Accepted Manuscript

A Modified Super-Efficiency in the Range Directional Model

Adel Hatami-Marbini, Jafar Pourmahmoud, Elnaz Babazadeh

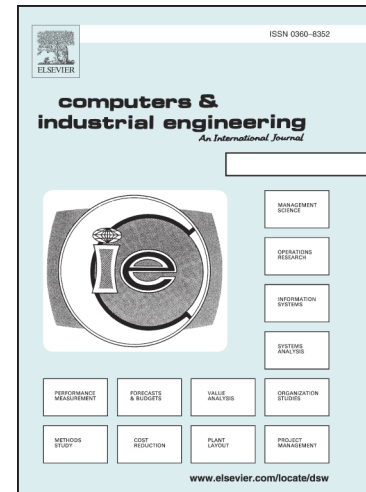# A Modified Super-Efficiency in the Range Directional Model

Adel Hatami-Marbini[*]
Department of Strategic Management and Marketing
Leicester Business School
De Montfort University
Hugh Aston Building, The Gateway, Leicester LE1 9BH, UK
E-mail: adel.hatamimarbini@dmu.ac.uk

Jafar Pourmahmoud
Department of Applied Mathematics
Azarbaijan Shahid Madani University, Tabriz, Iran
E-mail:pourmahmoud@azaruniv.edu

Elnaz Babazadeh
Department of Applied Mathematics
Azarbaijan Shahid Madani University, Tabriz, Iran
E-mail: elnaz.babazadeh@azaruniv.edu

[*]Corresponding Author

0

A Modified Super-Efficiency in the Range Directional Model

All additions and changes to the first revision are highlighted in this revision.

Abstract

The range directional model (RDM) relaxes the assumption of non-negativity of inputs and outputs in the conventional data envelopment analysis (DEA) with the aim of evaluating the efficiency of a decision-making unit (DMU) when some data are negative. Although the concept of super-efficiency in the RDM contributes to enhancing discriminatory power, the formulated model may lead to the infeasibility problem for some efficient DMUs. In this paper, we modify the super-efficiency RDM (SRDM) model to overcome the infeasibility problem occurring in such cases. Our method leads to a complete ranking of the DMUs with negative data for yielding valuable insights that aid decision makers to better understand the findings from a performance evaluation process. The contribution of this paper is fivefold: (1) we detect the source of infeasibility problems of SRDM in the presence of negative data, (2) the proposed model in this study yields the SRDM measures regardless of feasibility or infeasibility of the model, (3) when feasibility occurs, the modified SRDM model results in the scores that are the same as the original model, (4) we differentiate the efficient units to improve discriminatory power in SRDM, and (5) we provide two numerical examples to elucidate the details of the proposed method.

Keywords: *DEA; Super-efficiency; infeasibility; Negative data; RDM model.*

1. Introduction

Data envelopment analysis (DEA) is a powerful tool in the context of production management for performance measurement. The purpose of DEA is to measure the relative efficiency of a set of decision-making units (DMUs) where multiple inputs are converted into multiple outputs. In classical DEA models, the *Farrell output efficiency* of a firm among its peers measures how much it can proportionally expand all of its outputs and still use its inputs under a given technology (Farrell, 1954). Additionally, as a result of applying DEA, the DMUs can be divided into two groups: *efficient* and *inefficient* DMUs. Since the seminal

work of Charnes et al. (1978), DEA studies have been tremendously attracting both in modelling and applications in various disciplines. However, classical DEA models include two practical disadvantages. First, while a decision maker may desire a total ordering, many DMUs often belong to the efficient group without discriminating between efficient DMUs, particularly, when the number of DMUs is relatively small in comparison with the sum of the number of input and output variables (Cook et al. 2014; Adler and Yazhemsky, 2010). Second, in conventional DEA models, inputs and outputs are assumed to be non-negative while negative data may occur in some DEA applications such as the performance analysis of socially responsible and mutual funds (Basso and Funari, 2014) and the macroeconomic performance where "rate of growth of GDP per capita" can be either negative or positive (Lovell, 1995). As far as we know, the existing DEA software does not allow users to directly define negative outputs and/or inputs.

