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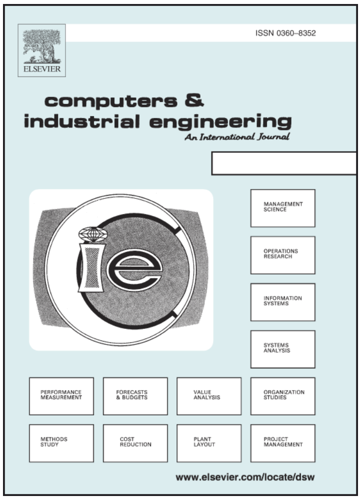


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# Measurement of Returns-to-Scale Using Interval Data Envelopment Analysis Models

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# Measurement of Returns-to-Scale using Interval Data Envelopment Analysis Models[1]

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**All additions and changes to the original manuscript are highlighted in this revision.**


## Abstract

The economic concept of Returns-to-Scale (RTS) has been intensively studied in the context of Data Envelopment Analysis (DEA). The conventional DEA models that are used for RTS classification require well-defined and accurate data whereas in reality observations gathered from production systems may be characterized by intervals. For instance, the heat losses of the combined production of heat and power (CHP) systems may be within a certain range, hinging on a wide variety of factors such as external temperature and real-time energy demand. Enriching the current literature independently tackling the two problems; interval data and RTS estimation; we develop an overarching evaluation process for estimating RTS of Decision Making Units (DMUs) in Imprecise DEA (IDEA) where the input and output data lie within bounded intervals. In the presence of interval data, we introduce six types of RTS involving increasing, decreasing, constant, non-increasing, non-decreasing and variable RTS. The situation for non-increasing (non-decreasing) RTS is then divided into two partitions; constant or decreasing (constant or increasing) RTS using sensitivity analysis. Additionally, the situation for variable RTS is split into three partitions consisting of constant, decreasing and increasing RTS using sensitivity analysis. Besides, we present the stability region of an observation while


[1] An earlier version of this paper previously circulated with the title "Estimating returns to scale in imprecise data envelopment analysis" in Hatami-Marbini et al. (2014).



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preserving its current RTS classification using the optimal values of a set of proposed DEA-based models. The applicability and efficacy of the developed approach is finally studied through two numerical examples and a case study.

***Keywords***: Returns-to-scale; Interval data; Data envelopment analysis.

**JEL Classification** C61 D24 D80

## 1. Introduction

Among the most important and highly discussed topics in the Data Envelopment Analysis (DEA) literature is the estimation of Returns-to-Scale (RTS) of individual Decision Making Units (DMUs) (i.e., observations) (Banker et al., 2004). Since the seminal work on most productive scale size by Banker (1984), a series of papers have been devoted to various aspects of RTS classification in different types of DEA models (as we will briefly review in Section 1.1 below). In short, RTS classification poses two challenges. The most straightforward challenge is to classify RTS of efficient DMUs which turn out to be closely related to the optimal solutions of the standard DEA models. The second challenge is to classify the RTS of inefficient DMUs which further requires a relevant projection onto the efficient frontier of the production possibility set.

In addition to the aforesaid challenges, this paper will consider *data imprecision* as another challenge. In many empirical cases, data is often subject to substantial imprecision that seriously questions the relevance of well-defined “crisp” data as required by the standard DEA framework (Hatami-Marbini et al., 2011). In such cases, interval data seem more appropriate as recognized by two important strands of literature: Imprecise DEA (IDEA) and Fuzzy DEA (FDEA).

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IDEA was originally presented by Cooper et al. (1999, 2001a, 2001b) when data sets include intervals, ordinal data and/or ratio bounds. Subsequently, a great deal of interest for interval DEA followed as briefly reviewed in Section 1.1 below.

The FDEA literature is recently surveyed in Emrouznejad et al. (2014) and Hatami-Marbini et al. (2011). Loosely speaking, fuzzy data can be seen as generalized intervals, but many of the relevant methods build on so-called α-level based approach, which are in fact interval data.

When data is in the form of intervals, it is no longer obvious what RTS means. In case of crisp data, Constant Returns-to-Scale (CRS) prevails when scaling up or down all inputs by a factor α scales up or down all outputs by exactly the same factor α. With inputs and outputs given in the form of intervals, we apply a straightforward generalization of CRS in the way that multiplying input intervals by a factor α, [α x^L, α x^U], leads to multiplying output intervals by a factor α as well, [α y^L, α y^U]. In terms of economics, this corresponds to looking at imprecision as a relative uncertainty surrounding the value of a given variable, say, +/- 10% of the crisp value.

Mimicking the standard methods for RTS classification in the conventional (crisp) case, we suggest using the approach of Wang et al. (2005) to determine the efficient frontier (in case of interval data). In particular, we propose six RTS classes involving increasing, decreasing, constant, non-increasing, non-decreasing and variable RTS. We then carry out the sensitivity analysis to divide the case of non-increasing RTS (non-decreasing RTS) into two partitions; constant or decreasing RTS (constant or increasing RTS) and to split the situation for variable RTS into three partitions consisting of constant, decreasing and increasing RTS. In the presence of bounded data, we present the stability region of a given DMU while preserving its current RTS classification using the optimal values of a set of proposed DEA-based models as in the approach of Seiford and Zhu (1999b).

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To sum up, we submit that the way of estimating a frontier of the production possibility set with interval valued data put forward in Wang et al., (2005) can be utilized in connection with conventional methods for RTS classification introduced in Seiford and Zhu (1999a, 1999b) in order to analyse RTS in case of interval valued production data. We demonstrate the usefulness of our approach through various numerical illustrations, including an example where the results are directly comparable with the conventional approaches for crisp data sets. We further consider an application involving undesirable outputs with interval data taken from a study of Iranian power plants by Khalili-Damghani et al. (2015). We demonstrate the ability of our suggested approach to classify these plants according to their RTS as well as supplementing these with ranges for which the classifications remain unchanged.

### *1.1. Related literature*

Banker (1984) initially discussed how to identify RTS in the CCR model, named after Charnes, Cooper and Rhodes (1978), and the BCC model, named after Banker, Charnes and Cooper (1984). Subsequently, Banker and Thrall (1992) and Zhu and Shen (1995) indicated some methods for estimating RTS when the BCC model encountered multiple optimal solutions. Banker et al. (1996) then proposed an alternative algorithm to specify RTS when the CCR model has alternative solutions but their method has high complexity.

Using the efficiency scores of DMUs, Färe et al. (1985, 1994) attempted to characterize types of RTS. In an interesting study conducted by Golany and Yu (1997), an algorithm was proposed to determine RTS of the efficient DMUs. Jahanshahloo et al. (2005) extended Golany and Yu (1997)'s method since their algorithm is limited to some cases. Also, Seiford and Zhu (1999a,

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1999b) proposed several DEA models for classifying RTS of DMUs in case of input and output orientation focusing on estimating stability regions of RTS.

In the non-radial models, the classification of RTS is further complicated due to the multiple projections for each inefficient DMU. In this regard, Sueyoshi and Sekitani (2007a) discussed RTS of the non–radial range-adjusted measure (RAM) model. They dealt with the RTS problem associated with non–radial DEA model by finding all the efficient DMUs that belong to the reference set. Subsequently, Sueyoshi and Sekitani (2007b) extended their model when the alternative optimal solutions occur in the reference set and supporting hyperplane. Fukuyama (2000) provided some mathematical properties of scale elasticity (SE) of the efficient and inefficient DMUs. Although Soleimani-damaneh and Mostafaee (2008) and Zhang (2008) claimed that Fukuyama (2000)'s results are incorrect, Fukuyama (2008) showed the correctness of the results.

Sueyoshi and Sekitani (2005) considered RTS in dynamic systems in which each DEA framework includes variable inputs and quasi-fixed inputs as two different types of inputs. Zarepisheh et al. (2006) introduced an algorithm to estimate the RTS of DMUs without chasing down alternative optimal solutions.  Førsund et al. (2007) presented two approaches for the specification of RTS. The first approach radially projected DMUs on the frontier. They then used the efficiency score and its dual variables for specifying RTS associated with DMUs. The second approach used the intersection of hyperplanes that passes through the DMU for distinguishing its RTS. Soleimani-damaneh et al. (2009) studied the relation between the RTS and scale elasticity (SE) when there are the alternative solutions. Zarepisheh and Soleimani-damaneh (2009) estimated RTS on the left and on the right by means of the dual simplex method. Soleimani-

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damaneh et al. (2010) explored the relation between the RTS and SE in the presence of weight restrictions and the alternative solutions.

Sueyoshi and Goto (2011) considered DEA for the environmental assessment with the desirable and undesirable outputs. They then used a RAM model to estimate RTS and damages to scale (DTS) by means of desirable and undesirable outputs, respectively. Sueyoshi and Goto (2012) first defined the natural and managerial disposability concepts for environmental assessment. Next, they proposed non-radial model for distinguishing RTS and DTS of the natural and managerial disposability. Their method was applied to petroleum firms. Sueyoshi and Goto (2012) proposed radial and non-radial model for distinguishing RTS and DTS of the natural and managerial disposability. The authors applied their models to U.S. fossil fuel power. Sueyoshi and Goto (2013) first developed a technique to estimate RTS and DTS. Next, they applied their method to the US coal-fired power plants.

Witte and Marques (2011), and Soleimani-damaneh and Mostafaee (2009) proposed models for non-convex production possibility set (PPS). Soleimani-damaneh and Mostafaee (2009) studied the RTS of free disposal hull (FDH) model according to the summation of lambda as well as providing an algorithm to calculate the stability region of RTS classification. Witte and Marques (2011) measured the RTS for a FDH model according to the most imprecise scale size and applied their model for data from the Portuguese drinking water sector.

Roughly speaking, there are many studies in the literature for determining the returns-to-scale properties of a projected point on the frontier (e.g., Førsund and Hjalmarsson 2004; Førsund et al. 2007; Podinovski et al. 2009; Podinovski and Førsund 2010)

Krivonozhko et al. (2012) argued that the returns-to-scale of efficient DMUs can be perceived by viewing the returns-to-scale characteristics of single interior points of these faces.

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Lately, Podinovski et al. (2016) contributed to the relevant DEA literature by developing a unified linear programming approach for the purpose of calculating scale elasticity and corresponding RTS characterization in any polyhedral technology including the standard VRS and CRS technologies, technologies with production trade-offs and weight restrictions, their variants with negative inputs and outputs, some technologies with weakly disposable undesirable outputs, and network DEA technologies.

Following the IDEA papers by Cooper et al. (1999, 2001a, 2001b), Entani et al. (2002) proposed a DEA model with interval and fuzzy data to measure the interval efficiencies of DMUs from the optimistic and pessimistic viewpoints. Despotis and Smirlis (2002) proposed a pair of DEA models to calculate the upper and lower efficiency when input and output data vary in intervals. They defined 2n PPS for evaluating n DMUs from the best and worst viewpoints. Wang et al. (2005) modified Despotis and Smirlis (2002)'s models by introducing a unified PPS in conjunction with n DMUs under assessment. Lee et al. (2002) presented a non-linear additive imprecise DEA (IDEA) model that was converted to the linear programming model. Kao (2006) expressed the imprecise DEA problem as a bi-level mathematical programming model when input and output data are characterized by the intervals. Park (2010) researched the relationship between primal and dual with imprecise data. Emrouznejad et al. (2011) developed the imprecise DEA models with interval data to estimate overall profit efficiency. The authors obtained the upper and lower bounds of the overall profit efficiency for DMUs and proposed a classifying the scores. Emrouznejad et al. (2012) developed general non-parametric corporate performance (GNCP) model and multiplicative non-parametric corporate performance (MNCP) model with interval ratio data. Shokouhi et al. (2010, 2014) used the robust optimization concept to propose



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two different robust data envelopment analysis (RDEA) models for dealing with data uncertainty.

The fuzzy DEA methods (as recently surveyed in Emrouznejad et al., 2014; Hatami-Marbini et al., 2011), can be classified into six main categories: (1) the tolerance approach, see, e.g. Sengupta (1992), (2) the α-level based approach, see, e.g. Triantis and Girod (1998), Kao and Liu (2000), Hatami-Marbini et al. (2010), (3) the fuzzy ranking approach, see, e.g. Guo and Tanaka (2001), (4) the possibility approach, see, e.g. Lertworasirikul et al. (2003), Tavana et al. (2012), (5) the fuzzy arithmetic, see, e.g. Wang et al. (2009), and (6) the fuzzy random/type-2, see, e.g. Qin and Liu (2010).

