 3, pp. 255-260, 1989.
[33] H.G. Perros, Open Queueing Networks with Blocking. North-Holland: Amsterdam, 1989.
[34] R.O. Onvural, “Survey of closed queueing networks with blocking,” ACM Computing Surveys, vol. 22, no. 2, pp. 83-121, 1990.
[35] S. Balsamo, “Properties and analysis of queueing network models with finite capacities,” Performance Evaluation of Computer and Communication Systems, vol. 729, pp. 21-52, 2005.
[36] N.G. Hall and C. Sriskandarajah, “A survey of machine scheduling problems with blocking and no-wait in process,” Operations Research, vol. 44, no.3, pp. 510-525, 1996.
[37] H.G. Perros, Queueing Networks with Blocking: Exact and Approximate Solutions. New York: Oxford University Press, 1994.
[38] S. Balsamo, V. de Nitto Persone, and Onvural, Analysis of Queueing Networks with Blocking. Boston: Kluwer Academic Publishers, 2001.
[38] G. Latouche and M.F. Neuts, “Efficient algorithmic solutions to exponential tandem queues with blocking,” SIAM Journal on Algebraic and Discrete Methods, vol. 1, no. 1, pp. 93-106, 1980.
[40] H.G. Perros and T. Altiok, Queueing Networks with Blocking. North-Holland: Amsterdam, 1989.
[41] F.S. Hillier and R.W. Boling, “Finite queues in series with exponential or Erlang service times – a numerical approach,” Operations Research, vol. 15, no. 2, pp. 286-303, 1967.
[42] H.G. Perros and T. Altiok, “Approximate analysis of open networks of queues with blocking: tandem configurations,” IEEE Transactions on Software Engineering,

vol. 33, no. 3, pp. 450-461, 1986.
[43] J. Abate, W. Whitt, “Transient behavior of the M/M/1 queue: starting at the origin,” Queueing Systems vol.2, no.1, pp. 41-65, 1987.
[44] J. Abate, W. Whitt, “Transient behavior of the M/M/1 queue via Laplace Transforms,” Advanced in Applied Probability vol. 20 no.1, pp. 145-178, 1988.
[45] D.J. Bertsimas, D. Nakazato, “Transient and busy period analysis of the GI/G/1 queue: the method of stages,” Queueing Systems vol.10, no.3 pp. 153-184, 1992.
[46] J. Abate, W. Whitt, “Transient behavior of the M/G/1 workload process,” Operations Research vol.42, no.4, pp. 750-764, 1994.
[47] G.L. Choudhury, D.M. Lucantoni, W. Whitt, “Multidimensional transform inversion with applications to the transient M/G/1 Queue,” The Annals of Applied Probability vol.4, no.3, pp. 719-740, 1994.
[48] W.H. Kaczynski, L.M. Leemis, J.H. Drew, “Transient queueing analysis,” INFORMS Journal on Computing, vol.24, no.1 pp. 10-28, 2012.
[49] S.H. Kim, W. Whitt, “Estimating waiting times with the time-varying Little’s Law,” Probability in the Engineering and Informational Sciences, vol.27, no.4, pp. 471-506, 2013.
[50] S.H. Kim, W. Whitt, “Statistical analysis with Little’s Law,” Operations Research, vol.61, no.4, pp. 1030-1045, 2013.
[51] Y.L. Tsai, D. Yanagisawa, K. Nishinari, “Transient analysis of of a series configuration queueing system,” Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Francisco, USA, pp747-753.
[52] M.F. Neuts, D.M. Lucantoni, “A Markovian queue with n servers subject to breakdowns and repairs,” Management Science vol. 25, no.9, pp. 849-861, 1979.
[53] H.T. Papadopoulos, C. Heavey, “Queueing theory in manufacturing systems analysis and design: A classification of models for production and transfer lines,” European Journal of Operational Research, vol.92, no.1, pp. 1-27, 1996.
[54] W.J. Gray, P.P. Wang, M. Scott, “A vacation queueing model with service breakdowns,” Applied Mathematical Modelling vol.24, no.5-6, pp. 391-400, 2000.
[55] N. Thomas, D. Thornley, H. Zatschler, Approximate solution of a class of queueing networks with breakdowns. European Simulation Multiconference, Nottingham, 2003.
[56] R. Chakka, E. Ever, O. Gemikonakli, Modelling open queueing networks with breakdowns, repairs and finite buffers, using an IPP departure process model. Proceedings of the 10th WSEAS International Conference on Computers, Greece, 2006.

[57] R. Chakka, E. Ever, O. Gemikonakli, Joint-State modeling for open queuing networks with breakdowns, repairs and finite buffers. The 15th International Symposium on Modeling, Analysis, and Simulation of Computer and Telecommunication Systems, Istanbul, 2007.

[58] M.F. Neuts, "Two queues in series with a finite, intermediate waiting room," Journal of Applied Probability, vol.5, no.1, pp. 123–142, 1968.