To deal with the former limitation in DEA models, many research studies have been carried out in the frontier analysis context and they can be partitioned into six distinct categories (Adler et al., 2002); (1) *cross-efficiency ranking* methods initially proposed by Sexton et al. (1986) in terms of both self and peer evaluation, (2) *benchmark ranking* methods initiated by Torgersen et al. (1996) where a total ordering of DMUs is obtained according to the share of total output increase (input decrease) achieved by DMUs for which the DMU is a peer, (3) *multivariate ranking* methods first proposed by Friedman and Sinuany (1997) where multivariate statistical tools such as canonical correlation analysis and discriminant analysis are used to rank the DMUs, (4) *the inefficiency-based ranking* methods that struggle to rank the inefficient DMUs (e.g., Bardhan et al. (1996)), (5) *DEA and MCDM* methods originally proposed by Golany (1988) with the aim of incorporating preference information into DEA models, and (6) *super-efficiency* method first developed by Andersen and Petersen (1993) where a DMU under analysis is excluded from the reference set so that the efficient DMUs can receive scores greater than or equal to the unity while the score for the inefficient DMUs do not change. Hinojosa et al. (2017) recently introduced three additional and independent categories in ranking DMUs in the literature; (*i*) *common weights* methods which make an attempt to rank all DMUs using a common set of weights (see e.g., Hosseinzadeh Lotfi et al. (2013); Hatami-Marbini et al. (2015)), (*ii*) *cross-influence* ranking methods which first disregard a DMU from the reference set like the super-efficiency method and then study its impact on all the DMUs (see e.g., Jahanshahloo et al. (2007)), and (*iii*) ranking methods based on the concept of *cooperative game theory* started off by Li et al. (2016) for ranking efficient DMUs in DEA.

2

To handle the negative data as the latter limitation of DEA, Lovell and Pastor (1995) and Pastor (1996) were the first by serving a translation invariance classification. That is, in light of the translation invariance property in basic DEA models such as the additive model, the original negative data can be equivalently converted to positive data by adding a constant number. However, many DEA models such as CCR may not have this property to be applied as a treatment of negative data (Ali and Seiford, 1990). A number of significant contributions have been developed in the DEA literature to address the occurrence of negative data (e.g., Seiford and Zhu, 2002; Silva Portela et al., 2004; Kerstens and Van de Woestyne, 2011).

Silva Portela et al. (2004) suggested working with some variations of the directional distance function. Kerstens and Van de Woestyne (2011) modified the traditional proportional distance function to treat negative data. Although Cheng et al. (2013) made an effort to propose a variant of the traditional input- or output-oriented radial efficiency measure to handle negative inputs and outputs, Kerstens and Van de Woestyne (2014) highlighted some shortcomings in their method by using a more general case of the directional distance function proposed by Kerstens and Van de Woestyne (2011). An overview of the various DEA modelling approaches can be found in Pastor and Ruiz (2007) and Pastor and Aparicio (2015).

The super-efficiency presents the possible capability of an efficient DMU in expanding its inputs and/or reducing its outputs without becoming inefficient (Chen et al., 2013). Banker and Chang (2006) exploited the super-efficiency model to detect and remove the outliers. Further, the super-efficiency DEA approach can be viewed as a tool for sensitivity analysis where a DMU under evaluation is excluded from reference set (see, e.g., Zhu 2001; Charnes et al. 1992; Rousseau and Semple 1995; Charnes et al., 1996). Whereas the classical super-efficiency model under constant returns to scale (CRS) does not suffer from the infeasibility problem[1], the super-efficiency model based upon the variable returns to scale (VRS) model of Banker et al. (1984) may be infeasible for a DMU under evaluation (see, e.g., Seiford and Zhu, 1999; Chen and Liang, 2011; Lee et al., 2011; Lee and Zhu 2012). Seiford and Zhu (1999) argued the necessary and sufficient conditions of infeasibility problem occurring in super-efficiency DEA models without solving the problem. Lovell and Rouse (2003) introduced a user-defined scaling factor to find a feasible solution for efficient DMUs that are infeasible in the standard VRS super-efficiency model. However, a user-defined scaling factor in Lovell and Rouse's method for all DMUs may have infeasible

---

[1] The CRS super-efficiency model may be also infeasible when the input or output value of an efficient DMU is zero (Thrall, 1996; Zhu, 1996; Lee and Zhu, 2012).

solutions as indicated in Cook et al. (2009). Chen (2005) further proposed the use of an integrated super-efficiency score that is obtained from both the input- and output-oriented VRS super-efficiency models. However, Chen's method will be unsuccessful once both the input- and output-oriented VRS super-efficiency models are infeasible. Cook et al. (2009) developed the modified input- and output- oriented VRS super-efficiency models to deal with the infeasibility trouble for efficient DMUs. Lee et al. (2011) suggested a two-stage process to treat the VRS infeasibility issue by defining a score that characterizes the super-efficiency in both inputs and outputs. Chen and Liang (2011) further simplified the two-stage process of Lee et al. (2011) by proposing a single linear program. Lee and Zhu (2012) first showed that Lee et al.'s model may be infeasible when some inputs are zero and then the authors proposed a modified model which is always feasible albeit data are non-negative.

