博士論文

Applications of max-plus algebra to scheduling problems
(マックスプラス代数のスケジューリング問題への応用)

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Introduction

1.1 Importance of scheduling

Scheduling is crucial in various industries such as manufacturing and service sectors. Companies have to operate their production systems efficiently and produce goods or services by due date to satisfy the customers.

Given tasks to carry out and resources available to process the tasks, scheduling problems are to determine the allocation of resources to tasks over time (see e.g. [1–3]). A number of problems in real world, for example improvement of production lines, gate assignments at an airport, etc., are scheduling problems. Since theory of scheduling were historically developed in problems arising from manufacturing, tasks and resources are usually called *jobs* and *machines*, respectively.

The concept of scheduling is not new in human history, but the theory of scheduling is comparably new. Some of the first publications in the research literature were done by Johnson [4], Jackson [5] and Smith [6] in the mid-1950s. Since then much research on a variety of scheduling problems have been done (see e.g. [2, 3]).

1.2 Classification of scheduling problems

1.2.1 Job shops, flow shops, and parallel-machine models

Scheduling problems contain n jobs and m machines available to process the jobs. The environment is called *job shops*. The thesis deals with some flow shops and parallel-machine models, which are derived as special cases of job shops. Flow shops and parallel-machine models are very prevalent in many manufacturing industries.

We consider deterministic models, where the job data, such as processing times and release dates, are known in advance.

In job shops, each job follows a predetermined route. A distinction is made between job shops in which there are no duplicate machines and in which there are duplicate machines.

In flow shops, each job follows the same route. Flow shops are a special case of job shops without duplicate machines, where there is the only route for all jobs (see Figs. 1.1 and 1.3).

job → machine 1 → machine 2 → ⋯ → machine m → (completion)

Figure 1.1: The flow of jobs in flow shops

In parallel-machine models, each job has to be processed on any one of the machines. Parallel-machine models are a special case of job shops with identical (duplicate) machines, where each job require a single operation (see Figs. 1.2 and 1.3).

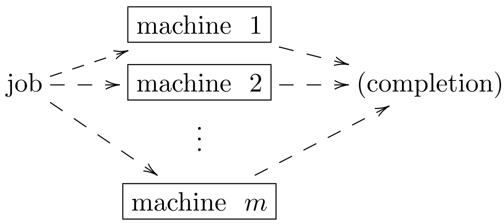


Figure 1.2: The flow of jobs in parallel-machine models

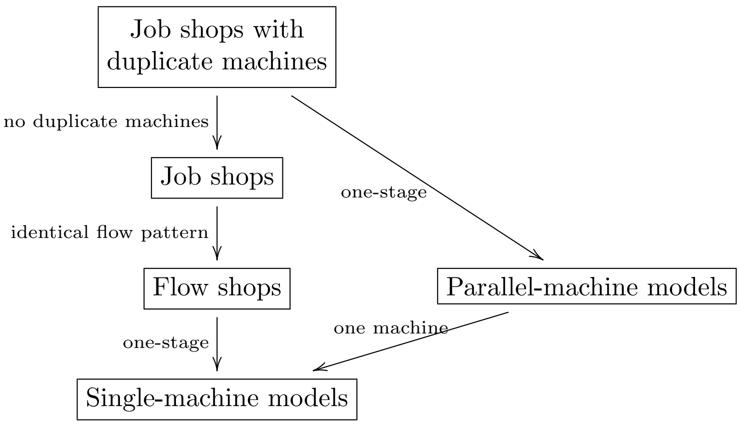


Figure 1.3: Relationships among job shops

1.2.2 Notations

We use the notation for scheduling problems suggested by Graham et al. [7]. A triplet $\alpha|\beta|\gamma$ describes a problem, where α denotes the machine environment, β

provides details of constraints, and γ describes the objective to be minimized. α, β and γ considered in the thesis are as follows.

The machine environments in the α field are:

- **Identical machines in parallel** (Pm). There are m parallel identical machines. Each job comes with one processing time and is processed on any one of the m machines.
- **Flow shop** (Fm). There are m different machines in series. Each job consists of m operations, each of which requires a different machine. All jobs have to be processed on the first machine, then on the second machine, and so on.

The constraints in the β field are:

- **Release dates** (r_j). r_j is the earliest time at which job j can be processed (see e.g. [2]). Without r_j in the field, all jobs are available at time zero.
- **Starting times** (s_i). s_i is the earliest time at which machine i can start processing (see e.g. [8, 9]). Without s_i in the field, all machines are available at time zero.

The following constraints may appear only in flow shops.

- **Permutation schedules** (prmu). The same sequence of jobs is maintained between machines throughout (see e.g. [1, 2]).
- **No-wait** (no-wait). A job must be processed without waiting between two successive machines (see e.g. [2, 10, 11]). Any no-wait schedule is a permutation schedule.
- **No-idle** (no-idle). A machine must process a job without idling, i.e., without waiting for the next job (see e.g. [12, 13])
- **Blocking** (blocking). A completed job at a machine has to remain on the machine until the downstream machine is available (see e.g. [2, 10]). Here suppose the *First In First Out* discipline, then any blocking schedule is a permutation schedule.
- **Busy** (busy). A machine completing a job has to have the job until the next job come (see e.g. [14]).

γ is the makespan ($C_{\max}$) or the total completion time ($\sum C_j$), where C_j denotes the completion time of job j. However, the case where γ itself is specified means that the objective is arbitrary.

1.3 Summary of previous research on scheduling problems

We review previous research on flow shops and a two-identical-parallel-machine problem, which we examine in the thesis.

1.3.1 Flow shops

Flow shop scheduling problems have been studied for several decades since Johnson's paper [4] on the two-machine flow shop problem. They have many applications in production lines such as pharmaceutical and agro-food industries. Since the late 1950s optimization procedures were applied to the problems. Linear programming formulations [15, 16], dynamic programming [17], and branch and bound methods [18] were used. Meanwhile, heuristic methods were developed due to computational difficulty [19].

In the about 1970s many theoretical research on flow shop problems has been done ad hoc and some solvable cases were found. Burns and Rooker [20, 21], Szwarc [22] and Achugbue and Chin [23] considered only three-machine flow shops ($F3$). Though Nabeshima [24,25] and Gupta [26] studied m-machine flow shops (Fm), the research needed complicated and laborious computations. On the other hand, it was revealed that most of the problems are NP-hard [27–30]. Most probably no fast (i.e., polynomial-time) optimal algorithm exists. Many heuristic approaches have been developed since then (see e.g. [31]).

As stated above a variety of methods were developed for solving flow shops. They are grouped into three classes [32]:

- **Efficient optimal methods**: These find an optimal schedule in polynomial time, i.e., its running time is bounded by a polynomial in the input size. The class of problems which these methods can be applied to is relatively narrow.
- **Enumerative optimal methods**: These also find an optimal schedule, but typically involve a partial enumeration of the set of all possible schedules, so its running time is not bounded by polynomial time. A lot of mathematical programming formulations (linear programming, dynamic programming, etc.), and branch and bound methods are included in this class.
- **Heuristic methods**: These approximate an optimal solution with some degree of closeness in polynomial time.

Nowadays heuristics methods are widely studied from practical point of view [33, 34].

1.3.2 A two-identical-parallel-machine problem

The two-identical-parallel-machine problem to minimize the makespan, $P2||C_{\max}$, is equivalent to the number partitioning problem (NPP), which is to find a set $S' \subset \{1, ..., n\}$ that minimize the discrepancy

$$\Delta = \left| \sum_{i \in S'} p_i - \sum_{i \in \{1,...,n\} \setminus S'} p_i \right|$$

for a given positive integer p_i for $i \in \{1, ..., n\}$. The problem has important applications such as VLSI chip production [35], choosing fair teams [36], etc. It has been studied from different perspectives, from its solution with heuristics or exact methods to statistical analyses.

This decision version is known to be NP-complete [37], so the optimal problem is NP-hard.

A variety of heuristic algorithms have been developed. A greedy algorithm, which sorts the given integers in decreasing order and assign each number to the subset with the smaller sum, is simple and intuitive. The set differencing method introduced by Karmarkar and Karp [38] is said to be the best polynomial-time heuristics. Metaheuristics such as simulated annealing [39], genetic algorithm [40, 41], tabu search [42], etc. were applied to the problem.

There are several optimal algorithms that give exact solutions in exponential time in n. The complete greedy algorithm and the complete Karmarkar-Karp algorithm, which produces the optimal solution in $\mathcal{O}(2^n)$, were developed by Korf [43]. A better exponential time algorithm which takes $\mathcal{O}(n2^{n/2})$ time was presented by Horowitz and Sahni [44]. A dynamic programming [28] solves the problem in pseudo-polynomial time. This means that it runs in time polynomial in the numeric value of the input (e.g., the sum of all given integers).

On the other hand, the solution structure has been studied based on the notion and tools from statistical mechanics [45–49].

1.4 Max-plus algebra

This section reviews known results in max-plus algebra.

1.4.1 Definitions

We use the following notation:

$$\mathbb{0} = -\infty,$$
$$\mathbb{1} = 0,$$

and

$$\bar{\mathbb{R}} = \mathbb{R} \cup \mathbb{0}.$$

For elements $a, b \in \bar{\mathbb{R}}$, we define the operations $\oplus$ and $\odot$ by

$$a \oplus b = \max[a, b]$$

and

$$a \odot b = a + b.$$

Note that by definition for $a \in \bar{\mathbb{R}}$

$$a \oplus \mathbb{0} = \mathbb{0} \oplus a = a$$

and

$$a \odot \mathbb{0} = \mathbb{0} \odot a = \mathbb{0}.$$

The first operator,$\oplus$, is idempotent, commutative, associative and has a neutral element $\mathbb{0}$. The second operator,$\odot$, is commutative, associative, distributive on $\oplus$ and has a neutral element $\mathbb{1}$.

The set $\bar{\mathbb{R}}$ with the operations $\oplus$ and $\odot$ is called *max-plus algebra* or also *tropical algebra*.

Every element, except $\mathbb{0}$, is invertible: the inverse of x is denoted by x^{-1} or $\frac{\mathbb{1}}{x}$. Note that in this thesis the two notations $\frac{X}{Y}$ and X/Y have different meanings and imply the quotient in max-plus algebra and the one in the conventional algebra respectively. For example, $\frac{1}{2} = 1 - 2 = -1$ and $1/2 = 0.5$.

Powers can be introduced as

$$A^{\alpha} = \begin{cases} \alpha A, & A, \alpha \in \mathbb{R}; \\ \mathbb{0}, & A = \mathbb{0}, \alpha > 0, \end{cases}$$

where αA is the product of α and A in the conventional algebra. For $\alpha > 0$ we have

$$A < (\leq) B \quad \Leftrightarrow \quad A^{\alpha} < (\leq) B^{\alpha}$$

in max-plus algebra.

Definition 1.4.1. A semiring is a set R together with two binary operations $\oplus_R$ and $\odot_R$ such that

- $\oplus_R$ is associative, commutative, and has zero element $\mathbb{0}_R$;
- $\odot_R$ is associative, distributive over $\oplus_R$, and has unit element $\mathbb{1}_R$;
- $\mathbb{0}_R$ is absorbing for $\odot_R$, that is, for $a \in R$, $a \odot_R \mathbb{0}_R = \mathbb{0}_R \odot_R a = \mathbb{0}_R$.

A *commutative semiring* is one whose multiplication is commutative. An *idempotent semiring* is one whose addition is idempotent, that is, for $a \in R$, $a \oplus_R a = a$.

Max-plus algebra is an idempotent and commutative semiring.

Lemma 1.4.2. *Idempotency of $\oplus$ in max-plus algebra implies that every element except for $\mathbb{0}$ does not have an additive inverse.*

Proof. Suppose that $a \neq \mathbb{0}$ had an additive inverse b. Then we would have

$$a \oplus b = \mathbb{0}.$$

Adding a to both sides from the left yields

$$\begin{aligned} a \oplus (a \oplus b) &= a \oplus \mathbb{0} \\ &= a. \end{aligned}$$

By associativity and idempotency of $\oplus$ we get

$$a = a \oplus (a \oplus b) = (a \oplus a) \oplus b = a \oplus b = \mathbb{0},$$

which is a contradiction since $a \neq \mathbb{0}$. □

1.4.2 Link with nonnegative numbers in the conventional algebra

Consider the equality

$$z = a \odot (b \oplus c) \odot d^{-1}.$$

Since

$$x \oplus y = \max[x, y] = \lim_{\varepsilon \to +0} \varepsilon \log \left(e^{x/\varepsilon} + e^{y/\varepsilon} \right)$$

and

$$x \odot y = x + y = \varepsilon \log \left(e^{x/\varepsilon} e^{y/\varepsilon} \right),$$

the equality can be approximated by

$$e^{z/\varepsilon} = e^{a/\varepsilon}(e^{b/\varepsilon} + e^{c/\varepsilon})e^{-d/\varepsilon}.$$

By the exponential transformation max-plus algebra can be mapped to a structure consisting of exponentials with the conventional addition and multiplication. This transformation maps $\mathbb{0}$ to 0 and $\mathbb{1}$ to 1. We can transform some results from the conventional algebra to max-plus algebra. Note that there does not exist an equivalent of the minus operator in max-plus algebra.

The relationship in the following table exists.

	max-plus algebra	the conventional algebra (nonnegative, without subtraction)
Addition	$\oplus$(max)	$+$
Multiplication	$\odot$(+)	$\cdot$

Table 1.1: The relationship between max-plus algebra and the conventional algebra

1.4.3 Matrices and vectors

Many features of linear algebra, such as eigenvalues, eigenvectors, the Cayley-Hamilton theorem and so on, were reproduced in max-plus algebra in Cuninghame-Green [50,51], Straubing [52], Olsder et al. [53], Elsner and van den Driessche [54], Butkovič and Murfitt [55], Burkard and Butkovič [56,57], and Binding and Volkmer [58]. We present only definitions here, since the features are not used in this thesis.

The two operators $\oplus$ and $\odot$ are extended to $m \times n$ matrices of elements of $\bar{\mathbb{R}}$. The element of a matrix $A \in \bar{\mathbb{R}}^{m\times n}$ in row i and column j is denoted by $(A)_{ij}$. The sum of matrices $A, B \in \bar{\mathbb{R}}^{m\times n}$ is defined as

$$(A \oplus B)_{ij} = (A)_{ij} \oplus (B)_{ij}$$

for all i, j. The product of $A \in \bar{\mathbb{R}}^{m\times l}$ and $B \in \bar{\mathbb{R}}^{l\times n}$ is defined as

$$(A \odot B)_{ij} = \bigoplus_{k=1}^{l} (A)_{ik} \odot (B)_{kj}$$

for all i, j.

The standard orders, $\leq$ and $\geq$, of real numbers is also extended to matrices (including vectors) componentwise, i.e., if A and B are of the same size then $A \leq (\geq) B$ means that $(A)_{ij} \leq (\geq)(B)_{ij}$ for all i, j.

Example 1.4.3.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \oplus \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a \oplus e & b \oplus f \\ c \oplus g & d \oplus h \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \odot x \oplus b \odot y \\ c \odot x \oplus d \odot y \end{pmatrix}.$$

1.4.4 Polynomials

Consider a max-plus algebraic analogue of polynomials, which we call tropical polynomials, or maxpolynomials. For tropical polynomials we use the notations by Carlsson and Kališnik [59].

Definitions

Definition 1.4.4. A *tropical polynomial expression* is a finite linear combination of tropical monomial expressions:

$$p\left(x_1, \ldots, x_N\right) = a_1 \odot x_1^{i_1^1} \odot \cdots \odot x_N^{i_N^1} \oplus \cdots \oplus a_m \odot x_1^{i_1^m} \odot \cdots \odot x_N^{i_N^m},$$

where $a_1, \ldots, a_m$ are real numbers and i_p^q for $p \in \{1, \ldots, N\}$ and $q \in \{1, \ldots, m\}$ is an integer.

Definition 1.4.5. *Tropical polynomials* are the semiring of equivalence classes of tropical polynomial expressions with respect to functional equivalence. In the case of N variables we denote it by $\mathrm{Trop}[x_1, \ldots, x_N]$.

Example 1.4.6. Consider the two tropical polynomial expressions

$$x_1^2 \oplus x_2^2 \oplus x_3^2 \quad \text{and} \quad x_1^2 \oplus x_2^2 \oplus x_3^2 \oplus x_1 \odot x_2 \oplus x_1 \odot x_3 \oplus x_2 \odot x_3.$$

The two expressions are functionally equivalent, that is,

$$x_1^2 \oplus x_2^2 \oplus x_3^2 = x_1^2 \oplus x_2^2 \oplus x_3^2 \oplus x_1 \odot x_2 \oplus x_1 \odot x_3 \oplus x_2 \odot x_3$$

for all $(x_1, x_2, x_3) \in \bar{\mathbb{R}}^3$, since $x_1 \odot x_2 \leq x_1^2 \oplus x_2^2$, $x_1 \odot x_3 \leq x_1^2 \oplus x_3^2$ and $x_2 \odot x_3 \leq x_2^2 \oplus x_3^2$. Therefore, the two expressions belong to the same tropical polynomial (the same equivalence class).

Definition 1.4.7. A tropical polynomial $p \in \mathrm{Trop}\left[x_1^{(1)}, ..., x_n^{(1)}, \ldots, x_1^{(r)}, ..., x_n^{(r)}\right]$ is *r-symmetric* if

$$p\left(x_1^{(1)}, ..., x_n^{(1)}, \ldots, x_1^{(r)}, ..., x_n^{(r)}\right) = p\left(x_{\pi(1)}^{(1)}, ..., x_{\pi(n)}^{(1)}, \ldots, x_{\pi(1)}^{(r)}, ..., x_{\pi(n)}^{(r)}\right)$$

for every permutation $\pi \in S_n$.

We use the definition especially in Appendix A.

Example 1.4.8. Let $n = 3$ and $r = 2$. Let us denote $x^{(1)}$ and $x^{(2)}$ by x and y, respectively. The tropical polynomial

$$x_1^2 \odot y_2 \oplus x_1^2 \odot y_3 \oplus x_2^2 \odot y_1 \oplus x_2^2 \odot y_3 \oplus x_3^2 \odot y_1 \oplus x_3^2 \odot y_2$$

is 2-symmetric.

Polynomial equations

Cuninghame-Green [60] studied general equations between two univariate tropical polynomials and presented an algorithm to determine the solutions.

We will first study the product form

$$\bigodot_{r=1,\ldots,n} (Z \oplus \beta_r)^{e_r},$$

where $\beta_r \in \bar{\mathbb{R}}$ and e_r is an integer. The constants β_r are called *corners*.

Theorem 1.4.9. *Every univariate tropical polynomial $P(Z) \in \mathrm{Trop}\,[Z]$ possesses a product form $P'(Z)$ such that $P(Z) = P'(Z)$ for all $Z \in \mathbb{R}$.*

Example 1.4.10. Consider the tropical polynomial

$$1 \oplus 2 \odot Z \oplus Z^3.$$

The product form is

$$(Z \oplus -1) \odot (Z \oplus 1)^2.$$

Theorem 1.4.11. *Let S be the solution set to the polynomial equation*

$$P(Z) = Q(Z),$$

where $P(Z)$ and $Q(Z)$ are univariate tropical polynomials. Then every boundary of S is a corner of $P(Z) \oplus Q(Z)$.

It is geometrically clear that S is the union of a finite number of closed intervals.

We define tropical algebraic equations as a special case of the tropical polynomial equations above. As a corollary of the theorem, it is revealed that tropical algebraic equations have a rich mathematical structure: the solution set is described explicitly under certain conditions and there is a relationship between the solutions and the coefficients of the equation through the elementary symmetric tropical polynomials [61].

Definition 1.4.12. Let n be a positive integer. A tropical equation in the form

$$\begin{cases} \bigoplus_{k=0}^{m} A_{2k} \odot Z^{n-2k} = \bigoplus_{k=0}^{m-1} A_{2k+1} \odot Z^{n-(2k+1)}, & n = 2m; \\ \bigoplus_{k=0}^{m} A_{2k} \odot Z^{n-2k} = \bigoplus_{k=0}^{m} A_{2k+1} \odot Z^{n-(2k+1)}, & n = 2m+1, \end{cases} \tag{1.1}$$

where $A_0 \in \mathbb{R}$ and $A_i \in \bar{\mathbb{R}}$ for $i \in \{1, ..., n\}$, is called a *tropical algebraic equation of the n-th degree.*

The equation is a tropicalization of the algebraic equation of the n-th degree

$$a_0 z^n - a_1 z^{n-1} + a_2 z^{n-2} - \cdots + (-1)^{n-1} a_{n-1} z + (-1)^n a_n = 0,$$
$$a_0 z^n + a_2 z^{n-2} \cdots = a_1 z^{n-1} + a_3 z^{n-3} + \cdots,$$

where $a_0 \neq 0$.

We suppose hereinafter that $A_0 = \mathbb{1}$ and $A_i \in \mathbb{R}$ for $i \in \{1, ..., n\}$.

Corollary 1.4.13. *Let n be an integer greater than 1. The equation (1.1) has n distinct solutions if the inequalities*

$$\frac{A_n}{A_{n-1}} < \cdots < \frac{A_2}{A_1} < A_1$$

hold. The solution set is

$$\left\{ \frac{A_n}{A_{n-1}}, ..., \frac{A_2}{A_1}, A_1 \right\}.$$

Definition 1.4.14. Given variables $X_1, X_2, ..., X_n$, the *elementary symmetric tropical polynomials* $\Sigma_1, ..., \Sigma_n$ are defined as follows:

$$\begin{aligned}
\Sigma_1(X_1, X_2, ..., X_n) &= X_1 \oplus X_2 \oplus \cdots \oplus X_n, \\
&\vdots \\
\Sigma_i(X_1, X_2, ..., X_n) &= \bigoplus_{1 \le j_1 < j_2 < \cdots < j_i \le n} X_{j_1} \odot X_{j_2} \odot \cdots \odot X_{j_i}, \\
&\vdots \\
\Sigma_n(X_1, X_2, ..., X_n) &= X_1 \odot X_2 \odot \cdots \odot X_n.
\end{aligned}$$

See Appendix A for an extension of the elementary symmetric tropical polynomials.

Corollary 1.4.15 (Vieta's Formulas)**.** *Let n be an integer greater than 1. Suppose that the inequalities*

$$\frac{A_n}{A_{n-1}} < \cdots < \frac{A_2}{A_1} < A_1$$

hold. Let the n distinct solutions to the equation (1.1) be $X_1, X_2, ..., X_n$. Then the following relations hold:

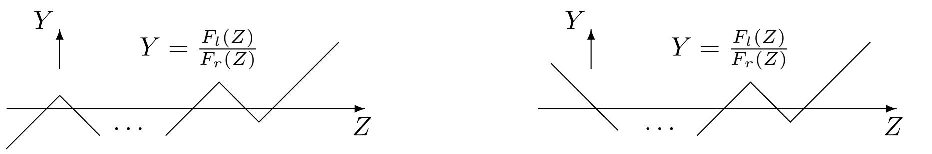
$$\begin{aligned}
A_1 &= \Sigma_1(X_1, X_2, ..., X_n), \\
&\vdots \\
A_i &= \Sigma_i(X_1, X_2, ..., X_n), \\
&\vdots \\
A_n &= \Sigma_n(X_1, X_2, ..., X_n).
\end{aligned}$$

Consider a geometric interpretation of the solutions of tropical algebraic equations.

Let the left-hand side and right-hand side of the tropical algebraic equation (1.1) be denoted by $F_l(Z)$ and $F_r(Z)$ respectively. Then the equation is equivalent to

$$\frac{F_l(Z)}{F_r(Z)} = \mathbb{1} = 0.$$

The solutions are the Z-intercepts of the graph of the function $\frac{F_l(Z)}{F_r(Z)}$. If the assumption of Corollary 1.4.13 holds, then the slopes are 1 or -1 and all the segments and half-lines cross the Z-axis (see Fig. 1.4).



n: odd $\qquad\qquad$ n: even

Figure 1.4: Graphs of the function $\frac{F_l(Z)}{F_r(Z)}$.

Example 1.4.16. Consider the tropical algebraic equation

$$\begin{aligned} & Z^4 \oplus 16{\odot}Z^2 \oplus 14 = 9{\odot}Z^3 \oplus 17{\odot}Z \\ \Leftrightarrow \quad & \max\,[4Z, 2Z+16, 14] = \max\,[3Z+9, Z+17] \quad \text{(in the conventional algebra).} \end{aligned}$$

The assumption of Corollary 1.4.13 is satisfied and the solution set is $\{-3, 1, 7, 9\}$. Let the solutions be X_1, X_2, X_3, X_4 in decreasing order. Then

$$\begin{aligned} \Sigma_1(X_1, X_2, X_3, X_4) &= X_1 = 9, \\ \Sigma_2(X_1, X_2, X_3, X_4) &= X_1{\odot}X_2 = 9{\odot}7 = 16, \\ \Sigma_3(X_1, X_2, X_3, X_4) &= X_1{\odot}X_2{\odot}X_3 = 9{\odot}7{\odot}1 = 17, \\ \Sigma_4(X_1, X_2, X_3, X_4) &= X_1{\odot}X_2{\odot}X_3{\odot}X_4 = 9{\odot}7{\odot}1{\odot}(-3) = 14. \end{aligned}$$

The solutions are the Z-intercepts of the graph of the function $\frac{Z^4 \oplus 16\odot Z^2 \oplus 14}{9\odot Z^3 \oplus 17\odot Z}$ (see Fig. 1.5).

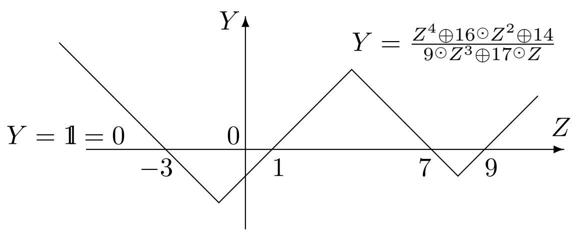


Figure 1.5: Graph of the function $\frac{Z^4 \oplus 16\odot Z^2 \oplus 14}{9\odot Z^3 \oplus 17\odot Z}$.

Corollary 1.4.17. *Let n be an integer greater than 1. If the inequalities*

$$\frac{A_n}{A_{n-1}} \leq \cdots \leq \frac{A_2}{A_1} \leq A_1$$

hold, then the solution set of the equation (1.1) is

$$\left\{ \frac{A_n}{A_{n-1}}, ..., \frac{A_2}{A_1}, A_1 \right\},$$

and the following hold:

$$A_1 = \Sigma_1(X_1, X_2, ..., X_n),$$
$$\vdots$$
$$A_i = \Sigma_i(X_1, X_2, ..., X_n),$$
$$\vdots$$
$$A_n = \Sigma_n(X_1, X_2, ..., X_n),$$

where $X_1, ..., X_n$ are the solutions (not necessarily distinct).

Example 1.4.18. Consider the tropical algebraic equation

$$\begin{aligned} & Z^5 \oplus 14 \odot Z^3 \oplus 22 \odot Z = 7 \odot Z^4 \oplus 21 \odot Z^2 \odot 23 \\ \Leftrightarrow \quad & \max[5Z, 3Z+14, Z+22] = \max[4Z+7, 2Z+21, 23] \\ & \qquad \text{(in the conventional algebra).} \end{aligned}$$

The assumption of Corollary 1.4.17 is satisfied and the solution set is $\{1, 1, 7, 7, 7\}(= \{1, 7\})$. Let the solutions be X_1, X_2, X_3, X_4, X_5 in non-increasing order. Then

$$\begin{aligned} \Sigma_1(X_1, X_2, X_3, X_4, X_5) &= X_1 = 7, \\ \Sigma_2(X_1, X_2, X_3, X_4, X_5) &= X_1 \odot X_2 = 7 \odot 7 = 14, \\ \Sigma_3(X_1, X_2, X_3, X_4, X_5) &= X_1 \odot X_2 \odot X_3 = 7 \odot 7 \odot 7 = 21, \\ \Sigma_4(X_1, X_2, X_3, X_4, X_5) &= X_1 \odot X_2 \odot X_3 \odot X_4 = 7 \odot 7 \odot 7 \odot 1 = 22, \\ \Sigma_5(X_1, X_2, X_3, X_4, X_5) &= X_1 \odot X_2 \odot X_3 \odot X_4 \odot X_5 = 7 \odot 7 \odot 7 \odot 1 \odot 1 = 23. \end{aligned}$$

The solutions are the Z-intercepts of the graph of the function $\frac{Z^5 \oplus 14 \odot Z^3 \oplus 22 \odot Z}{7 \odot Z^4 \oplus 21 \odot Z^2 \oplus 23}$ (see Fig. 1.6). The solution $Z = 1$ has multiplicity 2 and so the graph touches, but does not cross the Z-axis. The solution $Z = 7$ has multiplicity 3 and so the graph crosses the Z-axis.

In general, if the multiplicity of a solution is odd, then the graph crosses the Z-axis. If the multiplicity is even, then the graph only touches the Z-axis.