Summing up the above literature, for crisp data our understanding of RTS is well developed. The literature on IDEA and Fuzzy DEA further provide a number of relevant models and approaches to handle uncertain production data. Yet, our understanding of RTS in models with uncertain data is much more limited. Our contribution is a step in the direction of combining established methods from the literature on RTS with crisp data and the literature on IDEA and fuzzy DEA in order to understand RTS in models with interval valued production data and non-parametric estimation of the production possibility set.

### *1.2. Organization*

The rest of the paper is organized as follows: In Section 2, we first review the conventional precise and imprecise CCR models and then present an overview of the sensitivity analysis method for RTS estimation. In Section 3, we present a simple example to graphically show the concept of RTS with interval inputs and interval outputs. In Section 4, we develop a method to estimate RTS of DMUs with interval data followed by a discussion on stability of the RTS. We

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present an empirical example to illustrate the proposed method in Section 6. In section 7, we also study a case study of Iranian power plants to bespeak the ability of the approach to categorise the plants in terms of their RTS. Finally, we provide some concluding remarks and suggestions for future research in Section 8.

## 2. Preliminaries

In this section, we present the classic CCR model with precise data and review a central characterization result for determining Returns-to-Scale (RTS) of efficient DMUs. We then briefly review the Imprecise Data Envelopment Analysis (IDEA) model with interval data and the sensitivity analysis method for RTS estimation.

### *2.1. CCR model*

The problem of evaluating of the performance of $DMU_o$ can be formulated by linear programming. Suppose that we have *n* DMUs where each $DMU_j$, $j=1,\ldots,n,$ produces *s* outputs $y_{rj}$ ($r=1,\ldots,s$), using $m$ inputs $x_{ij}$ ($i=1,\ldots,m$). Charnes et al. (1978) present the following *envelopment form* of the "CCR model" for measuring the radial input-efficiency of $DMU_o$:

$$\begin{aligned} \min \quad & \theta \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io}, && i=1,\ldots,m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, && r=1,\ldots,s, \\ & \lambda_j \ge 0, && j=1,\ldots,n. \end{aligned} \tag{1}$$

where $\lambda_j$ (j=1,…,n) is the weight placed on each DMU for making up the efficient facet of $DMU_j$. A DMU is called "CCR efficient" if and only if the objective function of model (1) is

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equal to 1, i.e., if $\theta^* = 1$, otherwise, it is called "CCR inefficient". Note that the constraint space of (1) defines the production possibility set (PPS) as $T = \{(x, y) \mid x \geq \sum_j \lambda_j x_j, y \leq \sum_j \lambda_j y_j, \lambda_j \geq 0\}$. Within the DEA framework, the economic concept of RTS has received a great deal of attention (see, e.g., Banker, 1984; Banker et al. 1984). On the basis of model (1), Banker and Thrall (1992) proposed a theorem, so called *BT theorem,* characterizing RTS of given efficient DMUs (see appendix 1). It should be emphasized that BT theorem only holds for efficient DMUs. Therefore, the optimal value of $\theta^*$ from model (1) is recognized as a prerequisite for executing the RTS analysis. Banker at al. (1996) further developed a method to omit the examining all the alternate optima. The RTS classification of an inefficient unit can be identified via its projection onto the efficient frontier. Thereby, when studying the inefficient units, the input- and output-oriented CCR models may lead to different RTS classifications. In this paper, we shall focus on *the input oriented version* of the CCR model for identifying the RTS characterizations. It is worth noting that the research idea that will be presented in this study for the input oriented CCR model can straightforwardly extend to output oriented CCR model.

### *2.2. Imprecise CCR model*

We here briefly review the well-known imprecise (multiplier) DEA approach proposed by Despotis and Smirlis (2002).

Let *n* DMUs each produce s interval outputs $y_{rj} \in \left[ y_{rj}^L, y_{rj}^U \right]$ ($r$=1,…,$s$) using *m* interval inputs $x_{ij} \in [x_{ij}^L, x_{ij}^U]$ ($i$=1,…,$m$) where $y_{rj}^L, y_{rj}^U$, $x_{ij}^L$ and $x_{ij}^U$, are strictly positive. Despotis and Smirlis (2002) formulated a pair of imprecise technical efficiency (ITE) models [in multiplier form] to compute the lower bound (LB) and upper bound (UB) of technical efficiency, here presented in their envelopment forms:

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(2): ITE1-IN-CCR (LB)

$$
\begin{aligned}
&\theta^L = \min\ \theta \\
&s.t.\quad \sum_{\substack{j=1,\\ j\neq o}}^{n} \lambda_j x_{ij}^L + \lambda_o x_{io}^U \le \theta x_{io}^U, && i=1,\ldots,m, \\
&\qquad \sum_{\substack{j=1,\\ j\neq o}}^{n} \lambda_j y_{rj}^U + \lambda_o y_{ro}^L \ge y_{ro}^L, && r=1,\ldots,s, \\
&\qquad \lambda_j \ge 0\ , && j=1,\ldots,n.
\end{aligned}
$$

(3): ITE1-IN-CCR (UB)

$$
\begin{aligned}
&\theta^U = \min\ \theta \\
&s.t.\quad \sum_{\substack{j=1,\\ j\neq o}}^{n} \lambda_j x_{ij}^U + \lambda_o x_{io}^L \le \theta x_{io}^L, && i=1,\ldots,m, \\
&\qquad \sum_{\substack{j=1,\\ j\neq o}}^{n} \lambda_j y_{rj}^L + \lambda_o y_{ro}^U \ge y_{ro}^U, && r=1,\ldots,s, \\
&\qquad \lambda_j \ge 0\ , && j=1,\ldots,n.
\end{aligned}
$$

$\theta^U$ and $\theta^L$ are called the efficiency scores of DMU$_o$ in the best and worst conditions, respectively, treated in the input-oriented direction. As noticed by Wang et al. (2005) it is somewhat problematic that using (2) and (3) in effect implies that upper and lower efficiency scores are computed relative to different frontiers for the same DMU and hence introduces comparability issues. Consequently, Wang et al. (2005) developed a pair of multiplier DEA models using the same PPS to obtain the interval efficiency for each DMU. We apply the approach of Wang et al. (2005) to the envelopment CCR model resulting in the following models:

(4): ITE2-IN-CCR (LB)

$$
\begin{aligned}
&\bar{\theta}^L = \min\quad \theta \\
&s.t.\quad \sum_{j=1}^{n} \lambda_j x_{ij}^L \le \theta x_{io}^U, && i=1,\ldots,m, \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj}^U \ge y_{ro}^L, && r=1,\ldots,s, \\
&\qquad \lambda_j \ge 0\ , && j=1,\ldots,n.
\end{aligned}
$$

(5): ITE2-IN-CCR (UB)

$$
\begin{aligned}
&\bar{\theta}^U = \min\quad \theta \\
&s.t.\quad \sum_{j=1}^{n} \lambda_j x_{ij}^L \le \theta x_{io}^L, && i=1,\ldots,m, \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj}^U \ge y_{ro}^U, && r=1,\ldots,s, \\
&\qquad \lambda_j \ge 0\ , && j=1,\ldots,n.
\end{aligned}
$$

where $\bar{\theta}^L$ and $\bar{\theta}^U$ are the lower and upper bounds of efficiency for DMU$_o$.

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The only difference between the BCC and CCR models is the inclusion of the convexity constraint $\sum_{j=1}^{n} \lambda_j = 1$ in both models (4) and (5). A $DMU_o$ is called *efficient* if and only if the objective function of model (4) is equal to 1, otherwise, it is called *inefficient*.

Let us now graphically display the shortcomings of the Despotis and Smirlis (2002) approach using a simple example that consists of six DMUs, marked as A, B, C, D, E and F in Fig. 1 where these DMUs respectively produce one interval valued output [1, 2], [1, 3], [4, 6], [6.5, 7], [5, 6.5] and [1, 6.5] using one interval valued input [2.5, 4], [3, 6], [6, 10], [7.5, 9], [10, 12] and [7, 10]. Notice that the coordinates of each point in Fig. 1 are represented in the order (*x*, *y*).

**-------Insert Fig.1 here-------**

When calculating upper bound efficiency of the six DMUs under a Variable Returns-to-Scale (VRS) assumption each evaluated DMU autonomously uses its best situation and the worst situation for other DMUs to build the six different production frontiers.

For instance, let us focus on $DMU_A$ and $DMU_C$. When evaluating the upper efficiency bound of $DMU_A$, the points *k* and *t* in Fig. 1 as the best and worst situations are considered to form a piecewise DEA frontier *wkp* (black dashed line) that problematically disregards the *free disposability* and *convex hull* assumptions of standard DEA. Likewise, in assessing $DMU_C$ an alternative production frontier as the piecewise linear form *uthp* (blue dotted line) is established using the point *h* as the best setting of $DMU_C$ and the points *t* and *p* as the worst position of $DMU_A$ and $DMU_D$, respectively. Consequently, the PPS based on the Despotis and Smirlis (2002) approach is not only not unique (equals to twice the number of DMUs), but it also violates the fundamental DEA axioms, see e.g., Banker et al. (1984) and Wang et al. (2005).

To deal with these drawbacks, we can define a unique piecewise BCC frontier based on Wang et al. (2005)'s approach shown in models (4) and (5). The piecewise frontier here consists of *wk*,

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*kq*, *qh*, *hg* and *gb* (red dashed line) that starts at the vertical line (support at *k*) and ends through a horizontal line at *gb* (support at *g*), as shown in Fig. 1. To build the frontier, we use the best situation of all DMUs (i.e., the smallest and greatest value of inputs and outputs for each DMU), in other words, the data set {(2.5,2), (3,3), (6,6), (7.5,7), (10,6.5), (7,6.5)} is used to form the production frontier.

### *2.3. Sensitivity of RTS classifications in the standard DEA model*

The sensitivity of RTS classifications is an interesting and challenging research topic in the DEA literature. The estimation of RTS in DEA often takes into account the proportional change in all the outputs of $DMU_o$ derived from a proportional change in all its inputs. In the standard input-oriented DEA model the RTS classifications of a DMU is not changed by variation of input levels unless the DMU is on the efficient frontier (Seiford and Zhu, 1999b). Seiford and Zhu (1999b) present a sensitivity analysis framework by means of several linear programming models for exploring the stability of RTS classifications when the output levels are perturbed. In addition to the identification of the stability region for the RTS classifications (constant, increasing or decreasing returns-to-scale), the authors determined the RTS classification by the optimal values to a series of CCR-based models.

Within the input-oriented model, it is noticeable that if IRS prevails for a DMU, then its IRS cannot vary with decreases in outputs whereas if DRS prevail for a DMU, then its DRS remains unchanged with augmentations in outputs unless the DMU gets to the CCR frontier.

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The following pair of programs with different objective functions, i.e., $(\tau_o^*)^{-1} = \min \sum_{j \in E_o} \lambda_j$ and $(\sigma_o^*)^{-1} = \max \sum_{j \in E_o} \lambda_j$, can be used to detect the RTS classifications of $DMU_o$ (Seiford and Zhu, 1999b):

$$
\begin{aligned}
& (\tau_o^*)^{-1} = \min \sum_{j \in E_o} \lambda_j \qquad (\sigma_o^*)^{-1} = \max \sum_{j \in E_o} \lambda_j \\
& s.t. \quad \sum_{j \in E_o} \lambda_j x_{ij} \le \theta^* x_{io}, \quad i = 1, \dots, m, \\
& \qquad \sum_{j \in E_o} \lambda_j y_{rj} \ge y_{ro}, \quad r = 1, \dots, s, \\
& \qquad \lambda_j \ge 0, \qquad j \in E_o.
\end{aligned} \tag{6}
$$

where $\theta^*$ is the optimal value of the standard CCR model (1) and $E_o =${all efficient DMUs in model (1)}. The following theorem demonstrates how to determine the RTS classification using the above-defined models.

**Theorem 1. (Seiford and Zhu, 1999b)** Suppose that $\tau_o^*$ and $\sigma_o^*$ are the optimal values of models (6). The following conditions specify the RTS of $DMU_0$

(I) CRS prevail for $DMU_o$ if and only if $\sigma_o^* \le 1 \le \tau_o^*$,

(II) IRS prevail for $DMU_o$ if and only if $\sigma_o^* > 1$,

(III) DRS prevail for $DMU_o$ if and only if $\tau_o^* < 1$.

To implement the sensitivity analysis of RTS classification (CRS, IRS and DRS), the three following theorems were developed by Seiford and Zhu (1999b) to deal with situations when output perturbations occur in $DMU_o$ under the input-oriented DEA model.