[59] A. Brandwajn, Y.L.L. Jow, "An approximation method for tandem queues with blocking," Operations Research, vol.36, no.1, pp. 73–83, 1988.

[60] M. Vuuren, I. Adan, S.A.E.Resing-Sassen, "Performance analysis of multi-server tandem queues with finite buffers and blocking," OR Spectrum, vol. 27, no.2-3, pp. 315-338, 2005.

[61] J.B. Ke, Y.L. Tsai, "Measures of self-blocking system with infinite space," Journal of Physics: Conference Series 410, 012114, 2013.

[62] Y.L. Tsai, D. Yanagisawa, K. Nishinari, "General disposition strategy of self-blocking queueing system," Lecture Notes in Engineering and Computer Science: Proceedings of The International MultiConference of Engineers and Computer Scientists 2017, IMECS 2017, 15-17 March, 2017, Hong Kong, pp. 989-992.

A.1 The structure of the transition matrix Q and its sub-matrices for the system with three service stations

Finally, we represent transition matrix of the series configuration queueing system with three service stations as

$$Q=\begin{bmatrix} B_{0,0} & B_{0,1} & 0 & 0 & 0 & 0 & \cdots \\ B_{1,0} & A_1 & A_0 & 0 & 0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & \cdots \\ 0 & 0 & 0 & 0 & A_2 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The details of sub-matrices of the composition of the transition matrix corresponding to the quasi-birth-death process for the system with three service stations are given by

$$B_{0,0}=\begin{bmatrix} -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\lambda+\mu_1) & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\lambda+\mu_2) & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ \mu_3 & 0 & 0 & -(\lambda+\mu_3) & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(\lambda+\mu_1) & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1+\mu_2) & \mu_2 & 0 & 0 & \mu_1 \\ 0 & \mu_3 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1+\mu_3) & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 & -(\lambda+\mu_2+\mu_3) & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 & -(\lambda+\mu_3) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & -(\lambda+\mu_2) \end{bmatrix},$$

$$B_{1,0}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_3 \end{bmatrix}, \quad B_{0,1}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \end{bmatrix},$$

$$A_0 = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_3 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -(\lambda+\mu_1) & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\lambda+\mu_1+\mu_2) & \mu_2 & 0 & 0 & 0 & \mu_1 & 0 \\ 0 & 0 & -(\lambda+\mu_1+\mu_3) & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(\lambda+\mu_1+\mu_2+\mu_3) & \mu_2 & 0 & 0 & \mu_1 \\ 0 & 0 & 0 & 0 & -(\lambda+\mu_1+\mu_3) & \mu_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_3) & 0 & 0 \\ 0 & 0 & 0 & \mu_2 & 0 & 0 & -(\lambda+\mu_2) & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & -(\lambda+\mu_2+\mu_3) \end{bmatrix}$$

A1.2 Derivation of stability conditions of the system with three service stations

The stability conditions (23) and (24) in Section 2.2.4 of the system can be derived by following

$$\mathbf{P}_A A_0 \mathbf{1} < \mathbf{P}_A A_2 \mathbf{1},$$

We first evaluate the conservative stable matrix A

$$A = A_0 + A_1 + A_2$$

$$= \begin{bmatrix} -\mu_1 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\mu_1+\mu_2) & \mu_2 & 0 & 0 & 0 & \mu_1 & 0 \\ \mu_3 & 0 & -(\mu_1+\mu_3) & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & \mu_3 & 0 & -(\mu_1+\mu_2+\mu_3) & \mu_2 & 0 & 0 & \mu_1 \\ 0 & 0 & \mu_3 & 0 & -(\mu_1+\mu_3) & \mu_1 & 0 & 0 \\ 0 & 0 & 0 & \mu_3 & 0 & -\mu_3 & 0 & 0 \\ 0 & 0 & 0 & \mu_2 & 0 & 0 & -\mu_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_2 & \mu_3 & -(\mu_2+\mu_3) \end{bmatrix}$$

Then we obtain the steady-state probability vector

$\mathbf{P}_A = [P_{A,0}, P_{A,1}, P_{A,2}, P_{A,3}, P_{A,4}, P_{A,5}, P_{A,6}, P_{A,7}]$ by solving following system equations with normalization condition

$$\mathbf{P_A} A = \mathbf{0}, \tag{A1.1}$$

$$\sum_{i=0}^{7} P_{A,i} = 1. \tag{A1.2}$$

Finally, we obtain the stability condition of the system with three service stations

$$\mathbf{P}_A A_0 \mathbf{1} < \mathbf{P}_A A_2 \mathbf{1}, \tag{A1.3}$$

$$\lambda < \frac{N_3}{D_3}, \tag{A1.4}$$

where

$$N_3 = \mu_1\mu_2\mu_3(\mu_1+\mu_2)(\mu_2+\mu_3)(\mu_1^3+\mu_1^2\mu_2+3\mu_1^2\mu_3+\mu_1\mu_2\mu_3+3\mu_1\mu_3^2+\mu_2\mu_3^2+\mu_3^3),$$