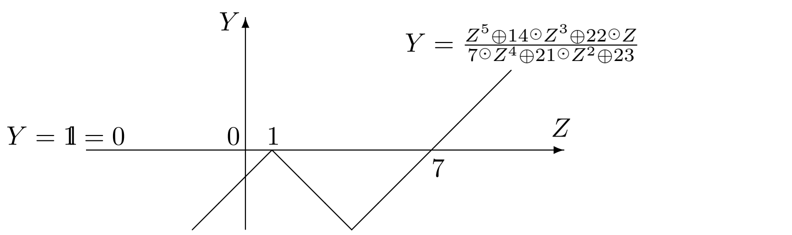


Figure 1.6: Graph of the function $\frac{Z^5 \oplus 14 \odot Z^3 \oplus 22 \odot Z}{7 \odot Z^4 \oplus 21 \odot Z^2 \oplus 23}$.

1.5 Background and motivation

As stated in Section 1.3, it was revealed that many scheduling problems, including flow shops and parallel-machine models, are NP-hard, that is, computationally intractable in the 1970s. Due to their computational complexity, most of research have recently been devoted to (meta-) heuristics, not exact solutions. In the marketplace companies tend to make schedules by empirical rule.

On the other hand, max-plus algebra, or tropical algebra, has been studied from the 1960s. A number of pioneering articles concerning links with linear algebra and analogues of algebraic equations were published. Note that max-plus algebra has no additive inverse and computations in the algebra requires different techniques.

To our knowledge, there are only a few papers concerning the application of max-plus algebra to flow shops and two-identical-parallel-machine problems. Max-plus algebra was applied to some flow shop problems [8, 9, 62, 63].

Our aim is to develop the basic theory of scheduling by using good properties of max-plus algebra. We investigate exact solutions of these problems. The study is important from theoretical and practical point of view. Our methods and results contribute to effectiveness in production systems as well as algorithm design, since the problems have many applications in real world.

1.6 Outline of the thesis

This thesis is organized as follows. Main results are described in Chapters 2 and 3. We in Chapter 2 present a new framework for flow shops. Using the framework, we present a new solvable condition. In Chapter 3 theoretical analysis on a two-identical-parallel-machine problem based on polynomial equations in max-plus algebra were presented. We show the mathematical structure of the NP-complete problem. Finally, Chapter 4 is devoted to concluding remarks.

Chapter 2

A New Framework for Flow Shops

2.1 Introduction

Methods for solving flow shops, as stated in Subsection 1.3.1, are grouped into three classes: efficient optimal methods, enumerative optimal methods, and heuristic methods. Nowadays heuristics methods are widely studied from practical point of view.

We reconsider the boundary between efficient optimal methods and enumerative ones, armed with max-plus algebra. It is ideal to obtain an optimal solution in polynomial time. In fact the cases where efficient optimal methods can be applied are demanded in real world. Hence, our study is of not only theoretical but also practical importance.

Section 2.2 introduces a basic flow shop problem. In Section 2.3, we present an easily verified sufficient condition for an extension of Johnson's rule. And using max-plus algebra we give a simple proof of the theorem that $Fm|$ no-wait $|C_{\max}$ can be formulated as a traveling salesman problem (TSP). We present in Section 2.4 a new theoretical framework which associates a machine with a matrix and is the dual of the existing approach based on job matrices. Using the framework, we present a new solvable condition which is an extension of known results. Moreover, we show duality relationships between some flow shops. Section 2.5 is devoted to links with linear algebra. Finally, we summarize this chapter in Section 2.6.

2.2 Problem formalization

A basic flow shop instance consists of m different machines, n jobs, and mn nonnegative vales $p_{i,j}$ ($i = 1, 2, ..., m$; $j = 1, 2, ..., n$), where $p_{i,j}$ specifies the time required by machine i for processing job j. Jobs flow from the first machine to the last (m-th) machine. Let $C_{i,j}$ be the completion time of job j at machine i.

Given a job sequence $\sigma = (\sigma(1), \sigma(2), ..., \sigma(n))$ for $Fm|$prmu, $r_j, s_i|\gamma$, the completion time $C_{i,j}$ can be computed through a set of recursive equations:

$$\begin{aligned}
C_{i,\sigma(k)} &= \max[C_{i-1,\sigma(k)}, C_{i,\sigma(k-1)}] + p_{i,\sigma(k)} && (i = 2, ..., m; k = 2, ..., n) && (2.1a)\\
C_{1,\sigma(k)} &= \max[r_{\sigma(k)}, C_{1,\sigma(k-1)}] + p_{1,\sigma(k)} && (k = 2, ..., n) && (2.1b)\\
C_{i,\sigma(1)} &= \max[C_{i-1,\sigma(1)}, s_i] + p_{i,\sigma(1)} && (i = 2, ..., m) && (2.1c)\\
C_{1,\sigma(1)} &= \max[r_{\sigma(1)}, s_1] + p_{1,\sigma(1)}. && && (2.1d)
\end{aligned}$$

2.3 Job matrices

2.3.1 Permutation flow shop problems

We give a new interpretation that an extension of Johnson's rule is understood as the magnitude relationship between the two products of two matrices and present a sufficient condition for the extension.

The recursion relations (2.1) for $Fm|\text{prmu}, s_i|\gamma$ are the following:

$$\begin{aligned}
C_{i,\sigma(k)} &= \left(C_{i-1,\sigma(k)} \oplus C_{i,\sigma(k-1)}\right) \odot p_{i,\sigma(k)} && (i = 2, ..., m; k = 2, ..., n)\\
C_{1,\sigma(k)} &= C_{1,\sigma(k-1)} \odot p_{1,\sigma(k)} && (k = 2, ..., n)\\
C_{i,\sigma(1)} &= \left(C_{i-1,\sigma(1)} \oplus s_i\right) \odot p_{i,\sigma(1)} && (i = 2, ..., m)\\
C_{1,\sigma(1)} &= s_1 \odot p_{1,\sigma(1)}
\end{aligned}$$

Let

$$\boldsymbol{C}'_j = \begin{pmatrix} C_{1,j} \\ C_{2,j} \\ \vdots \\ C_{m,j} \end{pmatrix}.$$

Then

$$\boldsymbol{C}'_{\sigma(n)} = J_{\sigma(n)} \odot \cdots \odot J_{\sigma(1)} \odot \boldsymbol{C}'_0, \tag{2.2}$$

where

$$J_j = \begin{pmatrix} p_{1,j} & \mathbb{0} & \mathbb{0} & \cdots & \mathbb{0} \\ p_{1,j} \odot p_{2,j} & p_{2,j} & \mathbb{0} & \cdots & \mathbb{0} \\ \vdots & & & \ddots & \vdots \\ p_{1,j} \odot \cdots \odot p_{m,j} & & & \cdots & p_{m,j} \end{pmatrix} \quad \text{and} \quad \boldsymbol{C}'_0 = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{pmatrix}.$$

This formalism is the same as the one by Vo and Lenté [8, 9] based on the research by Bouquard et al. [63]. Job j is associated with matrix J_j and we call the matrices and the formalism *job matrices* and *job representation*, respectively.

The makespan is the m-th (maximum) component of the vector $\boldsymbol{C}'_{\sigma(n)}$

Example 2.3.1. Consider an $F2|\text{prmu}|C_{\max}$ with two jobs. Let the job sequence be $(1, 2)$. Then the makespan $C_{2,2}$ is obtained as follows:

$$\begin{aligned}
C_{2,2} &= p_{1,1} + \max[p_{1,2}, p_{2,1}] + p_{2,2}\\
&= p_{1,1} \odot (p_{1,2} \oplus p_{2,1}) \odot p_{2,2}.
\end{aligned}$$

The Gantt chart is useful to understand a schedule. The chart displays the allocation of machines with a time scale shown along the horizontal axis (see Fig. 2.1).

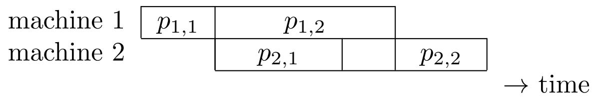


Figure 2.1: Gantt chart of a schedule of an $F2|\text{prmu}|C_{\max}$ with two jobs

Theorem 2.3.2. [64] *In an $Fm|prmu|C_{\max}$ problem, if $m(m-1)/2$ inequalities:*

$$\min_{i=l,..,k}[p_{l,j}\odot\cdots\odot p_{i-1,j}\odot p_{i+1,j'}\odot\cdots\odot p_{k,j'}]$$
$$\leq \min_{i=l,..,k}[p_{l,j'}\odot\cdots\odot p_{i-1,j'}\odot p_{i+1,j}\odot\cdots\odot p_{k,j}]$$
$$(l=1,...,m-1;\ k=l+1,...,m) \tag{2.3}$$

hold for two adjacent jobs j and j', then job j precedes job j' in an optimal sequence. If equality in all the inequalities holds, either ordering is possible.

Lemma 2.3.3. *The inequalities (2.3) are equivalent with the inequality of job matrices:*

$$J_{j'}J_j \leq J_jJ_{j'}.$$

Proof. Since

$$(J_j)_{kl} = \begin{cases} p_{l,j}\odot\cdots\odot p_{k,j} & (k\geq l) \\ \mathbb{0} & (k<l) \end{cases},$$

$$(J_{j'}J_j)_{kl} = \bigoplus_{i=1}^{m}(J_{j'})_{ki}\odot(J_j)_{il} = \begin{cases} \bigoplus_{i=l}^{k} p_{l,j}\odot\cdots\odot p_{i,j}\odot p_{i,j'}\odot\cdots\odot p_{k,j'} & (k\geq l) \\ \mathbb{0} & (k<l) \end{cases}.$$

Then $J_{j'}J_j \leq J_jJ_{j'}$ means that

$$\bigoplus_{i=l}^{k} p_{l,j}\odot\cdots\odot p_{i,j}\odot p_{i,j'}\odot\cdots\odot p_{k,j'} \leq \bigoplus_{i=l}^{k} p_{l,j'}\odot\cdots\odot p_{i,j'}\odot p_{i,j}\odot\cdots\odot p_{k,j} \quad (k\geq l).$$

Multiplying both sides by $(p_{l,j}\odot\cdots\odot p_{k,j}\odot p_{l,j'}\odot\cdots\odot p_{k,j'})^{-1}$, we have

$$\bigoplus_{i=l}^{k}(p_{i+1,j}\odot\cdots\odot p_{k,j}\odot p_{l,j'}\odot\cdots\odot p_{i-1,j'})^{-1}$$
$$\leq \bigoplus_{i=l}^{k}(p_{i+1,j'}\odot\cdots\odot p_{k,j'}\odot p_{l,j}\odot\cdots\odot p_{i-1,j})^{-1}.$$

Dividing both sides by the product of both sides this equals

$$\left(\bigoplus_{i=l}^{k}(p_{l,j}\odot\cdots\odot p_{i-1,j}\odot p_{i+1,j'}\odot\cdots\odot p_{k,j'})^{-1}\right)^{-1}$$
$$\leq \left(\bigoplus_{i=l}^{k}(p_{l,j'}\odot\cdots\odot p_{i-1,j'}\odot p_{i+1,j}\odot\cdots\odot p_{k,j})^{-1}\right)^{-1},$$

which is equivalent to

$$\min_{i=l,\ldots,k}[p_{l,j}\odot\cdots\odot p_{i-1,j}\odot p_{i+1,j'}\odot\cdots\odot p_{k,j'}]$$
$$\le \min_{i=l,\ldots,k}[p_{l,j'}\odot\cdots\odot p_{i-1,j'}\odot p_{i+1,j}\odot\cdots\odot p_{k,j}] \quad (k\ge l). \quad (2.4)$$

This is the same as the inequalities (2.3). Note that in the case $k=l$ equality always holds and

$$\min[a,b,c,...]=\left(a^{-1}\oplus b^{-1}\oplus c^{-1}\oplus\cdots\right)^{-1}.$$

□

We straightforwardly obtain the next corollary using Lemma 2.3.3.

Corollary 2.3.4. *In an $Fm|prmu|C_{\max}$ problem, if*

$$J_{j'}J_j \le J_jJ_{j'} \quad (2.5)$$

holds for two adjacent jobs j and j', then job j precedes job j' in an optimal sequence. If equality in the inequality holds, either ordering is possible.

In the case $m=2$,

$$\begin{aligned}
&J_{j'}J_j \le J_{j'}J_j\\
\Leftrightarrow\quad & (J_{j'}J_j)_{21} \le (J_{j'}J_j)_{21}\\
\Leftrightarrow\quad & p_{1,j'}\odot p_{2,j'}\odot p_{1,j}\oplus p_{2,j'}\odot p_{1,j}\odot p_{2,j} \le p_{1,j}\odot p_{2,j}\odot p_{1,j'}\oplus p_{2,j}\odot p_{1,j'}\odot p_{2,j'}\\
\Leftrightarrow\quad & \left(p_{1,j}^{-1}\oplus p_{2,j'}^{-1}\right)^{-1} \le \left(p_{1,j'}^{-1}\oplus p_{2,j}^{-1}\right)^{-1}\\
\Leftrightarrow\quad & \min[p_{1,j},p_{2,j'}] \le \min[p_{1,j'},p_{2,j}].
\end{aligned}$$

This is the well-known Johnson's rule [4], which satisfies the transitive property. The derivation from job matrices in max-plus algebra was obtained by Bouquard et al. [63]. Note that an $F2||C_{\max}$ is reduced to an $F2|\text{prmu}|C_{\max}$ (see e.g. [1]).

Example 2.3.5. Consider the $F2||C_{\max}$ with five jobs as described in the following table.

Job j	1	2	3	4	5
$p_{1,j}$	3	5	1	6	7
$p_{2,j}$	6	2	2	6	5

The job sequence $(3,1,4,5,2)$ is an optimal solution. Then the makespan is 24.

In general it is not easy to verify the inequality (2.5). We give a sufficient condition which can be verified relatively easily.

Theorem 2.3.6. *Let $m \geq 3$. If inequalities*

$$\min[p_{l,j}, p_{l+1,j'}] \leq \min[p_{l,j'}, p_{l+1,j}] \quad (l = 1, ..., m-1) \tag{2.6}$$

hold, and from the two inequalities

$$\min[p_{l,j}, p_{l+1,j'}] \leq \min[p_{l,j'}, p_{l+1,j}] \text{ and } \min[p_{l+1,j}, p_{l+2,j'}] \leq \min[p_{l+1,j'}, p_{l+2,j}]$$

for $l = 1, ..., m-2$, one is strict, i.e., for $l = 1, ..., m-2$, we have either

$$\min[p_{l,j}, p_{l+1,j'}] \leq \min[p_{l,j'}, p_{l+1,j}] \text{ and } \min[p_{l+1,j}, p_{l+2,j'}] < \min[p_{l+1,j'}, p_{l+2,j}]$$

or

$$\min[p_{l,j}, p_{l+1,j'}] < \min[p_{l,j'}, p_{l+1,j}] \text{ and } \min[p_{l+1,j}, p_{l+2,j'}] \leq \min[p_{l+1,j'}, p_{l+2,j}],$$

then

$$J_{j'}J_j \leq J_jJ_{j'}\,, \text{ while } J_{j'}J_j \neq J_jJ_{j'}.$$

We consider the inequality (2.4). Let the left-hand side and the right-hand side of (2.4) be $\tilde{d}_{k,l}$ and $d_{k,l}$, respectively. $\tilde{d}_{k,l} \leq d_{k,l}$ is equivalent to $(J_{j'}J_j)_{k,l} \leq (J_jJ_{j'})_{k,l}$. We give some relations existing between $\tilde{d}_{k,l}$s and $d_{k,l}$s at first.

Lemma 2.3.7. *Let*

$$\tilde{d}_{k,l} = \min_{i=l,\ldots,k} [p_{l,j} \odot \cdots \odot p_{i-1,j} \odot p_{i+1,j'} \odot \cdots \odot p_{k,j'}]$$

and

$$d_{k,l} = \min_{i=l,\ldots,k} [p_{l,j'} \odot \cdots \odot p_{i-1,j'} \odot p_{i+1,j} \odot \cdots \odot p_{k,j}]\,.$$

Then the followings hold:

$$\tilde{d}_{l+h,l} \odot \tilde{d}_{l+k',l+h} = \min\left[\tilde{d}_{l+k',l},\ \tilde{d}_{l+h-1,l} \odot p_{l+h,j'} \odot p_{l+h,j} \odot \tilde{d}_{l+k',l+h+1}\right]$$

and

$$d_{l+h,l} \odot d_{l+k',l+h} = \min\left[d_{l+k',l},\ d_{l+h-1,l} \odot p_{l+h,j} \odot p_{l+h,j'} \odot d_{l+k',l+h+1}\right]$$

for $1 \leq l \leq m-2,\ 2 \leq k' \leq m-l$ and $1 \leq h \leq k'-1$.

Proof. $\tilde{d}_{l+h,l}$ and $\tilde{d}_{l+k',l+h}$ imply

$$\min\Bigg[\min_{i=0,\ldots,h-1} [p_{l,j} \odot \cdots \odot p_{l+i-1,j} \odot p_{l+i+1,j'} \odot \cdots \odot p_{l+h-1,j'}] \odot p_{l+h,j'},$$
$$p_{l,j} \odot \cdots \odot p_{l+h-1,j}\Bigg]$$

and

$$\min\Bigg[p_{l+h+1,j'} \odot \cdots \odot p_{l+k',j'},$$
$$p_{l+h,j} \odot \min_{i=h+1,\ldots,k'} [p_{l+h+1,j} \odot \cdots \odot p_{l+i-1,j} \odot p_{l+i+1,j'} \odot \cdots \odot p_{l+k',j'}]\Bigg],$$

respectively. Hence we have

$$
\begin{aligned}
&\tilde{d}_{l+h,l} \odot \tilde{d}_{l+k',l+h} \\
&= \min\Bigg[\min_{i=0,\dots,h-1} [p_{l,j} \odot \cdots \odot p_{l+i-1,j} \odot p_{l+i+1,j'} \odot \cdots \odot p_{l+h-1,j'}] \odot p_{l+h,j'}, \\
&\qquad\qquad p_{l,j} \odot \cdots \odot p_{l+h-1,j} \Bigg] \\
&\quad \odot \min\Bigg[p_{l+h+1,j'} \odot \cdots \odot p_{l+k',j'}, \\
&\qquad\qquad p_{l+h,j} \odot \min_{i=h+1,\dots,k'} [p_{l+h+1,j} \odot \cdots \odot p_{l+i-1,j} \odot p_{l+i+1,j'} \odot \cdots \odot p_{l+k',j'}] \Bigg] \\
&= \min\left[\tilde{d}_{l+k',l},\ \tilde{d}_{l+h-1,l} \odot p_{l+h,j'} \odot p_{l+h,j} \odot \tilde{d}_{l+k',l+h+1} \right].
\end{aligned}
$$

Note that

$$\tilde{d}_{l+h-1,l} = \min_{i=0,\dots,h-1} [p_{l,j} \odot \cdots \odot p_{l+i-1,j} \odot p_{l+i+1,j'} \odot \cdots \odot p_{l+h-1,j'}]$$

and

$$\tilde{d}_{l+k',l+h+1} = \min_{i=h+1,\dots,k'} [p_{l+h+1,j} \odot \cdots \odot p_{l+i-1,j} \odot p_{l+i+1,j'} \odot \cdots \odot p_{l+k',j'}].$$

We can make a similar argument for $d_{l+h,l} \odot d_{l+k',l+h}$ □

Lemma 2.3.8.

1. *If*
$$\tilde{d}_{l+h-1,l} \odot p_{l+h,j'} < p_{l,j} \odot \cdots \odot p_{l+h-1,j} \quad (h = 1, \dots, k'-1),$$
hold for some $l\,(1 \le l \le m-1)$ *and* $k'\,(2 \le k' \le m-l+1)$*, then*
$$\tilde{d}_{l+k'-1,l} = p_{l+1,j'} \odot p_{l+2,j'} \odot \cdots \odot p_{l+k'-1,j'}.$$

2. *If*
$$p_{l+h,j} \odot \tilde{d}_{l+k',l+h+1} < p_{l+h+1,j'} \odot \cdots \odot p_{l+k',j'} \quad (h = 1, \dots, k'-1),$$
hold for some $l\,(1 \le l \le m-1)$ *and* $k'\,(2 \le k' \le m-l+1)$*, then*
$$\tilde{d}_{l+k',l+1} = p_{l+1,j} \odot p_{l+2,j} \odot \cdots \odot p_{l+k'-1,j}.$$

Proof. (1) The assumptions mean that

$$
\begin{aligned}
\tilde{d}_{l+k'-2,l} \odot p_{l+k'-1,j'} &< p_{l,j} \odot p_{l+1,j} \odot \cdots \odot p_{l+k'-3,j} \odot p_{l+k'-2,j}, \\
\tilde{d}_{l+k'-3,l} \odot p_{l+k'-2,j'} &< p_{l,j} \odot p_{l+1,j} \odot \cdots \odot p_{l+k'-3,j}, \\
&\vdots \\
\tilde{d}_{l+1,l} \odot p_{l+2,j'} &< p_{l,j} \odot p_{l+1,j}, \\
\tilde{d}_{l,l} \odot p_{l+1,j'} &< p_{l,j}.
\end{aligned}
$$

Using the inequalities, we obtain

$$\begin{aligned}
\tilde{d}_{l+k'-1,l} &= \min\left[\tilde{d}_{l+k'-2,l}\odot p_{l+k'-1,j'}, p_{l,j}\odot\cdots\odot p_{l+k'-2,j}\right]\\
&= \tilde{d}_{l+k'-2,l}\odot p_{l+k'-1,j'}\\
&= \min\left[\tilde{d}_{l+k'-3,l}\odot p_{l+k'-2,j'}, p_{l,j}\odot\cdots\odot p_{l+k'-3,j}\right]\odot p_{l+k'-1,j'}\\
&= \tilde{d}_{l+k'-3,l}\odot p_{l+k'-2,j'}\odot p_{l+k'-1,j'} = \cdots\\
&= \tilde{d}_{l+2,l}\odot p_{l+3,j'}\odot\cdots\odot p_{l+k'-1,j'}\\
&= \min\left[\tilde{d}_{l+1,l}\odot p_{l+2,j'}, p_{l,j}\odot p_{l+1,j}\right]\odot p_{l+3,j'}\odot\cdots\odot p_{l+k'-1,j'}\\
&= \tilde{d}_{l+1,l}\odot p_{l+2,j'}\odot p_{l+3,j'}\odot\cdots\odot p_{l+k'-1,j'}\\
&= \min\left[\tilde{d}_{l,l}\odot p_{l+1,j'}, p_{l,j}\right]\odot p_{l+2,j'}\odot\cdots\odot p_{l+k'-1,j'}\\
&= \tilde{d}_{l,l}\odot p_{l+1,j'}\odot p_{l+2,j'}\odot\cdots\odot p_{l+k'-1,j'} = p_{l+1,j'}\odot p_{l+2,j'}\odot\cdots\odot p_{l+k'-1,j'}.
\end{aligned}$$

(2) The assumptions mean that

$$\begin{aligned}
p_{l+1,j}\odot\tilde{d}_{l+k',l+2} &< p_{l+2,j'}\odot p_{l+3,j'}\odot\cdots\odot p_{l+k'-1,j'}\odot p_{l+k',j'},\\
p_{l+2,j}\odot\tilde{d}_{l+k',l+3} &< p_{l+3,j'}\odot\cdots\odot p_{l+k'-1,j'}\odot p_{l+k',j'},\\
&\vdots\\
p_{l+k'-2,j}\odot\tilde{d}_{l+k',l+k'-1} &< p_{l+k'-1,j'}\odot p_{l+k',j'},\\
p_{l+k'-1,j}\odot\tilde{d}_{l+k',l+k'} &< p_{l+k',j'}.
\end{aligned}$$

Using the inequalities, we obtain

$$\begin{aligned}
\tilde{d}_{l+k',l+1} &= \min\left[p_{l+1,j}\odot\tilde{d}_{l+k',l+2}, p_{l+2,j'}\odot\cdots\odot p_{l+k',j'}\right]\\
&= p_{l+1,j}\odot\tilde{d}_{l+k',l+2} = p_{l+1,j}\odot\min\left[p_{l+2,j}\odot\tilde{d}_{l+k',l+3}, p_{l+3,j'}\odot\cdots\odot p_{l+k',j'}\right]\\
&= p_{l+1,j}\odot p_{l+2,j}\odot\tilde{d}_{l+k',l+3} = \cdots\\
&= p_{l+1,j}\odot p_{l+2,j}\odot\cdots\odot p_{l+k'-1,j}.
\end{aligned}$$

□

Proof of Theorem 2.3.6. We show that for $l = 1, ..., m$ and $k' = 0, 1, ..., m-l$,

$$\tilde{d}_{l+k',l} \leq d_{l+k',l} \quad (k' = 0, 1), \tag{2.7a}$$

$$\tilde{d}_{l+k',l} < d_{l+k',l} \quad (k' = 2, 3, ...). \tag{2.7b}$$

In the case $k' = 0$, since $\tilde{d}_{l,l} = d_{l,l} = \mathbb{1}(= 0)$, the statement (2.7a) is true. In the case $k' = 1$, since $\tilde{d}_{l+1,l}$ and $d_{l+1,l}$ are just the right-hand side and the left-hand side of the assumption (2.6) of this theorem, the statement (2.7a) is true.

We show the statement (2.7b) by induction on k'. When $k' = 2$, suppose that $\tilde{d}_{l+2,l} \geq d_{l+2,l}$. From the two assumptions of the theorem, we obtain

$$\tilde{d}_{l+1,l}\odot\tilde{d}_{l+2,l+1} < d_{l+1,l}\odot d_{l+2,l+1} \ (l = 1, \ldots, m-2). \tag{2.8}$$

Note that from the latter assumptions of this theorem equality does not hold.

By Lemma 2.3.7 the inequality (2.8) is equivalent to

$$\min\left[\tilde{d}_{l+2,l},\ \tilde{d}_{l,l}\odot p_{l+1,j'}\odot p_{l+1,j}\odot\tilde{d}_{l+2,l+2}\right] < \min\left[d_{l+2,l},\ d_{l,l}\odot p_{l+1,j}\odot p_{l+1,j'}\odot d_{l+2,l+2}\right].$$

Since $\tilde{d}_{l+2,l} \geq d_{l+2,l}$, it is necessary that the inequality

$$\tilde{d}_{l,l}\odot p_{l+1,j'}\odot p_{l+1,j}\odot\tilde{d}_{l+2,l+2} < \tilde{d}_{l+2,l}$$

holds. The left-hand side equals $p_{l+1,j'}\odot p_{l+1,j}$, so the substitution

$$\tilde{d}_{l+2,l} = \min\left[p_{l+1,j'}\odot p_{l+2,j'},\ p_{l,j}\odot p_{l+2,j'},\ p_{l,j}\odot p_{l+1,j}\right]$$

reduces this inequality to

$$\begin{aligned} p_{l+1,j} &< p_{l+2,j'}, \\ p_{l+1,j'}\odot p_{l+1,j} &< p_{l,j}\odot p_{l+2,j'}, \\ p_{l+1,j'} &< p_{l,j}. \end{aligned}$$

Using the third and first inequalities and the inequality (2.7a), we obtain

$$\tilde{d}_{l+1,l} = \min\left[p_{l+1,j'},\ p_{l,j}\right] = p_{l+1,j'} \leq d_{l+1,l}, \tag{2.9a}$$

$$\tilde{d}_{l+2,l+1} = \min\left[p_{l+2,j'},\ p_{l+1,j}\right] = p_{l+1,j} \leq d_{l+2,l+1}, \tag{2.9b}$$

where either of the two inequalities is strict. On the other hand, the definitions of $d_{l+1,l}, d_{l+2,l+1}$ imply that

$$d_{l+1,l} = \min\left[p_{l+1,j},\ p_{l,j'}\right] \leq p_{l+1,j} \tag{2.10a}$$

$$d_{l+2,l+1} = \min\left[p_{l+2,j},\ p_{l+1,j'}\right] \leq p_{l+1,j'}. \tag{2.10b}$$

The inequalities (2.9) and (2.10) cannot hold simultaneously. Therefore, $\tilde{d}_{l+2,l} < d_{l+2,l}$.