**Theorem 2. (Seiford and Zhu, 1999b)** Let $DMU_o$ exhibit CRS. Then, its CRS classification remains unchanged for $\gamma \in R^{CRS} = \{\gamma \mid \min\{1, \sigma_o^*\} \le \gamma \le \max\{1, \tau_o^*\}\}$ where $\gamma$ indicates a



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proportional variation of all outputs, $\tilde{y}_{ro} = \gamma y_{ro} (r = 1,\ldots,s)$ and $\tau_o^*$ and $\sigma_o^*$ are obtained by models (6).

**Theorem 3. (Seiford and Zhu, 1999b)** Let DMU$_o$ exhibit IRS. Then, its IRS classification remains unchanged for $\alpha \in R^{IRS} = \{\alpha | 1 \le \alpha < \sigma_o^*\}$ where α is a proportional augmentation of all outputs $\tilde{y}_{ro} = \alpha y_{ro} (r = 1,\ldots,s)$ and $\sigma_o^*$ is calculated by model (6).

**Theorem 4. (Seiford and Zhu, 1999b)** Let DMU$_o$ exhibit DRS. Then, its DRS classification remains unchanged for $\beta \in R^{DRS} = \{\beta | \tau_o^* < \beta \le 1\}$ where $\beta$ indicates a proportional variation of all outputs $\tilde{y}_{ro} = \beta y_{ro} (r = 1,\ldots,s)$ and $\tau_o^*$ is the optimal objective function of model (6).

## 3. Motivating Example and a Few Preliminary Observations

This section makes an attempt to show the main motivation behind the mathematical modelling that will be presented in the ensuing section for the RTS classification of each DMU. Let us consider the simple example presented in Subsection 2.2. We take the unique frontier into account (red dashed line in Fig. 1) for determining the RTS classification of each DMU because this piecewise frontier envelops all the imprecise observations as tightly as possible and avoids the aforesaid shortcomings. The line segments *wk*, *kq*, *qh*, *hg* and *gb* present IRS, IRS, CRS, DRS, DRS. Ray *oqh* (black dash-dotted line) is the CCR efficient frontier giving the case of CRS. If a DMU is on or projected onto the CCR efficient frontier, then this DMU exhibits the condition for CRS. When we encounter two distinct RTSs at the intersections such as *q* and *h*, we give priority to CRS-viz., on the line segment *kq* IRS prevail to the left of *q*, whereas on the line segment *hg*, DRS prevail to the right of *h*.

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Unless a DMU lies on the production frontier, the interpretation of RTS is not straightforward. That is to say, the RTS for the inefficient DMUs can be identified after applying the projection. It should be noted that input- and output-oriented models may lead to different RTS classifications for the same DMU (Golany and Yu, 1994). We therefore limit our study to the input-oriented radial model (a similar approach can be developed for output efficiency with the straightforward changes).

Now, consider the interval data for $DMU_A$ that is presented by a rectangle. The left width of the rectangle of $DMU_A$ is on the line segments *wk* where IRS prevails. The input-oriented projection (input-reduction) of the residual points of the $DMU_A$ will be located on the line segment *wk* where IRS prevails. Therefore, the RTS classification for $DMU_A$ is IRS where the input and output varies in a range.

As a result, the two points AE=$(x_{io}^L, y_{ro}^L)$ and BE=$(x_{io}^L, y_{ro}^U)$ of each DMU enable us to determine the corresponding RTS classification. For instance, $DMU_A$ exhibits IRS because IRS prevails at the two points AE= (2.5,1) and BE=(2.5,2) where these points are shown in Fig. 1. As another example, consider $DMU_B$ in Fig. 1. Despite of the points AE and BE, it is clear that $DMU_B$ exhibits IRS and CRS simultaneously, which can be called Non-Decreasing Returns-to-Scale (NDRS). In other words, IRS prevails when the input and output lie within [3, 6] and [1, 3), respectively, and CRS prevails when the input lies within [3, 6] input and the output is 3. Interestingly, we can obtain the same result if the two points AE=(3,1) and BE=(3,3) of $DMU_B$ are used for RTS classification. It is obvious that there is no point of $DMU_B$ between points AE and BE that exhibits DRS. Resultantly, we have the following general observation.

**Observation 1:** If $DMU_o$ exhibits IRS (that is, both points AE and BE exhibit IRS), increases in outputs below the CRS output level cannot change its IRS classification.

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**Proof.** Let a DMU be associated with its (optimistic) point BE. Thus, let $\mathrm{DMU}_o = (x_o, y_o)$ exhibit IRS. By Theorem 3 (adapted from Seiford and Zhu, 1999b) the IRS classification remains unchanged for $\alpha \le \sigma_o^*$ as determined in the maximization model (6). Due to the inputs change does not alter the RTS, let us evaluate $(\sigma_o^* x_{io}, \sigma_o^* y_{ro})$ using the maximization model (6) as presented below:

$$(\bar{\sigma}_o)^{-1} = \max \sum_{j \in E_o} \hat{\lambda}_j^*$$

$$\sum_{j \in E_o} \hat{\lambda}_j^* x_{ij} \le \theta_o^* \left(\sigma_o^* x_{io}\right), \qquad i = 1, \ldots, m,$$

$$\sum_{j \in E_o} \hat{\lambda}_j^* y_{rj} \ge \sigma_o^* y_{ro}, \qquad r = 1, \ldots, s,$$

$$\hat{\lambda}_j^* \ge 0 \qquad j \in E_o.$$

where $\theta^*$ is the optimal objective function value of model (4) when evaluating $(\sigma_o^* x_{io}, \sigma_o^* y_{ro})$. Obviously, $\left(\hat{\lambda}_j^* / \sigma_o^*\right) \ge 0, j \in E_o$ is the feasible solution for the maximization model (6) when evaluating $(x_{io}, y_{ro})$. Due to $\left(\sum_{j \in E_o} \hat{\lambda}_j^* / \sigma_o^*\right) \le 1/\sigma_o^* < 1$ we have $\left(\sum_{j \in E_o} \hat{\lambda}_j^*\right) \le 1 < \sigma_o^*$. We already know $(x_{io}, \sigma_o^* y_{ro})$ is not IRS, therefore, $\sum_{j \in E_o} \hat{\lambda}_j^* = 1$ and the RTS of $\mathrm{DMU_o}$ is CRS. □

The rectangles associated with $\mathrm{DMU_C}$ and $\mathrm{DMU_D}$ exhibit CRS and DRS, respectively. $\mathrm{DMU_E}$ reveals DRS and CRS at the same time and it is consequently called Non-Increasing Returns-to-Scale (NIRS). Put differently, DRS prevails when the input and output lie within [10, 12] and (6,6.5], respectively, and CRS prevails when the input and output lies within [10,12] and [5,6], respectively.

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**Observation 2:** If $\mathrm{DMU_o}$ exhibits CRS (that is, both points AE and BE exhibit CRS), increases (decreases) in outputs cannot change its CRS classification unless for point BE, DRS (for point AE, IRS) prevails.

**Proof.** As in the case of Observation 1 above we can now use Theorem 2 (adapted from Seiford and Zhu, 1999b) with respect to points BE and AE respectively. □

The rectangle associated with $\mathrm{DMU_F}$ exhibits IRS, CRS and DRS at the same time. In other words, DRS prevails when the input and output lie within [7, 10] and (6,6.5], respectively, CRS prevails when the input and output lie within [7, 10] and [3,6], respectively and IRS prevails when the input and output lie within [7, 10] and [1,3), respectively. Therefore, the RTS classification of DMU is called Variable Returns-to-Scale (VRS). We obtain this result from the two points AE=(7, 1) and BE=(10, 6.5) which exhibit IRS and DRS, respectively. The DMU is thus classified into a VRS group according to Observation 3.

**Observation 3:** If $\mathrm{DMU_o}$ exhibits IRS (that is, point AE exhibits IRS), and increases in outputs change its IRS to DRS (that is, point BE exhibits DRS), then there is at least one potential output level for which CRS prevails. In other words, when $\mathrm{DMU}_o = (x_o, y_o)$ exhibits IRS, increase the outputs of $\mathrm{DMU}_o = (x_o, y_o)$ by $\gamma > 1$ such that $\mathrm{DMU}'_o = (x_o, \gamma y_o)$ exhibits DRS. Then $\mathrm{DMU}''_o = (x_o, \chi y_o)$ , $1 < \chi < \gamma$ exhibits CRS.

**Proof.** The proof is straightforward according to Observation 1 (omitted). □

## 4. RTS classification with interval data

As discussed and illustrated in the proceeding section, the two points AE= $(x_{io}^L, y_{ro}^L)$ and BE= $(x_{io}^L, y_{ro}^U)$ of each DMU are able to identify the RTS classification. In doing so, we first construct

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AE and BE as the substitutes for a DMU under evaluation and then calculate their best relative [technical] efficiencies using the following [input-oriented] models (8) and (9):

(8):TE-IN-CCR (AE)

$$\begin{aligned}
\min \quad & \theta_o^{AE} \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^L \le \theta_o^{AE} x_{io}^L, && i = 1,\ldots,m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^U \ge y_{ro}^L, && r = 1,\ldots,s, \\
& \lambda_j \ge 0 , && j = 1,\ldots,n.
\end{aligned}$$

(9): TE-IN-CCR (BE)

$$\begin{aligned}
\min \quad & \theta_o^{BE} \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^L \le \theta_o^{BE} x_{io}^L, && i = 1,\ldots,m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^U \ge y_{ro}^U, && r = 1,\ldots,s, \\
& \lambda_j \ge 0 , && j = 1,\ldots,n.
\end{aligned}$$

Each of the above models includes *n* DMUs and their PPSs are identical to Wang et al. (2005)'s models (see models (4) and (5)). In models (8) and (9), DMU$_o$ represents AE and BE, respectively, taken the role of a DMU under evaluation in which $x_{io}^L$ and $y_{ro}^L$ are the $i^{th}$ input and $r^{th}$ output for AE, and $x_{io}^L$ and $y_{ro}^U$ are the $i^{th}$ input and $r^{th}$ output for BE. The optimal solution of models (8) and (9) can be represented by $\theta_o^{AE*}$ and $\theta_o^{BE*}$, respectively, where $0 < \theta_o^{AE*} \le \theta_o^{BE*} \le 1$. Note that DMU$_o$ is efficient if $\theta_o^{BE*} = 1$ and, consequently, a set of efficient DMUs, denoted by $E_o$, can be defined.

Adapting the Seiford and Zhu (1999b) framework for each point of AE= $(x_{io}^L, y_{ro}^L)$ and BE= $(x_{io}^L, y_{ro}^U)$ results in the following pair of models:

(10):EFF-IN-CCR (AE)

$$\begin{aligned}
& (\sigma_o^{AE})^{-1} = \max \sum_{j \in E_o} \lambda_j \qquad (\tau_o^{AE})^{-1} = \min \sum_{j \in E_o} \lambda_j \\
s.t. \quad & \sum_{j \in E_o} \lambda_j x_{ij}^L \le \theta_o^{AE*} x_{io}^L, && i = 1,\ldots,m, \\
& \sum_{j \in E_o} \lambda_j y_{rj}^U \ge y_{ro}^L, && r = 1,\ldots,s, \\
& \lambda_j \ge 0 , && j \in E_o.
\end{aligned}$$

(11): EFF-IN-CCR (BE)

$$\begin{aligned}
& (\sigma_o^{BE})^{-1} = \max \sum_{j \in E_o} \lambda_j \qquad (\tau_o^{BE})^{-1} = \min \sum_{j \in E_o} \lambda_j \\
s.t. \quad & \sum_{j \in E_o} \lambda_j x_{ij}^L \le \theta_o^{B*} x_{io}^L, && i = 1,\ldots,m, \\
& \sum_{j \in E_o} \lambda_j y_{rj}^U \ge y_{ro}^U, && r = 1,\ldots,s, \\
& \lambda_j \ge 0 , && j \in E_o.
\end{aligned}$$

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where $\theta_o^{AE*}$ and $\theta_o^{BE*}$ are the optimal solution of models (8) and (9), respectively and $E_o = \{$all efficient DMUs in model (9)$\}$. It is worth noting that each of programs (10) and (11) include two separate linear programming models which possess the similar set of constraints, but the objective functions of (10) and (11) are independently maximized and minimized. The linear programs (10) and (11) have a unified PPS based on the set of efficient DMUs, $E_o$, subject to all DMUs consume the least inputs to secure the most outputs. However, programs (10) and (11) take the AE and BE, respectively, in lieu of the DMU under evaluation. At present, we customize Theorem 1 to propose the following definitions:

I. AE (or BE) exhibits IRS if and only if $\sigma_o^{AE*} > 1$ $(\sigma_o^{BE*} > 1)$.

II. AE (or BE) exhibits DRS if and only if $\tau_o^{AE*} < 1$ $(\tau_o^{BE*} < 1)$.

III. AE (or BE) exhibits CRS if and only if $\sigma_o^{*AE} \leq 1 \leq \tau_o^{*AE}$ $(\sigma_o^{*BE} \leq 1 \leq \tau_o^{*BE})$.