and

$$\begin{aligned} D_3 = {} & \mu_1^5(\mu_2^2+\mu_2\mu_3+\mu_3^2)+\mu_1^4(2\mu_2^3+5\mu_2^2\mu_3+5\mu_2\mu_3^2+3\mu_3^3) \\ & +\mu_1^3(\mu_2^4+5\mu_2^3\mu_3+8\mu_2^2\mu_3^2+7\mu_2\mu_3^3+3\mu_3^4) \\ & +\mu_1^2(\mu_2^4\mu_3+5\mu_2^3\mu_3^2+8\mu_2^2\mu_3^3+5\mu_2\mu_3^4+\mu_3^5) \\ & +\mu_1(\mu_2^4\mu_3^2+5\mu_2^3\mu_3^3+5\mu_2^2\mu_3^4+\mu_2\mu_3^5)+(\mu_2^4\mu_3^3+2\mu_2^3\mu_3^4+\mu_2^2\mu_3^5). \end{aligned}$$

For the system with same service rate, we set $\mu_1=\mu_2=\mu_3=\mu$

$$\lambda < \frac{22}{39}\mu, \tag{A1.5}$$

A.2 The structure of the transition matrix Q and its sub-matrices for the system with four service stations

We represent transition matrix of the series configuration queueing system with four service stations as

$$Q=\begin{bmatrix} B_{0,0} & B_{0,1} & 0 & 0 & 0 & 0 & \cdots \\ B_{1,0} & A_1 & A_0 & 0 & 0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & \cdots \\ 0 & 0 & 0 & 0 & A_2 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The details of sub-matrices of the composition of the transition matrix corresponding to the quasi-birth-death process for the system with four service stations are given by

$$B_{0,0}=\left[\begin{array}{cccccccccc}
-\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -(\lambda+\mu_1) & \mu_1 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & -(\lambda+\mu_2) & \mu_2 & 0 & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & -(\lambda+\mu_3) & \mu_3 & 0 & 0 & \lambda & 0 & 0 \\
\mu_4 & 0 & 0 & 0 & -(\lambda+\mu_4) & 0 & 0 & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1) & \mu_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1+\mu_2) & \mu_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1+\mu_3) & \mu_3 & 0 \\
0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1+\mu_4) & \mu_1 \\
0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_2+\mu_4) \\
0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.$$

$$
B_{0,0} = \left[\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4
\end{matrix}\right.
$$

$$
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu_1 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\
\mu_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\
-(\lambda+\mu_3+\mu_4) & \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \\
0 & -(\lambda+\mu_4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -(\lambda+\mu_2) & \mu_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -(\lambda+\mu_2+\mu_3) & \mu_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu_3 & 0 & 0 & 0 & -(\lambda+\mu_3) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1) & \mu_1 & 0 & 0 & 0 & 0
\end{matrix}
$$

$$
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1+\mu_2) & \mu_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1+\mu_3) & \mu_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1+\mu_4) & \mu_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1+\mu_2+\mu_4) & \mu_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1+\mu_3+\mu_4) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_3 \\
0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}
$$

$$
\left.
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}
\right],
$$

$$
\left.\begin{matrix}
0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_1 & 0 \\
\mu_3 & 0 & 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 \\
-(\lambda+\mu_1+\mu_4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_1 \\
0 & -(\lambda+\mu_2) & \mu_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -(\lambda+\mu_1+\mu_2+\mu_3) & \mu_1 & 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 \\
0 & 0 & 0 & -(\lambda+\mu_2+\mu_3) & \mu_2 & 0 & 0 & 0 & 0 & \mu_3 & 0 \\
0 & 0 & 0 & 0 & -(\lambda+\mu_3) & \mu_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_2+\mu_3+\mu_4) & \mu_2 & 0 & 0 & 0 & \mu_3 \\
0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_3+\mu_4) & \mu_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_4) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & -(\lambda+\mu_1+\mu_3) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 & -(\lambda+\mu_2+\mu_4) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & -(\lambda+\mu_2+\mu_4)
\end{matrix}\right]
$$

$$
B_{1,0} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

$$
\left.\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0
\end{matrix}\right],
$$

$$
B_{0,1} = \left[\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}\right.
$$

$$
B_{0,1} = \begin{bmatrix}
0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$\left.\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0
\end{matrix}\right],$$

$$\mathbf{A}_0 = \begin{bmatrix}
\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda
\end{bmatrix},$$