When $k' > 2$, assume that $\tilde{d}_{l+i,l} < d_{l+i,l}$ for $i = 2, ..., k'-1$, which are the induction hypotheses. We show that $\tilde{d}_{l+k',l} < d_{l+k',l}$. Suppose that

$$\tilde{d}_{l+k',l} \geq d_{l+k',l}.$$

From the induction hypotheses, we obtain

$$\tilde{d}_{l+h,l}\odot\tilde{d}_{l+k',l+h} < d_{l+h,l}\odot d_{l+k',l+h}\ (h = 1, \ldots, k'-1; l = 1, \ldots, m-k'). \tag{2.11}$$

Note that either of $\tilde{d}_{l+h,l} < d_{l+h,l}$ and $\tilde{d}_{l+k',l+h} < d_{l+k',l+h}$ holds since either of h and $k'-h$ is greater than one. By Lemma 2.3.7 the inequality (2.11) is equivalent to

$$\min\left[\tilde{d}_{l+k',l},\ \tilde{d}_{l+h-1,l}\odot p_{l+h,j'}\odot p_{l+h,j}\odot\tilde{d}_{l+k',l+h+1}\right] < \min\left[d_{l+k',l},\ d_{l+h-1,l}\odot p_{l+h,j}\odot p_{l+h,j'}\odot d_{l+k',l+h+1}\right].$$

Since $\tilde{d}_{l+k',l} \geq d_{l+k',l}$, it is necessary that the inequality

$$\tilde{d}_{l+h-1,l} \odot p_{l+h,j'} \odot p_{l+h,j} \odot \tilde{d}_{l+k',l+h+1} < \tilde{d}_{l+k',l}$$

holds. The substitution

$$\begin{aligned} \tilde{d}_{l+k',l} = \min \Big[& \tilde{d}_{l+h-1,l} \odot p_{l+h,j'} \odot p_{l+h+1,j'} \odot \cdots \odot p_{l+k',j'}, \\ & p_{l,j} \odot \cdots \odot p_{l+h-1,j} \odot p_{l+h+1,j'} \odot \cdots \odot p_{l+k',j'}, \\ & p_{l,j} \odot \cdots \odot p_{l+h-1,j} \odot p_{l+h,j} \odot \tilde{d}_{l+k',l+h+1} \Big], \end{aligned}$$

reduces this inequality to

$$\begin{aligned} p_{l+h,j} \odot \tilde{d}_{l+k',l+h+1} &< p_{l+h+1,j'} \odot \cdots \odot p_{l+k',j'}, \\ \tilde{d}_{l+h-1,l} \odot p_{l+h,j'} \odot p_{l+h,j} \odot \tilde{d}_{l+k',l+h+1} &< p_{l,j} \odot \cdots \odot p_{l+h-1,j} \odot p_{l+h+1,j'} \odot \cdots \odot p_{l+k',j'}, \\ \tilde{d}_{l+h-1,l} \odot p_{l+h,j'} &< p_{l,j} \odot \cdots \odot p_{l+h-1,j}. \end{aligned}$$

Each assumption of Lemma 2.3.8 is satisfied by the third and first inequalities, so we have

$$\begin{aligned} \tilde{d}_{l+k'-1,l} &= p_{l+1,j'} \odot p_{l+2,j'} \odot \cdots \odot p_{l+k'-1,j'}, \\ \tilde{d}_{l+k',l+1} &= p_{l+1,j} \odot p_{l+2,j} \odot \cdots \odot p_{l+k'-1,j}. \end{aligned}$$

From these and the induction hypotheses, it follow that

$$p_{l+1,j'} \odot \cdots \odot p_{l+k'-1,j'} = \tilde{d}_{l+k'-1,l} < d_{l+k'-1,l}, \tag{2.12a}$$

$$p_{l+1,j} \odot \cdots \odot p_{l+k'-1,j} = \tilde{d}_{l+k',l+1} < d_{l+k',l+1}. \tag{2.12b}$$

On the other hand, the definitions of $d_{l+k'-1,l}$ and $d_{l+k',l+1}$ imply that

$$d_{l+k'-1,l} \leq p_{l+1,j} \odot \cdots \odot p_{l+k'-1,j}, \tag{2.13a}$$

$$d_{l+k',l+1} \leq p_{l+1,j'} \odot \cdots \odot p_{l+k'-1,j'}. \tag{2.13b}$$

From the inequalities (2.12) and (2.13), we obtain

$$\begin{aligned} p_{l+1,j'} \odot \cdots \odot p_{l+k'-1,j'} < d_{l+k'-1,l} &\leq p_{l+1,j} \odot \cdots \odot p_{l+k'-1,j} \\ &< d_{l+k',l+1} \leq p_{l+1,j'} \odot \cdots \odot p_{l+k'-1,j'}, \end{aligned}$$

which is a contradiction. The proof is completed. □

The assumptions (2.6) of the theorem are correspondent to the magnitude relationship between the subdiagonal entries of $J_{j'}J_j$ and $J_jJ_{j'}$.The reason is as follows: for $i = 1, ..., m-1$

$$\begin{aligned} & (J_{j'}J_j)_{i+1,i} \leq (J_jJ_{j'})_{i+1,i} \\ \Leftrightarrow\ & p_{i,j} \odot p_{i,j'} \odot p_{i+1,j'} \oplus p_{i,j} \odot p_{i+1,j} \odot p_{i+1,j'} \leq p_{i,j'} \odot p_{i,j} \odot p_{i+1,j} \oplus p_{i,j'} \odot p_{i+1,j'} \odot p_{i+1,j} \\ \Leftrightarrow\ & (p_{i+1,j})^{-1} \oplus (p_{i,j'})^{-1} \leq (p_{i+1,j'})^{-1} \oplus (p_{i,j})^{-1} \\ \Leftrightarrow\ & \left((p_{i,j})^{-1} \oplus (p_{i+1,j'})^{-1}\right)^{-1} \leq \left((p_{i,j'})^{-1} \oplus (p_{i+1,j})^{-1}\right)^{-1}. \end{aligned}$$

The subdiagonal entries play an important role in the magnitude relationship between the job matrices.

2.3.2 No-wait flow shop problems

We reformulate the flow shop problems under no-wait condition using the job matrices, which is presented in [8], and give a new simple proof of the well-known theorem that the flow shop makespan problem with the no-wait condition can be formulated as a TSP [65].

No-wait processes are seen in the production of steel, in the plastic molding, in the chemical industry and so on, due to the temperature or other characteristics of the material [10].

Proposition 2.3.9. [8] *The makespan of a job sequence* $\sigma = (\sigma(1), ..., \sigma(n))$ *in* $Fm|no\text{-}wait, s_i|C_{\max}$ *is the* m*-th (maximum) component of the vector:*

$$\boldsymbol{C}'_{\sigma(n)} = J_{\sigma(n)} \odot \cdots \odot J_{\sigma(1)} \odot \boldsymbol{C}'_0, \tag{2.14}$$

where

$$J_j = \begin{pmatrix} p_{1,j} & \mathbb{1} & (p_{2,j})^{-1} & \cdots & (p_{2,j} \odot p_{3,j} \odot \cdots \odot p_{m-1,j})^{-1} \\ p_{1,j} \odot p_{2,j} & p_{2,j} & \mathbb{1} & \cdots & (p_{3,j} \odot \cdots \odot p_{m-1,j})^{-1} \\ p_{1,j} \odot p_{2,j} \odot p_{3,j} & p_{2,j} \odot p_{3,j} & p_{3,j} & \cdots & \\ \vdots & & & \ddots & \vdots \\ p_{1,j} \odot \cdots \odot p_{m,j} & & & \cdots & p_{m,j} \end{pmatrix}$$

and

$$\boldsymbol{C}'_0 = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{pmatrix}.$$

Proof. Under no-wait condition, the recursion relations are the following:

$$\begin{aligned} C_{i,\sigma(k)} &= C_{i-1,\sigma(k)} \odot p_{i,\sigma(k)} \quad (i = 2, ..., m; k = 1, 2, ..., n) \\ C_{1,\sigma(k)} &= \left(\bigoplus_{i=1}^{m} C_{i,\sigma(k-1)} \odot \left(p_{1,\sigma(k)} \odot \cdots \odot p_{i-1,\sigma(k)}\right)^{-1} \right) \odot p_{1,\sigma(k)} \quad (k = 2, ..., n) \\ C_{1,\sigma(1)} &= \left(\bigoplus_{i=1}^{m} s_i \odot \left(p_{1,\sigma(1)} \odot \cdots \odot p_{i-1,\sigma(1)}\right)^{-1} \right) \odot p_{1,\sigma(1)}. \end{aligned}$$

In max-plus algebra, this reads the given formula (2.14). □

The next lemma is used in the proof of the following theorem.

Lemma 2.3.10. *Let*

$$\boldsymbol{v} = \begin{pmatrix} p_1 \\ p_1 \odot p_2 \\ \vdots \\ p_1 \odot \cdots \odot p_m \end{pmatrix} \quad \textit{and} \quad \boldsymbol{u} = \begin{pmatrix} \mathbb{1} & p_1^{-1} & \ldots & (p_1 \odot \cdots \odot p_{m-1})^{-1} \end{pmatrix},$$

then

$$\boldsymbol{v}\odot\boldsymbol{u} = \begin{pmatrix} p_1 & \mathbb{1} & p_2^{-1} & \cdots & (p_2\odot p_3\odot\cdots\odot p_{m-1})^{-1} \\ p_1\odot p_2 & p_2 & \mathbb{1} & \cdots & (p_3\odot\cdots\odot p_{m-1})^{-1} \\ p_1\odot p_2\odot p_3 & p_2\odot p_3 & p_3 & \cdots & \\ \vdots & & & \ddots & \vdots \\ p_1\odot\cdots\odot p_m & & & \cdots & p_m \end{pmatrix}.$$

Proof. Straightforward. □

We provide a new simple proof of the next known theorem.

Theorem 2.3.11. [65] *An $Fm|no\text{-}wait|C_{\max}$ problem can be formulated as a Traveling Salesman Problem (TSP), of which the intercity costs are*

$$\begin{aligned} \delta_{j,j'} &= \bigoplus_{i=1}^{m} p_{1,j}\odot\cdots\odot p_{i-1,j}\odot p_{i,j}\odot(p_{1,j'}\odot\cdots\odot p_{i-1,j'})^{-1} && (j,j'=1,...,n), \\ \delta_{j,0} &= p_{1,j}\odot\cdots\odot p_{m,j} && (j=1,...,n), \\ \delta_{0,j} &= \mathbb{1}(=0) && (j=1,...,n). \end{aligned}$$

Proof. We consider the case $s_i = \mathbb{1}$ in the formula (2.14) of Proposition 2.3.9. Thus we have

$$\boldsymbol{C}'_{\sigma(n)} = J_{\sigma(n)}\odot J_{\sigma(n-1)}\odot\cdots\odot J_{\sigma(2)}\odot J_{\sigma(1)}\odot\boldsymbol{C}'_0.$$

By Lemma 2.3.10 this equals

$$\begin{aligned} &(\boldsymbol{v}_{\sigma(n)}\odot\boldsymbol{u}_{\sigma(n)})\odot(\boldsymbol{v}_{\sigma(n-1)}\odot\boldsymbol{u}_{\sigma(n-1)})\odot\cdots\odot(\boldsymbol{v}_{\sigma(2)}\odot\boldsymbol{u}_{\sigma(2)})\odot(\boldsymbol{v}_{\sigma(1)}\odot\boldsymbol{u}_{\sigma(1)})\odot\boldsymbol{C}'_0 \\ =&\boldsymbol{v}_{\sigma(n)}\odot(\boldsymbol{u}_{\sigma(n)}\odot\boldsymbol{v}_{\sigma(n-1)})\odot\cdots\odot(\boldsymbol{u}_{\sigma(3)}\odot\boldsymbol{v}_{\sigma(2)})\odot(\boldsymbol{u}_{\sigma(2)}\odot\boldsymbol{v}_{\sigma(1)}), \end{aligned}$$

where

$$\boldsymbol{v_j} = \begin{pmatrix} p_{1,j} \\ p_{1,j}\odot p_{2,j} \\ \vdots \\ p_{1,j}\odot\cdots\odot p_{m,j} \end{pmatrix} \quad \text{and} \quad \boldsymbol{u_j} = \begin{pmatrix} \mathbb{1} & p_{1,j}^{-1} & \ldots & (p_{1,j}\odot\cdots\odot p_{m-1,j})^{-1} \end{pmatrix}.$$

We can rewrite it as

$$\boldsymbol{v}_{\sigma(n)}\odot\delta_{\sigma(n-1),\sigma(n)}\odot\cdots\odot\delta_{\sigma(2),\sigma(3)}\odot\delta_{\sigma(1),\sigma(2)},$$

since $\delta_{j,j'} = \boldsymbol{u}_{j'}\odot\boldsymbol{v}_j$. The makespan is

$$\delta_{\sigma(n),0}\odot\delta_{\sigma(n-1),\sigma(n)}\odot\cdots\odot\delta_{\sigma(2),\sigma(3)}\odot\delta_{\sigma(1),\sigma(2)}\odot\delta_{0,\sigma(1)}.$$

Therefore, this makespan problem can be recast as a TSP. □

The proof is even simpler than the known one. This shows how useful the formulation based on max-plus algebra is.

2.3.3 Blocking flow shop problems

We present the job matrix of the flow shop problems under blocking condition, which is used to show a duality relationship in Section 2.4. This formulation is new.

The blocking environment is a lack of storage capacity between machines.

Proposition 2.3.12. *The makespan of a job sequence* $\sigma = (\sigma(1), ..., \sigma(n))$ *in* $Fm|blocking, s_i|C_{\max}$ *is the m-th (maximum) component of the vector:*

$$\boldsymbol{C}'_{\sigma(n)} = J_{\sigma(n)} \odot \cdots \odot J_{\sigma(1)} \odot \boldsymbol{C}'_0, \tag{2.15}$$

where

$$J_j = \begin{pmatrix} p_{1,j} & \mathbb{1} & \mathbb{0} & \cdots & \mathbb{0} \\ p_{1,j} \odot p_{2,j} & p_{2,j} & \mathbb{1} & \cdots & \mathbb{0} \\ p_{1,j} \odot p_{2,j} \odot p_{3,j} & p_{2,j} \odot p_{3,j} & p_{3,j} & \cdots & \mathbb{0} \\ \vdots & & & \ddots & \vdots \\ p_{1,j} \odot \cdots \odot p_{m,j} & & & \cdots & p_{m,j} \end{pmatrix} \quad \text{and} \quad \boldsymbol{C}'_0 = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{pmatrix}.$$

Proof. Under blocking condition, the recursion relations are the following:

$$\begin{aligned} C_{m,\sigma(k)} &= \left(C_{m,\sigma(k-1)} \oplus C_{m-1,\sigma(k)}\right) \odot p_{m,\sigma(k)} \quad (k = 2, ..., n) \\ C_{i,\sigma(k)} &= \left(C_{i,\sigma(k-1)} \oplus C_{i-1,\sigma(k)}\right) \odot p_{i,\sigma(k)} \oplus C_{i+1,\sigma(k-1)} \\ & \qquad\qquad\qquad\qquad (i = 2, ..., m-1; \quad k = 2, ..., n) \\ C_{1,\sigma(k)} &= C_{1,\sigma(k-1)} \odot p_{1,\sigma(k)} \oplus C_{2,\sigma(k-1)} \quad (k = 2, ..., n) \\ C_{m,\sigma(1)} &= \left(s_m \oplus C_{m-1,\sigma(1)}\right) \odot p_{m,\sigma(1)} \\ C_{i,\sigma(1)} &= \left(s_i \oplus C_{i-1,\sigma(1)}\right) \odot p_{i,\sigma(1)} \oplus s_{i+1} \quad (i = 2, ..., m-1) \\ C_{1,\sigma(1)} &= s_1 \odot p_{1,\sigma(1)} \oplus s_2. \end{aligned}$$

Here $C_{i,j}$ means the time at which machine i releases completed job j. In max-plus algebra, this reads the given formula (2.15). □

2.4 Machine matrices

2.4.1 Flow shop problems

We present a new framework for flow shop problems. The framework is the dual of the job representation (2.2). We identify a duality relationship between permutation flow shops and present a new solvable condition which includes two known conditions in m-machine permutation flow shops.

The recursion relations (2.1) for $Fm|\text{prmu}, r_j|\gamma$ are the following:

$$\begin{aligned} C_{i,\sigma(k)} &= \left(C_{i-1,\sigma(k)} \oplus C_{i,\sigma(k-1)}\right) \odot p_{i,\sigma(k)} & (i = 2, ..., m; k = 2, ..., n) \\ C_{1,\sigma(k)} &= \left(r_{\sigma(k)} \oplus C_{1,\sigma(k-1)}\right) \odot p_{1,\sigma(k)} & (k = 2, ..., n) \\ C_{i,\sigma(1)} &= C_{i-1,\sigma(1)} \odot p_{i,\sigma(1)} & (i = 2, ..., m) \\ C_{1,\sigma(1)} &= r_{\sigma(1)} \odot p_{1,\sigma(1)} \end{aligned}$$

Let

$$\boldsymbol{C}_i^\sigma = \begin{pmatrix} C_{i,\sigma(1)} \\ C_{i,\sigma(2)} \\ \vdots \\ C_{i,\sigma(n)} \end{pmatrix}.$$

be the completion time vector of a job sequence $\sigma = (\sigma(1), ..., \sigma(n))$ at machine i. Then

$$\boldsymbol{C}_m^\sigma = M_m^\sigma \odot \cdots \odot M_1^\sigma \odot \boldsymbol{C}_0^\sigma, \tag{2.16}$$

where

$$M_i^\sigma = \begin{pmatrix} p_{i,\sigma(1)} & \mathbb{0} & \mathbb{0} & \cdots & \mathbb{0} \\ p_{i,\sigma(1)} \odot p_{i,\sigma(2)} & p_{i,\sigma(2)} & \mathbb{0} & \cdots & \mathbb{0} \\ \vdots & & & \ddots & \vdots \\ p_{i,\sigma(1)} \odot \cdots \odot p_{i,\sigma(n)} & & & \cdots & p_{i,\sigma(n)} \end{pmatrix}$$

and

$$\boldsymbol{C}_0^\sigma = \begin{pmatrix} r_{\sigma(1)} \\ r_{\sigma(2)} \\ \vdots \\ r_{\sigma(n)} \end{pmatrix}.$$

Machine i is associated with the matrix M_i^σ and we call the matrices and the formalism *machine matrices* and *machine representation.*

The makespan is the n-th (maximum) component of the vector $\boldsymbol{C}_m^\sigma$ and the total completion time is the product (sum in the conventional algebra) of all the components of the the vector $\boldsymbol{C}_m^\sigma$, i.e.,

$$C_{m,\sigma(1)} \odot C_{m,\sigma(2)} \odot \cdots \odot C_{m,\sigma(n)} = C_{m,1} \odot C_{m,2} \odot \cdots \odot C_{m,n}.$$

Let $C_{\max}(\sigma, r, \alpha; [M_1, ..., M_m], s, \beta)$ denote the makespan of a sequence σ of jobs with the release dates r computed under the assumption of permutation schedules in machine environment $M_1, ..., M_m$ with the starting times s and the constraints α on jobs and β on machines. r, α, s and/or β may not be denoted.

For a given job sequence σ, define n artificial machines $\bar{M}_{\sigma(1)}$, $\bar{M}_{\sigma(2)}$,..., $\bar{M}_{\sigma(n)}$ in series, nm artificial processing times $\bar{p}_{k,l} = p_{l,k}\,(k = 1, \ldots, n;\, l = 1, \ldots, m)$ and n artificial starting times $\bar{s}_{\sigma(1)} = r_{\sigma(1)}$, $\bar{s}_{\sigma(2)} = r_{\sigma(2)}$,..., $\bar{s}_{\sigma(n)} = r_{\sigma(n)}$. Define also a sequence $id = (1, 2, ..., m)$ of m artificial jobs and m artificial release dates $\bar{r}_1 = s_1$, $\bar{r}_2 = s_2$,..., $\bar{r}_m = s_m$.

Theorem 2.4.1 (A duality relationship between flow shops)**.** *For every job sequence σ in $Fm|prmu, s_i|C_{\max}$ and $Fm|prmu, r_j|C_{\max}$,*

$$C_{\max}(\sigma; [M_1, ..., M_m], s) = C_{\max}(id, \bar{r}; [\bar{M}_{\sigma(1)}, ..., \bar{M}_{\sigma(n)}]); \tag{2.17a}$$

$$C_{\max}(\sigma, r; [M_1, ..., M_m]) = C_{\max}(id; [\bar{M}_{\sigma(1)}, ..., \bar{M}_{\sigma(n)}], \bar{s}). \tag{2.17b}$$

Proof. Note that the formulas (2.2) and (2.16) are similar.

The job representation (2.2) of an $Fm|$prmu, $s_i|C_{\max}$ can be interpreted as the machine representation of the $Fm|$prmu, $r_j|C_{\max}$ of a job sequence $id = (1, 2, ..., m)$ with machine series $\bar{M}_{\sigma(1)}$, $\bar{M}_{\sigma(2)}$,..., $\bar{M}_{\sigma(n)}$ and release dates $\bar{r}_1 = s_1$, $\bar{r}_2 = s_2$,..., $\bar{r}_m = s_m$. Hence, the equality (2.17a) holds.

On the other hand, the machine representation (2.16) of an $Fm|\text{prmu}, r_j|C_{\max}$ can be interpreted as the job representation of the $Fm|\text{prmu}, s_i|C_{\max}$ of a job sequence $id = (1, 2, ..., m)$ with machine series $\bar{M}_{\sigma(1)}$, $\bar{M}_{\sigma(2)}$,..., $\bar{M}_{\sigma(n)}$ and starting times $\bar{s}_{\sigma(1)}$, ..., $\bar{s}_{\sigma(n)}$. Hence, the equality (2.17b) holds. □

This theorem implies that the machine representation is the dual of the job representation.

The existing approaches based on max-plus algebra can be applied to only permutation schedules, but this framework is not limited to permutation schedules and the first one which can deal with non-permutation flow shop problems based on max-plus algebra. It is possible that the framework give a clue to tackle non-permutation flow shop problems. In $Fm||C_{\max}$ $(m \geq 4)$, non-permutation schedules must be considered (see e.g. [1]).

For example, suppose that there are two jobs in an $F4||C_{\max}$. Consider the schedule which has the job sequence $(1, 2)$ on machines 1 and 2, and the one $(2, 1)$ on machines 3 and 4. Then

$$\begin{pmatrix} C_{4,2} \\ C_{4,1} \end{pmatrix} = \begin{pmatrix} p_{4,2} & \mathbb{0} \\ p_{4,2} \odot p_{4,1} & p_{4,1} \end{pmatrix} \odot \begin{pmatrix} \mathbb{0} & p_{3,2} \\ p_{3,1} & p_{3,1} \odot p_{3,2} \end{pmatrix}$$
$$\odot \begin{pmatrix} p_{2,1} & \mathbb{0} \\ p_{2,1} \odot p_{2,2} & p_{2,2} \end{pmatrix} \odot \begin{pmatrix} p_{1,1} & \mathbb{0} \\ p_{1,1} \odot p_{1,2} & p_{1,2} \end{pmatrix} \odot \begin{pmatrix} \mathbb{1} \\ \mathbb{1} \end{pmatrix}.$$

Moreover, in terms of fluid dynamics, we may say that the job representation is Lagrange type since the representation pays attention to the flow of jobs. On the other hand, the machine representation is Euler type since the representation observes jobs from fixed machines.

The next theorem is important and used to prove the main theorem.

Theorem 2.4.2. *For an integer $l \geq 2$, let*

$$A_i = \begin{pmatrix} a_{i,1} & \mathbb{0} & \mathbb{0} & \cdots & \mathbb{0} \\ a_{i,1} \odot a_{i,2} & a_{i,2} & \mathbb{0} & \cdots & \mathbb{0} \\ \vdots & & & \ddots & \vdots \\ a_{i,1} \odot \cdots \odot a_{i,l} & & & \cdots & a_{i,l} \end{pmatrix}$$

and $A_{i'}$ defined similarly, with the condition $\min_{1 \leq k \leq l}[a_{i,k}] \geq \max_{1 \leq k \leq l}[a_{i',k}]$, *then*

$$A_i \odot A_{i'} = A_i \odot D(A_{i'}) \quad \textit{and} \quad A_{i'} \odot A_i = D(A_{i'}) \odot A,$$

where $D(A_i)$ is the diagonal matrix composed of diagonal entries of A_i, i.e.,

$$D(A_i) = \begin{pmatrix} a_{i,1} & \mathbb{0} & \mathbb{0} & \cdots & \mathbb{0} \\ \mathbb{0} & a_{i,2} & \mathbb{0} & \cdots & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & a_{i,3} & \cdots & \mathbb{0} \\ \vdots & & & \ddots & \vdots \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \cdots & a_{i,l} \end{pmatrix}.$$

Proof. Since

$$(A_i)_{\alpha\beta} = \begin{cases} \mathbb{0} & (\alpha < \beta) \\ a_{i,\beta} \odot \cdots \odot a_{i,\alpha} & (\alpha \geq \beta) \end{cases},$$

$$(A_i \odot A_{i'})_{\alpha\beta}$$
$$= \bigoplus_{\gamma=1}^{l} (A_i)_{\alpha\gamma} \odot (A_{i'})_{\gamma\beta} = \begin{cases} \mathbb{0} & (\alpha < \beta) \\ \bigoplus_{\gamma=\beta}^{\alpha} a_{i,\gamma} \odot \cdots \odot a_{i,\alpha} \odot a_{i',\beta} \odot \cdots \odot a_{i',\gamma} & (\alpha \geq \beta) \end{cases}.$$

When $\min_{1 \leq k \leq l}[a_{i,k}] \geq \max_{1 \leq k \leq l}[a_{i',k}]$,

$$(A_i \odot A_{i'})_{\alpha\beta} = \begin{cases} \mathbb{0} & (\alpha < \beta) \\ a_{i,\beta} \odot \cdots \odot a_{i,\alpha} \odot a_{i',\beta} & (\alpha \geq \beta) \end{cases}.$$

Therefore,

$$(A_i \odot A_{i'})_{\alpha\beta} = (A_i)_{\alpha\beta} \odot a_{i',\beta} = (A_i \odot D(A_{i'}))_{\alpha\beta}.$$

The proof of $A_{i'} \odot A_i$ is similar. □

The next theorem is our new result. It is difficult to discover the theorem without using max-plus algebra. The theorem is a fruit of the algebraic modelization.

Theorem 2.4.3. *In an $Fm|prmu|C_{\max}$ problem, if*

$$\max_j[p_{i,j}] \leq \min_j[p_{i+1,j}] \qquad (1 \leq i \leq g-1), \tag{2.18}$$

$$\min_j[p_{i,j}] \geq \max_j[p_{i+1,j}] \qquad (g \leq i \leq f-1), \tag{2.19}$$

$$\max_j[p_{i,j}] \leq \min_j[p_{i+1,j}] \qquad (f+1 \leq i \leq h-1), \tag{2.20}$$

$$\min_j[p_{i,j}] \geq \max_j[p_{i+1,j}] \qquad (h \leq i \leq m-1) \tag{2.21}$$

hold for $f, g, h\,(g < h; g, h = 1, ..., m; f = g, ..., h-1)$, then an optimal sequence is obtained in polynomial time.

Proof. Since there is no release date r_j, we may suppose that $\boldsymbol{C}_0^\sigma = (\mathbb{1}\ \mathbb{0} \cdots \mathbb{0})^T$ in the formula (2.16). Using the assumptions (2.18), (2.19), (2.20), (2.21) and Theorem 2.4.2, we have the completion vector of machine m for a job sequence σ as follows:

$$\begin{aligned} \boldsymbol{C}_m^\sigma &= M_m^\sigma \odot \cdots \odot M_h^\sigma \cdots \odot M_g^\sigma \cdots \odot M_1^\sigma \odot \boldsymbol{C}_0^\sigma \\ &= D(M_m^\sigma) \odot \cdots \odot D(M_{h+1}^\sigma) \odot M_h^\sigma \odot D(M_{h-1}^\sigma) \odot \cdots \odot D(M_{g+1}^\sigma) \\ &\quad \odot M_g^\sigma \odot D(M_{g-1}^\sigma) \odot \cdots \odot D(M_1^\sigma) \odot \boldsymbol{C}_0^\sigma. \end{aligned}$$

Hence,

$$\begin{aligned} C_{\max}(\sigma; [M_1, \ldots, M_m]) &= p_{m,\sigma(n)} \odot \cdots \odot p_{h+1,\sigma(n)} \\ &\odot (M_h^\sigma \odot D(M_{h-1}^\sigma) \odot \cdots \odot D(M_{g+1}^\sigma) \odot M_g^\sigma)_{n,1} \odot p_{g-1,\sigma(1)} \odot \cdots \odot p_{1,\sigma(1)}. \end{aligned}$$

Here let $q_j = p_{g+1,j} \odot \cdots \odot p_{h-1,j}$. Then

$$\begin{aligned} &(M_h^\sigma \odot D(M_{h-1}^\sigma) \odot \cdots \odot D(M_{g+1}^\sigma) \odot M_g^\sigma)_{n,1} \\ &= \left(p_{h,\sigma(1)} \odot \cdots \odot p_{h,\sigma(n)} \ \ \cdots \ \ p_{h,\sigma(n-1)} \odot p_{h,\sigma(n)} \ \ p_{h,\sigma(n)}\right) \\ &\quad \odot \begin{pmatrix} q_{\sigma(1)} & & & \\ & q_{\sigma(2)} & & \\ & & \ddots & \\ & & & q_{\sigma(n)} \end{pmatrix} \odot \begin{pmatrix} p_{g,\sigma(1)} \\ p_{g,\sigma(1)} \odot p_{g,\sigma(2)} \\ \vdots \\ p_{g,\sigma(1)} \odot \cdots \odot p_{g,\sigma(n)} \end{pmatrix}. \end{aligned}$$

Since we can rewrite the diagonal matrix as

$$\left(q_{\sigma(1)}\odot\cdots\odot q_{\sigma(n)}\right)^{-1}\odot\begin{pmatrix} q_{\sigma(1)}\odot\cdots\odot q_{\sigma(n)} & & & \\ & q_{\sigma(2)}\odot\cdots\odot q_{\sigma(n)} & & \\ & & \ddots & \\ & & & q_{\sigma(n)} \end{pmatrix}$$
$$\odot\begin{pmatrix} q_{\sigma(1)} & & & \\ & q_{\sigma(1)}\odot q_{\sigma(2)} & & \\ & & \ddots & \\ & & & q_{\sigma(1)}\odot\cdots\odot q_{\sigma(n)} \end{pmatrix},$$

we obtain

$$\begin{aligned}
&(M_h^\sigma\odot D(M_{h-1}^\sigma)\odot\cdots\odot D(M_{g+1}^\sigma)\odot M_g^\sigma)_{n,1}\\
&=\left(q_{\sigma(1)}\odot\cdots\odot q_{\sigma(n)}\right)^{-1}\\
&\quad\odot\big(p_{h,\sigma(1)}\odot q_{\sigma(1)}\odot\cdots\odot p_{h,\sigma(n)}\odot q_{\sigma(n)}\ \ \cdots\\
&\qquad p_{h,\sigma(n-1)}\odot q_{\sigma(n-1)}\odot p_{h,\sigma(n)}\odot q_{\sigma(n)}\ \ p_{h,\sigma(n)}\odot q_{\sigma(n)}\big)\\
&\quad\odot\begin{pmatrix} p_{g,\sigma(1)}\odot q_{\sigma(1)}\\ p_{g,\sigma(1)}\odot q_{\sigma(1)}\odot p_{g,\sigma(2)}q_{\sigma(2)}\\ \vdots\\ p_{g,\sigma(1)}\odot q_{\sigma(1)}\odot\cdots\odot p_{g,\sigma(n)}\odot q_{\sigma(n)}\end{pmatrix}\\
&=(q_{\sigma(1)}\odot\cdots\odot q_{\sigma(n)})^{-1}\odot C_{\max}(\sigma;[\tilde{M}_g,\tilde{M}_h]),
\end{aligned}$$

where artificial machines $\tilde{M}_g$ and $\tilde{M}_h$ have the processing times $p_{g,j}\odot q_j$ and $q_j\odot p_{h,j}$, respectively. The factor $(q_{\sigma(1)}\odot\cdots\odot q_{\sigma(n)})^{-1}$ is independent of σ.