Let us return to the earlier example in Section 3 to illustrate the above step of the proposed method. To identify RTS of AE= (2.5, 1) and BE=(2.5, 2) associated with $DMU_A$ we use the optimal solution of models (10) and (11) which are $\sigma_o^{AE*} = 3 > 1$ $(\sigma_o^{BE*} = 1.5 > 1)$. Thereby, AE and BE exhibit IRS. Table 1 is summarized the findings for all the DMUs.

-------**Insert Table 1 here**-------

The following conditions determine the RTS classification of $DMU_o$ in terms of its RTS estimation for the two points AE and BE:

***Con. 1.*** $DMU_o$ exhibits IRS iff BE exhibits IRS.

***Con. 2.*** $DMU_o$ exhibits DRS iff AE exhibits DRS.

***Con. 3.*** $DMU_o$ exhibits CRS iff AE and BE exhibit CRS.

***Con. 4.*** $DMU_o$ exhibits NDRS iff AE and BE exhibit IRS and CRS, respectively.

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***Con. 5.*** $\text{DMU}_o$ exhibits NIRS iff AE and BE exhibit CRS and DRS, respectively.

***Con. 6.*** $\text{DMU}_o$ exhibits VRS iff AE and BE exhibit IRS and DRS, respectively.

We draw the attention to the fact that the following conditions *never* occurs for $\text{DMU}_o$:

***Con. 7.*** AE and BE exhibit DRS and IRS, respectively.

***Con. 8.*** AE and BE exhibit DRS and CRS, respectively.

***Con. 9.*** AE and BE exhibit CRS and IRS, respectively.

For example, $\text{DMU}_A$ exhibits IRS because of Con. 1. The last column of Table 1 reports the RTS estimation of all the DMUs in terms of the above conditions.

As illustrated in Section 3, NDRS, NIRS and VRS individually encompass different RTS (CRS, IRS or DRS). Therefore, when $\text{DMU}_o$ according to Con. 4, 5, or 6 is classified as having either NDRS, NIRS or VRS, we present three propositions regarding sensitivity analysis to identify its different partitions.

**Proposition 1.** If $\text{DMU}_o$ exhibits NDRS (involving CRS and IRS) derived from Con. 4, then its CRS and IRS classifications are unaltered under the following conditions, respectively:

(I) IRS prevails when the inputs and outputs of $\text{DMU}_o$ lie within $[x_{io}^L, x_{io}^U]$ and $[y_{ro}^L, y'_{ro})$ where $y'_{ro} = \min\{\sigma_o^{AE*} y_{ro}^L, y_{ro}^U\}$ and $\sigma_o^{AE*}$ is the optimal solution of (10).

(II) CRS prevails when the inputs and outputs of $\text{DMU}_o$ lie within $[x_{io}^L, x_{io}^U]$ and $[y'_{ro}, y_{ro}^U]$ where $y'_{ro} = \min\{\sigma_o^{AE*} y_{ro}^L, y_{ro}^U\}$ and $\sigma_o^{AE*}$ is the optimal solution of (10).

**Proof (I).** Assume that $\text{DMU}'_o = (x_{io}^L, \alpha y_{ro}^L)$ where $\alpha \in [1, \sigma_o^{AE*})$ and its RTS is not IRS. Therefore, the RTS on $\text{DMU}'_o$ will be either DRS or CRS. In this regard,

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$\mathrm{DMU}''_o = (\alpha\theta_o^{*AE} x_{io}^L, \alpha y_{ro}^L)$ will be either DRS or CRS because the inputs change does not impact on RTS. Therefore, we have:

$$\begin{aligned}
&\sum_{j\in E_o} \lambda_j^* x_{ij}^L \le \varphi^* \left(\alpha\theta_o^{*AE} x_{io}^L\right) \le \alpha\theta_o^{*AE} x_{io}^L, \qquad i = 1,\ldots,m,\\
&\sum_{j\in E_o} \lambda_j^* y_{rj}^U \ge \left(\alpha y_{ro}^L\right), \qquad r = 1,\ldots,s,\\
&\sum_{j\in E_o} \lambda_j^* \ge 1, \qquad\qquad (i)\\
&\lambda_j^* \ge 0, \qquad j \in E_o.
\end{aligned}$$

where $\varphi^*$ is the optimal objective function value of model (8) when assessing $\mathrm{DMU}''_o$. Evidently, $\left(\lambda_j^*/\alpha\right) \ge 0 \ \left(j \in E_o\right)$ is the feasible solution for model (10). According to the above equation ($i$), $\left(\sum_{j\in E_o} \lambda_j^*/\alpha\right) \ge 1/\alpha > 1/\sigma_o^{AE*}$ failing the optimality of (10).□

**Proof (II).** We need to prove that $(x_{io}^L, \alpha y_{ro}^U), \left(\sigma_o^{AE*} y_{ro}^L / y_{ro}^U \le \alpha \le 1\right)$ exhibits CRS where $\left(x_{io}^L, y_{ro}^U\right)$ is CRS. Obviously, the RTS of $(x_{io}^L, \alpha y_{ro}^U)$ and $(\alpha x_{io}^L, \alpha y_{ro}^U)$ are identical. In addition, $\left(\left(\sigma_o^{AE*} y_{ro}^L / y_{ro}^U\right) x_{io}^L, \left(\sigma_o^{AE*} y_{ro}^L / y_{ro}^U\right) y_{ro}^U\right)$ exhibits CRS since $\left(x_{io}^L, y_{ro}^U\right)$ is CRS (see Thrall and Banker, 1992). As a result, $(x_{io}^L, \alpha y_{ro}^U), \left(\sigma_o^{AE*} y_{ro}^L / y_{ro}^U \le \alpha \le 1\right)$ exhibits CRS.□

**Proposition 2.** If $\mathrm{DMU_o}$ exhibits NIRS (involving CRS and DRS) derived from Con. 5, then its CRS and DRS classifications are unaltered under the following conditions, respectively:

(I) DRS prevails when the inputs and outputs of $\mathrm{DMU_o}$ lie within $[x_{io}^L, x_{io}^U]$ and $(y''_{ro}, y_{ro}^U]$ where $y''_{ro} = \max\{\tau_o^{BE*} y_{ro}^U, y_{ro}^L\}$ and $\tau_o^{BE*}$ is the optimal solution of (11).

(II) CRS prevails when the inputs and outputs of $\mathrm{DMU_o}$ lie within $[x_{io}^L, x_{io}^U]$ and $[y_{ro}^L, y''_{ro}]$ where $y''_{ro} = \max\{\tau_o^{BE*} y_{ro}^U, y_{ro}^L\}$ and $\tau_o^{BE*}$ is the optimal solution of (11).

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**Proof (I).** Assume that $\text{DMU}'_o = (x_{io}^L, \alpha y_{ro}^L)$ where $\alpha \in (\tau_o^{BE*}, 1]$ and its RTS is not DRS. Therefore, the RTS on $\text{DMU}'_o$ will be either IRS or CRS. In this regard, $\text{DMU}''_o = (\alpha\theta_o^{BE*} x_{io}^L, \alpha y_{ro}^U)$ will be either IRS or CRS because the inputs change does not impact on RTS. Therefore, we have:

$$
\begin{aligned}
&\sum_{j\in E_o} \lambda_j^* x_{ij}^L \le \varphi^*\left(\alpha\theta_o^{BE*} x_{io}^L\right) \le \alpha\theta_o^{BE*} x_{io}^L, && i = 1,\ldots,m,\\
&\sum_{j\in E_o} \lambda_j^* y_{rj}^U \ge \alpha y_{ro}^U, && r = 1,\ldots,\text{s},\\
&\sum_{j\in E_o} \lambda_j^* \le 1, && (i)\\
&\lambda_j^* \ge 0, && j \in E_o.
\end{aligned}
$$

where $\varphi^*$ is the optimal objective function value of model (9) when assessing $\text{DMU}''_{\text{o}}$. Evidently, $\left(\lambda_j^*/\alpha\right) \ge 0$ $\left(j \in E_o\right)$ is the feasible solution for model (11). According to the above equation ($i$), $\left(\sum_{j\in E_o} \lambda_j^*/\alpha\right) \le 1/\alpha < 1/\tau_o^{\text{BE*}}$ failing the optimality of (11).□

**Proof (II).** We need to prove that $(x_{io}^L, \alpha y_{ro}^L)$, $\left(1 \le \alpha \le \tau_o^{BE*} y_{ro}^U / y_{ro}^L\right)$ exhibits CRS where $\left(x_{io}^L, y_{ro}^L\right)$ is CRS. Obviously, the RTS of $(x_{io}^L, \alpha y_{ro}^L)$ and $(\alpha x_{io}^L, \alpha y_{ro}^L)$ are identical. In addition, $\left(\left(\tau_o^{BE*} y_{ro}^U / y_{ro}^L\right) x_{io}^L, \left(\tau_o^{BE*} y_{ro}^U / y_{ro}^L\right) y_{ro}^L\right)$ exhibits CRS since $\left(x_{io}^L, y_{ro}^L\right)$ is CRS (see Thrall and Banker, 1992). As a result, $(x_{io}^L, \alpha y_{ro}^L)$, $\left(1 \le \alpha \le \tau_o^{\text{BE*}} y_{ro}^U / y_{ro}^L\right)$ exhibits CRS.□

**Proposition 3.** If $\text{DMU}_\text{o}$ exhibits VRS (involving CRS, IRS and DRS) derived from Con. 6, its CRS, IRS and DRS classifications are unaltered under the following conditions, respectively:

(I) DRS prevails when the inputs and outputs of $\text{DMU}_\text{o}$ lie within $[x_{io}^L, x_{io}^U]$ and $(\tau_o^{\text{BE*}} y_{ro}^U, y_{ro}^U]$ and $\tau_o^{\text{BE*}}$ is the optimal solution of (11).

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(II) CRS prevails when the inputs and outputs of $\mathrm{DMU_o}$ lie within $[x_{io}^{L}, x_{io}^{U}]$ and $[\sigma_o^{AE*} y_{ro}^{L}, \tau_o^{\mathrm{BE}*} y_{ro}^{U}]$ and $\sigma_o^{AE*}$ is the optimal solution of (10).

(III) IRS prevails when the inputs and outputs of $\mathrm{DMU_o}$ lie within $[x_{io}^{L}, x_{io}^{U}]$ and $[y_{ro}^{L}, \sigma_o^{AE*} y_{ro}^{L}]$ and $\sigma_o^{AE*}$ is the optimal solution of (10).

**Proof (I).** The proof is straightforward according to Proposition 2 (omitted). □

**Proof (II).** The proof is straightforward according to Propositions 1 and 2 (omitted). □

**Proof (III).** The proof is straightforward according to Proposition 1 (omitted). □

Returning to the example in Section 3 we now consider, $\mathrm{DMU_B}$, $\mathrm{DMU_E}$ and $\mathrm{DMU_F}$, that exhibit NDRS, NIRS and VRS and explain how to implement a sensitivity analysis as above. The set of efficient DMUs is $E_o$ ={$\mathrm{DMU_B}$, $\mathrm{DMU_C}$} since their upper efficiency scores derived from model (9) are equal to unity (see the 8th column of Table 1). As can be seen in the last column of Table 1, $\mathrm{DMU_B}$, $\mathrm{DMU_E}$ and $\mathrm{DMU_F}$ display NDRS, NIRS and VRS, respectively. Model (10) for $\mathrm{DMU_B}$ with the situation for NDRS is formulated as follows:

$$
\begin{aligned}
(\sigma_B^{AE})^{-1} &= \max \sum_{j=1}^{2} \hat{\lambda}_j \\
s.t. \quad & 3\lambda_1 + 6\lambda_2 \le 0.33 \times 3 \\
& 3\lambda_1 + 6\lambda_2 \ge 1 \\
& \lambda_1, \lambda_2 \ge 0.
\end{aligned}
$$

where AE=(3,1), $\theta_B^{AE*} = 0.33$ and the optimal value of the above model, $(\sigma_B^{AE})^{-1}$, for $\mathrm{DMU_B}$ is 0.167. According to Proposition 1, the region of IRS and CRS for $\mathrm{DMU_B}$ can be obtained as:

(I) IRS prevails at $\mathrm{DMU_B}$ when its input and output vary within [3, 6] and [1, 3), respectively. Note that $y'_{rB} = \min\{5.988, 3\} = 3$.