$$
A_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$
A_1 = \left[\begin{matrix}
-(\lambda+\mu_1) & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -(\lambda+\mu_1+\mu_2) & \mu_2 & 0 & 0 & 0 & 0 & \mu_1 & 0 & 0 \\
0 & 0 & -(\lambda+\mu_1+\mu_3) & \mu_3 & 0 & 0 & 0 & 0 & \mu_1 & 0 \\
0 & 0 & 0 & -(\lambda+\mu_1+\mu_4) & \mu_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -(\lambda+\mu_1+\mu_2+\mu_4) & \mu_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1+\mu_3+\mu_4) & \mu_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1+\mu_4) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_2) & \mu_2 & 0 \\
0 & 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 & -(\lambda+\mu_1+\mu_2+\mu_3) & \mu_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_2+\mu_3) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}\right.
$$

$$
\left.
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu_2 & 0 & 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 & 0 \\
-(\lambda+\mu_3) & \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -(\lambda+\mu_1+\mu_2+\mu_3+\mu_4) & \mu_2 & 0 & 0 & 0 & \mu_3 & \mu_1 & 0 & 0 & 0 \\
0 & 0 & -(\lambda+\mu_1+\mu_3+\mu_4) & \mu_3 & 0 & 0 & 0 & 0 & 0 & \mu_1 & 0 \\
0 & 0 & 0 & -(\lambda+\mu_1+\mu_4) & 0 & 0 & 0 & 0 & \mu_1 & 0 & 0 \\
\mu_1 & 0 & 0 & 0 & -(\lambda+\mu_1+\mu_3) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu_2 & 0 & 0 & 0 & -(\lambda+\mu_2+\mu_4) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_2 & 0 & 0 & -(\lambda+\mu_1+\mu_2+\mu_4) & 0 & 0 & 0 & \mu_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_2+\mu_3+\mu_4) & 0 & \mu_2 & \mu_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_4) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_3 & -(\lambda+\mu_3+\mu_4) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & -(\lambda+\mu_2+\mu_4)
\end{matrix}
\right].
$$

A3.2 Derivation of stability conditions of the system with four service stations

The stability conditions (35) and (36) of the systems can be derived by following

$$\mathbf{P}_A A_0 \mathbf{1} < \mathbf{P}_A A_2 \mathbf{1},$$

We first evaluate the conservative stable matrix A

$$
A = A_0 + A_1 + A_2
$$

$$
= \left[
\begin{matrix}
-\mu_1 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -(\mu_1+\mu_2) & \mu_2 & 0 & 0 & 0 & 0 & \mu_1 & 0 & 0 \\
0 & 0 & -(\mu_1+\mu_3) & \mu_3 & 0 & 0 & 0 & 0 & \mu_1 & 0 \\
\mu_4 & 0 & 0 & -(\mu_1+\mu_4) & \mu_1 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu_4 & 0 & 0 & -(\mu_1+\mu_2+\mu_4) & \mu_2 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu_4 & 0 & 0 & -(\mu_1+\mu_3+\mu_4) & \mu_3 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_4 & 0 & 0 & -(\mu_1+\mu_4) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_2 & \mu_2 & 0 \\
0 & 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 & -(\mu_1+\mu_2+\mu_3) & \mu_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\mu_2+\mu_3) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}
\right.
$$

$$\left.\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu_2 & 0 & 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 & 0 \\
-\mu_3 & \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -(\mu_1+\mu_2+\mu_3+\mu_4) & \mu_2 & 0 & 0 & 0 & \mu_3 & \mu_1 & 0 & 0 & 0 \\
0 & 0 & -(\mu_1+\mu_3+\mu_4) & \mu_3 & \mu_4 & 0 & 0 & 0 & 0 & \mu_1 & 0 \\
0 & 0 & 0 & -(\mu_1+\mu_4) & 0 & 0 & 0 & 0 & \mu_1 & 0 & 0 \\
\mu_1 & 0 & 0 & 0 & -(\mu_1+\mu_3) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu_2 & 0 & 0 & 0 & -(\mu_2+\mu_4) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_2 & 0 & 0 & -(\mu_1+\mu_2+\mu_4) & 0 & 0 & 0 & \mu_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\mu_2+\mu_3+\mu_4) & 0 & \mu_2 & \mu_3 \\
0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_4 & 0 & 0 \\
\mu_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_3 & -(\mu_3+\mu_4) & 0 \\
0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 & \mu_2 & 0 & -(\mu_2+\mu_4)
\end{matrix}\right]$$

Then we obtain the steady-state probability vector

$\mathbf{P}_A = [P_{A,0}, P_{A,1}, P_{A,2}, P_{A,3}, P_{A,4}, P_{A,5}, P_{A,6}, P_{A,7}, P_{A,8}, P_{A,9}, P_{A,10}, P_{A,11}, P_{A,12}, P_{A,13}, P_{A,14}, P_{A,15}, P_{A,16}, P_{A,17}, P_{A,18}, P_{A,19}, P_{A,20}]$

by solving following system equations with normalization condition

$$\mathbf{P_A} A = \mathbf{0}, \tag{A3.1}$$

$$\sum_{i=0}^{20} P_{A,i} = 1. \tag{A3.2}$$

Finally, we obtain the stability condition of the system with four service stations

$$\mathbf{P}_A A_0 \mathbf{1} < \mathbf{P}_A A_2 \mathbf{1}, \tag{A3.3}$$

$$\lambda < \frac{N_4}{D_4}, \tag{A3.4}$$

where N_4 and D_4 are shown in supplementary document I.