Denote an optimal sequence for the two artificial machines alone by $\tau=(j_1,\ldots,j_n)$. For each $k,l\,(k\neq l;k,l=1,...,n)$, let $\tau_{k,l}=(j_k,j_1,...,j_n,j_l)$ be a job sequence obtained from the sequence τ only by shifting k-th job j_k to the top and l-th job j_l to the end.

Considering the job representation of the two-machine flow shop problem, for every job sequence σ we obtain

$$C_{\max}(\sigma;[\tilde{M}_g,\tilde{M}_h])\geq C_{\max}(\tau_{\sigma(1),\sigma(n)};[\tilde{M}_g,\tilde{M}_h]).$$

Therefore,

$$C_{\max}(\sigma;[M_1,...,M_m])\geq C_{\max}(\tau_{\sigma(1),\sigma(n)};[M_1,...,M_m]).$$

This means that an optimal sequence is included in $n(n-1)$ job sequences $\tau_{k,l}$. □

The case ($g=1$ and $h=m$) and the case $g=h-1$ are in [24] and [25], respectively.

Example 2.4.4. Consider the $F6|\text{prmu}|C_{\max}$ with five jobs as described in the following table.

The assumptions of Theorem 2.4.3 are satisfied ($g=2,f=3,h=5$).

Step 1

Job j	1	2	3	4	5
$p_{1,j}$	4	5	9	8	1
$p_{2,j}$	14	12	12	15	10
$p_{3,j}$	4	8	2	6	7
$p_{4,j}$	10	2	8	7	8
$p_{5,j}$	14	16	17	14	18
$p_{6,j}$	10	5	8	6	13

Job j	1	2	3	4	5
$\tilde{p}_{1,j}$ (product of machines 2-4)	28	22	22	28	25
$\tilde{p}_{2,j}$ (product of machines 3-5)	28	26	27	27	33

Consider the artificial $F2||C_{\max}$ with five jobs as described in the following table.

The job sequence $\tau = (2, 3, 5, 1, 4)$ is one of optimal solutions.

Step 2

Compute the makespans for all the job sequences that are obtained from τ by shifting a job to the top and another job to the end (20 sequences such as $(1, 3, 5, 4, 2), (1, 2, 5, 4, 3)$ and so on.). We obtain the job sequence $(5, 3, 1, 4, 2)$ that minimize the makespan.

2.4.2 No-idle flow shop problems

We present the machine matrix of the flow shop problems under no-idle condition. By this formulation, we can easily show a duality relationship between "no-wait" and "no-idle" constraints and calculate the total completion time in permutation flow shops.

The no-idle condition is especially important when the using cost of a machine determined by the actual time consumption is very high.

Proposition 2.4.5. *The makespan of a job sequence $\sigma = (\sigma(1), ..., \sigma(n))$ in $Fm|prmu, no\text{-}idle, r_j|C_{\max}$ is the n-th (maximum) component of the vector:*

$$\boldsymbol{C}_m^\sigma = M_m^\sigma \odot \cdots \odot M_1^\sigma \odot \boldsymbol{C}_0^\sigma, \tag{2.22}$$

where

$$M_i^\sigma = \begin{pmatrix} p_{i,\sigma(1)} & \mathbb{1} & (p_{i,\sigma(2)})^{-1} & \cdots & (p_{i,\sigma(2)} \odot \cdots \odot p_{i,\sigma(n-1)})^{-1} \\ p_{i,\sigma(1)} \odot p_{i,\sigma(2)} & p_{i,\sigma(2)} & \mathbb{1} & \cdots & \\ \vdots & & & \ddots & \vdots \\ p_{i,\sigma(1)} \odot \cdots \odot p_{i,\sigma(n)} & & & \cdots & p_{i,\sigma(n)} \end{pmatrix}$$

and

$$\boldsymbol{C}_0^\sigma = \begin{pmatrix} r_{\sigma(1)} \\ r_{\sigma(2)} \\ \vdots \\ r_{\sigma(n)} \end{pmatrix}.$$

Proof. Under no-idle condition and the assumption of permutation schedules, the recursion relations are the following:

$$\begin{aligned} C_{i,\sigma(k)} &= C_{i,\sigma(k-1)} \odot p_{i,\sigma(k)} \quad (i=1,2,...,m;\ k=2,...,n) \\ C_{i,\sigma(1)} &= \left(\bigoplus_{k=1}^{n} C_{i-1,\sigma(k)} \odot \left(p_{1,\sigma(1)} \odot \cdots \odot p_{1,\sigma(k-1)} \right)^{-1} \right) \odot p_{i,\sigma(1)} \quad (i=2,...,m) \\ C_{1,\sigma(1)} &= \left(\bigoplus_{k=1}^{n} r_{\sigma(k)} \odot \left(p_{1,\sigma(1)} \odot \cdots \odot p_{1,\sigma(k-1)} \right)^{-1} \right) \odot p_{1,\sigma(1)}. \end{aligned}$$

In max-plus algebra, this reads the given formula (2.22). □

Theorem 2.4.6 (A duality relationship between "no-wait" and "no-idle" constraints)**.** *For every job sequence σ in $Fm|$no-wait, $s_i|C_{\max}$ and $Fm|$prmu, no-idle, $r_j|C_{\max}$,*

$$\begin{aligned} C_{\max}(\sigma, \textit{no-wait}; [M_1, ..., M_m], s_i) &= C_{\max}(id, \bar{r}; [\bar{M}_{\sigma(1)}, ..., \bar{M}_{\sigma(n)}], \textit{no-idle}); \\ C_{\max}(\sigma, r_j; [M_1, ..., M_m], \textit{no-idle}) &= C_{\max}(id, \textit{no-wait}; [\bar{M}_{\sigma(1)}, ..., \bar{M}_{\sigma(n)}], \bar{s}). \end{aligned}$$

Proof. Note that the formula (2.14) of Proposition 2.3.9 and the formula (2.22) of Proposition 2.4.5 are similar.

The proof is obtained similarly as in Theorem 2.4.1. □

The case without s_i and r_j has been studied in [14].

The next theorem is also a fruit of the machine representation.

Theorem 2.4.7. *The total completion time of a job sequence σ in $Fm|$prmu, no-idle$|\sum C_j$ is*

$$\begin{aligned} &\Big(C_{\max}(\sigma; [M_1, M_2]) \odot \left(p_{2,\sigma(1)} \odot \cdots \odot p_{2,\sigma(n)} \right)^{-1} \odot \cdots \\ &\qquad \odot C_{\max}(\sigma; [M_{m-1}, M_m]) \odot \left(p_{m,\sigma(1)} \odot \cdots \odot p_{m,\sigma(n)} \right)^{-1} \Big)^{n} \\ &\odot p_{m,\sigma(1)}^{n} \odot p_{m,\sigma(2)}^{n-1} \odot \cdots \odot p_{m,\sigma(n)}. \end{aligned}$$

Proof. We consider the case $r_j = \mathbb{1}$ in the formula (2.22) of Proposition 2.4.5. Similarly as in Lemma 2.3.10,

$$M_i^{\sigma} = \tilde{\boldsymbol{v}}_i^{\sigma} \odot \tilde{\boldsymbol{u}}_i^{\sigma},$$

where

$$\tilde{\boldsymbol{v}}_i^{\sigma} = \begin{pmatrix} p_{i,\sigma(1)} \\ p_{i,\sigma(1)} \odot p_{i,\sigma(2)} \\ \vdots \\ p_{i,\sigma(1)} \odot \cdots \odot p_{i,\sigma(n)} \end{pmatrix}$$

and

$$\tilde{\boldsymbol{u}}_i^{\sigma} = \begin{pmatrix} \mathbb{1} & p_{i,\sigma(1)}^{-1} & \cdots & (p_{i,\sigma(1)} \odot \cdots \odot p_{i,\sigma(n-1)})^{-1} \end{pmatrix}.$$

Then

$$\tilde{\boldsymbol{u}}_{i+1}^{\sigma} \odot \tilde{\boldsymbol{v}}_i^{\sigma} = C_{\max}(\sigma; [M_i, M_{i+1}]) \odot \left(p_{i+1,\sigma(1)} \odot \cdots \odot p_{i+1,\sigma(n)} \right)^{-1}.$$

Hence,

$$
\begin{aligned}
\boldsymbol{C}_m^\sigma &= M_m^\sigma \odot M_{m-1}^\sigma \odot \cdots \odot M_2^\sigma \odot M_1^\sigma \odot \boldsymbol{C}_0^\sigma \\
&= (\tilde{\boldsymbol{v}}_m^\sigma \odot \tilde{\boldsymbol{u}}_m^\sigma) \odot \left(\tilde{\boldsymbol{v}}_{m-1}^\sigma \odot \tilde{\boldsymbol{u}}_{m-1}^\sigma\right) \odot \cdots \odot (\tilde{\boldsymbol{v}}_2^\sigma \odot \tilde{\boldsymbol{u}}_2^\sigma) \odot \tilde{\boldsymbol{v}}_1^\sigma \\
&= \tilde{\boldsymbol{v}}_m^\sigma \odot \left(\tilde{\boldsymbol{u}}_m^\sigma \odot \tilde{\boldsymbol{v}}_{m-1}^\sigma\right) \odot \cdots \odot (\tilde{\boldsymbol{u}}_3^\sigma \odot \tilde{\boldsymbol{v}}_2^\sigma) \odot (\tilde{\boldsymbol{u}}_2^\sigma \odot \tilde{\boldsymbol{v}}_1^\sigma) \\
&= \begin{pmatrix} p_{m,\sigma(1)} \\ p_{m,\sigma(1)} \odot p_{m,\sigma(2)} \\ \vdots \\ p_{m,\sigma(1)} \odot \cdots \odot p_{m,\sigma(n)} \end{pmatrix} \\
&\quad \odot \Big(C_{\max}(\sigma; [M_1, M_2]) \odot \left(p_{2,\sigma(1)} \odot \cdots \odot p_{2,\sigma(n)}\right)^{-1} \odot \cdots \\
&\qquad \odot C_{\max}(\sigma; [M_{m-1}, M_m]) \odot \left(p_{m,\sigma(1)} \odot \cdots \odot p_{m,\sigma(n)}\right)^{-1} \Big).
\end{aligned}
$$

The total completion time is the product (sum in the conventional algebra) of all the components of the vector $\boldsymbol{C}_m^\sigma$. □

2.4.3 Busy flow shop problems

We present the machine matrix of the flow shop problems under busy condition. By this formulation, we can easily show a duality relationship between "blocking" and "busy" constraints.

Proposition 2.4.8. *The makespan of a job sequence $\sigma = (\sigma(1), ..., \sigma(n))$ in $Fm|prmu, busy, r_j\,|C_{\max}$ is the n-th (maximum) component of the vector:*

$$\boldsymbol{C}_m^\sigma = M_m^\sigma \odot \cdots \odot M_1^\sigma \odot \boldsymbol{C}_0^\sigma, \tag{2.23}$$

where

$$
M_i^\sigma = \begin{pmatrix}
p_{i,\sigma(1)} & \mathbb{1} & \mathbb{0} & \cdots & \mathbb{0} \\
p_{i,\sigma(1)} \odot p_{i,\sigma(2)} & p_{i,\sigma(2)} & \mathbb{1} & \cdots & \mathbb{0} \\
p_{i,\sigma(1)} \odot p_{i,\sigma(2)} \odot p_{i,\sigma(3)} & p_{i,\sigma(2)} \odot p_{i,\sigma(3)} & p_{i,\sigma(3)} & \cdots & \mathbb{0} \\
\vdots & & & \ddots & \vdots \\
p_{i,\sigma(1)} \odot \cdots \odot p_{i,\sigma(n)} & & & \cdots & p_{i,\sigma(n)}
\end{pmatrix}
$$

and

$$
\boldsymbol{C}_0^\sigma = \begin{pmatrix} r_{\sigma(1)} \\ r_{\sigma(2)} \\ \vdots \\ r_{\sigma(n)} \end{pmatrix}.
$$

Proof. Under busy condition and the assumption of permutation schedules, the recursion relations are the following:

$$\begin{aligned}
C_{i,\sigma(n)} &= \big(C_{i-1,\sigma(n)} \oplus C_{i,\sigma(n-1)}\big) \odot p_{i,\sigma(n)} \quad (i = 2, ..., m) \\
C_{i,\sigma(k)} &= \big(C_{i-1,\sigma(k)} \oplus C_{i,\sigma(k-1)}\big) \odot p_{i,\sigma(k)} \oplus C_{i,\sigma(k+1)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad (i = 2, ..., m; k = 2, ..., n-1) \\
C_{i,\sigma(1)} &= C_{i-1,\sigma(1)} \odot p_{i,\sigma(1)} \oplus C_{i,\sigma(2)} \quad (i = 2, ..., m) \\
C_{1,\sigma(n)} &= \big(r_{\sigma(n)} \oplus C_{1,\sigma(n-1)}\big) \odot p_{1,\sigma(n)} \\
C_{1,\sigma(k)} &= \big(r_{\sigma(k)} \oplus C_{1,\sigma(k-1)}\big) \odot p_{1,\sigma(k)} \oplus r_{\sigma(k+1)} \quad (k = 2, ..., n-1) \\
C_{1,\sigma(1)} &= r_{\sigma(1)} \odot p_{1,\sigma(1)} \oplus r_{\sigma(2)}.
\end{aligned}$$

In max-plus algebra, this reads the given formula (2.23). □

Theorem 2.4.9 (A duality relationship between "blocking" and "busy" constraints). *For every job sequence σ in $Fm|blocking, s_i|C_{\max}$ and $Fm|prmu,$ busy, $r_j|C_{\max}$,*

$$\begin{aligned}
&C_{\max}(\sigma, blocking; [M_1, ..., M_m], s_i) = C_{\max}(id, \bar{r}; [\bar{M}_{\sigma(1)}, ..., \bar{M}_{\sigma(n)}], busy); \\
&C_{\max}(\sigma, r_j; [M_1, ..., M_m], busy) = C_{\max}(id, blocking; [\bar{M}_{\sigma(1)}, ..., \bar{M}_{\sigma(n)}], \bar{s}).
\end{aligned}$$

Proof. Note that the formula (2.15) of Proposition 2.3.12 and the formula (2.23) of Proposition 2.4.8 are similar.

The proof is obtained similarly as in Theorems 2.4.1 and 2.4.6. □

The case without s_i and r_j has been studied in [14].

2.5 Links with linear algebra

2.5.1 Background of Theorem 2.3.6

Readers may think that Theorem 2.3.6 is not natural, but the theorem is conjectured from the next proposition in linear algebra.

Proposition 2.5.1. *For an integer $l \geq 2$ and positive numbers $x_k, y_k (k = 1, ..., l)$, let*

$$X = \begin{pmatrix} x_1 & 0 & 0 & \cdots & 0 \\ x_1 x_2 & x_2 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ x_1 \cdots x_l & & & \cdots & x_l \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_1 & 0 & 0 & \cdots & 0 \\ y_1 y_2 & y_2 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ y_1 \cdots y_l & & & \cdots & y_l \end{pmatrix}.$$

If

$$\big(x_k^{-1} + y_{k+1}^{-1}\big)^{-1} \leq \big(y_k^{-1} + x_{k+1}^{-1}\big)^{-1} \quad (k = 1, ..., l-1), \tag{2.24}$$

then $YX \leq XY$.

Proof.

$$X^{-1}Y^{-1} = \begin{pmatrix} x_1^{-1} & 0 & 0 & \cdots & 0 \\ -1 & x_2^{-1} & 0 & \cdots & 0 \\ 0 & -1 & x_3^{-1} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_l^{-1} \end{pmatrix} \begin{pmatrix} y_1^{-1} & 0 & 0 & \cdots & 0 \\ -1 & y_2^{-1} & 0 & \cdots & 0 \\ 0 & -1 & y_3^{-1} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & y_l^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} x_1^{-1}y_1^{-1} & 0 & 0 & \cdots & 0 \\ -y_1^{-1} - x_2^{-1} & x_2^{-1}y_2^{-1} & 0 & \cdots & 0 \\ 1 & -y_2^{-1} - x_3^{-1} & x_3^{-1}y_3^{-1} & \cdots & 0 \\ 0 & 1 & -y_3^{-1} - x_4^{-1} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_l^{-1}y_l^{-1} \end{pmatrix}.$$

Therefore, if

$$-y_k^{-1} - x_{k+1}^{-1} \geq -x_k^{-1} - y_{k+1}^{-1} \quad (k = 1, ..., l-1),$$

i.e.,

$$\left(x_k^{-1} + y_{k+1}^{-1}\right)^{-1} \leq \left(y_k^{-1} + x_{k+1}^{-1}\right)^{-1} \quad (k = 1, ..., l-1),$$

then

$$X^{-1}Y^{-1} \geq Y^{-1}X^{-1}.$$

Since $X^{-1}Y^{-1} - Y^{-1}X^{-1} \geq 0, X \geq 0$ and $Y \geq 0$,

$$XY(X^{-1}Y^{-1} - Y^{-1}X^{-1})YX = XY - YX \geq 0.$$

□

Substituting adding and multiplying with $\oplus$ and $\odot$ in the assumption (2.24), we obtain $\min[x_k, y_{k+1}] \leq \min[y_k, x_{k+1}]$, which is correspondent to the inequality (2.6). To our knowledge, we do not know if the proposition is new or known, but we identified a link between the proposition and the condition in m-machine permutation flow shop problems. In fact, we conjectured Theorem 2.3.6 by considering a max-algebraic analogue of the proposition. We could not have found the theorem without using max-plus algebra. We, however, cannot prove the theorem similarly with the proposition since an additive inverse is not defined in max-plus algebra.

2.5.2 Problems in linear algebra

In Sections 2.3 and 2.4 a variety of flow shop problems are represented as minimizing the maximum component of a vector in max-plus algebra. Due to the relationship between max-plus algebra and the conventional algebra, there exist problems in linear algebra that are correspond to flow shop problems. We here present the problems. If the problems were solved, the methods for the solutions would be applied to flow shop problems, i.e., NP-hard problems for $m \geq 3$.

In this subsection, let $x_{i,j}, s_i, r_j$ for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ be real numbers larger than or equal to 1.

The problem that corresponds to $Fm|\mathrm{prmu}, s_i|C_{\max}$

Problem 2.5.2. *Find a permutation $\sigma \in S_n$ for minimizing the p-norm of the vector*

$$X_{\sigma(n)} \cdots X_{\sigma(1)} \boldsymbol{C}'_0,$$

where

$$X_j = \begin{pmatrix} x_{1,j} & 0 & 0 & \cdots & 0 \\ x_{1,j}x_{2,j} & x_{2,j} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ x_{1,j}\cdots x_{m-1,j} & & & x_{m-1,j} & 0 \\ x_{1,j}\cdots x_{m-1,j}x_{m,j} & & \cdots & & x_{m,j} \end{pmatrix} \quad \textit{and} \quad \boldsymbol{C}'_0 = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{pmatrix}.$$

For $p \geq 1$, the p-norm $||\boldsymbol{V}||_p$ of the vector $\boldsymbol{V} = (v_1, v_2, \ldots, v_m)$ is

$$\left(|v_1|^p + |v_2|^p + \cdots + |v_m|^p\right)^{1/p}.$$

Since the components of the vector are sorted in increasing order, minimizing the p-norm of the vector is correspondent to minimizing the m-th (maximum) component in flow shops.

In the case without the starting times (s_i) of machines, we may set $\boldsymbol{C}'_0 = (1, 0, \ldots, 0)^T$. The cases of the following problems are similar.

The problem that corresponds to $Fm|\text{no-wait}, s_i|C_{\max}$

Problem 2.5.3. *Find a permutation $\sigma \in S_n$ for minimizing the p-norm of the vector*

$$X_{\sigma(n)} \cdots X_{\sigma(1)} \boldsymbol{C}'_0,$$

where

$$X_j = \begin{pmatrix} x_{1,j} & 1 & x_{2,j}^{-1} & \cdots & (x_{2,j}x_{3,j}\cdots x_{m-1,j})^{-1} \\ x_{1,j}x_{2,j} & x_{2,j} & 1 & \cdots & (x_{3,j}\cdots x_{m-1,j})^{-1} \\ x_{1,j}x_{2,j}x_{3,j} & x_{2,j}x_{3,j} & x_{3,j} & \cdots & \\ \vdots & & & \ddots & \vdots \\ x_{1,j}\cdots x_{m,j} & & & \cdots & x_{m,j} \end{pmatrix}$$

and

$$\boldsymbol{C}'_0 = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{pmatrix}.$$

The problem that corresponds to $Fm|\text{blocking}, s_i|C_{\max}$

Problem 2.5.4. *Find a permutation $\sigma \in S_n$ for minimizing the p-norm of the vector*

$$X_{\sigma(n)} \cdots X_{\sigma(1)} \boldsymbol{C}'_0,$$

where

$$X_j = \begin{pmatrix} x_{1,j} & 1 & 0 & \cdots & 0 \\ x_{1,j}x_{2,j} & x_{2,j} & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ x_{1,j}\cdots x_{m-1,j} & \cdots & & x_{m-1,j} & 1 \\ x_{1,j}\cdots x_{m-1,j}x_{m,j} & \cdots & & & x_{m,j} \end{pmatrix}$$

and

$$\boldsymbol{C}'_0 = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{pmatrix}.$$

The problem that corresponds to $Fm|\text{prmu}, r_j|C_{\max}$

Problem 2.5.5. *Find a permutation* $\sigma \in S_n$ *for minimizing the p-norm of the vector*

$$X_m^\sigma \cdots X_1^\sigma \boldsymbol{C}_0^\sigma,$$

where

$$X_i^\sigma = \begin{pmatrix} x_{i,\sigma(1)} & 0 & 0 & \cdots & 0 \\ x_{i,\sigma(1)}x_{i,\sigma(2)} & x_{i,\sigma(2)} & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ x_{i,\sigma(1)}\cdots x_{i,\sigma(n)} & & & \cdots & x_{i,\sigma(n)} \end{pmatrix} \text{ and } \boldsymbol{C}_0^\sigma = \begin{pmatrix} r_{\sigma(1)} \\ r_{\sigma(2)} \\ \vdots \\ r_{\sigma(n)} \end{pmatrix}.$$

Since the components of the vector are sorted in increasing order, minimizing the p-norm of the vector is correspondent to minimizing the n-th (maximum) component in flow shops.

In the case without the release times (r_j) of jobs, we may set $\boldsymbol{C}_0^\sigma = (1, 0, \ldots, 0)^T$. The cases of the following problems are similar.

The problem that corresponds to $Fm|\text{prmu, no-idle}, r_j|C_{\max}$

Problem 2.5.6. *Find a permutation* $\sigma \in S_n$ *for minimizing the p-norm of the vector*

$$X_m^\sigma \cdots X_1^\sigma \boldsymbol{C}_0^\sigma,$$

where

$$X_i^\sigma = \begin{pmatrix} x_{i,\sigma(1)} & 1 & x_{i,\sigma(2)}^{-1} & \cdots & (x_{i,\sigma(2)}\cdots x_{i,\sigma(n-1)})^{-1} \\ x_{i,\sigma(1)}x_{i,\sigma(2)} & x_{i,\sigma(2)} & 1 & \cdots & \\ \vdots & & \ddots & & \vdots \\ x_{i,\sigma(1)}\cdots x_{i,\sigma(n)} & & \cdots & & x_{i,\sigma(n)} \end{pmatrix}$$

and

$$\boldsymbol{C}_0^\sigma = \begin{pmatrix} r_{\sigma(1)} \\ r_{\sigma(2)} \\ \vdots \\ r_{\sigma(n)} \end{pmatrix}.$$

The problem that corresponds to $Fm|\text{prmu, busy}, r_j|C_{\max}$

Problem 2.5.7. *Find a permutation $\sigma \in S_n$ for minimizing the p-norm of the vector*

$$X_m^\sigma \cdots X_1^\sigma \boldsymbol{C}_0^\sigma,$$

where

$$X_i^\sigma = \begin{pmatrix} x_{i,\sigma(1)} & 1 & 0 & \cdots & 0 \\ x_{i,\sigma(1)}x_{i,\sigma(2)} & x_{i,\sigma(2)} & 1 & \cdots & \vdots \\ \vdots & & \ddots & & \\ x_{i,\sigma(1)}\cdots x_{i,\sigma(n-1)} & & \cdots & x_{i,\sigma(n-1)} & 1 \\ x_{i,\sigma(1)}\cdots x_{i,\sigma(n-1)}x_{i,\sigma(n)} & & \cdots & & x_{i,\sigma(n)} \end{pmatrix}$$

and

$$\boldsymbol{C}_0^\sigma = \begin{pmatrix} r_{\sigma(1)} \\ r_{\sigma(2)} \\ \vdots \\ r_{\sigma(n)} \end{pmatrix}.$$

2.6 Summary

We presented a new solvable condition in m-machine permutation flow shop problems. This is derived using a new framework, the machine representation, and difficult to derive without max-plus algebra. The result enables us to easily obtain an optimal job sequence in more complicated production lines.

The new framework associates a machine with a matrix and is the dual of the existing max-plus approach associating a job with a matrix by Bouquard et al. The framework is the first one which can deal with non-permutation flow shop problems based on max-plus algebra. Moreover, using the framework, we provided new simple proofs of some known results such as duality relationships between some flow shops and the reduction of a no-wait flow shop to a traveling salesman problem (TSP).

We investigated links with linear algebra. We presented a sufficient condition for an extension of Johnson's rule by considering a max-plus algebraic analogue of a proposition in linear algebra, and the problems in linear algebra that are correspondent to some flow shop problems.

Max-plus algebra is very useful for flow shop problems. We expect that the two dual frameworks based on max-plus algebra contribute to the development of the theory of flow shops, especially non-permutation flow shops. It is a future problem to formalize various flow shop problems, including stochastic models, in the job representation and/or the machine representation and to analyze them. Furthermore, since there exist a lot of results in linear algebra, it is also an interesting problem to investigate the application of the results to flow shop problems.

Chapter 3

Theoretical Analysis on a Two-identical-parallel-machine Problem

3.1 Introduction

The two-identical-parallel-machine problem to minimize the makespan, $P2||C_{\max}$, which is equivalent to the number partitioning problem (NPP), is to find a set $S' \subset \{1, ..., n\}$ that minimize the discrepancy

$$\Delta = \left| \sum_{i \in S'} p_i - \sum_{i \in \{1,...,n\} \setminus S'} p_i \right|$$

for a given positive integer p_i for $i \in \{1, ..., n\}$. This decision version is known to be NP-complete, so the problem is NP-hard.

A variety of heuristic algorithms as well as optimal algorithms have been developed. On the other hand, the solution structure has been studied based on the notion and tools from statistical mechanics. We examine the solution structure directly using max-plus algebra.