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(II) CRS prevails at $DMU_B$ when its inputs of $DMU_o$ vary within [3, 6] and its output is 3. Note that $[y'_{rB}, y^U_{rB}] = [3,3]$ where $y'_{rB} = \min\{5.988, 3\} = 3$.

Model (11) for $DMU_E$ with the situation for NIRS is described as follows:

$$
\begin{aligned}
(\tau_E^{BE})^{-1} &= \min \sum_{j=1}^{2} \hat{\lambda}_j \\
s.t. \quad & 3\lambda_1 + 6\lambda_2 \le 0.65 \times 10 \\
& 3\lambda_1 + 6\lambda_2 \ge 6.5 \\
& \lambda_1, \lambda_2 \ge 0.
\end{aligned}
$$

where BE=(10, 6.5), $\theta_E^{BE*} = 0.65$ and the optimal value of the above model, $(\tau_E^{BE})^{-1}$, for $DMU_E$ is 1.0833. According to Proposition 2, the region of DRS and CRS for $DMU_E$ can be determined as:

(I) DRS prevails at $DMU_E$ when its input and output vary within [10,12] and (6,6.5], respectively. Note that $y''_{rE} = \max\{6, 6\} = 6$.

(II) CRS prevails at $DMU_E$ when its input and output vary within [10,12] and [5,6], respectively. Note that $y''_{rE} = \max\{6, 6\} = 6$.

According to the proposed algorithm, we use models (10) and (11) for $DMU_F$ with the situation for VRS as follows:

$$
\begin{aligned}
(\sigma_F^{AE})^{-1} &= \max \sum_{j=1}^{2} \hat{\lambda}_j \\
s.t. \quad & 3\lambda_2 + 6\lambda_3 \le 0.142 \times 7 \\
& 3\lambda_2 + 6\lambda_3 \ge 1 \\
& \lambda_1, \lambda_2 \ge 0,
\end{aligned}
\qquad
\begin{aligned}
(\tau_F^{BE})^{-1} &= \min \sum_{j=1}^{2} \hat{\lambda}_j \\
s.t. \quad & 3\lambda_2 + 6\lambda_3 \le 0.92 \times 7 \\
& 3\lambda_2 + 6\lambda_3 \ge 6.5 \\
& \lambda_1, \lambda_2 \ge 0.
\end{aligned}
$$

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where AE=(7,1), BE=(7, 6.5), $\theta_F^{AE*} = 0.142$, $\theta_F^{BE*} = 0.928$ and the optimal values of the above models are $(\sigma_F^{AE})^{-1} = 0.3333$ and $(\tau_F^{BE})^{-1} = 1.0833$. According to Proposition 3, the region of CRS, IRS and DRS for $DMU_F$ can be determined as:

(I) DRS prevails at $DMU_F$ when the inputs and outputs of $DMU_o$ lie within [7,10] and (6,6.5], respectively.

(II) CRS prevails at $DMU_F$ when the inputs and outputs of $DMU_o$ lie within [7,10] and [3,6], respectively.

(III) IRS prevails at $DMU_F$ when the inputs and outputs of $DMU_o$ lie within [7,10] and [1, 3), respectively.

## 5. Stability of the RTS classification with interval data

This section presents the stability regions of RTS classification when inputs and outputs are given by intervals. Given the identification of the RTS of DMUs in the input-oriented perspective, the input perturbations cannot change the RTS classification and we only need to study the output perturbations (Seiford and Zhu, 1999b)[2].

Again, consider the example in Section 3 and focus on $DMU_E$ that is classified as NIRS (i.e., its points BE and AE are DRS and CRS, respectively). The output increase of $DMU_E$ expresses the output increase[3] of point BE in $DMU_E$ and this DMU still exhibits DRS. Therefore, the RTS of $DMU_E$ remains unchanged since the RTS of point BE is unchanged with the output augmentation. The output reduction in $DMU_E$ prompts the output reduction[4] of point AE in $DMU_E$ and this change is able to turn RTS into VRS, but it definitely depends on the amount of the output reduction. As a result, the impact of the output perturbation on two points AE and BE

[2] Note that in the case of the output-oriented perspective, the output perturbations cannot alter the RTS nature of DMU under evaluation and it is only essential to scrutinize the input perturbations for the RTS estimation of DMUs.
[3] The output increase means an increase in the upper bound of the output.
[4] The output reduction means a reduction in the lower bound of the output.



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of the DMU under evaluation enable us to find the stability of RTS classifications. We here explain how to identify the stability region of six types of RTS:

(1) *A DMU with IRS*: If a DMU exhibits IRS, then the reduction of its outputs does not alter the RTS classification and only the output augmentation has the capability to change the RTS classification. We identify the stability region for the virtual point BE using model (11) and Theorem 3 when its outputs are increased. In this regard, the IRS of the point BE remains unaltered for $1 \leq \alpha \leq \sigma_o^{BE*}$ where α is a proportional increase in all outputs of the virtual point BE $(y_r^{new(BE)} = \alpha y_r^{BE})$. Therefore, the stability region of IRS for the DMU is $[y_{ro}^L, \alpha y_{ro}^U]$ where $1 \leq \alpha < \sigma_o^{BE*}$.

(2) *A DMU with DRS*: If a DMU exhibits DRS, then the augmentation of its outputs does not alter the RTS classification and only the output reduction is able to change the RTS classification. We identify the stability region for the virtual point AE using model (10) and Theorem 4 when its outputs are reduced. In this regard, the DRS of the point AE remains unaltered for $\tau_o^{AE*} \leq \beta \leq 1$ where $\beta$ is a proportional reduction in all outputs of the virtual point AE ( $y_r^{new(AE)} = \beta y_r^{(AE)}$ ). Therefore, the stability region of DRS for the DMU is $[\beta y_{ro}^L, y_{ro}^U]$ where $\tau_o^{AE*} < \beta \leq 1$.

(3) *A DMU with CRS*: If a DMU exhibits CRS then the variation of its outputs (increase and/or decrease) may alter the RTS classification. We identify the stability region for the virtual points AE and BE using models (10) and (11) as well as Theorem 2 when its outputs are varied. The stability region of CRS for the DMU is $[\beta y_{ro}^L, \alpha y_{ro}^U]$ where $(\min\{1, \sigma_o^{AE*}\} \leq \beta \leq 1)$ and $(1 \leq \alpha \leq \max\{1, \tau_o^{BE*}\})$.

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(4) *A DMU with NDRS*: If a DMU exhibits NDRS, then the reduction of its outputs does not alter the RTS classification and only the output augmentation has the capability to change the RTS classification. We identify the stability region for the virtual point BE using model (11) and Theorem 2 when its outputs are increased. The stability region of NDRS for the DMU is $[y_{ro}^{L}, \alpha y_{ro}^{U}]$ where $(1 \le \alpha < \max\{1, \tau_{o}^{BE*}\})$.

(5) *A DMU with NIRS*: If a DMU exhibits NIRS, then the augmentation of its outputs does not alter the RTS classification and only the output reduction is able to change the RTS classification. We identify the stability region for the virtual point AE using model (10) and Theorem 2 when its outputs are reduced. The stability region of NIRS for the DMU is $[\beta y_{ro}^{L}, y_{ro}^{U}]$ where $(\min\{1, \sigma_{o}^{AE*}\} < \beta \le 1)$.

(6) *A DMU with VRS*: If a DMU is VRS, then the variation of its outputs (increase and/or decrease) does not alter the RTS classification.

Let us consider all six DMUs of an earlier example in Section 3 (see Fig. 1) to detail the formulation and solution to the sensitivity analysis of RTS classification.

The set of efficient DMUs is $E_o$ ={$DMU_B$, $DMU_C$} because their upper efficiency scores flowed from the maximization model (6) are equal to 1.

(1) $DMU_A$ shows IRS. We evaluate the point BE=(2.5,2) using model (11) as follows:

$$
\begin{aligned}
&(\sigma_{A}^{BE})^{-1} = \max \sum_{j=1}^{2} \hat{\lambda}_{j} \\
&s.t. \quad 3\lambda_2 + 6\lambda_3 \le 0.8 \times 2.5 \\
&\qquad\ \ 3\lambda_2 + 6\lambda_3 \ge 2 \\
&\qquad\ \ \lambda_1, \lambda_2 \ge 0.
\end{aligned}
$$

The optimal solution of the above model is $(\sigma_{A}^{BE*})^{-1} = 0.666$ and the RTS keeps unchanged when the output of $DMU_A$ varies within [1, 3).

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(2) $\mathrm{DMU_B}$ exhibits NDRS. We evaluate the point BE=(3,3) using model (11) as follows:

$$
\begin{aligned}
(\sigma_B^{BE})^{-1} &= \max \sum_{j=1}^{2} \hat{\lambda}_j \\
s.t. \quad & 3\lambda_2 + 6\lambda_3 \le 3 \\
& 3\lambda_2 + 6\lambda_3 \ge 3 \\
& \lambda_1, \lambda_2 \ge 0.
\end{aligned}
$$

The optimal solution of the above model is $(\sigma_B^{BE*})^{-1} = 1$ and the RTS keeps unchanged when the output of $\mathrm{DMU_B}$ varies within [1, 3].

(3) $\mathrm{DMU_C}$ exhibits CRS. We assess the points BE=(6,6) and AE=(6,4) using models (11) and (10), respectively, as follows:

$$
\begin{aligned}
(\tau_C^{BE})^{-1} &= \min \sum_{j=1}^{2} \hat{\lambda}_j \\
s.t. \quad & 3\lambda_2 + 6\lambda_3 \le 6 \\
& 3\lambda_2 + 6\lambda_3 \ge 6 \\
& \lambda_1, \lambda_2 \ge 0.
\end{aligned}
\qquad
\begin{aligned}
(\sigma_C^{AE})^{-1} &= \max \sum_{j=1}^{2} \hat{\lambda}_j \\
s.t. \quad & 3\lambda_2 + 6\lambda_3 \le 0.33 \times 6 \\
& 3\lambda_2 + 6\lambda_3 \ge 4 \\
& \lambda_1, \lambda_2 \ge 0.
\end{aligned}
$$

The optimal solution of the above models are $(\tau_C^{BE*})^{-1} = 1$ and $(\sigma_C^{AE*})^{-1} = 1.3333$. Therefore, the RTS keeps unchanged when the output of $\mathrm{DMU_C}$ varies within [3, 6].

(4) $\mathrm{DMU_D}$ exhibits DRS. We evaluate the point AE=(7.5,6.5) using model (10) as follows:

$$
\begin{aligned}
(\tau_D^{AE})^{-1} &= \min \sum_{j=1}^{2} \hat{\lambda}_j \\
s.t. \quad & 3\lambda_2 + 6\lambda_3 \le 0.86 \times 7.5 \\
& 3\lambda_2 + 6\lambda_3 \ge 6.5 \\
& \lambda_1, \lambda_2 \ge 0,
\end{aligned}
$$

The optimal solution of the above model is $(\tau_D^{AE*})^{-1} = 1.0833$. Therefore, the RTS keeps unchanged when the output of $\mathrm{DMU_D}$ varies within (6, 7].

(5) $\mathrm{DMU_E}$ exhibits NIRS. We evaluate the point AE=(10,5) using model (10) as follows:

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$$
\begin{aligned}
&(\sigma_E^{AE})^{-1} = \max \sum_{j=1}^{2} \hat{\lambda}_j \\
&s.t. \quad 3\lambda_2 + 6\lambda_3 \le 0.5 \times 10 \\
&\qquad\;\; 3\lambda_2 + 6\lambda_3 \ge 5 \\
&\qquad\;\; \lambda_1, \lambda_2 \ge 0,
\end{aligned}
$$

The optimal solution of the above model is $(\sigma_E^{AE*})^{-1} = 1.6666$. Therefore, the RTS keeps unchanged when the output of $DMU_E$ varies within [3,6.5].

(6) $DMU_F$ exhibits VRS and its RTS preserves fixed with any output variations.

## 6. Illustrative Example

In order to illustrate our method as well as to compare with the conventional approach, this section presents a real-world data set with 28 Chinese cities (DMUs) with three inputs and three outputs in 1983 from Charnes et al. (1989). Seiford and Zhu (1999b) used the data set to implement their RTS sensitivity analysis on the inefficient DMUs without considering imprecision in data (see the 12$^{th}$ column of Table 3). We think of imprecision as a variable having a "true" value added a percentage of uncertainty, +/- 5%, in data of 28 DMUs in 1983, as represented in Table 2. For instance, $DMU_1$ consumes three interval inputs [463.068, 511.812],[1514338, 1673742] and [683005.4, 754900.7] to produce three interval outputs [7071515, 7815885],[1607495, 1776705] and [1257610, 1389990].