In a recent paper, Hadi-Vencheh and Esmaeilzadeh (2013) made an attempt to develop a super-efficiency model based on the RDM model in the presence of negative data. However, Hadi-Vencheh and Esmaeilzadeh's model suffers from the common infeasibility and unboundedness problems (Pourmahmoud et al., 2016). Pourmahmoud et al. (2016) showed that the RDM super-efficiency model will be always feasible when all range of possible improvements are strictly positive. In addition, they defined four cases in which the envelopment form of the RDM super-efficiency model is infeasible. In general, the infeasibility occurs when (i) there exists zero range of possible improvements in inputs and/or outputs of the evaluated DMU and (ii) the corresponding inputs (outputs) of the DMU under evaluation with a zero amount of improvement are outside of the production possibility set (PPS) spanned by the inputs (outputs) of the remaining DMUs. Apart from Hadi-Vencheh and Esmaeilzadeh (2013), super-efficiency models with negative data have received no attention in the literature. In this study, we first investigate whether a RDM super-efficiency model is infeasible, and then calculate a super-efficiency score when infeasibility occurs. In the case of feasibility, the proposed RDM super-efficiency scores are identical to the results obtained from the standard RDM super-efficiency model. Our proposed model has an intuitive capability to provide a complete ranking of all DMUs.

The rest of the paper is outlined as follows. Section 2 presents RDM model, super-efficiency RDM model and our motivation. In Section 3, we develop our new RDM super-efficiency model in the presence of infeasibility. In the penultimate section, our proposed model is applied to two numerical examples and finally Section 5 concludes the paper.

2. Background and Motivation

In this section, we first review a certain DEA model with negative data as well as its super-efficiency model, and then discuss our research motivation.

2.1. The Range Directional Model (RDM)

To deal with the negative data in the conventional DEA models, Silva Portela et al. (2004) used a directional distance model of Chambers et al. (1996, 1998) to propose the range directional model (RDM) for evaluating the performance of production units. In addition, the RDM model that originally introduced under the variable returns to scale (VRS) presents closer targets compared to the existing models in the literature.

Consider a set of n observed DMUs, $\{DMU_j \ (j = 1,2,...,n)\}$ where each observation transforms m inputs, $x_{ij} \ (i = 1,2,...,m)$, into s outputs, $y_{rj} \ (r = 1,2,...,s)$. Let (x_{io}, y_{ro}) denote a DMU$_o$ under evaluation amongst n observations. Furthermore, assume that some data can take negative values. By the use of *convexity* and *free disposability* of inputs and outputs, and *VRS* assumptions, the *technology* or *production possibility set (PPS)*, $T_o(x, y)$ from the observed input-output data for n DMUs can be defined as follows:

$$T_o(x,y) = \left\{ (x,y) : x \geq \sum_{j=1}^{n} \lambda_j x_j; \ y \leq \sum_{j=1}^{n} \lambda_j y_j; \ \sum_{j=1}^{n} \lambda_j = 1 ; \lambda_j \geq 0; (j = 1,2,...,n) \right\}$$

The RDM model in terms of the directional distance function and $T_o(x, y)$ can be expressed as follows (Silva Portela et al. 2004):

$$
\begin{aligned}
\min \quad & 1 - \beta \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} - \beta P_{io}^-, \quad \forall i, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} + \beta P_{ro}^+, \quad \forall r, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0, \quad \forall j, \\
& \beta \ free.
\end{aligned}
\tag{1}
$$

where $P_{io}^- = x_{io} - \min_j\{x_{ij} : j = 1,2,...,n\}, i = 1,2,...,m$; $P_{ro}^+ = \max_j\{y_{rj} : j = 1,2,...,n\} - y_{ro}$, $r = 1,2,...,s$ are always non-negative and called a *range of possible improvement* of DMU$_o$. The bundle (P_{io}^-, P_{ro}^+) defines the ideal directional vectors for input and output levels. Model (1) combines the features of both an input- and output-oriented models in which each input and output of the unit under assessment are respectively lessened and increased at the same time by the same portion β. The factors β can be considered as a surrogate for *technical*

5

inefficiency of the DMU in order to define its *efficiency* as 1-β. If $P_{rj}^+ = 0$ ($r = 1,2, \dots, s; j = 1,2, \dots, n$) or $P_{ij}^- = 0$ ($i = 1,2, \dots, m; j = 1,2, \dots, n$), model (1) is transformed to input- or output-oriented models, respectively. The RDM model (1) takes advantage of the desirable properties of translation and unit invariance.