Example 3.1.1. Consider the $P2||C_{\max}$ with ten jobs as described in the following table.

Job j	$\underline{1}$	2	$\underline{3}$	4	5	$\underline{6}$	$\underline{7}$	8	9	$\underline{10}$
p_j	2	2	3	4	4	5	7	8	9	10

The minimum of the discrepancy is 0, since letting $S' = \{1, 3, 6, 7, 10\}$ (with underlines in the table) the makespan is 27. This is easy to solve.

Next consider the $P2||C_{\max}$ with ten jobs as described in the following table.

Job j	$\underline{1}$	2	3	4	5	6	$\underline{7}$	8	$\underline{9}$	$\underline{10}$
p_j	50	61	307	336	495	630	633	760	946	960

The minimum of the discrepancy is also 0, but it is not easy to find the optimal subset $S' = \{1, 7, 9, 10\}$ (with underlines in the table).

Hereinafter we refer to $P2||C_{\max}$ as the NPP.

Section 3.2 is devoted to the proof of a proposition which is necessary for theoretical analysis. In Section 3.3 we show that the decision version of the NPP is equivalent to deciding whether one of given integers $p_1, \ldots, p_n$ is a solution of a tropical algebraic equation with coefficients composed of other integers. For n up to 6 we derive concretely and explicitly the equation and its solution set. Moreover, we consider the balanced NPP (BalNPP), where the difference of cardinalities of two subsets is at most one. Finally, we summarize this chapter in Section 3.4.

3.2 Preliminaries

We prove a proposition which is useful for computations in theoretical analysis. Moreover, we present a simple procedure to write every symmetric tropical polynomial in terms of the elementary symmetric ones. Carlsson and Kališnik [59] showed the fact, but did not present the algorithm.

Proposition 3.2.1. *Let n be a positive integer and let a_i and b_i be nonnegative numbers for $i \in \{1, ..., n\}$. A necessary and sufficient condition that*

$$X_1^{a_1} \odot \cdots \odot X_n^{a_n} \leq X_1^{b_1} \odot \cdots \odot X_n^{b_n}$$

for any nonnegative numbers $X_1, \ldots, X_n$ such that $X_1 \geq \cdots \geq X_n$ is that the followings hold:

$$\begin{aligned} a_1 &\leq b_1, \\ &\vdots \\ a_1 \odot \cdots \odot a_n &\leq b_1 \odot \cdots \odot b_n. \end{aligned}$$

Proof. The necessity of the condition is obvious.

We prove the sufficiency by induction on n. The base case $n = 1$ is straightforward. Assume that for $n - 1 (\geq 1)$ the condition is sufficient. If $a_n \leq b_n$, then

$$\begin{aligned} \frac{X_1^{b_1} \odot \cdots \odot X_{n-1}^{b_{n-1}} \odot X_n^{b_n}}{X_1^{a_1} \odot \cdots \odot X_{n-1}^{a_{n-1}} \odot X_n^{a_n}} &\geq \mathbb{1} \odot \frac{X_n^{b_n}}{X_n^{a_n}} \quad \text{(by induction assumption)} \\ &\geq \mathbb{1}. \end{aligned}$$

If $a_n > b_n$, then

$$\begin{aligned} \frac{X_1^{b_1} \odot \cdots \odot X_{n-1}^{b_{n-1}} \odot X_n^{b_n}}{X_1^{a_1} \odot \cdots \odot X_{n-1}^{a_{n-1}} \odot X_n^{a_n}} &\geq \frac{X_1^{b_1} \odot \cdots \odot X_{n-1}^{b_{n-1}} \odot X_{n-1}^{b_n}}{X_1^{a_1} \odot \cdots \odot X_{n-1}^{a_{n-1}} \odot X_{n-1}^{a_n}} = \frac{X_1^{b_1} \odot \cdots \odot X_{n-1}^{b_{n-1} \odot b_n}}{X_1^{a_1} \odot \cdots \odot X_{n-1}^{a_{n-1} \odot a_n}} \\ &\geq \mathbb{1} \quad \text{(by induction assumption)}. \end{aligned}$$

The proof is complete. □

Simplifying symmetric tropical polynomials by using the proposition and substituting Σ_1, $\frac{\Sigma_2}{\Sigma_1}$, ..., $\frac{\Sigma_n}{\Sigma_{n-1}}$ for X_1, $X_2, \ldots, X_n$ respectively, we get expressions of the tropical polynomials in terms of the elementary symmetric ones.

Example 3.2.2. Consider the symmetric tropical polynomial

$$X_1^3 \odot X_2 \odot X_3 \oplus X_1 \odot X_2^3 \odot X_3 \oplus X_1 \odot X_2 \odot X_3^3 \oplus X_1^2 \odot X_2^2 \odot X_3^2.$$

Suppose that $X_1 \geq X_2 \geq X_3$. Then the polynomial equals $X_1^3 \odot X_2 \odot X_3 \oplus X_1^2 \odot X_2^2 \odot X_3^2$. By substituting Σ_1, $\frac{\Sigma_2}{\Sigma_1}$, $\frac{\Sigma_3}{\Sigma_2}$ for X_1, X_2, X_3 respectively, we can rewrite it as follows:

$$\Sigma_1^3 \odot \frac{\Sigma_2}{\Sigma_1} \odot \frac{\Sigma_3}{\Sigma_2} \oplus \Sigma_1^2 \odot \left(\frac{\Sigma_2}{\Sigma_1}\right)^2 \odot \left(\frac{\Sigma_3}{\Sigma_2}\right)^2 = \Sigma_1^2 \odot \Sigma_3 \oplus \Sigma_3^2.$$

3.3 Theoretical analysis based on max-plus algebra

We show that the decision version of the NPP is reduced to solving a tropical algebraic equation, and derive concretely and explicitly the equation and the solution set for n up to 6 as examples.

The problem is to decide whether there is a set $S' \subset \{1, ..., n\}$ such that

$$\sum_{i \in S'} p_i = \sum_{i \in \{1,\ldots,n\} \setminus S'} p_i = \left(\sum_{i=1}^{n} p_i\right)/2 \tag{3.1}$$

for a given positive integer p_i for $i \in \{1, ..., n\}$. The case $S' = \emptyset$ or $S' = \{1, \ldots, n\}$ clearly does not satisfy the condition (3.1). Our strategy is to investigate whether the minimum of

$$\max\left[\sum_{i \in S'} p_i, \sum_{i \in \{1,\ldots,n\} \setminus S'} p_i\right] \tag{3.2}$$

among all possible subsets equals $\left(\sum_{i=1}^{n} p_i\right)/2$. Using max-plus algebra we rewrite the expression (3.2) as

$$p_1^{a_1} \odot \cdots \odot p_n^{a_n} \oplus p_1^{1-a_1} \odot \cdots \odot p_n^{1-a_n},$$

where $a_i \in \{0, 1\}$. We may assume that $a_1 = 1$ and thus it is sufficient to consider $(2^{n-1} - 1)$ subsets. The minimum C is defined as follows:

$$C(p_1, \ldots, p_n) = \min\left[p_1 \oplus p_2 \odot \cdots \odot p_n, p_1 \odot p_2 \oplus p_3 \odot \cdots \odot p_n, \ldots\right].$$

Since the identity

$$\min[x_1, x_2, \ldots, x_M] = \frac{\Sigma_M(x_1, x_2, \ldots, x_M)}{\Sigma_{M-1}(x_1, x_2, \ldots, x_M)}$$

holds, where Σ_i is the elementary symmetric tropical polynomial, the minimum C can be expressed as the ratio of two tropical polynomials N and D, that is,

$$C(p_1, \ldots, p_n) = \frac{N(p_1, \ldots, p_n)}{D(p_1, \ldots, p_n)}.$$

After all, the problem is reduced to solving the equation

$$\begin{aligned}
& \frac{N(p_1,\dots,p_n)}{D(p_1,\dots,p_n)} = (p_1 \odot \cdots \odot p_n)^{1/2} \\
\Leftrightarrow\quad & \frac{N(p_1,\dots,p_n)^2}{D(p_1,\dots,p_n)^2} = p_1 \odot \cdots \odot p_n \\
\Leftrightarrow\quad & N\left(p_1^2,\dots,p_n^2\right) = p_1 \odot \cdots \odot p_n \odot D\left(p_1^2,\dots,p_n^2\right). \qquad (3.3)
\end{aligned}$$

Note that the Frobenius identity holds for all powers n: $(A \oplus B)^n = A^n \oplus B^n$. In fact, this equation is a tropical algebraic equation we defined. Squaring both sides plays an important role as we show in the next proposition.

Proposition 3.3.1. *The equation (3.3) is a tropical algebraic equation in p_1 of at most the $(2^n - 2)$-th degree.*

Proof. Let $L = 2^{n-1} - 1$. It is sufficient to consider L subsets, so we have

$$N(p_1,\dots,p_n) = \Sigma_L = \bigodot_{\substack{a_2,\dots,a_n \in \{0,1\} \\ (a_2,\dots,a_n) \neq (1,\dots,1)}} \left(p_1 \odot p_2^{a_2} \odot \cdots \odot p_n^{a_n} \oplus p_2^{1-a_2} \odot \cdots \odot p_n^{1-a_n}\right).$$

N is thus rewritten as

$$\bigoplus_{k=0}^{L} N_k(p_2,\dots,p_n) \odot p_1^{L-k},$$

where N_k is a tropical polynomial. Similarly we have

$$D(p_1,\dots,p_n) = \Sigma_{L-1} = \bigoplus_{k=0}^{L-1} D_k(p_2,\dots,p_n) \odot p_1^{L-1-k},$$

where D_k is a tropical polynomial. Therefore, the equation (3.3) is as follows:

$$\begin{aligned}
& \bigoplus_{k=0}^{L} N_k\left(p_2^2,\dots,p_n^2\right) \odot p_1^{2L-2k} = p_1 \odot \cdots \odot p_n \odot \bigoplus_{k=0}^{L-1} D_k\left(p_2^2,\dots,p_n^2\right) \odot p_1^{2L-2-2k} \\
\Leftrightarrow\quad & \bigoplus_{k=0}^{L} N_k\left(p_2^2,\dots,p_n^2\right) \odot p_1^{2L-2k} = \bigoplus_{k=0}^{L-1} D'_k\left(p_2^2,\dots,p_n^2\right) \odot p_1^{2L-(2k+1)},
\end{aligned}$$

where $D'_k = p_2 \odot \cdots \odot p_n \odot D_k$. The equation is a tropical algebraic equation of the $2L$-th degree. The proof is complete. $\square$

Without loss of generality we may assume that $p_1 \geq p_2 \geq \cdots \geq p_n$. As examples we derive concretely and explicitly the tropical algebraic equation and the solution set for n up to 6, but the tropical algebraic equations can be computed automatically for n lager than 6. In order to express the equations in terms of the elementary symmetric tropical polynomials, let $s_i = \Sigma_i(p_1,\dots,p_n)$ for $i \in \{1,\dots,n\}$.

The multiplication sign $\odot$ is omitted hereafter.

3.3.1 The case $n = 2$

Since

$$C(p_1, p_2) = p_1 \oplus p_2,$$

we have the equation

$$p_1^2 \oplus p_2^2 = p_1 p_2,$$

which can be regarded as a tropical quadratic equation in p_1. Simplifying the equation using Proposition 3.2.1 we get

$$p_1 = p_2. \tag{3.4}$$

The solution set is

$$\{p_2\}.$$

This implies that it is sufficient to consider only the partition:

$$(\{p_1\}, \{p_2\}).$$

Expressing the equation (3.4) in terms of the elementary symmetric tropical polynomials we have the equation

$$s_1^2 = s_2.$$

3.3.2 The case $n = 3$

Since

$$\begin{aligned} C(p_1, p_2, p_3) &= \min[p_1 \oplus p_2 p_3,\ p_2 \oplus p_1 p_3,\ p_3 \oplus p_1 p_2] \\ &= \frac{p_2 p_3 p_1^3 \oplus p_2^2 p_3^2 p_1^2 \oplus \left(p_2^3 p_3 \oplus p_2 p_3^3\right) p_1 \oplus p_2^2 p_3^2}{p_2 p_3 p_1^2 \oplus (p_2^2 p_3 \oplus p_2 p_3^2)\, p_1 \oplus p_2^2 p_3 \oplus p_2 p_3^2}, \end{aligned}$$

we have the equation

$$\begin{aligned} &p_2^2 p_3^2 p_1^6 \oplus p_2^4 p_3^4 p_1^4 \oplus \left(p_2^6 p_3^2 \oplus p_2^2 p_3^6\right) p_1^2 \oplus p_2^4 p_3^4 \\ &\qquad = p_2^3 p_3^3 p_1^5 \oplus \left(p_2^5 p_3^3 \oplus p_2^3 p_3^5\right) p_1^3 \oplus \left(p_2^5 p_3^3 \oplus p_2^3 p_3^5\right) p_1, \end{aligned}$$

which can be regarded as a tropical algebraic equation of the sixth degree in p_1. The equation is reduced to

$$p_2^2 p_3^2 p_1^4 \oplus p_2^4 p_3^4 p_1^2 \oplus \left(p_2^6 p_3^2 \oplus p_2^2 p_3^6\right) = p_2^3 p_3^3 p_1^3 \oplus \left(p_2^5 p_3^3 \oplus p_2^3 p_3^5\right) p_1,$$

since p_i is positive. Moreover, simplifying the equation using Proposition 3.2.1 we get

$$p_1^2 \oplus p_2^2 p_3^2 = (p_2 p_3)\, p_1. \tag{3.5}$$

This can be regarded as a tropical quadratic equation in p_1. The assumption of Corollary 1.4.17 is satisfied and the solution set is

$$\{p_2 p_3\}.$$

This implies that it is sufficient to consider only the partition:

$$(\{p_1\}, \{p_2, p_3\}).$$

Expressing the equation (3.5) in terms of the elementary symmetric tropical polynomials we have the equation

$$s_1^4 \oplus s_3^2 = s_1^2 s_3.$$

3.3.3 The case $n = 4$

The minimum C is as follows:

$$C(p_1, p_2, p_3, p_4) = \min[p_1 \oplus p_2p_3p_4,\ p_1p_2 \oplus p_3p_4,\ p_1p_3 \oplus p_2p_4,\\ p_1p_2p_3 \oplus p_4,\ p_1p_4 \oplus p_2p_3,\ p_1p_2p_4 \oplus p_3,\ p_1p_3p_4 \oplus p_2].$$

Simplifying this using Proposition 3.2.1 we get

$$C(p_1, p_2, p_3, p_4) = \frac{p_1^2 \oplus p_1p_2p_3p_4 \oplus p_2^2p_3^2}{p_1 \oplus p_2p_3}.$$

Therefore, we have the equation

$$p_1^4 \oplus p_1^2\left(p_2^2p_3^2p_4^2\right) \oplus p_2^4p_3^4 = p_1^3\left(p_2p_3p_4\right) \oplus p_1\left(p_2^3p_3^3p_4\right). \tag{3.6}$$

This can be regarded as a tropical quartic equation in p_1. The assumption of Corollary 1.4.17 is satisfied and the solution set is

$$\left\{p_2p_3p_4,\ p_2p_3p_4^{-1}\right\}.$$

This implies that it is sufficient to consider only the partitions:

$$(\{p_1\}, \{p_2, p_3, p_4\}), \quad (\{p_1, p_4\}, \{p_2, p_3\}).$$

Table 3.1 shows the solutions. The solutions to the BalNPP are straightforwardly obtained from these solutions (Table 3.2).

Rank in decreasing order	Solution
1	$p_2p_3p_4$
2	$p_2p_3p_4^{-1}$

Table 3.1: The solutions to the NPP $(n = 4)$

Rank in decreasing order	Solution
1	$p_2p_3p_4^{-1}$

Table 3.2: The solution to the BalNPP $(n = 4)$

Expressing the equation (3.6) in terms of the elementary symmetric tropical polynomials we have the equation

$$s_1^8 \oplus s_1^4s_4^2 \oplus s_3^4 = s_1^6s_4 \oplus s_1^2s_3^2s_4.$$

3.3.4 The case $n = 5$

The minimum C is as follows:

$$C(p_1, p_2, p_3, p_4, p_5) = \min[p_1 \oplus p_2p_3p_4p_5,\ \ldots,\ p_1p_3p_5 \oplus p_2p_4,\ p_1p_4p_5 \oplus p_2p_3].$$

Simplifying this using Proposition 3.2.1 we get

$$C(p_1,p_2,p_3,p_4,p_5) = \frac{N(p_1,p_2,p_3,p_4,p_5)}{D(p_1,p_2,p_3,p_4,p_5)},$$

where

$$\begin{aligned} N =& p_1^5 \oplus p_1^4p_2p_3p_4p_5 \oplus p_1^3p_2^2p_3^2p_4^2 \oplus p_1^2p_2^3p_3^3p_4p_5 \oplus p_1p_2^4p_3^4 \\ &\oplus p_1p_2^4p_3^2p_4^2p_5^2 \oplus p_2^3p_3^3p_4^3p_5^3, \\ D =& p_1^4 \oplus p_1^3p_2p_3p_4 \oplus p_1^2p_2^2p_3^2p_4 \oplus p_1p_2^3p_3^3 \oplus p_1p_2^3p_3^2p_4p_5 \oplus p_2^3p_3^2p_4^2p_5^2. \end{aligned}$$

Therefore, we have the equation

$$\begin{aligned} & p_1^{10} \oplus p_1^8\left(p_2^2p_3^2p_4^2p_5^2\right) \oplus p_1^6\left(p_2^4p_3^4p_4^4\right) \oplus p_1^4\left(p_2^6p_3^6p_4^2p_5^2\right) \\ & \oplus p_1^2\left(p_2^8p_3^8 \oplus p_2^8p_3^4p_4^4p_5^4\right) \oplus p_2^6p_3^6p_4^6p_5^6 \\ =& p_1^9\left(p_2p_3p_4p_5\right) \oplus p_1^7\left(p_2^3p_3^3p_4^3p_5\right) \oplus p_1^5\left(p_2^5p_3^5p_4^3p_5\right) \\ & \oplus p_1^3\left(p_2^7p_3^7p_4p_5 \oplus p_2^7p_3^5p_4^3p_5^3\right) \oplus p_1\left(p_2^7p_3^5p_4^5p_5^5\right). \end{aligned} \tag{3.7}$$

This can be regarded as a tropical algebraic equation of the tenth degree in p_1.

When $p_3 \leq p_4p_5$, the assumption of Corollary 1.4.17 is satisfied and the solution set is

$$\left\{p_2p_3p_4p_5,\ p_2p_3p_4p_5^{-1},\ p_2p_3p_4^{-1}p_5,\ p_2p_3^{-1}p_4p_5,\ p_2^{-1}p_3p_4p_5\right\}.$$

This implies that it is sufficient to consider only the partitions:

$(\{p_1\},\{p_2,p_3,p_4,p_5\})$, $(\{p_1,p_5\},\{p_2,p_3,p_4\})$, $(\{p_1,p_4\},\{p_2,p_3,p_5\})$,
$(\{p_1,p_3\},\{p_2,p_4,p_5\})$, $(\{p_1,p_2\},\{p_3,p_4,p_5\})$.

When $p_3 > p_4p_5$, the assumption is not satisfied, but considering the graph of the function $N^2(p_1p_2p_3p_4p_5D^2)^{-1}$ we easily obtain the solution set

$$\left\{p_2p_3p_4p_5,\ p_2p_3p_4p_5^{-1},\ p_2p_3p_4^{-1}p_5,\ p_2p_3p_4^{-1}p_5^{-1}\right\}.$$

This implies that it is sufficient to consider only the partitions:

$(\{p_1\},\{p_2,p_3,p_4,p_5\})$, $(\{p_1,p_5\},\{p_2,p_3,p_4\})$,
$(\{p_1,p_4\},\{p_2,p_3,p_5\})$, $(\{p_1,p_4,p_5\},\{p_2,p_3\})$.

Table 3.3 shows the solutions. The solutions to the BalNPP are straightforwardly obtained from these solutions (Table 3.4).

Expressing the equation (3.7) in terms of the elementary symmetric tropical polynomials we have the equation

$$\begin{aligned} & s_1^{16} \oplus s_1^{12}s_5^2 \oplus s_1^8s_4^4 \oplus s_1^4s_3^4s_5^2 \oplus s_3^8 \oplus s_2^4s_5^4 \oplus s_5^6 \\ & \qquad = s_1^{14}s_5 \oplus s_1^{10}s_4^2s_5 \oplus s_1^6s_3^2s_4^2s_5 \oplus s_1^2s_3^6s_5 \oplus s_1^2s_2^2s_3^2s_5^3 \oplus s_2^2s_5^5. \end{aligned}$$

Rank in decreasing order	Solution	
	$p_3 \le p_4p_5$	$p_3 > p_4p_5$
1	$p_2p_3p_4p_5$	
2	$p_2p_3p_4p_5^{-1}$	
3	$p_2p_3p_4^{-1}p_5$	
4	$p_2p_3^{-1}p_4p_5$	$p_2p_3p_4^{-1}p_5^{-1}$
5	$p_2^{-1}p_3p_4p_5$	$-$

Table 3.3: The solutions to the NPP ($n = 5$)

Rank in decreasing order	Solution	
	$p_3 \le p_4p_5$	$p_3 > p_4p_5$
1	$p_2p_3p_4p_5^{-1}$	
2	$p_2p_3p_4^{-1}p_5$	
3	$p_2p_3^{-1}p_4p_5$	$p_2p_3p_4^{-1}p_5^{-1}$
4	$p_2^{-1}p_3p_4p_5$	$-$

Table 3.4: The solutions to the BalNPP ($n = 5$)

3.3.5 The case $n = 6$

The minimum C is as follows:

$$\begin{aligned}&C(p_1, p_2, p_3, p_4, p_5, p_6)\\ &\quad = \min[p_1 \oplus p_2p_3p_4p_5p_6, \ldots, p_1p_4p_6 \oplus p_2p_3p_5,\ p_1p_5p_6 \oplus p_2p_3p_4].\end{aligned}$$

Simplifying this using Proposition 3.2.1 we get

$$C(p_1, p_2, p_3, p_4, p_5, p_6) = \frac{N(p_1, p_2, p_3, p_4, p_5, p_6)}{D(p_1, p_2, p_3, p_4, p_5, p_6)},$$

where

$$\begin{aligned}
N =&p_1^{10} \oplus p_1^9p_2p_3p_4p_5p_6 \oplus p_1^8p_2^2p_3^2p_4^2p_5^2 \oplus p_1^7p_2^3p_3^3p_4^3p_5p_6\\
&\oplus p_1^6(p_2^4p_3^4p_4^4 \oplus p_2^4p_3^4p_4^2p_5^2p_6^2) \oplus p_1^5(p_2^5p_3^5p_4^3p_5p_6 \oplus p_2^5p_3^3p_4^3p_5^3p_6^3)\\
&\oplus p_1^4(p_2^6p_3^6p_4^2p_5^2 \oplus p_2^6p_3^4p_4^4p_5^2p_6^2 \oplus p_2^4p_3^4p_4^4p_5^4p_6^4)\\
&\oplus p_1^3(p_2^7p_3^7p_4p_5p_6 \oplus p_2^7p_3^5p_4^3p_5^3p_6 \oplus p_2^5p_3^5p_4^5p_5^3p_6^3)\\
&\oplus p_1^2(p_2^8p_3^8 \oplus p_2^8p_3^6p_4^2p_5^2p_6^2 \oplus p_2^8p_3^4p_4^4p_5^4 \oplus p_2^6p_3^6p_4^4p_5^4p_6^2)\\
&\oplus p_1(p_2^7p_3^7p_4^3p_5^3p_6^3 \oplus p_2^7p_3^5p_4^5p_5^5p_6) \oplus p_2^6p_3^6p_4^6p_5^6,\\
D =&p_1^9 \oplus p_1^8p_2p_3p_4p_5 \oplus p_1^7p_2^2p_3^2p_4^2p_5 \oplus p_1^6(p_2^3p_3^3p_4^2p_5p_6 \oplus p_2^3p_3^3p_4^3)\\
&\oplus p_1^5(p_2^4p_3^3p_4^2p_5^2p_6^2 \oplus p_2^4p_3^4p_4^3 \oplus p_2^4p_3^4p_4^2p_5p_6)\\
&\oplus p_1^4(p_2^4p_3^3p_4^3p_5^3p_6^3 \oplus p_2^5p_3^4p_4^3p_5p_6 \oplus p_2^5p_3^5p_4^2p_5 \oplus p_2^5p_3^3p_4^3p_5^2p_6^2)\\
&\oplus p_1^3(p_2^5p_3^4p_4^4p_5^2p_6^2 \oplus p_2^6p_3^5p_4^2p_5^2 \oplus p_2^6p_3^6p_4p_5 \oplus p_2^6p_3^4p_4^3p_5^2p_6 \oplus p_2^4p_3^4p_4^4p_5^3p_6^3)\\
&\oplus p_1^2(p_2^6p_3^5p_4^3p_5^3p_6 \oplus p_2^7p_3^6p_4p_5p_6 \oplus p_2^7p_3^7 \oplus p_2^7p_3^4p_4^3p_5^3 \oplus p_2^7p_3^5p_4^2p_5^2p_6\\
&\qquad \oplus p_2^5p_3^5p_4^4p_5^3p_6^2)\\
&\oplus p_1(p_2^7p_3^6p_4^2p_5^2p_6^2 \oplus p_2^7p_3^4p_4^4p_5^4 \oplus p_2^6p_3^5p_4^4p_5^4p_6 \oplus p_2^6p_3^6p_4^3p_5^3p_6^2) \oplus p_2^6p_3^5p_4^5p_5^5.
\end{aligned}$$

Therefore, we have the equation

$$\begin{aligned}
&q_0p_1^{20} \oplus q_2p_1^{18} \oplus q_4p_1^{16} \oplus q_6p_1^{14} \oplus q_8p_1^{12} \oplus q_{10}p_1^{10} \\
&\quad \oplus q_{12}p_1^{8} \oplus q_{14}p_1^{6} \oplus q_{16}p_1^{4} \oplus q_{18}p_1^{2} \oplus q_{20} \\
=&q_1p_1^{19} \oplus q_3p_1^{17} \oplus q_5p_1^{15} \oplus q_7p_1^{13} \oplus q_9p_1^{11} \oplus q_{11}p_1^{9} \\
&\quad \oplus q_{13}p_1^{7} \oplus q_{15}p_1^{5} \oplus q_{17}p_1^{3} \oplus q_{19}p_1,
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
q_0 &= \mathbb{1}, \\
q_1 &= p_2p_3p_4p_5p_6, \\
q_2 &= p_2^2p_3^2p_4^2p_5^2p_6^2, \\
q_3 &= p_2^3p_3^3p_4^3p_5^3p_6, \\
q_4 &= p_2^4p_3^4p_4^4p_5^4, \\
q_5 &= p_2^5p_3^5p_4^5p_5^3p_6, \\
q_6 &= p_2^6p_3^6p_4^6p_5^2p_6^2, \\
q_7 &= p_2^7p_3^7p_4^5p_5^3p_6^3 \oplus p_2^7p_3^7p_4^7p_5p_6, \\
q_8 &= p_2^8p_3^8p_4^8 \oplus p_2^8p_3^8p_4^4p_5^4p_6^4, \\
q_9 &= p_2^9p_3^7p_4^5p_5^5p_6^5 \oplus p_2^9p_3^9p_4^7p_5p_6 \oplus p_2^9p_3^9p_4^5p_5^3p_6^3, \\
q_{10} &= p_2^{10}p_3^{10}p_4^6p_5^2p_6^2 \oplus p_2^{10}p_3^6p_4^6p_5^6p_6^6, \\
q_{11} &= p_2^9p_3^7p_4^7p_5^7p_6^7 \oplus p_2^{11}p_3^9p_4^7p_5^3p_6^3 \oplus p_2^{11}p_3^{11}p_4^5p_5^3p_6 \oplus p_2^{11}p_3^7p_4^7p_5^5p_6^5, \\
q_{12} &= p_2^{12}p_3^{12}p_4^4p_5^4 \oplus p_2^{12}p_3^8p_4^8p_5^4p_6^4 \oplus p_2^8p_3^8p_4^8p_5^8p_6^8, \\
q_{13} &= p_2^{11}p_3^9p_4^9p_5^5p_6^5 \oplus p_2^{13}p_3^{11}p_4^5p_5^5p_6 \oplus p_2^{13}p_3^{13}p_4^3p_5^3p_6 \oplus p_2^{13}p_3^9p_4^7p_5^5p_6^3 \oplus p_2^9p_3^9p_4^9p_5^7p_6^7, \\
q_{14} &= p_2^{14}p_3^{14}p_4^2p_5^2p_6^2 \oplus p_2^{14}p_3^{10}p_4^6p_5^6p_6^2 \oplus p_2^{10}p_3^{10}p_4^{10}p_5^6p_6^6, \\
q_{15} &= p_2^{13}p_3^{11}p_4^7p_5^7p_6^3 \oplus p_2^{15}p_3^{13}p_4^3p_5^3p_6^3 \oplus p_2^{15}p_3^{15}p_4p_5p_6 \oplus p_2^{15}p_3^9p_4^7p_5^7p_6 \\
&\quad \oplus p_2^{15}p_3^{11}p_4^5p_5^5p_6^3 \oplus p_2^{11}p_3^{11}p_4^9p_5^7p_6^5, \\
q_{16} &= p_2^{16}p_3^{16} \oplus p_2^{16}p_3^{12}p_4^4p_5^4p_6^4 \oplus p_2^{16}p_3^8p_4^8p_5^8 \oplus p_2^{12}p_3^{12}p_4^8p_5^8p_6^4, \\
q_{17} &= p_2^{15}p_3^{13}p_4^5p_5^5p_6^5 \oplus p_2^{15}p_3^9p_4^9p_5^9p_6 \oplus p_2^{13}p_3^{11}p_4^9p_5^9p_6^3 \oplus p_2^{13}p_3^{13}p_4^7p_5^7p_6^5, \\
q_{18} &= p_2^{14}p_3^{14}p_4^6p_5^6p_6^6 \oplus p_2^{14}p_3^{10}p_4^{10}p_5^{10}p_6^2, \\
q_{19} &= p_2^{13}p_3^{11}p_4^{11}p_5^{11}p_6, \\
q_{20} &= p_2^{12}p_3^{12}p_4^{12}p_5^{12}.
\end{aligned}$$

This can be regarded as a tropical algebraic equation of the twentieth degree in p_1. We can identify that $\frac{q_{i+1}}{q_i} \leq \frac{q_i}{q_{i-1}}$ for $i = 1, 2, ..., 15, 16$ and $\frac{q_{i+1}}{q_i} = \frac{q_i}{q_{i-1}}$ for $i = 1, 3, 5, ..., 13, 15$

When $p_3p_6 \leq p_4p_5$, the assumption of Corollary 1.4.17 is satisfied, that is, $\frac{q_{i+1}}{q_i} \leq \frac{q_i}{q_{i-1}}$ for $i = 17, 18, 19$ and the solution set is

$$\begin{aligned}
&\{p_2p_3p_4p_5p_6,\ p_2p_3p_4p_5p_6^{-1},\ p_2p_3p_4p_5^{-1}p_6,\ p_2p_3p_4^{-1}p_5p_6,\ p_2p_3^{-1}p_4p_5p_6, \\
&\ p_2^{-1}p_3p_4p_5p_6,\ p_2p_3p_4p_5^{-1}p_6^{-1},\ p_2p_3p_4^{-1}p_5p_6^{-1},\ p_2p_3^{-1}p_4p_5p_6^{-1},\ p_2^{-1}p_3p_4p_5p_6^{-1}\}.
\end{aligned}$$

This implies that it is sufficient to consider only the partitions:

$(\{p_1\},\{p_2,p_3,p_4,p_5,p_6\})$, $(\{p_1,p_6\},\{p_2,p_3,p_4,p_5\})$, $(\{p_1,p_5\},\{p_2,p_3,p_4,p_6\})$, $(\{p_1,p_4\},\{p_2,p_3,p_5,p_6\})$, $(\{p_1,p_3\},\{p_2,p_4,p_5,p_6\})$, $(\{p_1,p_2\},\{p_3,p_4,p_5,p_6\})$, $(\{p_1,p_5,p_6\},\{p_2,p_3,p_4\})$, $(\{p_1,p_4,p_6\},\{p_2,p_3,p_5\})$, $(\{p_1,p_3,p_6\},\{p_2,p_4,p_5\})$, $(\{p_1,p_2,p_6\},\{p_3,p_4,p_5\})$.