**-------Insert Table 2 here-------**

We first calculate the efficiency of points AE and BE for each DMU using models (8) and (9) as reported in the 2$^{nd}$ and 3$^{rd}$ columns in Table 3. The DMUs {$DMU_1$, $DMU_{11}$, $DMU_{19}$, $DMU_{21}$, $DMU_{22}$, $DMU_{23}$, $DMU_{24}$, $DMU_{25}$, $DMU_{26}$, $DMU_{28}$} are efficient on account of $\theta_j^{BE*} = 1$. The 4$^{th}$

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column of Table 3 presents the set of DMUs whose $\lambda$s are strictly positive in model (4). To determine the RTS of DMU we solve models (10) and (11) with respect to the efficiency of points AE and BE as its the optimal objective function value are reported in 5th, 6th, 8th and 9th columns of Table 3. The 7th and 10th columns of Table 3 represent the RTS of points AE and BE while the 11th column indicates the RTS of DMU under assessment in terms of $(\sigma_o^{AE*})^{-1}, (\tau_o^{BE*})^{-1}$ , $(\sigma_o^{AE*})^{-1}$ and conditions 1-6.

**-------Insert Table 3 here-------**

As a result, DMUs {$DMU_1$, $DMU_{11}$, $DMU_{19}$, $DMU_{21}$, $DMU_{22}$, $DMU_{23}$, $DMU_{24}$, $DMU_{25}$, $DMU_{26}$, $DMU_{28}$} exhibit NDRS where all of them are a composite of two partitions IRS and CRS, DMUs {$DMU_6$} exhibit VRS where all of them are a mix of three partitions: CRS, IRS and DRS, DMUs {$DMU_2$, $DMU_3$, $DMU_4$, $DMU_5$} exhibit DRS, and the remaining DMUs exhibit IRS. Table 4 shows the different RTS regions of DMUs whose RTS exhibit NDRS and VRS in terms of their outputs. Notice that the change in inputs does not influence the RTS classification.

**-------Insert Table 4 here-------**

The stability region of DMUs with the interval observations keeping their present RTS classifications are identified by means of the discussion in Section 5. The results are presented in the last column on Table 3 where "N.G." or (Non-Change) presents the DMUs {$DMU_1$, $DMU_6$, $DMU_{11}$, $DMU_{19}$, $DMU_{22}$, $DMU_{23}$, $DMU_{24}$, $DMU_{25}$, $DMU_{26}$, $DMU_{28}$} which do not alter the RTS classification with the output reduction or outputs augmentation.

Even though we have introduced imprecision in the data set (+/- 5%), the overall structure of the RTS classification remains almost the same as under the conventional approach with precise data. The main difference is concerned with $DMU_8$ where its RTS classification is IRS and CRS for imprecise and precise scenarios, respectively.  Adding data imprecision, we obtain a more

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nuanced picture where, for instance, $DMU_1$ is now classified as NDRS including IRS and CRS regions (see Table 4) whereas with precise data it was classified as CRS and $DMU_6$ is now VRS including IRS, CRS and DRS regions (see Table 4) versus CRS with precise data etc. Further nuances are illustrated by the stability regions (column 13 in Table 3) and mapping of RTS regions in Table 4.

It is important to emphasize that our approach does not generate useless vague results where most DMUs are classified as exhibiting every kind of RTS despite adding imprecision to the data set. Clearly this is caused by the fact that we still refer to a single common efficient frontier.

## 7. Case study

Every combined cycle power plant includes gas and steam turbines for producing electricity in which the gas turbine consumes fossil fuels such as gas and gasoline to produce electricity and at the same time the steam turbine uses the leftover (exhaust) heat released from the gas turbine to produce electricity. While the purpose of combined cycle power plants is to produce electricity, some emissions and pollutions are unavoidably and undesirably produced through the production processes. In this section, we determine RTS classification and stability region of seventeen combined cycle power plants in Iran over six years in the presence of undesirable variables (pollution) and interval data. The production process includes one input; *Fossil fuel* ($m^3$), and fours undesirable outputs $CO_2$ (ton), $SO_2$(ton), $SO_3$(ton), and $NO_x$(ton); as well as a single desirable output; *Electricity* (thousand kilo-watts per hour). The structural pattern of the combined cycle power plant is depicted in Fig. 2.

**-------Insert Fig. 2 here-------**

We use a dataset consisting of interval data for 17 combined cycle power plants from Khalili-Damghani et al. (2015). Data are reported in Table 5, referring to Khalili-Damghani et al. (2015)

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for further data description. Let us first get to grips with undesirable outputs which are often observed through environmental efficiencies. To address this challenge, several approaches have been developed and discussed in the DEA literature. Dyckhoff and Allen (2001) organised the most approaches for tackling undesirable outputs into three categories: (i) making use of the reciprocal of the undesirable outputs in a way that is changed to the desirable one (Scheel, 2001), (ii) making use of a multi-criteria approach in a way that is regarded as an input (Rheinhard et al., 1999), and (iii) making use of the translation property observed in BCC and additive DEA models in a way that a positive scalar can be added to the reciprocal additive transformation of the undesirable output (Ali and Seiford, 1990). Other than the last category that is appropriate for special DEA models, considering an undesirable output as an input or utilising its reciprocal seems to be straightforward. In the present study, we therefore consider the four undesirable outputs as inputs of the process.

-------**Insert Table 5 here**-------

We calculate the efficiency of points AE and BE for each combined cycle power plant using models (8) and (9) as reported in the 2nd and 3rd columns of Table 6. Power plant 12 is known as an efficient unit on account of $\theta_{12}^{B*} = 1$, and it can be recognised as the reference set for all other inefficient plants. To identify the RTS of power plants, models (10) and (11) are first solved by making use of the efficiency of points AE and BE.

-------**Insert Table 6 here**-------

The optimal values of objective functions for models (10) and (11) are reported in the 4th and 5th columns of Table 6 when our focus is on point AE, and the 7th and 8th columns of Table 6 are allocated to the optimal values of objective functions for models (10) and (11) when one thinks of point BE. Thereafter, the RTS of points AE and BE can be determined as represented in the

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$6^{th}$ and $9^{th}$ columns of Table 6. Concerning point AE, the power plants {1,2,3,4,6,9,10,13,17} all exhibit DRS and the remaining plants reveal IRS. For point BE, all plants exhibit DRS, except for plants {11, 12} that exhibits CRS. At present, we are able to specify the RTS of all the power plants by the use of the RTS of points AE and BE (see Conditions 1-6 in Section 4). The results including three partitions; DRS, VRS and NDRS appear in the $10^{th}$ column of Table 6. In the case of DRS, the smaller firms outperform the bigger ones. Thereby, the power plants {1,2,3,4,6,9,10,13,17} are required to scale down their production. The power plants {5,7,8,14,15,16} and {11,12} exhibit VRS and NDRS, respectively, whereby the VRS situation subsumes a mix of CRS, IRS and DRS regions, and the NDRS situation subsumes a combination of CRS and IRS regions (see Table 7). It is worth noting that those power plants which exhibit VRS and NDRS will grab the utmost situation if the value of their output (electricity power) is aligned in terms of the CRS partition defined in the $4^{th}$ column of Table 7.

-------**Insert Table 7 here**-------

With reference to the last column of Table 6, we scrutinise the stability region of the RTS classification (argued in Section 5) to determine in which situation the present RTS classification of power plants remain unchanged.

## 8. Final remarks

In the present paper, we have shown how one could can extend the RTS classification of standard DEA models to Imprecise DEA where the input and output data take the form of intervals. In short, our idea relates RTS classification to one common frontier for all DMUs and that frontier becomes the frontier spanned by the most optimistic data for all DMUs. Once this frontier is in place we can utilize RTS characterizations of standard (crisp) DEA models and

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analyse the sensitivity of these using the Seiford-Zhu approach (Seiford and Zhu, 1999b). We illustrate our approach on a well know data set from Charnes et al. (1989).

Fundamentally, our approach builds on the assumption that the data imprecision represented by the interval data can be seen as a relative uncertainty around some “true” value of the variables. However, many other forms of imprecision can be imagined and we leave for future research a deeper analysis of the connection between imprecision measures and RTS classification. Another future research opportunity would be to dictate how the proposed approach can be applied to solve a practical case in the presence of imprecise data.

**Appendix 1**

**BT theorem . *(Banker and Thrall, 1992)*** Suppose that $(\hat{x}_o, \hat{y}_o)$ is CCR efficient and $\lambda_j^*$ is an optimal solution of model (1).

(I) Increasing returns-to-scale (IRS) prevail at $(\hat{x}_o, \hat{y}_o)$ if $\sum_{j=1}^{n} \lambda_j^* < 1$ for all alternate optimal solutions.

(II) Decreasing returns-to-scale (DRS) prevail at $(\hat{x}_o, \hat{y}_o)$ if $\sum_{j=1}^{n} \lambda_j^* > 1$ for all alternate optimal solutions.

(III) Constant returns-to-scale (CRS) prevail at $(\hat{x}_o, \hat{y}_o)$ if $\sum_{j=1}^{n} \lambda_j^* = 1$ in some optimal solutions.

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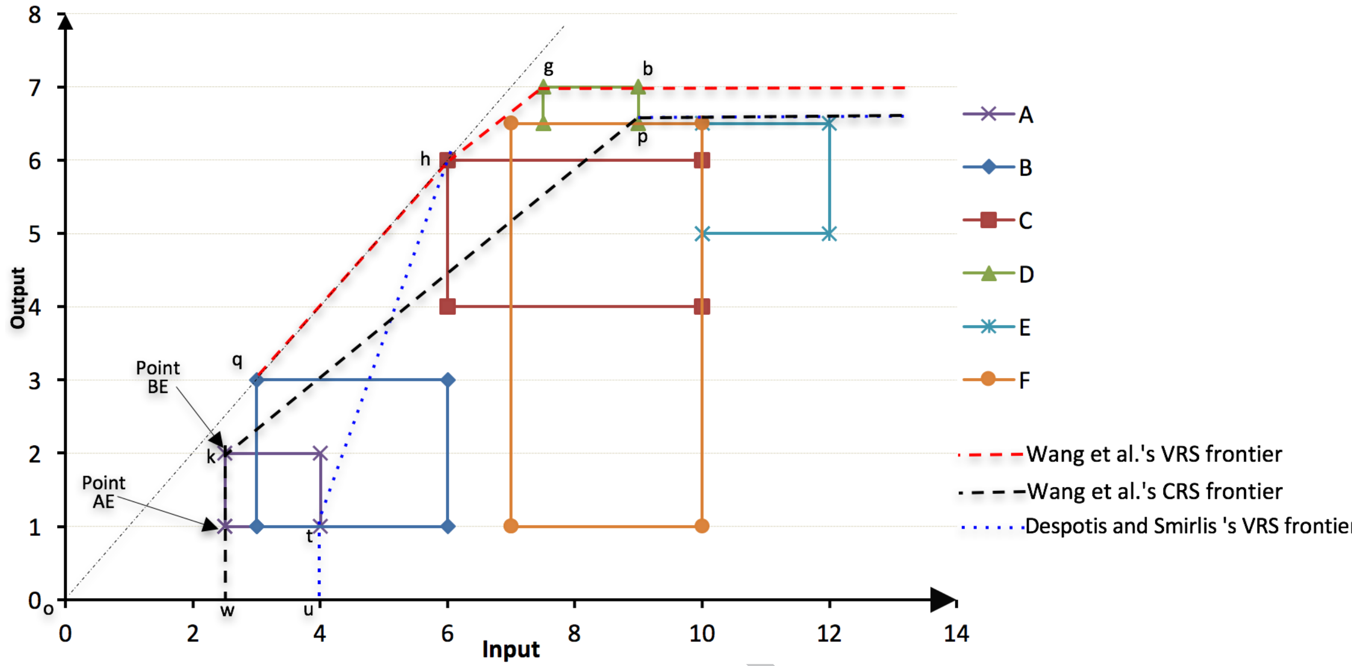