For the special case, we set $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu$,

$$\lambda < \frac{4024}{7817}\mu, \tag{A3.5}$$

A.3 The structure of the transition matrix Q and its sub-matrices for the system with two service stations subject to breakdowns and repairs

We provide transition matrix of the series configuration queueing system with two service stations subject to breakdowns and repairs as

$$Q=\begin{bmatrix} B_{0,0} & A_0 & 0 & 0 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & 0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & \cdots \\ 0 & 0 & 0 & 0 & A_2 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The details of sub-matrices of the composition of the transition matrix corresponding to the quasi-birth-death process for the system are given by

$$B_{0,0}=\begin{bmatrix} -(\lambda+\alpha_1+\alpha_2) & 0 & 0 & \alpha_1 & 0 & 0 & \alpha_2 & 0 & 0 \\ \mu_2 & -(\lambda+\mu_2+\alpha_1+\alpha_2) & 0 & 0 & \alpha_1 & 0 & 0 & \alpha_2 & 0 \\ 0 & \mu_2 & -(\lambda+\mu_2+\alpha_1+\alpha_2) & 0 & 0 & \alpha_1 & 0 & 0 & \alpha_2 \\ \beta_1 & 0 & 0 & -(\lambda+\beta_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_1 & 0 & \mu_2 & -(\lambda+\mu_2+\beta_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & \mu_2 & -(\lambda+\mu_2+\beta_1) & 0 & 0 & 0 \\ \beta_2 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\beta_2) & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\beta_2) & 0 \\ 0 & 0 & \beta_2 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\beta_2) \end{bmatrix},$$

$$A_0=\begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix}, \quad A_2=\begin{bmatrix} 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_1=\begin{bmatrix} -(\lambda+\mu_1+\alpha_1+\alpha_2) & 0 & 0 & \alpha_1 & 0 & 0 & \alpha_2 & 0 & 0 \\ \mu_2 & -(\lambda+\mu_1+\mu_2+\alpha_1+\alpha_2) & 0 & 0 & \alpha_1 & 0 & 0 & \alpha_2 & 0 \\ 0 & \mu_2 & -(\lambda+\mu_2+\alpha_1+\alpha_2) & 0 & 0 & \alpha_1 & 0 & 0 & \alpha_2 \\ \beta_1 & 0 & 0 & -(\lambda+\beta_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_1 & 0 & \mu_2 & -(\lambda+\mu_2+\beta_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & \mu_2 & -(\lambda+\mu_2+\beta_1) & 0 & 0 & 0 \\ \beta_2 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1+\beta_2) & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1+\beta_2) & 0 \\ 0 & 0 & \beta_2 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\beta_2) \end{bmatrix}.$$

A3.2 Derivation of stability conditions of the series configuration system subject to breakdowns and repairs

The stability conditions (14) and (15) in Section 4.2.4 of the system can be derived by following

$$\mathbf{P}_A A_0 \mathbf{1} < \mathbf{P}_A A_2 \mathbf{1},$$

We first evaluate the conservative stable matrix A

$$A = A_0 + A_1 + A_2$$

$$= \begin{bmatrix} -(\mu_1+\alpha_1+\alpha_2) & \mu_1 & 0 & \alpha_1 & 0 & 0 & \alpha_2 & 0 & 0 \\ \mu_2 & -(\mu_1+\mu_2+\alpha_1+\alpha_2) & \mu_1 & 0 & \alpha_1 & 0 & 0 & \alpha_2 & 0 \\ 0 & \mu_2 & -(\mu_2+\alpha_1+\alpha_2) & 0 & 0 & \alpha_1 & 0 & 0 & \alpha_2 \\ \beta_1 & 0 & 0 & -\beta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_1 & 0 & \mu_2 & -(\mu_2+\beta_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & \mu_2 & -(\mu_2+\beta_1) & 0 & 0 & 0 \\ \beta_2 & 0 & 0 & 0 & 0 & 0 & -(\mu_1+\beta_2) & \mu_1 & 0 \\ 0 & \beta_2 & 0 & 0 & 0 & 0 & 0 & -(\mu_1+\beta_2) & \mu_1 \\ 0 & 0 & \beta_2 & 0 & 0 & 0 & 0 & 0 & -\beta_2 \end{bmatrix}.$$