Note that $\frac{q_{i+1}}{q_i} = \frac{q_i}{q_{i-1}}$ for $i = 17, 19$.

When $p_3p_6 > p_4p_5$, the assumption is not satisfied. Considering the graph of the function $N^2(p_1p_2p_3p_4p_5p_6D^2)^{-1}$ we easily obtain the solution set. If $p_3p_6 > p_4p_5$ and $p_3 \le p_4p_5p_6$, then $\frac{q_{i+1}}{q_i} \le \frac{q_i}{q_{i-1}}$ for $i = 17$ and the solution set is

$$\{p_2p_3p_4p_5p_6,\ p_2p_3p_4p_5p_6^{-1},\ p_2p_3p_4p_5^{-1}p_6,\ p_2p_3p_4^{-1}p_5p_6,\ p_2p_3^{-1}p_4p_5p_6,\ p_2^{-1}p_3p_4p_5p_6,\ p_2p_3p_4p_5^{-1}p_6^{-1},\ p_2p_3p_4^{-1}p_5p_6^{-1},\ p_2p_3p_4^{-1}p_5^{-1}p_6\}.$$

This implies that it is sufficient to consider only the partitions:

$(\{p_1\},\{p_2,p_3,p_4,p_5,p_6\})$, $(\{p_1,p_6\},\{p_2,p_3,p_4,p_5\})$, $(\{p_1,p_5\},\{p_2,p_3,p_4,p_6\})$, $(\{p_1,p_4\},\{p_2,p_3,p_5,p_6\})$, $(\{p_1,p_3\},\{p_2,p_4,p_5,p_6\})$, $(\{p_1,p_2\},\{p_3,p_4,p_5,p_6\})$, $(\{p_1,p_5,p_6\},\{p_2,p_3,p_4\})$, $(\{p_1,p_4,p_6\},\{p_2,p_3,p_5\})$, $(\{p_1,p_4,p_5\},\{p_2,p_3,p_6\})$.

Note that $\frac{q_{18}}{q_{17}} = \frac{q_{17}}{q_{16}}$. If $p_3 > p_4p_5p_6$, then the solution set is

$$\{p_2p_3p_4p_5p_6,\ p_2p_3p_4p_5p_6^{-1},\ p_2p_3p_4p_5^{-1}p_6,\ p_2p_3p_4^{-1}p_5p_6,\ p_2p_3p_4p_5^{-1}p_6^{-1},\ p_2p_3p_4^{-1}p_5p_6^{-1},\ p_2p_3p_4^{-1}p_5^{-1}p_6,\ p_2p_3p_4^{-1}p_5^{-1}p_6^{-1}\}.$$

This implies that it is sufficient to consider only the partitions:

$(\{p_1\},\{p_2,p_3,p_4,p_5,p_6\})$, $(\{p_1,p_6\},\{p_2,p_3,p_4,p_5\})$, $(\{p_1,p_5\},\{p_2,p_3,p_4,p_6\})$, $(\{p_1,p_4\},\{p_2,p_3,p_5,p_6\})$, $(\{p_1,p_5,p_6\},\{p_2,p_3,p_4\})$, $(\{p_1,p_4,p_6\},\{p_2,p_3,p_5\})$, $(\{p_1,p_4,p_5\},\{p_2,p_3,p_6\})$, $(\{p_1,p_4,p_5,p_6\},\{p_2,p_3\})$.

Tables 3.5, 3.6, and 3.7 show the solutions for the cases $p_3p_6 \le p_4p_5$, ($p_3p_6 > p_4p_5$, $p_3 \le p_4p_5p_6$), and $p_3 > p_4p_5p_6$, respectively. Note that we write $\frac{i_2 \ldots i_k}{i_{k+1} \ldots i_n}$ for $\frac{p_{i_2} \cdots p_{i_k}}{p_{i_{k+1}} \cdots p_{i_n}}$. The solutions to the BalNPP are straightforwardly obtained from these solutions (Table 3.8).

<table>
<tr><th rowspan="2">Rank in decreasing order</th><th colspan="14">Solution</th></tr>
<tr><th>(1)</th><th>(2)</th><th>(3)</th><th>(4)</th><th>(5)</th><th>(6)</th><th>(7)</th><th>(8)</th><th>(9)</th><th>(10)</th><th>(11)</th><th>(12)</th><th>(13)</th><th>(14)</th></tr>
<tr><td>1</td><td colspan="14">23456</td></tr>
<tr><td>2</td><td colspan="14">$\frac{2345}{6}$</td></tr>
<tr><td>3</td><td colspan="14">$\frac{2346}{5}$</td></tr>
<tr><td>4</td><td colspan="5">$\frac{234}{56}$</td><td colspan="9">$\frac{2356}{4}$</td></tr>
<tr><td>5</td><td colspan="5">$\frac{2356}{4}$</td><td colspan="5">$\frac{234}{56}$</td><td colspan="4">$\frac{2456}{3}$</td></tr>
<tr><td>6</td><td colspan="2">$\frac{235}{46}$</td><td colspan="3">$\frac{2456}{3}$</td><td colspan="2">$\frac{235}{46}$</td><td colspan="3">$\frac{2456}{3}$</td><td colspan="3">$\frac{234}{56}$</td><td>$\frac{3456}{2}$</td></tr>
<tr><td>7</td><td colspan="2">$\frac{2456}{3}$</td><td colspan="2">$\frac{235}{46}$</td><td>$\frac{3456}{2}$</td><td colspan="2">$\frac{2456}{3}$</td><td colspan="2">$\frac{235}{46}$</td><td>$\frac{3456}{2}$</td><td colspan="2">$\frac{235}{46}$</td><td>$\frac{3456}{2}$</td><td>$\frac{234}{56}$</td></tr>
<tr><td>8</td><td>$\frac{245}{36}$</td><td>$\frac{3456}{2}$</td><td>$\frac{245}{36}$</td><td>$\frac{3456}{2}$</td><td>$\frac{235}{46}$</td><td>$\frac{245}{36}$</td><td>$\frac{3456}{2}$</td><td>$\frac{245}{36}$</td><td>$\frac{3456}{2}$</td><td>$\frac{235}{46}$</td><td>$\frac{245}{36}$</td><td>$\frac{3456}{2}$</td><td colspan="2">$\frac{235}{46}$</td></tr>
<tr><td>9</td><td>$\frac{3456}{2}$</td><td>$\frac{245}{36}$</td><td>$\frac{3456}{2}$</td><td colspan="2">$\frac{245}{36}$</td><td>$\frac{3456}{2}$</td><td>$\frac{245}{36}$</td><td>$\frac{3456}{2}$</td><td colspan="2">$\frac{245}{36}$</td><td>$\frac{3456}{2}$</td><td colspan="3">$\frac{245}{36}$</td></tr>
<tr><td>10</td><td colspan="14">$\frac{345}{26}$</td></tr>
</table>

(1) $p_5p_6 \le p_4,\ p_4p_6 \le p_3,\ p_3p_6 \le p_2$
(2) $p_5p_6 \le p_4,\ p_4p_6 \le p_3,\ p_2 \le p_3p_6$
(3) $p_5p_6 \le p_4,\ p_3 \le p_4p_6,\ p_3p_6 \le p_2$
(4) $p_5p_6 \le p_4,\ p_3 \le p_4p_6 \le p_2 \le p_3p_6$
(5) $p_5p_6 \le p_4,\ p_2 \le p_4p_6$
(6) $p_4 \le p_5p_6,\ p_4p_6 \le p_3,\ p_3p_6 \le p_2$
(7) $p_4 \le p_5p_6,\ p_4p_6 \le p_3,\ p_2 \le p_3p_6$
(8) $p_4 \le p_5p_6 \le p_3 \le p_4p_6,\ p_3p_6 \le p_2$
(9) $p_4 \le p_5p_6 \le p_3 \le p_4p_6 \le p_2 \le p_3p_6$
(10) $p_4 \le p_5p_6 \le p_3,\ p_2 \le p_4p_6$
(11) $p_3 \le p_5p_6,\ p_3p_6 \le p_2$
(12) $p_3 \le p_5p_6,\ p_4p_6 \le p_2 \le p_3p_6$
(13) $p_3 \le p_5p_6 \le p_2 \le p_4p_6$
(14) $p_2 \le p_5p_6$

Table 3.5: The solutions to the NPP ($n = 6,\ p_3p_6 \le p_4p_5$)

<table>
<tr><th rowspan="2">Rank in decreasing order</th><th colspan="16">Solution</th></tr>
<tr><th>(1)</th><th>(2)</th><th>(3)</th><th>(4)</th><th>(5)</th><th>(6)</th><th>(7)</th><th>(8)</th><th>(9)</th><th>(10)</th><th>(11)</th><th>(12)</th><th>(13)</th><th>(14)</th><th>(15)</th><th>(16)</th></tr>
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<tr><td>4</td><td colspan="6">$\frac{234}{56}$</td><td colspan="10">$\frac{2356}{4}$</td></tr>
<tr><td>5</td><td colspan="6">$\frac{2356}{4}$</td><td colspan="6">$\frac{234}{56}$</td><td colspan="4">$\frac{2456}{3}$</td></tr>
<tr><td>6</td><td colspan="3">$\frac{235}{46}$</td><td colspan="3">$\frac{2456}{3}$</td><td colspan="3">$\frac{235}{46}$</td><td colspan="3">$\frac{2456}{3}$</td><td colspan="3">$\frac{234}{56}$</td><td>$\frac{3456}{2}$</td></tr>
<tr><td>7</td><td>$\frac{236}{45}$</td><td colspan="2">$\frac{2456}{3}$</td><td colspan="2">$\frac{235}{46}$</td><td>$\frac{3456}{2}$</td><td>$\frac{236}{45}$</td><td colspan="2">$\frac{2456}{3}$</td><td colspan="2">$\frac{235}{46}$</td><td>$\frac{3456}{2}$</td><td colspan="2">$\frac{235}{46}$</td><td>$\frac{3456}{2}$</td><td>$\frac{234}{56}$</td></tr>
<tr><td>8</td><td>$\frac{2456}{3}$</td><td>$\frac{236}{45}$</td><td>$\frac{3456}{2}$</td><td>$\frac{236}{45}$</td><td>$\frac{3456}{2}$</td><td>$\frac{235}{46}$</td><td>$\frac{2456}{3}$</td><td>$\frac{236}{45}$</td><td>$\frac{3456}{2}$</td><td>$\frac{236}{45}$</td><td>$\frac{3456}{2}$</td><td>$\frac{235}{46}$</td><td>$\frac{236}{45}$</td><td>$\frac{3456}{2}$</td><td colspan="2">$\frac{235}{46}$</td></tr>
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</table>

(1) $p_5p_6 \leq p_4,\ p_4p_5 \leq p_3$
(2) $p_5p_6 \leq p_4,\ p_4p_6 \leq p_3 \leq p_4p_5 \leq p_2$
(3) $p_5p_6 \leq p_4,\ p_4p_6 \leq p_3,\ p_2 \leq p_4p_5$
(4) $p_5p_6 \leq p_4,\ p_3 \leq p_4p_6,\ p_4p_5 \leq p_2$
(5) $p_5p_6 \leq p_4,\ p_3 \leq p_4p_6 \leq p_2 \leq p_4p_5$
(6) $p_5p_6 \leq p_4,\ p_2 \leq p_4p_6$
(7) $p_4 \leq p_5p_6,\ p_4p_5 \leq p_3$
(8) $p_4 \leq p_5p_6,\ p_4p_6 \leq p_3 \leq p_4p_5 \leq p_2$
(9) $p_4 \leq p_5p_6,\ p_4p_6 \leq p_3,\ p_2 \leq p_4p_5$
(10) $p_4 \leq p_5p_6 \leq p_3 \leq p_4p_6,\ p_4p_5 \leq p_2$
(11) $p_4 \leq p_5p_6 \leq p_3 \leq p_4p_6 \leq p_2 \leq p_4p_5$
(12) $p_4 \leq p_5p_6 \leq p_3,\ p_2 \leq p_4p_6$
(13) $p_3 \leq p_5p_6,\ p_4p_5 \leq p_2$
(14) $p_3 \leq p_5p_6,\ p_4p_6 \leq p_2 \leq p_4p_5$
(15) $p_3 \leq p_5p_6 \leq p_2 \leq p_4p_6$
(16) $p_2 \leq p_5p_6$

Table 3.6: The solutions to the NPP ($n = 6$, $p_3p_6 > p_4p_5$, $p_3 \leq p_4p_5p_6$)

Rank in decreasing order	Solution	
	$p_5p_6 \leq p_4$	$p_4 \leq p_5p_6$
1	23456	
2	$\frac{2345}{6}$	
3	$\frac{2346}{5}$	
4	$\frac{234}{56}$	$\frac{2356}{4}$
5	$\frac{2356}{4}$	$\frac{234}{56}$
6	$\frac{235}{46}$	
7	$\frac{236}{45}$	
8	$\frac{23}{456}$	

Table 3.7: The solutions to the NPP ($n = 6$, $p_3 > p_4p_5p_6$)

Rank in decreasing order	Solution	
	$p_3p_6 \leq p_4p_5$	$p_3p_6 > p_4p_5,$
1	$p_2p_3p_4p_5^{-1}p_6^{-1}$	
2	$p_2p_3p_4^{-1}p_5p_6^{-1}$	
3	$p_2p_3^{-1}p_4p_5p_6^{-1}$	$p_2p_3p_4^{-1}p_5^{-1}p_6$
4	$p_2^{-1}p_3p_4p_5p_6^{-1}$	$-$

Table 3.8: The solutions to the BalNPP ($n = 6$)

Expressing the equation (3.8) in terms of the elementary symmetric tropical polynomials we have the equation

$$\begin{aligned}
&s_1^{32} \oplus s_1^{28}s_6^2 \oplus s_1^{24}s_5^4 \oplus s_1^{20}s_4^4s_6^2 \oplus s_1^{16}s_4^8 \oplus s_1^{16}s_3^4s_6^4 \oplus s_1^{12}s_3^4s_4^4s_6^2 \oplus s_1^{12}s_2^4s_6^6 \\
&\oplus s_1^8s_3^8s_5^4 \oplus s_1^8s_2^4s_4^4s_6^4 \oplus s_1^{12}s_6^8 \oplus s_1^8s_4^4s_6^6 \oplus s_1^4s_3^{12}s_6^2 \oplus s_1^4s_2^4s_3^4s_5^4s_6^2 \oplus s_1^4s_3^4s_5^4s_6^4 \\
&\oplus s_3^{16} \oplus s_2^4s_3^8s_6^4 \oplus s_2^8s_5^8 \oplus s_3^8s_6^6 \oplus s_2^4s_5^8s_6^2 \oplus s_5^{12} \\
=&s_1^{30}s_6 \oplus s_1^{26}s_5^2s_6 \oplus s_1^{22}s_4^2s_5^2s_6 \oplus s_1^{18}s_3^2s_4^2s_6^3 \oplus s_1^{18}s_4^6s_6 \oplus s_1^{14}s_2^2s_3^2s_6^5 \oplus s_1^{14}s_3^2s_4^6s_6 \\
&\oplus s_1^{14}s_3^4s_4^2s_6^3 \oplus s_1^{12}s_2^2s_6^7 \oplus s_1^{10}s_2^2s_3^2s_4^4s_6^3 \oplus s_1^{10}s_3^6s_4^2s_5^2s_6 \oplus s_1^{10}s_2^4s_4^2s_6^5 \oplus s_1^{10}s_4^2s_6^7 \\
&\oplus s_1^8s_2^2s_4^4s_6^5 \oplus s_1^6s_2^2s_3^6s_5^4s_6 \oplus s_1^6s_3^{10}s_5^2s_6 \oplus s_1^6s_2^4s_3^2s_4^2s_5^2s_6^3 \oplus s_1^6s_3^2s_4^2s_5^2s_6^5 \\
&\oplus s_1^4s_2^2s_3^4s_5^4s_6^3 \oplus s_1^2s_2^2s_3^{10}s_6^3 \oplus s_1^2s_3^{14}s_6 \oplus s_1^2s_2^6s_3^2s_5^6s_6 \oplus s_1^2s_2^4s_3^6s_5^2s_6^3 \\
&\oplus s_1^2s_2^2s_3^2s_5^6s_6^3 \oplus s_1^2s_3^6s_5^2s_6^5 \oplus s_2^2s_3^8s_6^5 \oplus s_2^6s_5^8s_6 \oplus s_2^2s_5^{10}s_6.
\end{aligned}$$

3.4 Summary

We found that tropical algebraic equations are useful in theoretical studies in the NPP. The decision version of the NPP, which is NP-complete, is equivalent to deciding whether one of given integers is a solution of a tropical algebraic equation with coefficients composed of other integers.

Our new approach gives all possible partitions such that the two sums are equal. The computations are simple but cumbersome. In fact, we get a tropical algebraic equation of the forty-fourth degree in p_1 for $n = 7$. It is laborious to derive the equation and the solution set for larger n. We also obtained expressions for the equations in terms of the elementary symmetric tropical polynomials. Through the expressions the theory of algebraic geometry may be applied to the problem.

Furthermore, our approach provides a new view point for the P versus NP problem. The cardinality of the solution set seems to be $\mathcal{O}(2^n)$. Since the solutions are sorted, by the binary search algorithm it is determined whether one of given integers is equal to a solution of the equation in polynomial steps. Therefore, if a solution set (a column in the table) concerned is specified in polynomial time, it is expected that P equals NP.

For the optimal version of the NPP we need to consider the graph of the function derived from the equation, so the optimal NPP is more difficult. This is a future problem.

Chapter 4

Conclusion

In this thesis, the new methods to tackle flow shops and a two-identical-parallel-machine problem based on max-plus algebra were presented.

First, we presented a new framework for flow shops. The framework, the machine representation, which associates a machine with a matrix is the dual of the existing approach associating a job with a matrix by Bouquard et al. and the first one which can deal with non-permutation flow shops based on max-plus algebra. Using the framework we found a new solvable condition in m-machine permutation flow shop problems to minimize makespan ($Fm|\mathrm{prmu}|C_{\max}$), which is an extension of known results. The result implies that an optimal job sequence is easily obtained in more complicated production lines. And we provided new simple proofs of some known results such as the duality between no-wait and no-idle constraints and the reduction of a no-wait flow shop to a traveling salesman problem. Moreover, we found a sufficient condition for an extension of Johnson's rule by considering a max-plus algebraic analogue of a proposition in linear algebra and presented the problems in linear algebra that correspond to some flow shops.

Max-plus algebra is very useful for flow shop problems. We expect that the two dual frameworks based on max-plus algebra contribute to the development of the theory of flow shops, especially non-permutation flow shops. It is a future problem to formalize various flow shops, including stochastic models, in the job representation and/or the machine representation and to analyze them. Furthermore, since there exist a lot of results in linear algebra, it is also an interesting problem to investigate the application of the results to flow shop problems.

Secondly, we presented a new approach to reveal the mathematical structure of the decision version of the two-identical-parallel-machine problem, or the number partition problem (NPP), which is NP-complete. We showed that the problem is equivalent to deciding whether one of given integers is a solution of a tropical algebraic equation with coefficients composed of other integers. Our approach gives all possible partitions such that the two sums are equal. The computations are simple but cumbersome. In fact, we get a tropical algebraic equation of the forty-fourth degree in p_1 for $n = 7$. It is laborious to derive the equation and the solution set for larger n. We also obtained expressions for the equations in terms of the elementary symmetric tropical polynomials. Through the expressions the theory of algebraic geometry may be applied to the problem.

Furthermore, our approach provides a new view point for the P versus NP problem. The cardinality of the solution set seems to be $\mathcal{O}(2^n)$. Since the solutions are sorted, by the binary search algorithm it is determined whether one of given integers is equal to a solution of the equation in polynomial steps. Therefore, if a solution set (a column in the table) concerned is specified in polynomial time, it is expected that P equals NP.

Our methods and results are important from theoretical and practical point of view. It can be expected that the development of our methods will generate new results and contribute to higher efficiency in production systems in the future.

Appendix A

An Extension of the Elementary Symmetric Tropical Polynomials

A.1 Introduction

We consider r-symmetric tropical polynomials in nr variables. We define basic r-symmetric tropical polynomials based on Matsui [66]. The basic polynomials are tropicalizations of the symmetric polynomials presented by Weyl [67] and a part of the elementary r-symmetric tropical polynomials defined by Carlsson and Kališnik [59].

The basic 2-symmetric tropical polynomials are not independent and many inequalities between the polynomials hold. Using the inequalities we show that the basic 2-symmetric polynomials give coordinates on $\mathbb{R}^{2n}/S_n$ (Theorem A.3.4). This result is better than the one by Carlsson and Kališnik. Moreover, for vectors such that at least a vector (Proposition A.4.1) or a difference vector (Proposition A.4.2) has distinct components, the basic polynomials separate orbits even for $r \geq 3$.

A.2 Definition of basic r-symmetric tropical polynomials

In order to define basic r-symmetric tropical polynomials we prepare the forms:

$$\begin{aligned}
\Phi_1\left(u^{(\alpha_1)}\right) &= \bigoplus_{k_1} u_{k_1}^{(\alpha_1)}, \\
\Phi_2\left(u^{(\alpha_1)}, u^{(\alpha_2)}\right) &= \bigoplus_{k_1 \neq k_2} u_{k_1}^{(\alpha_1)} \odot u_{k_2}^{(\alpha_2)}, \\
\Phi_3\left(u^{(\alpha_1)}, u^{(\alpha_2)}, u^{(\alpha_3)}\right) &= \bigoplus_{k_1, k_2, k_3 \text{ all} \neq} u_{k_1}^{(\alpha_1)} \odot u_{k_2}^{(\alpha_2)} \odot u_{k_3}^{(\alpha_3)}, \\
&\vdots \\
\Phi_n\left(u^{(\alpha_1)}, u^{(\alpha_2)}, \dots, u^{(\alpha_n)}\right) &= \bigoplus_{k_1, k_2, \dots, k_n \text{ all} \neq} u_{k_1}^{(\alpha_1)} \odot u_{k_2}^{(\alpha_2)} \odot \cdots \odot u_{k_n}^{(\alpha_n)},
\end{aligned}$$

where $u^{(\alpha)} = \left(u_1^{(\alpha)}, u_2^{(\alpha)}, \dots, u_n^{(\alpha)}\right)^{\mathrm{t}}$.

Definition A.2.1. Let $x^{(1)}, \dots, x^{(r)}$ be n-dimensional vectors of variables. The basic r-symmetric tropical polynomials in nr variables $x_1^{(1)}, \dots, x_n^{(1)}, \dots, x_1^{(r)}, \dots, x_n^{(r)}$ are defined by the formulas

$$\begin{gathered}
\Phi_0, \\
\Phi_1\left(x^{(\alpha_1)}\right), \quad \alpha_1 = 1, \dots, r, \\
\vdots \\
\Phi_l\left(x^{(\alpha_1)}, \dots, x^{(\alpha_l)}\right), \quad \alpha_1, \dots, \alpha_l = 1, \dots, r, \\
\vdots \\
\Phi_n\left(x^{(\alpha_1)}, x^{(\alpha_2)}, \dots, x^{(\alpha_n)}\right), \quad \alpha_1, \alpha_2, \dots, \alpha_n = 1, \dots, r,
\end{gathered}$$

where $\Phi_0 = \mathbb{1} (= 0)$.

The basic r-symmetric tropical polynomials are tropicalizations of the typical basic invariants in Weyl's book [67].

Example A.2.2. Let $r = 1$ and let us write x instead of $x^{(1)}$. The basic 1-symmetric tropical polynomials are as follows:

$$\begin{aligned}
\Phi_0 &= \mathbb{1}, \\
\Phi_1(x) &= x_1 \oplus x_2 \oplus \cdots \oplus x_n, \\
\Phi_2(x, x) &= x_1 \odot x_2 \oplus x_1 \odot x_3 \oplus \cdots \oplus x_{n-1} \odot x_n, \\
&\vdots \\
\Phi_n(x, x, \dots, x) &= x_1 \odot x_2 \odot \cdots \odot x_n.
\end{aligned}$$

These are the same as the elementary symmetric tropical polynomials defined by Carlsson and Kališnik [59] except for Φ_0.