**Fig. 1. Different production possibility frontiers for six DMUs**

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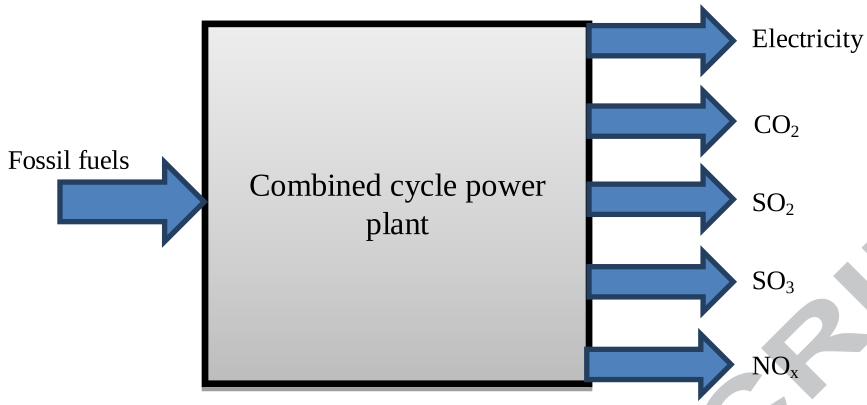


**Fig. 2. A production system**

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**Table 1. Efficiency and RTS classification of points AE and BE, and RTS classification of DMUs**

| DMU | AE | $\theta^{*AE}$ maximization model (6) | $(\sigma_o^{AE*})^{-1}$ | $(\tau_o^{AE*})^{-1}$ | RTS of AE | BE | $\theta^{*BE}$ minimization model (6) | $(\sigma_o^{BE*})^{-1}$ | $(\tau_o^{BE*})^{-1}$ | RTS of BE | RTS of DMU |
|---|---|---|---|---|---|---|---|---|---|---|---|
| A | (2.5,1) | 0.400 | 0.333 | 0.166 | IRS | (2.5,2) | 0.800 | 0.666 | 0.333 | IRS | IRS |
| B | (3,1) | 0.333 | 0.333 | 0.166 | IRS | (3,3) | 1.000 | 1.000 | 0.500 | CRS | NDRS |
| C | (6,4) | 0.666 | 1.333 | 0.666 | CRS | (6,6) | 1.000 | 2.000 | 1.000 | CRS | CRS |
| D | (7.5,6.5) | 0.866 | 2.166 | 1.083 | DRS | (7.5,7) | 0.933 | 2.333 | 1.166 | DRS | DRS |
| E | (10,5) | 0.500 | 1.666 | 0.833 | CRS | (11,6.5) | 0.650 | 2.166 | 1.083 | DRS | NIRS |
| F | (7,1) | 0.142 | 0.333 | 0.166 | IRS | (7,6.5) | 0.928 | 2.166 | 1.083 | DRS | VRS |

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**Table 2. Imprecise data for 28 Chinese cities**

| DMU | Lower bound of inputs | | | Upper bound of inputs | | | Lower bound of outputs | | | Upper bound of outputs | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $I^L(1)$ | $I^L(2)$ | $I^L(3)$ | $I^U(1)$ | $I^U(2)$ | $I^U(3)$ | $O^L(1)$ | $O^L(2)$ | $O^L(3)$ | $O^U(1)$ | $O^U(2)$ | $O^U(3)$ |
| 1 | 463.068 | 1514338 | 683005.4 | 511.812 | 1673742 | 754900.7 | 7071515 | 1607495 | 1257610 | 7815885 | 1776705 | 1389990 |
| 2 | 356.6965 | 907367.8 | 495930.4 | 394.2435 | 1002880 | 548133.6 | 2676340 | 560120 | 965770 | 2958060 | 619080 | 1067430 |
| 3 | 260.0055 | 743455.8 | 352748.3 | 287.3745 | 821714.3 | 389879.7 | 2389155 | 421420 | 537890 | 2640645 | 465780 | 594510 |
| 4 | 198.379 | 465192.2 | 131116.2 | 219.261 | 514159.8 | 144917.9 | 1270245 | 154565 | 389785 | 1403955 | 170835 | 430815 |
| 5 | 189.9905 | 493701.7 | 135553.6 | 209.9895 | 545670.3 | 149822.4 | 1308720 | 221255 | 363755 | 1446480 | 244545 | 402045 |
| 6 | 172.805 | 456372.4 | 246137.4 | 190.995 | 504411.6 | 272046.6 | 1266920 | 199215 | 583775 | 1400280 | 220185 | 645225 |
| 7 | 143.3835 | 390190.7 | 90986.25 | 158.4765 | 431263.4 | 100563.8 | 720999.7 | 97748.35 | 283634.9 | 796894.4 | 108037.7 | 313491.2 |
| 8 | 178.8185 | 446598.8 | 127329.5 | 197.6415 | 493609.2 | 140732.6 | 1099693 | 158629.1 | 392341.5 | 1215451 | 175326.9 | 433640.6 |
| 9 | 120.156 | 281707.3 | 123808.8 | 132.804 | 311360.7 | 136841.3 | 925253.5 | 157753.2 | 280180.7 | 1022649 | 174358.8 | 309673.4 |
| 10 | 116.565 | 318562.6 | 100200.3 | 128.835 | 352095.5 | 110747.7 | 634701.7 | 79548.25 | 236246 | 701512.4 | 87921.75 | 261114 |
| 11 | 126.4735 | 318824.8 | 98259.45 | 139.7865 | 352385.3 | 108602.6 | 792870 | 122032.3 | 814024.6 | 876330 | 134877.8 | 899711.4 |
| 12 | 103.8445 | 263692.5 | 62610.7 | 114.7755 | 291449.6 | 69201.3 | 513406.6 | 81973.6 | 283272.9 | 567449.4 | 90602.4 | 313091.1 |
| 13 | 89.7275 | 194748.1 | 123831.6 | 99.1725 | 215247.9 | 136866.5 | 514826.9 | 82931.2 | 138110.1 | 569019.2 | 91660.8 | 152648 |
| 14 | 107.502 | 294290.1 | 109612 | 118.818 | 325268 | 121150.1 | 872582.6 | 160985.1 | 275993.1 | 964433.4 | 177930.9 | 305045 |
| 15 | 83.163 | 198737.2 | 61657.85 | 91.917 | 219656.9 | 68148.15 | 807458.2 | 122231.8 | 261232.9 | 892453.8 | 135098.3 | 288731.1 |
| 16 | 70.015 | 238030.1 | 81742.75 | 77.385 | 263085.9 | 90347.25 | 513814.2 | 124106.1 | 134623.6 | 567899.9 | 137169.9 | 148794.5 |
| 17 | 73.758 | 172013.7 | 50215.1 | 81.522 | 190120.4 | 55500.9 | 518761.8 | 77955.1 | 182470.3 | 573368.3 | 86160.9 | 201677.7 |
| 18 | 70.4805 | 150907.5 | 50079.25 | 77.8995 | 166792.5 | 55350.75 | 519765.9 | 133115.9 | 114702.1 | 574478.1 | 147128.1 | 126776 |
| 19 | 85.2055 | 193534 | 72751 | 94.1745 | 213906 | 80409 | 652063.9 | 191232.2 | 145237 | 720702.2 | 211361.9 | 160525.1 |
| 20 | 71.744 | 175442.2 | 74751.7 | 79.296 | 193909.8 | 82620.3 | 428205.9 | 84579.45 | 162566.9 | 473280.2 | 93482.55 | 179679.2 |
| 21 | 67.1935 | 129459.4 | 12752.8 | 74.2665 | 143086.7 | 14095.2 | 876443.4 | 58115.3 | 242900.8 | 968700.6 | 64232.7 | 268469.3 |
| 22 | 64.695 | 698517.9 | 11746.75 | 71.505 | 772046.1 | 12983.25 | 958299.2 | 130252.6 | 283781.2 | 1059173 | 143963.4 | 313652.9 |
| 23 | 55.651 | 90926.4 | 7081.3 | 61.509 | 100497.6 | 7826.7 | 631212.3 | 59479.5 | 206676.3 | 697655.7 | 65740.5 | 228431.7 |
| 24 | 65.8065 | 165994.5 | 13210.7 | 72.7335 | 183467.6 | 14601.3 | 1039188 | 92964.15 | 203374.1 | 1148576 | 102749.9 | 224781.9 |
| 25 | 45.5715 | 105994.4 | 9976.9 | 50.3685 | 117151.7 | 11027.1 | 673814.1 | 65875.85 | 142634.9 | 744741.9 | 72810.15 | 157649.1 |
| 26 | 64.3815 | 99821.25 | 9801.15 | 71.1585 | 110328.8 | 10832.85 | 659580.3 | 36103.8 | 243238 | 729009.8 | 39904.2 | 268842 |
| 27 | 19.0665 | 52614.8 | 1754.65 | 21.0735 | 58153.2 | 1939.35 | 154331.3 | 12198.95 | 27588.95 | 170576.7 | 13483.05 | 30493.05 |
| 28 | 68.7515 | 127134.7 | 4105.9 | 75.9885 | 140517.3 | 4538.1 | 341958.2 | 36925.55 | 191705.3 | 377953.8 | 40812.45 | 211884.8 |



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**Table 3. RTS classification and stability region**

| DMU | $\theta_o^{AE*}$ | $\theta_o^{BE*}$ | $E_o$ | $(\sigma_o^{AE})^{-1}$ | $(\tau_o^{AE})^{-1}$ | RTS AE | $(\sigma_o^{BE})^{-1}$ | $(\tau_o^{BE})^{-1}$ | RTS BE | RTS of DMU | RTS (Charnes et al., 1989) | Stability region |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.905 | 1 | 1 | 1.105 | 1.105 | IRS | 1 | 1 | CRS | NDRS | CRS | N.G.* |
| 2 | 0.657 | 0.727 | 1,11,23 | 0.478 | 0.478 | DRS | 0.418 | 0.476 | DRS | DRS | DRS | $0.478<\beta\le 1$ |
| 3 | 0.583 | 0.645 | 1,11,23,24 | 0.887 | 0.887 | DRs | 0.772 | 0.818 | DRS | DRS | DRS | $0.887<\beta\le 1$ |
| 4 | 0.462 | 0.511 | 1,11,23,24 | 0.850 | 0.850 | DRS | 0.767 | 0.770 | DRS | DRS | DRS | $0.850<\beta\le 1$ |
| 5 | 0.505 | 0.558 | 1,11,22,23 | 0.952 | 0.952 | DRS | 0.802 | 0.875 | DRS | DRS | DRS | $0.952<\beta\le 1$ |
| 6 | 0.637 | 0.705 | 1,11,23,24 | 1.067 | 1.067 | IRS | 0.947 | 0.973 | DRS | VRS | CRS | N.G. |
| 7 | 0.399 | 0.441 | 1,11,22,24,25 | 1.434 | 1.434 | IRS | 1.292 | 1.340 | IRS | IRS | IRS | $1\le\alpha\le 1.292$ |
| 8 | 0.475 | 0.525 | 1,11,23,24 | 1.125 | 1.125 | IRS | 1.121 | 1.121 | IRS | IRS | CRS | $1\le\alpha<1.121$ |
| 9 | 0.595 | 0.657 | 1,11,23 | 1.274 | 1.274 | IRS | 1.124 | 1.228 | IRS | IRS | IRS | $1\le\alpha<1.124$ |
| 10 | 0.415 | 0.459 | 1,11,22,24 | 1.933 | 1.933 | IRS | 1.641 | 1.819 | IRS | IRS | IRS | $1\le\alpha<1.640$ |
| 11 | 0.905 | 1 | 11 | 1.105 | 1.105 | IRS | 1 | 1 | CRS | NDRS | IRS | N.G. |
| 12 | 0.515 | 0.569 | 1,11,22,23 | 1.662 | 1.662 | IRS | 1.441 | 1.516 | IRS | IRS | IRS | $1\le\alpha<1.441$ |
| 13 | 0.437 | 0.483 | 1,23 | 2.247 | 2.247 | IRS | 1.987 | 2.087 | IRS | IRS | IRS | $1\le\alpha<1.987$ |
| 14 | 0.598 | 0.661 | 1,11,23,24 | 2.133 | 2.133 | IRS | 1.882 | 1.953 | IRS | IRS | IRS | $1\le\alpha<1.882$ |
| 15 | 0.733 | 0.810 | 1,11,23,24 | 1.500 | 1.500 | IRS | 1.353 | 1.359 | IRS | IRS | IRS | $1\le\alpha<1.353$ |
| 16 | 0.552 | 0.610 | 1,11,22,23 | 4.671 | 4.671 | IRS | 4.223 | 4.227 | IRS | IRS | IRS | $1\le\alpha<4.223$ |
| 17 | 0.555 | 0.613 | 1,11,23,24 | 2.190 | 2.190 | IRS | 1.974 | 1.985 | IRS | IRS | IRS | $1\le\alpha<1.974$ |
| 18 | 0.850 | 0.940 | 19,23 | 1.293 | 1.293 | IRS | 1.168 | 1.188 | IRS | IRS | IRS | $1\le\alpha<1.168$ |
| 19 | 0.905 | 1 | 19 | 1.105 | 192.926 | IRS | 1 | 1 | CRS | NDRS | IRS | N.G. |
| 20 | 0.535 | 0.591 | 1,11,23 | 2.647 | 2.647 | IRS | 2.393 | 2.401 | IRS | IRS | IRS | $1\le\alpha<2.393$ |
| 21 | 0.905 | 1 | 21 | 1.105 | 1.105 | IRS | 1 | 1 | CRS | NDRS | CRS | $1\le\alpha\le 1$ |
| 22 | 0.905 | 1 | 22 | 1.105 | 1.105 | IRS | 1 | 1 | CRS | NDRS | IRS | N.G. |
| 23 | 0.905 | 1 | 23 | 1.105 | 1.105 | IRS | 1 | 1 | CRS | NDRS | CRS | N.G. |
| 24 | 0.905 | 1 | 24 | 1.105 | 1.105 | IRS | 1 | 1 | CRS | NDRS | CRS | N.G. |
| 25 | 0.905 | 1 | 25 | 1.105 | 1.105 | IRS | 1 | 1 | CRS | NDRS | CRS | N.G. |
| 26 | 0.905 | 1 | 26 | 1.105 | 1.105 | IRS | 1 | 1 | CRS | NDRS | CRS | N.G. |
| 27 | 0.893 | 0.987 | 23 | 4.520 | 4.520 | IRS | 2.323 | 4.140 | IRS | IRS | IRS | $1\le\alpha<2.323$ |
| 28 | 0.905 | 1 | 28 | 1.105 | 1.105 | IRS | 1 | 1 | CRS | NDRS | IRS | N.G. |