Then we obtain the steady-state probability vector

$\mathbf{P}_A = [P_{A,0}, P_{A,1}, P_{A,2}, P_{A,3}, P_{A,4}, P_{A,5}, P_{A,6}, P_{A,7}, P_{A,8}]$ by solving following system equations

with normalization condition

$$\mathbf{P_A} A = \mathbf{0}, \qquad \text{(A3.1)}$$

$$\sum_{i=0}^{8} P_{A,i} = 1. \qquad \text{(A3.2)}$$

Finally, we obtain the stability condition of the system

$$\mathbf{P}_A A_0 \mathbf{1} < \mathbf{P}_A A_2 \mathbf{1}, \qquad \text{(A3.3)}$$

$$\lambda < \frac{N_{breakdo}}{D_{breakdo}}, \qquad \text{(A3.4)}$$

where

$$\begin{aligned} N_{breakdown} = {} & \beta_1\beta_2\mu_1\mu_2(\alpha_1\alpha_2^2\beta_1\mu_1 + \alpha_2^2\beta_1^2\mu_1 + 2\alpha_1\alpha_2\beta_1\beta_2\mu_1 + 2\alpha_2\beta_1^2\beta_2\mu_1 + \alpha_1\beta_1\beta_2^2\mu_1 + \beta_1^2\beta_2^2\mu_1 \\ & + 2\alpha_1\alpha_2\beta_1\mu_1^2 + 2\alpha_2\beta_1^2\mu_1^2 + 2\alpha_1\beta_1\beta_2\mu_1^2 + 2\beta_1^2\beta_2\mu_1^2 + \alpha_1\beta_1\mu_1^3 + \beta_1^2\mu_1^3 + \alpha_1^2\alpha_2\beta_2\mu_2 \\ & + 2\alpha_1\alpha_2\beta_1\beta_2\mu_2 + \alpha_2\beta_1^2\beta_2\mu_2 + \alpha_1^2\beta_2^2\mu_2 + 2\alpha_1\beta_1\beta_2^2\mu_2 + \beta_1^2\beta_2^2\mu_2 + 2\alpha_1^2\alpha_2\mu_1\mu_2 + 2\alpha_1\alpha_2^2\mu_1\mu_2 \\ & + 4\alpha_1\alpha_2\beta_1\mu_1\mu_2 + 2\alpha_2^2\beta_1\mu_1\mu_2 + 2\alpha_2\beta_1^2\mu_1\mu_2 + 2\alpha_1^2\beta_2\mu_1\mu_2 + 4\alpha_1\alpha_2\beta_2\mu_1\mu_2 + 4\alpha_1\beta_1\beta_2\mu_1\mu_2 \\ & + 4\alpha_2\beta_1\beta_2\mu_1\mu_2 + 2\beta_1^2\beta_2\mu_1\mu_2 + 2\alpha_1\beta_2^2\mu_1\mu_2 + 2\beta_1\beta_2^2\mu_1\mu_2 + \alpha_1^2\mu_1^2\mu_2 + 4\alpha_1\alpha_2\mu_1^2\mu_2 + 2\alpha_1\beta_1\mu_1^2\mu_2 \\ & + 4\alpha_2\beta_1\mu_1^2\mu_2 + \beta_1^2\mu_1^2\mu_2 + 4\alpha_1\beta_2\mu_1^2\mu_2 + 4\beta_1\beta_2\mu_1^2\mu_2 + 2\alpha_1\mu_1^3\mu_2 + 2\beta_1\mu_1^3\mu_2 + 2\alpha_1\alpha_2\beta_2\mu_2^2 \\ & + 2\alpha_2\beta_1\beta_2\mu_2^2 + 2\alpha_1\beta_2^2\mu_2^2 + 2\beta_1\beta_2^2\mu_2^2 + 4\alpha_1\alpha_2\mu_1\mu_2^2 + \alpha_2^2\mu_1\mu_2^2 + 4\alpha_2\beta_1\mu_1\mu_2^2 + 4\alpha_1\beta_2\mu_1\mu_2^2 \\ & + 2\alpha_2\beta_2\mu_1\mu_2^2 + 4\beta_1\beta_2\mu_1\mu_2^2 + \beta_2^2\mu_1\mu_2^2 + 2\alpha_1\mu_1^2\mu_2^2 + 2\alpha_2\mu_1^2\mu_2^2 + 2\beta_1\mu_1^2\mu_2^2 + 2\beta_2\mu_1^2\mu_2^2 \\ & + \mu_1^3\mu_2^2 + \alpha_2\beta_2\mu_2^3 + \beta_2^2\mu_2^3 + 2\alpha_2\mu_1\mu_2^3 + 2\beta_2\mu_1\mu_2^3 + \mu_1^2\mu_2^3), \end{aligned}$$