Example A.2.3. Let $n = 3$ and $r = 3$, and let us write x, y, z instead of $x^{(1)}, x^{(2)}, x^{(3)}$, respectively. The basic 3-symmetric tropical polynomials are as follows:

$$
\begin{aligned}
\Phi_0 &= \mathbb{1}, \\
\Phi_1(x) &= x_1 \oplus x_2 \oplus x_3, \\
\Phi_1(y) &= y_1 \oplus y_2 \oplus y_3, \\
\Phi_1(z) &= z_1 \oplus z_2 \oplus z_3, \\
\Phi_2(x, x) &= x_1 \odot x_2 \oplus x_1 \odot x_3 \oplus x_2 \odot x_3, \\
\Phi_2(y, y) &= y_1 \odot y_2 \oplus y_1 \odot y_3 \oplus y_2 \odot y_3, \\
\Phi_2(z, z) &= z_1 \odot z_2 \oplus z_1 \odot z_3 \oplus z_2 \odot z_3, \\
\Phi_2(x, y) &= x_1 \odot y_2 \oplus x_1 \odot y_3 \oplus x_2 \odot y_1 \oplus x_2 \odot y_3 \oplus x_3 \odot y_1 \oplus x_3 \odot y_2, \\
\Phi_2(x, z) &= x_1 \odot z_2 \oplus x_1 \odot z_3 \oplus x_2 \odot z_1 \oplus x_2 \odot z_3 \oplus x_3 \odot z_1 \oplus x_3 \odot z_2, \\
\Phi_2(y, z) &= y_1 \odot z_2 \oplus y_1 \odot z_3 \oplus y_2 \odot z_1 \oplus y_2 \odot z_3 \oplus y_3 \odot z_1 \oplus y_3 \odot z_2, \\
\Phi_3(x, x, x) &= x_1 \odot x_2 \odot x_3, \\
\Phi_3(y, y, y) &= y_1 \odot y_2 \odot y_3, \\
\Phi_3(z, z, z) &= z_1 \odot z_2 \odot z_3, \\
\Phi_3(x, x, y) &= x_1 \odot x_2 \odot y_3 \oplus x_1 \odot x_3 \odot y_2 \oplus x_2 \odot x_3 \odot y_1, \\
\Phi_3(x, x, z) &= x_1 \odot x_2 \odot z_3 \oplus x_1 \odot x_3 \odot z_2 \oplus x_2 \odot x_3 \odot z_1, \\
\Phi_3(y, y, x) &= y_1 \odot y_2 \odot x_3 \oplus y_1 \odot y_3 \odot x_2 \oplus y_2 \odot y_3 \odot x_1, \\
\Phi_3(y, y, z) &= y_1 \odot y_2 \odot z_3 \oplus y_1 \odot y_3 \odot z_2 \oplus y_2 \odot y_3 \odot z_1, \\
\Phi_3(z, z, x) &= z_1 \odot z_2 \odot x_3 \oplus z_1 \odot z_3 \odot x_2 \oplus z_2 \odot z_3 \odot x_1, \\
\Phi_3(z, z, y) &= z_1 \odot z_2 \odot y_3 \oplus z_1 \odot z_3 \odot y_2 \oplus z_2 \odot z_3 \odot y_1, \\
\Phi_3(x, y, z) &= x_1 \odot y_2 \odot z_3 \oplus x_1 \odot y_3 \odot z_2 \oplus x_2 \odot y_1 \odot z_3 \oplus x_2 \odot y_3 \odot z_1 \\
&\quad \oplus x_3 \odot y_1 \odot z_2 \oplus x_3 \odot y_2 \odot z_1.
\end{aligned}
$$

Proposition A.2.4. *Let* $x^{(1)} = \left(x_1^{(1)}, ..., x_n^{(1)}\right)^{\mathrm{t}}, \ldots, x^{(r)} = \left(x_1^{(r)}, ..., x_n^{(r)}\right)^{\mathrm{t}}$ *and* l *be an integer between 1 and* n. *The basic* r*-symmetric tropical polynomials in* nr *variables can be written in terms of those in* $(n-1)r$ *variables as follows:*

$$
\begin{aligned}
\Phi_l\left(x^{(\alpha_1)}, \ldots, x^{(\alpha_l)}\right) &= \Phi_l\left(x'^{(\alpha_1)}, \ldots, x'^{(\alpha_l)}\right) \\
&\quad \oplus \bigoplus_{1 \le j \le l} x_n^{(\alpha_j)} \odot \Phi_{l-1}\left(x'^{(\alpha_1)}, \ldots, x'^{(\alpha_{j-1})}, x'^{(\alpha_{j+1})}, \ldots, x'^{(\alpha_l)}\right),
\end{aligned}
$$

where $x'^{(\alpha)} = \left(x_1^{(\alpha)}, \ldots, x_{n-1}^{(\alpha)}\right)^{\mathrm{t}}$ *and* $\Phi_n\left(x'^{(\alpha_1)}, \ldots, x'^{(\alpha_n)}\right) = \mathbb{0}$.

Proof. By definition we have

$$
\Phi_l\left(x^{(\alpha_1)}, \ldots, x^{(\alpha_l)}\right) = \bigoplus_{k_1, \ldots, k_l \text{ all} \neq} x_{k_1}^{(\alpha_1)} \odot \cdots \odot x_{k_l}^{(\alpha_l)}. \tag{A.1}
$$

For $1 \leq l \leq n-1$ we can rewrite this as

$$\bigoplus_{\substack{k_1,\dots,k_l \leq n-1 \\ \text{and all} \neq}} x_{k_1}^{(\alpha_1)} \odot \cdots \odot x_{k_l}^{(\alpha_l)}$$
$$\oplus \bigoplus_{1 \leq j \leq l} \bigoplus_{\substack{k_1,\dots,k_{j-1},k_{j+1},\dots,k_l \leq n-1 \\ \text{and all} \neq}} x_{k_1}^{(\alpha_1)} \odot \cdots \odot x_{k_{j-1}}^{(\alpha_{j-1})} \odot x_n^{(\alpha_j)} \odot x_{k_{j+1}}^{(\alpha_{j+1})} \odot \cdots \odot x_{k_l}^{(\alpha_l)},$$

which equals

$$\Phi_l\left(x'^{(\alpha_1)}, \dots, x'^{(\alpha_l)}\right)$$
$$\oplus \bigoplus_{1 \leq j \leq l} x_n^{(\alpha_j)} \odot \Phi_{l-1}\left(x'^{(\alpha_1)}, \dots, x'^{(\alpha_{j-1})}, x'^{(\alpha_{j+1})}, \dots, x'^{(\alpha_l)}\right).$$

For $l = n$ we can rewrite the right-hand side of the formula (A.1) as

$$\bigoplus_{1 \leq j \leq n} \bigoplus_{\substack{k_1,\dots,k_{j-1},k_{j+1},\dots,k_n \leq n-1 \\ \text{and all} \neq}} x_{k_1}^{(\alpha_1)} \odot \cdots \odot x_{k_{j-1}}^{(\alpha_{j-1})} \odot x_n^{(\alpha_j)} \odot x_{k_{j+1}}^{(\alpha_{j+1})} \odot \cdots \odot x_{k_n}^{(\alpha_n)},$$

which equals

$$\bigoplus_{1 \leq j \leq n} x_n^{(\alpha_j)} \odot \Phi_{n-1}\left(x'^{(\alpha_1)}, \dots, x'^{(\alpha_{j-1})}, x'^{(\alpha_{j+1})}, \dots, x'^{(\alpha_n)}\right).$$

These complete the proof. □

Let us introduce a notation for the basic r-symmetric tropical polynomials. We write $b_{q_1,q_2,\dots,q_r}\left(x^{(1)}, \dots, x^{(r)}\right)$ (or simply $b_{q_1,q_2,\dots,q_r}$ when the arguments are clear from the context) instead of

$$\Phi_k\left(\underbrace{x^{(1)}, \dots, x^{(1)}}_{q_1}, \underbrace{x^{(2)}, \dots, x^{(2)}}_{q_2}, \dots, \underbrace{x^{(r)}, \dots, x^{(r)}}_{q_r}\right),$$

where $q_1 + q_2 + \cdots + q_r = k$ and the index q_α for $\alpha \in \{1, \dots, r\}$ indicates the number of arguments $x^{(\alpha)}$. For example, we write $b_{2,1,0}(x^{(1)}, x^{(2)}, x^{(3)})$ instead of $\Phi_3\left(x^{(1)}, x^{(1)}, x^{(2)}\right)$.

Corollary A.2.5. *Let $x = (x_1, \dots, x_n)^{\mathrm{t}}$, $y = (y_1, \dots, y_n)^{\mathrm{t}}$, and p and q be integers. The basic 2-symmetric tropical polynomials $b_{p,q}$ in $2n$ variables can be written in terms of those $b'_{r,s}$ in $2(n-1)$ variables $x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}$ as follows:*

$$b_{p,q} = b'_{p,q} \oplus x_n \odot b'_{p-1,q} \oplus y_n \odot b'_{p,q-1},$$

where by definition $b_{p,q} = \mathbb{0}$ when $p < 0$, $q < 0$ or $p + q > n$.

Proof. For p and q such that $p \geq 0$, $q \geq 0$ and $p + q \leq n$, straightforward by Proposition A.2.4.

If $p < 0$, $q < 0$ or $p + q > n$, then both sides are $\mathbb{0}$. □

A.3 Properties of basic 2-symmetric tropical polynomials

Let $r = 2$ in this section. We present inequalities between the basic 2-symmetric tropical polynomials. Using the inequalities, we show that the basic polynomials give coordinates on $\mathbb{R}^{2n}/S_n$, which is the main result in this paper.

Proposition A.3.1. *Let $n \geq 1$, p and q be integers, and $b_{p,q}$ be the basic 2-symmetric tropical polynomial in $2n$ variables. Then the following inequalities hold:*

$$b_{p,q-1} \odot b_{p-1,q} \geq b_{p,q} \odot b_{p-1,q-1}, \tag{A.2a}$$

$$b_{p,q} \odot b_{p,q-1} \geq b_{p+1,q-1} \odot b_{p-1,q}, \tag{A.2b}$$

$$b_{p,q} \odot b_{p-1,q} \geq b_{p-1,q+1} \odot b_{p,q-1}. \tag{A.2c}$$

Proof. Let $2n$ variables be denoted by (x, y), where $x = (x_1, \ldots, x_n)^{\mathrm{t}}$, $y = (y_1, \ldots, y_n)^{\mathrm{t}}$. By exchanging the roles of x and y in the inequality (A.2b) we obtain the inequality (A.2c). Therefore, we show that the inequalities (A.2a) and (A.2b) hold.

We will prove the inequalities by induction on n. The base case is $n = 1$. For p and q such that $p \leq 0$, $q \leq 0$ or $p + q > 1$, that is, any integers p and q, the right-hand sides of (A.2a) and (A.2b) are $\mathbb{0}$, so the two inequalities hold.

Now assume that the proposition is true for $n - 1$, that is,

$$b'_{p,q-1} \odot b'_{p-1,q} \geq b'_{p,q} \odot b'_{p-1,q-1}, \tag{A.3a}$$

$$b'_{p,q} \odot b'_{p,q-1} \geq b'_{p+1,q-1} \odot b'_{p-1,q}, \tag{A.3b}$$

$$b'_{p,q} \odot b'_{p-1,q} \geq b'_{p-1,q+1} \odot b'_{p,q-1},$$

where each b' means the basic 2-symmetric tropical polynomial in $2(n-1)$ variables. Using the inequalities we prepare some inequalities. Replacing p by $p - 1$ in the inequalities (A.3), we can write the inequalities in the forms

$$b'_{p-1,q-1} \odot b'_{p-2,q} \geq b'_{p-1,q} \odot b'_{p-2,q-1}, \tag{A.4a}$$

$$b'_{p-1,q} \odot b'_{p-1,q-1} \geq b'_{p,q-1} \odot b'_{p-2,q}, \tag{A.4b}$$

$$b'_{p-1,q} \odot b'_{p-2,q} \geq b'_{p-2,q+1} \odot b'_{p-1,q-1}. \tag{A.4c}$$

Similarly replacing q by $q - 1$ in the inequalities (A.3) we have

$$b'_{p,q-2} \odot b'_{p-1,q-1} \geq b'_{p,q-1} \odot b'_{p-1,q-2}, \tag{A.5a}$$

$$b'_{p,q-1} \odot b'_{p,q-2} \geq b'_{p+1,q-2} \odot b'_{p-1,q-1}, \tag{A.5b}$$

$$b'_{p,q-1} \odot b'_{p-1,q-1} \geq b'_{p-1,q} \odot b'_{p,q-2}. \tag{A.5c}$$

Multiplying (Adding in the conventional algebra) up the two inequalities (A.3a) and (A.4a), (A.3b) and (A.4b), (A.3a) and (A.5a), (A.3a) and (A.3b), (A.4a) and (A.4b), (A.5a) and (A.5c), (A.5b) and (A.5c), and (A.4b) and (A.5b), we obtain

$$
\begin{aligned}
b'_{p,q-1}\odot b'_{p-2,q} &\geq b'_{p,q}\odot b'_{p-2,q-1}, && \text{(A.6a)}\\
b'_{p,q}\odot b'_{p-1,q-1} &\geq b'_{p+1,q-1}\odot b'_{p-2,q}, && \text{(A.6b)}\\
b'_{p,q-2}\odot b'_{p-1,q} &\geq b'_{p,q}\odot b'_{p-1,q-2}, && \text{(A.6c)}\\
b'^{2}_{p,q-1} &\geq b'_{p+1,q-1}\odot b'_{p-1,q-1}, && \text{(A.6d)}\\
b'^{2}_{p-1,q-1} &\geq b'_{p,q-1}\odot b'_{p-2,q-1}, && \text{(A.6e)}\\
b'^{2}_{p-1,q-1} &\geq b'_{p-1,q}\odot b'_{p-1,q-2}, && \text{(A.6f)}\\
b'^{2}_{p,q-1} &\geq b'_{p+1,q-2}\odot b'_{p-1,q}, && \text{(A.6g)}\\
b'_{p-1,q}\odot b'_{p,q-2} &\geq b'_{p+1,q-2}\odot b'_{p-2,q}. && \text{(A.6h)}
\end{aligned}
$$

Now let us consider the inequality (A.2a). It follows from Corollary A.2.5 that

$$
\begin{aligned}
&b_{p,q-1}\odot b_{p-1,q}\\
&= \left(b'_{p,q-1}\oplus x_n\odot b'_{p-1,q-1}\oplus y_n\odot b'_{p,q-2}\right)\odot\left(b'_{p-1,q}\oplus x_n\odot b'_{p-2,q}\oplus y_n\odot b'_{p-1,q-1}\right)\\
&= b'_{p,q-1}\odot b'_{p-1,q}\oplus x_n\odot\left(b'_{p,q-1}\odot b'_{p-2,q}\oplus b'_{p-1,q-1}\odot b'_{p-1,q}\right)\\
&\quad\oplus y_n\odot\left(b'_{p,q-1}\odot b'_{p-1,q-1}\oplus b'_{p,q-2}\odot b'_{p-1,q}\right)\oplus x_n^2\odot b'_{p-1,q-1}\odot b'_{p-2,q}\\
&\quad\oplus y_n^2\odot b'_{p,q-2}\odot b'_{p-1,q-1}\oplus x_n\odot y_n\odot\left(b'^{2}_{p-1,q-1}\oplus b'_{p,q-2}\odot b'_{p-2,q}\right), \qquad \text{(A.7a)}
\end{aligned}
$$

and

$$
\begin{aligned}
&b_{p,q}\odot b_{p-1,q-1}\\
&= \left(b'_{p,q}\oplus x_n\odot b'_{p-1,q}\oplus y_n\odot b'_{p,q-1}\right)\odot\left(b'_{p-1,q-1}\oplus x_n\odot b'_{p-2,q-1}\oplus y_n\odot b'_{p-1,q-2}\right)\\
&= b'_{p,q}\odot b'_{p-1,q-1}\oplus x_n\odot\left(b'_{p,q}\odot b'_{p-2,q-1}\oplus b'_{p-1,q-1}\odot b'_{p-1,q}\right)\\
&\quad\oplus y_n\odot\left(b'_{p,q-1}\odot b'_{p-1,q-1}\oplus b'_{p,q}\odot b'_{p-1,q-2}\right)\oplus x_n^2\odot b'_{p-1,q}\odot b'_{p-2,q-1}\\
&\quad\oplus y_n^2\odot b'_{p,q-1}\odot b'_{p-1,q-2}\oplus x_n\odot y_n\odot\left(b'_{p,q-1}\odot b'_{p-2,q-1}\oplus b'_{p-1,q}\odot b'_{p-1,q-2}\right). \qquad \text{(A.7b)}
\end{aligned}
$$

The expressions (A.7a) and (A.7b) are tropical polynomials of x_n and y_n. By comparing the coefficients of $\mathbb{1}, x_n, y_n, x_n^2, y_n^2,\ x_n\odot y_n$ in both expressions, we observe that all the coefficients of (A.7a) are greater than or equal to those of (A.7b), because (A.3a), (A.6a), (A.6c), (A.4a), (A.5a), (A.6e), and (A.6f) hold. Hence, the inequality (A.2a) holds.

Let us consider the inequality (A.2b). We similarly have

$$
\begin{aligned}
&b_{p,q}\odot b_{p,q-1}\\
&= b'_{p,q}\odot b'_{p,q-1}\oplus x_n\odot\left(b'_{p,q}\odot b'_{p-1,q-1}\oplus b'_{p-1,q}\odot b'_{p,q-1}\right)\\
&\quad\oplus y_n\odot\left(b'_{p,q}\odot b'_{p,q-2}\oplus b'^{2}_{p,q-1}\right)\oplus x_n^2\odot b'_{p-1,q}\odot b'_{p-1,q-1}\\
&\quad\oplus y_n^2\odot b'_{p,q-1}\odot b'_{p,q-2}\oplus x_n\odot y_n\odot\left(b'_{p-1,q}\odot b'_{p,q-2}\oplus b'_{p,q-1}\odot b'_{p-1,q-1}\right), \qquad \text{(A.8a)}\\
&b_{p+1,q-1}\odot b_{p-1,q}\\
&= b'_{p+1,q-1}\odot b'_{p-1,q}\oplus x_n\odot\left(b'_{p+1,q-1}\odot b'_{p-2,q}\oplus b'_{p-1,q}\odot b'_{p,q-1}\right)\\
&\quad\oplus y_n\odot\left(b'_{p+1,q-1}\odot b'_{p-1,q-1}\oplus b'_{p+1,q-2}\odot b'_{p-1,q}\right)\oplus x_n^2\odot b'_{p,q-1}\odot b'_{p-2,q}\\
&\quad\oplus y_n^2\odot b'_{p+1,q-2}\odot b'_{p-1,q-1}\oplus x_n\odot y_n\odot\left(b'_{p+1,q-2}\odot b'_{p-2,q}\oplus b'_{p,q-1}\odot b'_{p-1,q-1}\right). \qquad \text{(A.8b)}
\end{aligned}
$$

By comparing the coefficients of $\mathbb{1}, x_n, y_n, x_n^2, y_n^2, x_n \odot y_n$, we observe that all the coefficients of (A.8a) are greater than or equal to those of (A.8b), because (A.3b), (A.6b), (A.6d), (A.6g), (A.4b), (A.5b), and (A.6h) hold. Hence, the inequality (A.2b) holds, completing the proof. □

Corollary A.3.2. *Let $n \geq 1$, p and q be integers, and $b_{p,q}$ be the basic 2-symmetric tropical polynomial in $2n$ variables. Then the following inequalities hold:*

$$\begin{aligned} b_{p,q-1}^2 &\geq b_{p+1,q-1} \odot b_{p-1,q-1}, && \text{(A.9a)} \\ b_{p-1,q}^2 &\geq b_{p-1,q+1} \odot b_{p-1,q-1}, && \text{(A.9b)} \\ b_{p,q}^2 &\geq b_{p+1,q-1} \odot b_{p-1,q+1}, && \text{(A.9c)} \\ b_{p-1,q}^3 &\geq b_{p,q} \odot b_{p-2,q+1} \odot b_{p-1,q-1}, && \text{(A.9d)} \\ b_{p,q-1}^3 &\geq b_{p,q} \odot b_{p+1,q-2} \odot b_{p-1,q-1}, && \text{(A.9e)} \\ b_{p,0} \odot b_{0,1} &\geq b_{p,1}, && \text{(A.9f)} \\ b_{0,q} \odot b_{1,0} &\geq b_{1,q}. && \text{(A.9g)} \end{aligned}$$

Proof. Let $2n$ variables be denoted by (x, y), where x and y are n-dimensional column vectors. By exchanging the roles of x and y in the inequalities (A.9a), (A.9d) and (A.9f), we obtain the inequalities (A.9b), (A.9e) and (A.9g), respectively. Therefore, we show that the inequalities (A.9a), (A.9c), (A.9d) and (A.9f) hold.

First we prove the inequalities (A.9a) and (A.9c). Multiplying (Adding in the conventional algebra) up the two inequalities (A.2a) and (A.2b), and (A.2b) and (A.2c), we obtain them.

Next we prove the inequality (A.9d). Replacing p by $p-1$ in the inequality (A.9c), we have

$$b_{p-1,q}^2 \geq b_{p,q-1} \odot b_{p-2,q+1}.$$

Multiplying up this and the inequality (A.2a), we obtain it.

Finally we prove the inequality (A.9f). If $p < 0$ or $p \geq n$, then the right-hand side is equal to $\mathbb{0}$, so the inequality holds. Consider the inequalities obtained by letting $q = 1$ in the inequality (A.2a) for p between 0 and $(n-1)$. Multiplying up the inequalities

$$\begin{aligned} b_{p,0} \odot \cancel{b_{p-1,1}} &\geq b_{p,1} \odot \cancel{b_{p-1,0}}, \\ \cancel{b_{p-1,0}} \odot \cancel{b_{p-2,1}} &\geq \cancel{b_{p-1,1}} \odot \cancel{b_{p-2,0}}, \\ &\vdots \\ \cancel{b_{2,0}} \odot \cancel{b_{1,1}} &\geq \cancel{b_{2,1}} \odot \cancel{b_{1,0}}, \\ \cancel{b_{1,0}} \odot b_{0,1} &\geq \cancel{b_{1,1}} \odot b_{0,0}, \end{aligned}$$

we have $b_{p,0} \odot b_{0,1} \geq b_{p,1} \odot b_{0,0} = b_{p,1}$. □

In order to prove our main theorem, we prepare the next proposition.

Hereinafter we use $+$ or $-$ instead of $\odot$.

Proposition A.3.3. *Let*

$$z \oplus a_1 = c_1, \ldots, \; z \oplus a_m = c_m$$

be equations in z. If $z \geq a_j$ for some $j \in \{1, \ldots, m\}$, then

$$z = \min[c_1, \ldots, c_m].$$

Proof. Since $z \leq c_i$ for every $i \in \{1, \ldots, m\}$, we have $z \leq \min[c_1, \ldots, c_m]$. Suppose that $z \geq a_j$ for some $j \in \{1, \ldots, m\}$. Then $z = c_j$ and we have $\min[c_1, \ldots, c_m] \leq c_j = z$. Therefore, we obtain $z = \min[c_1, \ldots, c_m]$. □

Note that if $mz \geq a_1 + \cdots + a_m$, where mz is the product of m and z in the conventional algebra, then the assumption of Proposition A.3.3 is satisfied.

Theorem A.3.4. *Let p and q be integers and $b_{p,q}$ be the basic 2-symmetric tropical polynomial in $2n$ variables. Let $x, y, \bar{x}, \bar{y}$ be n-dimensional vectors, and $[(x, y)]$ and $[(\bar{x}, \bar{y})]$ be two orbits under the row permutation action on $\mathbb{R}^{2n}$. If*

$$b_{p,q}(x, y) = b_{p,q}(\bar{x}, \bar{y})$$

for every p and q, then

$$[(x, y)] = [(\bar{x}, \bar{y})].$$

Proof. Let $x = (x_1, \ldots, x_n)^{\mathrm{t}}, y = (y_1, \ldots, y_n)^{\mathrm{t}}, \bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^{\mathrm{t}}$, and $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n)^{\mathrm{t}}$, such that $b_{p,q}(x, y) = b_{p,q}(\bar{x}, \bar{y})$ for every p and q. Suppose that $x_1 \geq x_2 \geq \cdots \geq x_n,\ \bar{x}_1 \geq \bar{x}_2 \geq \cdots \geq \bar{x}_n$, if $x_i = x_{i+1}$ for some $i \in \{1, \ldots, n-1\}$, then $y_i \leq y_{i+1}$ and if $\bar{x}_i = \bar{x}_{i+1}$ for some $i \in \{1, \ldots, n-1\}$, then $\bar{y}_i \leq \bar{y}_{i+1}$. We will prove that $x = \bar{x}$ and $y = \bar{y}$.

We use induction on n. When $n = 1$, we have

$$x_1 = b_{1,0}(x, y) = b_{1,0}(\bar{x}, \bar{y}) = \bar{x}_1,$$
$$y_1 = b_{0,1}(x, y) = b_{0,1}(\bar{x}, \bar{y}) = \bar{y}_1.$$

Hence, the theorem is true.

Now assume that the theorem is true for $n - 1$. First we show that $x_1 = \bar{x}_1$, $x_2 = \bar{x}_2, \ldots, x_n = \bar{x}_n$. Applying $b_{1,0}$, we get $x_1 = \bar{x}_1$. Applying $b_{2,0}$, we get $x_1 + x_2 = \bar{x}_1 + \bar{x}_2$, and from here $x_2 = \bar{x}_2$ and so on. Finally, applying $b_{n,0}$ yields $x_n = \bar{x}_n$.

Let $s = \max\{i \in \{1, \ldots, n\} \colon b_{i-1,1} = b_{i-1,0} + b_{0,1}\}$. Next we show that

$$y_s = b_{0,1} = \bar{y}_s.$$

To do so, we show that $\bigoplus_{s \leq k \leq n} y_k = b_{0,1}$ at first. When $s = 1$, this is obvious. Suppose that $s \geq 2$. By definition of $b_{p,q}$ we have

$$b_{s-1,1} = \bigoplus_{k_1, \ldots, k_s \text{ all } \neq} \left(x_{k_1} + \cdots + x_{k_{s-1}} + y_{k_s}\right).$$

Using the ordering of x we rewrite it as

$$\left(x_1 + \ldots + x_{s-1} + \bigoplus_{s \leq k \leq n} y_k\right) \oplus \bigoplus_{1 \leq k \leq s-1} \left(x_1 + \cdots + x_{k-1} + x_{k+1} + \cdots x_s + y_k\right).$$

On the other hand, by definition of s we have

$$b_{s-1,1} = b_{s-1,0} + b_{0,1} = x_1 + \ldots + x_{s-1} + b_{0,1}.$$

Therefore, the following holds:

$$\begin{aligned}
&\left(x_1 + \ldots + x_{s-1} + \bigoplus_{s\le k\le n} y_k\right) \\
&\quad \oplus \bigoplus_{1\le k\le s-1} (x_1 + \cdots + x_{k-1} + x_{k+1} + \cdots x_s + y_k) \\
&= x_1 + \ldots + x_{s-1} + b_{0,1}.
\end{aligned}$$

If $\bigoplus_{s\le k\le n} y_k < b_{0,1}$, then

$$\bigoplus_{1\le k\le s-1} (x_1 + \cdots + x_{k-1} + x_{k+1} + \cdots x_s + y_k) = x_1 + \ldots + x_{s-1} + b_{0,1}.$$

Hence, for some $l \in \{1, \ldots, s-1\}$

$$x_1 + \cdots + x_{l-1} + x_{l+1} + \cdots x_s + y_l = x_1 + \ldots + x_{s-1} + b_{0,1},$$

and, thus, $x_s + y_l = x_l + b_{0,1}$, which implies $x_l = x_{l+1} = \cdots = x_s$ and $y_l = b_{0,1}$ since $x_s \le x_l$ and $y_l \le b_{0,1}$. It follows from this and the ordering of y that $y_s = b_{0,1}$, which a contradiction. Thus, $\bigoplus_{s\le k\le n} y_k = b_{0,1}$. If $y_s < b_{0,1}$, then $\bigoplus_{s+1\le k\le n} y_k = b_{0,1}$ and we observe that

$$\begin{aligned}
b_{s,1} &= \left(x_1 + \cdots + x_s + \bigoplus_{s+1\le k\le n} y_k\right) \\
&\quad \oplus \bigoplus_{1\le k\le s} (x_1 + \cdots + x_{k-1} + x_{k+1} + \cdots x_{s+1} + y_k) \\
&= x_1 + \cdots + x_s + b_{0,1}.
\end{aligned}$$

This contradicts the definition of s. Therefore, we get $y_s = b_{0,1}$. A similar argument proves that $\bar{y}_s = b_{0.1}$. We conclude that $y_s = b_{0.1} = \bar{y}_s$.