* Non-Change



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**Table 4. RTS regions for NDRS and VRS**

| DMU | RTS | IRS | | | CRS | | | DRS | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | O(1) | O(2) | O(3) | O(1) | O(2) | O(3) | O(1) | O(2) | O(3) |
| 1 | NDRS | [7071515, 7815885) | [1607495, 1776705) | [1257610, 1389990) | [7815885, 7815885] | [1776705, 1776705] | [1389990, 1389990] | - | - | - |
| 6 | VRS | [1266920, 1351292) | [199215, 212482) | [583775, 622652.3) | [1351292, 1363114] | [212482, 214340.9] | [622652.3, 628099.592] | (1363114, 1400280] | (214340.9, 220185] | (628099.592, 645225] |
| 11 | NDRS | [792870, 876330) | [122032.3, 134877.8) | [814024.6, 899711.4) | [792870, 876330] | [122032.3, 134877.8] | [814024.6, 899711.4] | - | - | - |
| 19 | NDRS | [652063.9, 720702.2) | [191232.2, 211361.9) | [145237, 160525.1) | [720702.2, 720702.2] | [211361.9, 211361.9] | [160525.1, 160525.1] | - | - | - |
| 21 | NDRS | [876443.4, 968700.6) | [58115.3, 64232.7) | [242900.8, 268469.3) | [968700.6, 968700.6] | [64232.7, 64232.7] | [968700.6, 968700.6] | - | - | - |
| 22 | NDRS | [958299.2, 1059173) | [130252.6, 143963.4) | [283781.2, 313652.9) | [1059173, 1059173) | [143963.4, 143963.4) | [313652.9, 313652.9] | - | - | - |
| 23 | NDRS | [631212.3, 697655.7) | [59479.5, 65740.5) | [206676.3, 228431.7) | [697655.7, 697655.7] | [65740.5, 65740.5] | [228431.7, 228431.7] | - | - | - |
| 24 | NDRS | [1039188, 1148245) | [92964.15, 102720.2) | [203374.1, 224717) | [1148245, 1148576] | [102720.2, 102749.9] | [224717, 224781.9] | - | - | - |
| 25 | NDRS | [673814.1, 744741.9) | [65875.85, 72810.15) | [142634.9, 157649.1) | [744741.9, 744741.9] | [72810.15, 72810.15] | [157649.1, 157649.1] | - | - | - |
| 26 | NDRS | [659580.3, 728934.4) | [36103.8, 39900.07) | [243238, 268814.2) | [728934.4, 729009.8] | [39900.07, 39904.2] | [268814.2, 268842] | - | - | - |
| 28 | NDRS | [341958.2, 377953.8) | [36925.55, 40812.45) | [191705.3, 211884.8) | [377953.8, 377953.8] | [40812.45, 40812.45] | [211884.8, 211884.8] | - | - | - |



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**Table 5. Input and output data for the 17 power plants.**

| DMU | Input | | Output | | Undesirable output (Ton) | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Fuel ($M^3$) | | Electricity power (1000KW/Hr) | | $No_x$ | | $SO_2$ | | $CO_2$ | | $SO_3$ | |
| | Lower bound | Upper bound | Lower bound | Upper bound | Lower bound | Upper bound | Lower bound | Upper bound | Lower bound | Upper bound | Lower bound | Upper bound |
| 1 | 1,002,243 | 1,534,381 | 4,663,820 | 5,948,123 | 3.6 | 5 | 1.8 | 6.6 | 2338 | 3015 | 0 | 0.1 |
| 2 | 971,509 | 1,298,112 | 4,821,296 | 5,657,392 | 3.7 | 4.4 | 2.1 | 4.7 | 2367 | 2727 | 0 | 0.1 |
| 3 | 1,331,457 | 1,831,098 | 7,220,851 | 7,699,512 | 5.3 | 5.8 | 2.8 | 6.7 | 3478 | 3631 | 0 | 0.1 |
| 4 | 766,658 | 1,117,322 | 3,781,843 | 4,628,520 | 2.8 | 3.7 | 1.7 | 4.8 | 1779 | 2250 | 0 | 0.1 |
| 5 | 24,213 | 1,060,942 | 356,963 | 3,184,631 | 0.6 | 3.2 | 0.4 | 2.2 | 318 | 2119 | 0 | 0 |
| 6 | 1,045,455 | 1,283,541 | 5,339,780 | 5,975,686 | 3.8 | 4.4 | 1.3 | 3 | 2545 | 2806 | 0 | 0 |
| 7 | 412,442 | 758,142 | 1,925,856 | 2,631,210 | 1.7 | 2.3 | 0.1 | 1.1 | 1052 | 1557 | 0 | 0 |
| 8 | 446,094 | 1,017,339 | 1,836,793 | 4,289,004 | 1.8 | 3.6 | 1.1 | 4.2 | 1089 | 2229 | 0 | 0.1 |
| 9 | 1,244,520 | 1,820,737 | 4,222,796 | 7,935,571 | 4.3 | 8 | 0.2 | 12.5 | 2806 | 4788 | 0 | 0.2 |
| 10 | 1,056,182 | 1,410,680 | 5,126,256 | 6,213,138 | 3.4 | 4.4 | 0.2 | 3 | 2262 | 2802 | 0 | 0 |
| 11 | 311,239 | 635,257 | 1,820,209 | 2,106,015 | 1.6 | 1.9 | 1 | 3.3 | 979 | 1091 | 0 | 0.1 |
| 12 | 204 | 796,605 | 515 | 2,128,410 | 0 | 2.6 | 0 | 5 | 1 | 1595 | 0 | 0.1 |
| 13 | 1,234,922 | 2,303,468 | 4,500,169 | 9,886,102 | 5.2 | 8.3 | 4.1 | 11.4 | 3209 | 4993 | 0.1 | 0.2 |
| 14 | 422,191 | 905,874 | 1,770,332 | 2,761,553 | 1.9 | 2.9 | 0.4 | 2.4 | 1222 | 1848 | 0 | 0 |
| 15 | 147,683 | 2,769,634 | 5,008,772 | 1,030,008 | 5.5 | 8.5 | 3 | 9.2 | 3546 | 5535 | 0 | 0.1 |
| 16 | 161,614 | 928,637 | 1,258,570 | 2,678,996 | 1.9 | 4.5 | 1.4 | 7.7 | 985 | 2661 | 0 | 0.1 |
| 17 | 1,298,688 | 1,961,314 | 4,785,753 | 5,898,717 | 5.3 | 5.7 | 0.5 | 4 | 3382 | 3828 | 0 | 0.1 |

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**Table 6. RTS classification and stability region**

| DMU | $\theta_o^{A*}$ | $\theta_o^{B*}$ | $(\tau_o^A)^{-1}$ | $(\sigma_o^A)^{-1}$ | RTS A | $(\tau_o^B)^{-1}$ | $(\sigma_o^B)^{-1}$ | RTS B | RTS | Stability region |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.00094 | 0.00119 | 2.19122 | 2.19141 | DRS | 2.79461 | 2.80560 | DRS | DRS | 0.4564<β≤1 |
| 2 | 0.00096 | 0.00112 | 2.26521 | 2.26521 | DRS | 2.65738 | 3.07710 | DRS | DRS | 0.4414<β≤1 |
| 3 | 0.00098 | 0.00104 | 3.39260 | 3.39262 | DRS | 3.61717 | 3.82580 | DRS | DRS | 0.2947<β≤1 |
| 4 | 0.00100 | 0.00122 | 1.77684 | 1.77685 | DRS | 2.17442 | 2.31270 | DRS | DRS | 0.5627<β≤1 |
| 5 | 0.00141 | 0.01261 | 0.16771 | 0.16771 | IRS | 1.49620 | 1.50738 | DRS | VRS | N.G. |
| 6 | 0.00099 | 0.00110 | 2.50881 | 2.50884 | DRS | 2.80720 | 3.05400 | DRS | DRS | 0.3985<β≤1 |
| 7 | 0.00086 | 0.00118 | 0.90483 | 0.90484 | IRS | 1.23619 | 1.26240 | DRS | VRS | N.G. |
| 8 | 0.00079 | 0.00185 | 0.86299 | 0.86299 | IRS | 2.01504 | 2.06910 | DRS | VRS | N.G. |
| 9 | 0.00071 | 0.00133 | 1.98401 | 1.98404 | DRS | 3.72809 | 3.92840 | DRS | DRS | 0.5040<β≤1 |
| 10 | 0.00106 | 0.00129 | 2.40849 | 2.40851 | DRS | 2.91911 | 2.94060 | DRS | DRS | 0.4151<β≤1 |
| 11 | 0.00087 | 0.00101 | 0.85520 | 0.85521 | IRS | 0.98934 | 1.07690 | CRS | NDRS | 1< α ≤1.0107 |
| 12 | 0.00024 | 1 | 0.00024 | 0.00024 | IRS | 1 | 1 | CRS | NDRS | N.G. |
| 13 | 0.00066 | 0.00145 | 2.11433 | 2.11435 | DRS | 4.64457 | 4.81350 | DRS | DRS | 0.4729<β≤1 |
| 14 | 0.00068 | 0.00106 | 0.83176 | 0.83177 | IRS | 1.29740 | 1.34420 | DRS | VRS | N.G. |
| 15 | 0.00067 | 0.00325 | 0.48393 | 0.48394 | IRS | 2.35314 | 2.38899 | DRS | VRS | N.G. |
| 16 | 0.00075 | 0.00159 | 0.59132 | 0.59133 | IRS | 1.25865 | 1.26756 | DRS | VRS | N.G. |
| 17 | 0.00066 | 0.00082 | 2.24851 | 2.24852 | DRS | 2.77099 | 3.04380 | DRS | DRS | 0.4447<β≤1 |



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**Table 7. RTS regions for NDRS and VRS**

| DMU | RTS | IRS | CRS | DRS |
|---|---|---|---|---|
| | | $[O^L,O^U]$* | $[O^L,O^U]$* | $[O^L,O^U]$* |
| 5 | VRS | [356963 ,2128400.6641) | [2128400.6641,2128477.0371] | (2128477.03771, 3184631] |
| 7 | VRS | [1925856, 2128404.03780) | [2128404.03780,2128480.6397] | (2128480.6397, 2631210] |
| 8 | VRS | [1836793, 2128408.5054) | [2128408.5054, 2128499.3957] | (2128499.3957,4289004] |
| 11 | NDRS | [1820209, 2106015) | [2106015,2106015] | - |
| 12 | NDRS | [515, 2128363.0202) | [2128363.0202,2128410] | - |
| 14 | VRS | [1770332, 2128400.1669) | [2128400.1669, 2128530.1924] | (2128530.1924, 2761553] |
| 15 | VRS | [1030008, 2128393.0937) | [2128393.0937, 2128546.7187] | (2128546.7187,5008772] |
| 16 | VRS | [1258570, 2128389.5322) | [2128389.5322, 2128473.5607] | (2128473.5607,2678996] |

* Electricity power

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# Highlights

- We estimate returns-to-scale of firms in imprecise data envelopment analysis
- We present the stability regions of RTS classification with interval data.
- Two examples are presented for illustrating the proposed approach.

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