and

$$
\begin{aligned}
D_{\text{breakdown}} = {} & (\alpha_2\beta_1 + \alpha_1\beta_2 + \beta_1\beta_2)(\alpha_2^2\beta_1^2\mu_1^2 + 2\alpha_2\beta_1^2\beta_2\mu_1^2 + \beta_1^2\beta_2^2\mu_1^2 + 2\alpha_2\beta_1^2\mu_1^3 + 2\beta_1^2\beta_2\mu_1^3 + \beta_1^2\mu_1^4 \\
& + \alpha_1\alpha_2\beta_1\beta_2\mu_1\mu_2 + \alpha_2\beta_1^2\beta_2\mu_1\mu_2 + \alpha_1\beta_1\beta_2^2\mu_1\mu_2 + \beta_1^2\beta_2^2\mu_1\mu_2 + 2\alpha_1\alpha_2\beta_1\mu_1^2\mu_2 + 2\alpha_2^2\beta_1\mu_1^2\mu_2 \\
& + 2\alpha_2\beta_1^2\mu_1^2\mu_2 + 2\alpha_1\beta_1\beta_2\mu_1^2\mu_2 + 4\alpha_2\beta_1\beta_2\mu_1^2\mu_2 + 2\beta_1^2\beta_2\mu_1^2\mu_2 + 2\beta_1\beta_2^2\mu_1^2\mu_2 + \alpha_1\beta_1\mu_1^3\mu_2 \\
& + 4\alpha_2\beta_1\mu_1^3\mu_2 + \beta_1^2\mu_1^3\mu_2 + 4\beta_1\beta_2\mu_1^3\mu_2 + 2\beta_1\mu_1^4\mu_2 + \alpha_1^2\beta_2^2\mu_2^2 + 2\alpha_1\beta_1\beta_2^2\mu_2^2 + \beta_1^2\beta_2^2\mu_2^2 \\
& + 2\alpha_1^2\beta_2\mu_1\mu_2^2 + 2\alpha_1\alpha_2\beta_2\mu_1\mu_2^2 + 4\alpha_1\beta_1\beta_2\mu_1\mu_2^2 + 2\alpha_2\beta_1\beta_2\mu_1\mu_2^2 + 2\beta_1^2\beta_2\mu_1\mu_2^2 + 2\alpha_1\beta_2^2\mu_1\mu_2^2 \\
& + 2\beta_1\beta_2^2\mu_1\mu_2^2 + \alpha_1^2\mu_1^2\mu_2^2 + 2\alpha_1\alpha_2\mu_1^2\mu_2^2 + \alpha_2^2\mu_1^2\mu_2^2 + 2\alpha_1\beta_1\mu_1^2\mu_2^2 + 4\alpha_2\beta_1\mu_1^2\mu_2^2 + \beta_1^2\mu_1^2\mu_2^2 \\
& + 4\alpha_1\beta_2\mu_1^2\mu_2^2 + 2\alpha_2\beta_2\mu_1^2\mu_2^2 + 4\beta_1\beta_2\mu_1^2\mu_2^2 + \beta_2^2\mu_1^2\mu_2^2 + 2\alpha_1\mu_1^3\mu_2^2 + 2\alpha_2\mu_1^3\mu_2^2 + 2\beta_1\mu_1^3\mu_2^2 \\
& + 2\beta_2\mu_1^3\mu_2^2 + \mu_1^4\mu_2^2 + 2\alpha_1\beta_2^2\mu_2^3 + 2\beta_1\beta_2^2\mu_2^3 + 4\alpha_1\beta_2\mu_1\mu_2^3 + \alpha_2\beta_2\mu_1\mu_2^3 + 4\beta_1\beta_2\mu_1\mu_2^3 + \beta_2^2\mu_1\mu_2^3 \\
& + 2\alpha_1\mu_1^2\mu_2^3 + 2\alpha_2\mu_1^2\mu_2^3 + 2\beta_1\mu_1^2\mu_2^3 + 2\beta_2\mu_1^2\mu_2^3 + \mu_1^3\mu_2^3 + \beta_2^2\mu_2^4 + 2\beta_2\mu_1\mu_2^4 + \mu_1^2\mu_2^4).
\end{aligned}
$$

For the special case, we set $\mu_1 = \mu_2 = \mu$, $\alpha_1 = \alpha_2 = \alpha$, and $\beta_1 = \beta_2 = \beta$,

$$\lambda < \frac{2\beta\mu(\alpha+\beta+\mu)^2}{(2\alpha+\beta)(3\alpha\beta+3\beta^2+2\alpha\mu+6\beta\mu+3\mu^2)}. \tag{A3.5}$$

A.4 The structure of the transition matrix Q and its sub-matrices for the self-blocking system with three service stations