Let $x', y', \bar{x}', \bar{y}'$ be the $(n-1)$-dimensional vectors obtained from $x, y, \bar{x}, \bar{y}$ by removing the s-th component respectively. Finally in order to use the induction hypothesis we show that

$$b_{p,0}(x', y') = b_{p,0}(\bar{x}', \bar{y}'), \qquad 1 \le p \le n-1, \tag{A.10a}$$

$$b_{p,q}(x', y') = b_{p,q}(\bar{x}', \bar{y}'), \qquad p \ge 1,\ q \ge 1,\ p+q \le n-1, \tag{A.10b}$$

$$b_{0,q}(x', y') = b_{0,q}(\bar{x}', \bar{y}'), \qquad 1 \le q \le n-1. \tag{A.10c}$$

We at first show that

$$b_{p,0}(x', y') = \min\left[b_{p,0}(x, y),\ b_{p+1,0}(x, y) - x_s\right], \quad 1 \le p \le n-1, \tag{A.11a}$$

$$b_{p,q}(x', y') = \min\left[b_{p,q}(x, y),\ b_{p+1,q}(x, y) - x_s,\ b_{p,q+1}(x, y) - y_s\right], \quad p \ge 1,\ q \ge 1,\ p+q \le n-1, \tag{A.11b}$$

$$b_{0,q}(x', y') = \min\left[b_{0,q}(x, y),\ b_{0,q+1}(x, y) - y_s\right], \quad 1 \le q \le n-1. \tag{A.11c}$$

Let us write $b'_{p,q}$ instead of $b_{p,q}(x', y')$. We show the formula (A.11a). By Corollary A.2.5 we have

$$\begin{aligned}
b_{p,0} &= b'_{p,0} \oplus \left(b'_{p-1,0} + x_s\right), \qquad && 1 \le p \le n, \\
b_{p+1,0} - x_s &= b'_{p,0} \oplus \left(b'_{p+1,0} - x_s\right), \qquad && 0 \le p \le n-1,
\end{aligned}$$

which are equations in $b'_{p,0}$. We claim that the assumption of Proposition A.3.3 is satisfied. We observe that

$$\left(b'_{p-1,0} + x_s\right) + \left(b'_{p+1,0} - x_s\right) = b'_{p-1,0} + b'_{p+1,0} \leq 2b'_{p,0}.$$

The inequality is obtained by letting $q = 1$ in the inequality (A.9a). Hence, by Proposition A.3.3, we have the formula (A.11a). By exchanging the role of x and y in the formula (A.11a) we obtain the formula (A.11c). Next we show the formula (A.11b). By Corollary A.2.5 we have

$$\begin{aligned}
b_{p,q} &= b'_{p,q} \oplus \left(b'_{p-1,q} + x_s\right) \oplus \left(b'_{p,q-1} + y_s\right), \quad p \geq 0,\, q \geq 0,\, p+q \leq n, \\
b_{p,q+1} - y_s &= b'_{p,q} \oplus \left(b'_{p,q+1} - y_s\right) \oplus \left(b'_{p-1,q+1} + x_s - y_s\right), \\
&\qquad\qquad\qquad\qquad\qquad\qquad p \geq 0,\, q \geq 0,\, p+q \leq n-1, \\
b_{p+1,q} - x_s &= b'_{p,q} \oplus \left(b'_{p+1,q} - x_s\right) \oplus \left(b'_{p+1,q-1} + y_s - x_s\right), \\
&\qquad\qquad\qquad\qquad\qquad\qquad p \geq 0,\, q \geq 0,\, p+q \leq n-1,
\end{aligned}$$

which are equations in $b'_{p,q}$. We claim that the assumption of Proposition A.3.3 is satisfied. Suppose that the assumption is not satisfied, that is, for $p \geq 1$, $q \geq 1$, $p+q \leq n-1$,

$$b'_{p,q} < \left(b'_{p-1,q} + x_s\right) \oplus \left(b'_{p,q-1} + y_s\right), \tag{A.12a}$$

$$b'_{p,q} < \left(b'_{p,q+1} - y_s\right) \oplus \left(b'_{p-1,q+1} + x_s - y_s\right), \tag{A.12b}$$

$$b'_{p,q} < \left(b'_{p+1,q} - x_s\right) \oplus \left(b'_{p+1,q-1} + y_s - x_s\right). \tag{A.12c}$$

We can eliminate y_s by writing the two inequalities (A.12b) and (A.12c) in the forms:

$$\begin{aligned}
b'_{p+1,q-1} + y_s + b'_{p,q} &< \left(b'_{p,q+1} + b'_{p+1,q-1}\right) \oplus \left(b'_{p+1,q-1} + b'_{p-1,q+1} + x_s\right), \\
x_s + 2b'_{p,q} &< \left(b'_{p,q} + b'_{p+1,q}\right) \oplus \left(b'_{p+1,q-1} + y_s + b'_{p,q}\right).
\end{aligned}$$

Eliminating x_s similarly, we thus obtain the two inequalities

$$\begin{aligned}
x_s + 2b'_{p,q} &< \left(b'_{p,q} + b'_{p+1,q}\right) \oplus \left(b'_{p,q+1} + b'_{p+1,q-1}\right) \oplus \left(b'_{p+1,q-1} + b'_{p-1,q+1} + x_s\right), \\
y_s + 2b'_{p,q} &< \left(b'_{p,q} + b'_{p,q+1}\right) \oplus \left(b'_{p+1,q} + b'_{p-1,q+1}\right) \oplus \left(b'_{p+1,q-1} + b'_{p-1,q+1} + y_s\right).
\end{aligned}$$

Since the inequality (A.9c) holds, we can neglect the third term in the right-hand side of each inequality. Hence, we have

$$\begin{aligned}
x_s + 2b'_{p,q} &< \left(b'_{p,q} + b'_{p+1,q}\right) \oplus \left(b'_{p,q+1} + b'_{p+1,q-1}\right), \\
y_s + 2b'_{p,q} &< \left(b'_{p,q} + b'_{p,q+1}\right) \oplus \left(b'_{p+1,q} + b'_{p-1,q+1}\right).
\end{aligned}$$

Using the inequalities we can eliminate x_s and y_s from (A.12a) and obtain the inequality

$$\begin{aligned}
3b'_{p,q} < {} &\left(b'_{p,q} + b'_{p+1,q} + b'_{p-1,q}\right) \oplus \left(b'_{p,q+1} + b'_{p+1,q-1} + b'_{p-1,q}\right) \\
&\oplus \left(b'_{p,q} + b'_{p,q+1} + b'_{p,q-1}\right) \oplus \left(b'_{p+1,q} + b'_{p-1,q+1} + b'_{p,q-1}\right).
\end{aligned}$$

Since (A.9a), (A.9e), (A.9b) and (A.9d) hold, the inequality does not hold. This is a contradiction, so the assumption of Proposition A.3.3 is satisfied and by Proposition A.3.3 we have the formula (A.11b).

We similarly have for $\bar{x}', \bar{y}'$

$$b_{p,0}(\bar{x}',\bar{y}') = \min\left[b_{p,0}(\bar{x},\bar{y}),\ b_{p+1,0}(\bar{x},\bar{y}) - \bar{x}_s\right], \quad 1 \le p \le n-1, \tag{A.13a}$$
$$b_{p,q}(\bar{x}',\bar{y}') = \min\left[b_{p,q}(\bar{x},\bar{y}),\ b_{p+1,q}(\bar{x},\bar{y}) - \bar{x}_s,\ b_{p,q+1}(\bar{x},\bar{y}) - \bar{y}_s\right], \quad p \ge 1,\ q \ge 1,\ p+q \le n-1, \tag{A.13b}$$
$$b_{0,q}(\bar{x}',\bar{y}') = \min\left[b_{0,q}(\bar{x},\bar{y}),\ b_{0,q+1}(\bar{x},\bar{y}) - \bar{y}_s\right], \quad 1 \le q \le n-1. \tag{A.13c}$$

Hence the equalities (A.10) hold.

By the induction hypothesis, we have $x' = \bar{x}'$ and $y' = \bar{y}'$. Therefore, $x = \bar{x}$ and $y = \bar{y}$, which complete the proof. □

By Theorem A.3.4 it is clear that the basic 2-symmetric tropical polynomials give coordinates on $\mathbb{R}^{2n}/S_n$.

Example A.3.5. Let $n = 4$ and $x = (2,1,1,0)^{\mathrm{t}}$, $y = (2,2,3,1)^{\mathrm{t}}$. Then

$$\begin{aligned}
&b_{1,0}(x,y) = 2, \quad b_{0,1}(x,y) = 3,\\
&b_{2,0}(x,y) = 3, \quad b_{1,1}(x,y) = 5, \quad b_{0,2}(x,y) = 5,\\
&b_{3,0}(x,y) = 4, \quad b_{2,1}(x,y) = 6, \quad b_{1,2}(x,y) = 7, \quad b_{0,3}(x,y) = 7,\\
&b_{4,0}(x,y) = 4, \quad b_{3,1}(x,y) = 6, \quad b_{2,2}(x,y) = 7, \quad b_{1,3}(x,y) = 8, \quad b_{0,4}(x,y) = 8.
\end{aligned}$$

To the contrary, we show how the orbit is determined by the values of the basic 2-symmetric tropical polynomials according to the proof of Theorem A.3.4. Suppose that $x_1 \ge x_2 \ge x_3 \ge x_4$ and if $x_i = x_{i+1}$ for some $i \in \{1,2,3\}$, then $y_i \le y_{i+1}$. First we obtain

$$x_1 = b_{1,0} = 2, \quad x_2 = b_{2,0} - b_{1,0} = 1, \quad x_3 = b_{3,0} - b_{2,0} = 1, \quad x_4 = b_{4,0} - b_{3,0} = 0.$$

We observe that

$$s = \max\{i \in \{1,2,3,4\} \colon b_{i-1,1} = b_{i-1,0} + b_{0,1}\} = 3,$$

since $b_{2,1} = b_{2,0} + b_{0,1}$ and $b_{3,1} < b_{3,0} + b_{0,1}$. Hence, $(x_3, y_3) = (1, b_{0,1}) = (1,3)$.

Let $x' = (2,1,0)^{\mathrm{t}}$, $y' = (2,2,1)^{\mathrm{t}}$. Then

$$\begin{aligned}
b_{1,0}(x',y') &= \min[b_{1,0}(x,y), b_{2,0}(x,y) - x_3] = 2,\\
b_{0,1}(x',y') &= \min[b_{0,1}(x,y), b_{0,2}(x,y) - y_3] = 2,\\
b_{2,0}(x',y') &= \min[b_{2,0}(x,y), b_{3,0}(x,y) - x_3] = 3,\\
b_{1,1}(x',y') &= \min[b_{1,1}(x,y), b_{2,1}(x,y) - x_3, b_{1,2}(x,y) - y_3] = 4,\\
b_{0,2}(x',y') &= \min[b_{0,2}(x,y), b_{0,3}(x,y) - y_3] = 4,\\
b_{3,0}(x',y') &= \min[b_{3,0}(x,y), b_{4,0}(x,y) - x_3] = 3,\\
b_{2,1}(x',y') &= \min[b_{2,1}(x,y), b_{3,1}(x,y) - x_3, b_{2,2}(x,y) - y_3] = 4,\\
b_{1,2}(x',y') &= \min[b_{1,2}(x,y), b_{2,2}(x,y) - x_3, b_{1,3}(x,y) - y_3] = 5,\\
b_{0,3}(x',y') &= \min[b_{0,3}(x,y), b_{0,4}(x,y) - y_3] = 5.
\end{aligned}$$

Therefore, the rest is reduced to the case $n = 3$.

A.4 Properties of basic r-symmetric tropical polynomials

We present special cases for $r \geq 3$ where the basic r-symmetric tropical polynomials separate orbits.

Proposition A.4.1. *Let $r \geq 3$ and there be r n-dimensional vectors, at least one of which has distinct components. Then the basic r-symmetric tropical polynomials separate orbits.*

Proof. Let r n-dimensional vectors be denoted by $x^{(1)}, x^{(2)}, \dots, x^{(r)}$. Suppose that the components of the vector $x^{(1)}$ are all distinct and that $x_1^{(1)} > x_2^{(1)} > \cdots > x_n^{(1)}$. We will show that $\left(x^{(1)}, x^{(2)}, \dots, x^{(r)}\right)$ is uniquely determined by the values of the basic r-symmetric tropical polynomials.

By Theorem A.3.4 we obtain $\left(x^{(1)}, x^{(2)}\right)$. Similarly we obtain $\left(x^{(1)}, x^{(3)}\right)$, $\dots$, $\left(x^{(1)}, x^{(r)}\right)$. Since all the components of $x^{(1)}$ are distinct, we can uniquely merge $\left(x^{(1)}, x^{(2)}\right)$, $\dots$, $\left(x^{(1)}, x^{(r)}\right)$ by $x^{(1)}$ and obtain $\left(x^{(1)}, x^{(2)}, \dots, x^{(r)}\right)$. □

Proposition A.4.2. *Let $r \geq 3$ and there be r n-dimensional vectors, such that at least one of the differences of all pairs of these vectors has distinct components. Then the basic r-symmetric tropical polynomials separate orbits.*

Proof. Let r n-dimensional vectors be denoted by $x^{(1)}, x^{(2)}, \dots, x^{(r)}$. Suppose that the components of the vector $x^{(2)} - x^{(1)}$ are all distinct. We will show that $\left(x^{(1)}, x^{(2)}, \dots, x^{(r)}\right)$ is uniquely determined by the values of the basic r-symmetric tropical polynomials $b_{q_1,q_2,\dots,q_r}\left(x^{(1)}, x^{(2)}, \dots, x^{(r)}\right)$.

First we show that each $b_{p_1,\dots,p_{r-1}}\left(x^{(2)} - x^{(1)}, \dots, x^{(r)} - x^{(1)}\right)$ is written in terms of the basic polynomials $b_{q_1,q_2,\dots,q_r}\left(x^{(1)}, x^{(2)}, \dots, x^{(r)}\right)$.

Let $q_1 + q_2 + \cdots + q_r = n$. Then by definition we have

$$
\begin{aligned}
&b_{q_1,q_2,\dots,q_r}\left(x^{(1)}, x^{(2)}, \dots, x^{(r)}\right) - b_{n,0,\dots,0}\left(x^{(1)}, x^{(2)}, \dots, x^{(r)}\right) \\
&= \bigoplus_{\sigma \in S_n} \Big(x^{(1)}_{\sigma(1)} + \cdots + x^{(1)}_{\sigma(q_1)} + x^{(2)}_{\sigma(q_1+1)} + \cdots + x^{(2)}_{\sigma(q_1+q_2)} + \cdots \\
&\qquad + x^{(r)}_{\sigma(q_1+\cdots+q_{r-1}+1)} + \cdots + x^{(r)}_{\sigma(n)} \Big) - \left(x^{(1)}_1 + x^{(1)}_2 + \cdots + x^{(1)}_n \right) \\
&= \bigoplus_{\sigma \in S_n} \Big(x^{(2)}_{\sigma(q_1+1)} - x^{(1)}_{\sigma(q_1+1)} + \cdots + x^{(2)}_{\sigma(q_1+q_2)} - x^{(1)}_{\sigma(q_1+q_2)} + \cdots \\
&\qquad + x^{(r)}_{\sigma(q_1+\cdots+q_{r-1}+1)} - x^{(1)}_{\sigma(q_1+\cdots+q_{r-1}+1)} + \cdots + x^{(r)}_{\sigma(n)} - x^{(1)}_{\sigma(n)} \Big),
\end{aligned}
$$

since $x^{(1)}_1 + x^{(1)}_2 + \cdots + x^{(1)}_n = x^{(1)}_{\sigma(1)} + x^{(1)}_{\sigma(2)} + \cdots + x^{(1)}_{\sigma(n)}$. Hence, we obtain the formula

$$
\begin{aligned}
&b_{q_2,\dots,q_r}\left(x^{(2)} - x^{(1)}, \dots, x^{(r)} - x^{(1)}\right) \\
&\qquad = b_{q_1,q_2,\dots,q_r}\left(x^{(1)}, x^{(2)}, \dots, x^{(r)}\right) - b_{n,0,\dots,0}\left(x^{(1)}, x^{(2)}, \dots, x^{(r)}\right).
\end{aligned}
$$

Using the formula and Proposition A.4.1 we obtain $\left(x^{(2)} - x^{(1)}, \dots, x^{(r)} - x^{(1)}\right)$, since the components of the vector $x^{(2)} - x^{(1)}$ are all distinct.

On the other hand, we obtain $\left(x^{(1)}, x^{(2)}\right)$ from $b_{q_1,q_2,0,\ldots,0}\left(x^{(1)}, x^{(2)}, \ldots, x^{(r)}\right)$. Thus we get $\left(x^{(1)}, x^{(2)} - x^{(1)}\right)$. Since all the components of $x^{(2)} - x^{(1)}$ are distinct, we can uniquely merge $\left(x^{(1)}, x^{(2)} - x^{(1)}\right)$ and $\left(x^{(2)} - x^{(1)}, \ldots, x^{(r)} - x^{(1)}\right)$ by $x^{(2)} - x^{(1)}$. Therefore, we get $\left(x^{(1)}, x^{(2)} - x^{(1)}, \ldots, x^{(r)} - x^{(1)}\right)$ and thus $\left(x^{(1)}, x^{(2)}, \ldots, x^{(r)}\right)$. □

A.5 Discussion

We defined the basic r-symmetric tropical polynomials and showed that the basic 2-symmetric tropical polynomials give coordinates on $\mathbb{R}^{2n}/S_n$. Moreover, we presented special cases where the basic polynomials separate orbits even for $r \geq 3$.

The basic 2-symmetric tropical polynomials separate orbits more efficiently than the elementary 2-symmetric tropical polynomials defined by Carlsson and Kališnik [59] in the sense that the set of the basic polynomials is a proper subset of the set of the elementary polynomials. In addition, Carlsson and Kališnik alleged that the elementary polynomials generate the r-symmetric tropical rational functions, but the proof is false, because the authors made a mistake in page 3626, line 7 (by replacing $\oplus$ with $\odot$).

It is interesting to investigate whether the basic polynomials separate orbits for $r \geq 3$ in general and whether they are generators for the set of r-symmetric tropical rational functions. It would be suitable to call the basic polynomials elementary depending on the results.

Bibliography

[1] R. W. Conway, W. L. Maxwell, and L. W. Miller, *Theory of Scheduling.* Reading, Mass.: Addison-Wesley, 1967.

[2] M. L. Pinedo, *Scheduling: Theory, Algorithms, and Systems.* New York, N. Y.: Springer, 2012.

[3] K. R. Baker and D. Trietsch, *Principles of Sequencing and Scheduling.* Hoboken, N. J.: John Wiley and Sons, 2013.

[4] S. M. Johnson, "Optimal two- and three-stage production schedules with setup times included," *Naval Research Logistics Quarterly*, vol. 1, no. 1, pp. 61–68, 1954.

[5] J. R. Jackson, "An extension of Johnson's results on job lot scheduling," *Naval Research Logistics*, vol. 3, no. 3, pp. 201–203, 1956.

[6] W. E. Smith, "Various optimizers for single-stage production," *Naval Research Logistics Quarterly*, vol. 3, no. 1-2, pp. 59–66, 1956.

[7] R. L. Graham, E. L. Lawler, J. K. Lenstra, and A. H. G. Rinnooy Kan, "Optimization and approximation in deterministic sequencing and scheduling: a survey," *Annals of Discrete Mathematics*, vol. 5, pp. 287–326, 1979.

[8] N.-V. Vo and C. Lenté, "Equivalence between two flowshop problems - maxplus approach," in *the 2nd International Conference on Operations Research and Enterprise Systems*, pp. 322–325, 2013.

[9] N.-V. Vo and C. Lenté, "From maxplus algebra to general lower bounds for the total weighted completion time in flowshop scheduling problems," *Lecture Notes in Management Science*, vol. 6, pp. 128–137, 2014.

[10] N. G. Hall and C. Sriskandarajah, "A survey of machine scheduling problems with blocking and no-wait in process," *Operations Research*, vol. 44, no. 3, pp. 510–525, 1996.

[11] S. S. Reddi and C. V. Ramamoorthy, "On the flow-shop sequencing problem with no wait in process," *Operational Research Quarterly*, pp. 323–331, 1972.

[12] I. Adiri and D. Pohoryles, "Flowshop/no-idle or no-wait scheduling to minimize the sum of completion times," *Naval Research Logistics Quarterly*, vol. 29, no. 3, pp. 495–504, 1982.

[13] J. Kamburowski, “More on three-machine no-idle flow shops,” *Computers and Industrial Engineering*, vol. 46, no. 3, pp. 461–466, 2004.

[14] P. J. Kalczynski and J. Kamburowski, “On no-wait and no-idle flow shops with makespan criterion,” *European Journal of Operational Research*, vol. 178, no. 3, pp. 677–685, 2007.

[15] H. M. Wagner, “An integer linear-programming model for machine scheduling,” *Naval Research Logistics Quarterly*, vol. 6, pp. 131–140, 1958.

[16] E. H. Bowmann, “The schedule-sequencing problem,” *Operations Research*, vol. 7, pp. 621–624, 1959.

[17] B. D. Corwin and A. O. Esogbue, “Two machine flow shop scheduling problems with sequence dependent setup times: A dynamic programming approach,” *Naval Research Logistics Quarterly*, vol. 21, no. 3, pp. 515–524, 1974.

[18] E. Ignall and L. Schrage, “Application of the branch and bound technique to some flow-shop scheduling problems,” *Operations Research*, vol. 13, pp. 400–412, 1965.

[19] D. S. Palmer, “Sequencing jobs through a multi-stage process in the minimum total time–a quick method of obtaining a near optimum,” *Operational Research Quarterly*, vol. 16, pp. 101–107, 1965.

[20] F. Burns and J. Rooker, “Johnson’s three-machine flow-shop conjecture,” *Operations Research*, vol. 24, no. 3, pp. 578–580, 1976.

[21] F. Burns and J. Rooker, “Three-stage flow-shops with recessive second stage,” *Operations Research*, vol. 26, no. 1, pp. 207–208, 1978.

[22] W. Szwarc, “Optimal two-machine orderings in the 3 x n flow-shop problem,” *Operations Research*, vol. 25, no. 1, pp. 70–77, 1977.

[23] J. O. Achugbue and F. Y. Chin, “Complexity and solutions of some three-stage flow shop scheduling problems,” *Mathematics of Operations Research*, vol. 7, no. 4, pp. 532–544, 1982.

[24] I. Nabeshima, “The order of n items processed on m machines. (II),” *Journal of the Operations Research Society of Japan*, vol. 4, pp. 1–8, 1961.

[25] I. Nabeshima, “Notes on the analytical results in flow shop scheduling: part 1 and 2,” *Reports of the University of Electro-Communications*, vol. 27, no. 2, pp. 245–257, 1977.

[26] J. N. D. Gupta, “Optimal schedules for special structure flowshops,” *Naval Research Logistics Quarterly*, vol. 22, no. 2, pp. 255–269, 1975.

[27] M. R. Garey, D. S. Johnson, and R. Shthi, “The complexity of flowshop and jobshop scheduling,” *Mathematics of Operations Research*, vol. 1, no. 2, pp. 117–129, 1976.

[28] M. R. Garey and D. S. Johnson, *Computers and intractability: a guide to the theory of NP-completeness.* San Francisco, L. A.: Freeman, 1979.

[29] J. K. Lenstra, A. H. G. Rinnooy Kan, and P. Brucker, "Complexity of machine scheduling problems," *Annals of Discrete Mathematics*, vol. 1, pp. 343–362, 1977.

[30] T. Gonzalez and S. Sahni, "Flowshop and jobshop schedules: complexity and approximation," *Operations Research*, vol. 26, no. 1, pp. 36–52, 1978.

[31] R. Ruiz and C. Maroto, "A comprehensive review and evaluation of permutation flowshop heuristics," *European Journal of Operational Research*, vol. 165, pp. 479–494, 2005.

[32] B. L. MacCarthy and J. Liu, "Addressing the gap in scheduling research: a review of optimization and heuristic methods in production scheduling," *International Journal of Production Research*, vol. 31, no. 1, pp. 59–79, 1993.

[33] F. L. Rossi, M. S. Nagano, R. Fernandes, and T. Neto, "Evaluation of high performance constructive heuristics for the flow shop with makespan minimization," *The International Journal of Advanced Manufacturing Technology*, pp. 125–136, 2016.

[34] J. Dubois-lacoste, F. Pagnozzi, and T. Stützle, "An iterated greedy algorithm with optimization of partial solutions for the makespan permutation flowshop problem," *Computers and Operations Research*, vol. 81, pp. 160–166, 2017.

[35] L.-H. Tsai, "Asymptotic analysis of an algorithm for balanced processor scheduling," *SIAM Journal on Computing*, vol. 21, no. 1, pp. 59–64, 1992.

[36] B. Hayes, "The easiest hard problem," *American Scientist*, vol. 90, no. 2, pp. 113–117, 2002.

[37] R. M. Karp, "Reducibility among combinatorial problems," in *Complexity of Computer Computations* (R. E. Miller and J. W. Thatcher, eds.), (New York), pp. 85–103, Plenum Press, 1972.

[38] N. Karmarkar and R. M. Karp, "The differencing method of set partitioning," *Technical Report, Computer Science Division, University of California, Berkeley*, vol. 82/113, 1982.

[39] D. S. Johnson, C. R. Aragon, L. A. McGeoch, and C. Schevon, "Optimization by simulated annealing: an experimental evaluation; part II, graph coloring and number partitioning," *Operations Research*, vol. 39, no. 3, pp. 378–406, 1991.

[40] W. Ruml, J. T. Ngo, J. Marks, and S. M. Shieber, "Easily searched encodings for number partitioning," *Journal of Optimization Theory and Applications*, vol. 89, no. 2, pp. 251–291, 1996.

[41] L. Fuksz and P. C. Pop, "A hybrid genetic algorithm with variable neighborhood search approach to the number partitioning problem," in *8th International Conference on Hybrid Artificial Intelligence Systems*, pp. 649–658, 2013.

[42] B. Alidaee, F. Glover, G. A. Kochenberger, and C. Rego, "A new modeling and solution approach for the number partitioning problem," *Journal of Applied Mathematics and Decision Sciences*, vol. 2, pp. 113–121, 2005.

[43] R. E. Korf, "A complete anytime algorithm for number partitioning," *Artificial Intelligence*, vol. 106, pp. 181–203, 1998.

[44] E. Horowitz and S. Sahni, "Computing partitions with applications to the knapsack problem," *Journal of the ACM*, vol. 21, no. 2, pp. 277–292, 1974.

[45] I. P. Gent and T. Walsh, "Phase transitions and annealed theories: Number partitioning as a case study," in *Proceedings of 12th European Conference on Artificial Intelligence*, pp. 170–174, 1996.

[46] S. Mertens, "Phase transition in the number partitioning problem," *Physical Review Letters*, vol. 81, no. November, pp. 4281–4284, 1998.

[47] S. Mertens, "A physicist's approach to number partitioning," *Theoretical Computer Science*, vol. 265, pp. 79–108, 2001.

[48] S. Mertens, "The easiest hard problem: number partitioning," in *Computational Complexity and Statistical Physics*, pp. 125–139, New York: Oxford University Press, 2006.

[49] A. K. Hartmann, A. Mann, and W. Radenbach, "Solution-space structure of (some) optimization problems," *Journal of Physics: Conference Series*, vol. 95, p. 012011, 2008.

[50] R. A. Cuninghame-Green, "Projections in minimax algebra," *Mathematical Programming*, vol. 10, pp. 111–123, 1976.

[51] R. A. Cuninghame-Green, "The characteristic maxpolynomial of a matrix," *Journal of Mathematical Analysis and Application*, vol. 95, pp. 110–116, 1983.

[52] H. Straubing, "A combinatorial proof of the Cayley-Hamilton theorem," *Discrete Mathematics*, vol. 43, pp. 273–279, 1983.

[53] G. J. Olsder, C. Roos, and P. O. Box, "Cramer and Cayley-Hamilton in the max algebra," *Linear Algebra and its Applications*, vol. 101, pp. 87–108, 1988.

[54] L. Elsner and P. van den Driessche, "On the power method in max algebra," *Linear Algebra and its Applications*, vol. 302-303, pp. 17–32, 1999.

[55] P. Butkovič and L. Murfitt, "Calculating essential terms of a characteristic maxpolynomial," *Central European Journal of Operations Research*, vol. 8, pp. 237–246, 2000.

[56] R. E. Burkard and P. Butkovič, "Finding all essential terms of a characteristic maxpolynomial," *Discrete Applied Mathematics*, vol. 130, no. 3, pp. 367–380, 2003.

[57] R. E. Burkard and P. Butkovič, "Max algebra and the linear assignment problem," *Mathematical Programming*, vol. 98, no. 1-3, pp. 415–429, 2003.

[58] P. A. Binding and H. Volkmer, "A generalized eigenvalue problem in the max algebra," *Linear Algebra and its Applications*, vol. 422, no. 2-3, pp. 360–371, 2007.

[59] G. Carlsson and S. V. Mališnik, "Symmetric and r-symmetric tropical polynomials and rational functions," *Journal of Pure and Applied Algebra*, vol. 220, no. 11, pp. 3610–3627, 2016.

[60] R. A. Cuninghame-Green, "Maxpolynomial equations," *Fuzzy Sets and Systems*, vol. 75, no. 2, pp. 179–187, 1995.

[61] F. Baccelli, G. Cohen, G. J. Olsder, and J.-P. Quadrat, *Synchronization and Linearity: an Algebra for Discrete Event Systems.* Chichester, N. Y.: Wiley, 1992.

[62] J.-L. Bouquard and C. Lenté, "Two-machine flow shop scheduling problems with minimal and maximal delays," *4OR*, vol. 4, no. 1, pp. 15–28, 2006.

[63] J.-L. Bouquard, C. Lenté, and J.-C. Billaut, "Application of an optimization problem in max-plus algebra to scheduling problems," *Discrete Applied Mathematics*, vol. 154, no. 15, pp. 2064–2079, 2006.

[64] I. Nabeshima, "The order of n items processed on m machines [III]," *Journal of the Operations Research Society of Japan*, vol. 16, no. 3, pp. 163–185, 1973.

[65] D. A. Wismer, "Solution of the flowshop-scheduling problem with no intermediate queues," *Operations Research*, vol. 20, no. 3, pp. 689–697, 1972.

[66] K. Matsui, "Order statistics and ultra-discretization," in *the 28th Symposium of Japanese Society of Applied Statistics (Japanese only)*, 2006.

[67] H. Weyl, *The Classical Groups: Their Invariants and Representations.* Princeton, N.J.: Princeton University Press, 1939.

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