We provide transition matrix of the series configuration queueing system with two service stations subject to breakdowns and repairs as

$$Q=\begin{bmatrix} B_{0,0} & B_{0,1} & 0 & 0 & 0 & 0 & \cdots \\ B_{1,0} & A_1 & A_0 & 0 & 0 & 0 & \cdots \\ B_{2,0} & A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ B_{3,0} & A_3 & A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & A_4 & A_3 & A_2 & A_1 & A_0 & \cdots \\ 0 & 0 & A_4 & A_3 & A_2 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The details of sub-matrices of the composition of the transition matrix corresponding to the quasi-birth-death process for the self-blocking system with three service stations are given by

$$B_{0,0}=\begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 \\ \mu_3 & -(\lambda+\mu_3) & 0 & \lambda & 0 \\ \mu_2 & 0 & -(\lambda+\mu_2) & 0 & 0 \\ 0 & 0 & \mu_3 & -(\lambda+\mu_2+\mu_3) & \mu_2 \\ \mu_3 & 0 & 0 & 0 & -(\lambda+\mu_3) \end{bmatrix},\ B_{1,0}=\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \mu_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_{2,0}=\begin{bmatrix} 0 & \mu_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ B_{3,0}=\begin{bmatrix} 0 & 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ B_{0,1}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -(\lambda+\mu_1) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\lambda+\mu_2) & 0 & 0 & 0 & 0 & 0 \\ \mu_2 & \mu_1 & -(\lambda+\mu_1+\mu_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_3 & -(\lambda+\mu_1+\mu_2+\mu_3) & \mu_2 & 0 & \mu_1 \\ \mu_3 & 0 & 0 & 0 & -(\lambda+\mu_1+\mu_3) & \mu_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_3) & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_2 & -(\lambda+\mu_2+\mu_3) \end{bmatrix},$$

$$A_0 = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; A_4 = \begin{bmatrix} 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A4.2 Derivation of stability conditions of the self-blocking system with three service stations

The stability conditions (14) and (15) of the system in the Section 5.2.4 can be derived by following

$$\mathbf{P}_A A_0 \mathbf{1} < \mathbf{P}_A A_2 \mathbf{1} + 2(\mathbf{P}_A A_3 \mathbf{1}) + 3(\mathbf{P}_A A_4 \mathbf{1}),$$

We first evaluate the conservative stable matrix A

$$A = A_0 + A_1 + A_2 + A_3 + A_4$$
$$= \begin{bmatrix} -\mu_1 & 0 & 0 & \mu_1 & 0 & 0 & 0 \\ 0 & -\mu_2 & 0 & \mu_2 & 0 & 0 & 0 \\ \mu_2 & \mu_1 & -(\mu_1+\mu_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_3 & -(\mu_1+\mu_2+\mu_3) & \mu_2 & 0 & \mu_1 \\ \mu_3 & 0 & 0 & 0 & -(\mu_1+\mu_3) & \mu_1 & 0 \\ 0 & 0 & 0 & \mu_3 & 0 & -\mu_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_2 & -(\mu_2+\mu_3) \end{bmatrix}.$$

Then we obtain the steady-state probability vector

$\mathbf{P}_{A} = [P_{A,0}, P_{A,1}, P_{A,2}, P_{A,3}, P_{A,4}, P_{A,5}, P_{A,6}]$ by solving following system equations

with normalization condition

$$\mathbf{P_A} A = \mathbf{0}, \tag{A4.1}$$

$$\sum_{i=0}^{6} P_{A,i} = 1. \tag{A4.2}$$

Finally, we obtain the stability condition of the self-blocking system with three service stations

$$\mathbf{P}_{A} A_0 \mathbf{1} < \mathbf{P}_{A} A_2 \mathbf{1} + 2(\mathbf{P}_{A} A_3 \mathbf{1}) + 3(\mathbf{P}_{A} A_4 \mathbf{1}), \tag{A4.3}$$

$$\lambda < \frac{N_{self-blocking}}{D_{self-blocking}}, \tag{A4.4}$$

where

$$N_{self-blocking} = 3\mu_1\mu_2\mu_3(\mu_1+\mu_2)(\mu_1+\mu_3)(\mu_2+\mu_3)(\mu_1+\mu_2+\mu_3),$$

and

$$\begin{aligned} D_{self-blocking} &= \mu_1^4(\mu_2^2+\mu_2\mu_3+\mu_3^2)+\mu_1^3(2\mu_2^3+4\mu_2^2\mu_3+4\mu_2\mu_3^2+2\mu_3^3) \\ &+\mu_1^2(\mu_2^4+4\mu_2^3\mu_3+5\mu_2^2\mu_3^2+4\mu_2\mu_3^3+\mu_3^4) \\ &+\mu_1(\mu_2^4\mu_3+4\mu_2^3\mu_3^2+4\mu_2^2\mu_3^3+\mu_2\mu_3^4)+\mu_2^2(\mu_2^2\mu_3^2+2\mu_2\mu_3^3+\mu_3^4). \end{aligned}$$

For the special case, we set, $\mu_1 = \mu_2 = \mu_3 = \mu$

$$\lambda < \frac{18}{11}\mu. \tag{A4.5}$$