To further exemplify the RDM model, let us consider a simple numerical example in Figure 1 where eight DMUs {*A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*} consume two inputs (-6,5), (-6,3), (-5,-2), (-2,-5), (2,-6), (-3.5,3.5), (6.5,-3) and (5,2), respectively, to produce the same amount of a single output[2]. The performance of DMUs is therefore assessed using an input-oriented RDM model thanks to $P_{rj}^+ = 0$ ($r = 1,2, \dots, s; j = 1,2, \dots, n$). The ideal point (minimum inputs) is $(x_1^{ideal}, x_2^{ideal})$=(-6, -6) indicated by *I* in Figure 1. The segments connecting DMUs *A*, *B*, *C*, *D* and *E* form the efficient frontier. The region bounded by the frontier line *ABCDE*, the horizontal line passing through the point *E* and the vertical line through the point *A* is the *PSS* or *technology* where all the observed points (the coordinates of any point) are enveloped within all four quadrants. The efficiency measure of DMU$_A$, $(1 - \beta_A^*)$, in the RDM model equals to 0.8182 since it is apparent that x_2 of DMU$_A$ can be reduced from 5 to 3 so that DMU$_A$ coincides with DMU$_B$ that is fully efficient. DMU$_A$ is therefore weakly efficient while DMUs *B*, *C*, *D* and *E* are fully efficient (i.e., $\beta^* = 0$). The RDM-efficiency of DMU$_H$ that is placed in an inefficient portion of the technology is calculated by the ratio $1 - \beta_H^* = 1 - \frac{|PH|}{|IH|} = \frac{|IP|}{|IH|}$=(|-6-(-3.1048)|)/(|5-(-6)|)=0.2632. Analogously, the RDM-efficiency of DMU$_F$ and DMU$_G$ are 0.4167 and 0.3265, respectively.

Despite the reference point of the RDM-efficiency, it can be viewed the close affinity between the RDM efficiency measure and conventional radial efficiency measure. That is, the origin is regarded as the reference point in conventional radial DEA models while the RDM model exploits the ideal point in lieu of the origin to measure the efficiencies.

----Insert Figure 1 Here----

It should be noted that, due to non-directional-slack of the RDM model, the projection may not be possessed of Pareto-efficient frontier. To project units onto the Pareto-efficient

---

[2] It should be pointed out that although the input-oriented RDM model with a single constant output is equivalent to a model without outputs which clearly appears awkward to be justified from an economic viewpoint, our aim is only to underline the characteristics of the RDM and its extension to the super-efficiency model by using a graphical representation.

frontier, the [weighted] additive model can be solved in a second phase (Asmild and Pastor (2010)).

2.2. The Super-Efficiency RDM (SRDM)

The RDM model measures the technical efficiency (1-β) of a DMU relative to the others to discriminate between efficient and inefficient DMUs. Inefficient DMUs can be simply ranked in terms of their different measures whereas we face with the lack of discrimination among efficient DMUs. The super-efficiency method suggested by Andersen and Petersen (1993) ranks efficient DMUs that are determined using the standard CCR model. The underlying idea is to exclude the DMU under analysis from the technology (reference set) so that efficient DMUs may have the capability to augment efficiency scores (>=1) hinging on the DEA model orientation while the measure of inefficient DMUs remains the same as those obtained from the CCR model. The technology of super-efficiency for n DMUs, $T_o^s(x, y)$, can be defined as follows:

$$T_o^s(x,y) = \left\{ (x,y): x \geq \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j x_j;\ y \leq \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j y_j;\ \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j = 1;\ \lambda_j \geq 0;\ (j = 1,2,\dots,n; j \neq o) \right\}$$

The super-efficiency RDM (SRDM) of DMU_o apropos to the technology $T_o^s(x, y)$ can be formulated as:

$$
\begin{aligned}
\min\quad & 1 - \beta \\
\text{s.t.}\quad & \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j x_{ij} \leq x_{io} - \beta P_{io}^-, \quad \forall i, \\
& \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j y_{rj} \geq y_{ro} + \beta P_{ro}^+, \quad \forall r, \\
& \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0, \quad \forall j (j \neq o), \\
& \beta\ free.
\end{aligned}
\tag{2}
$$

Model (2) is solved for a set of efficient DMUs obtained from the RDM model, i.e., $\beta^* = 0$ in model (1). In the case of super-efficiency of DMU_o, β^* is less than zero, meaning that the outputs are scaled down while its inputs are scaled up so as to move onto the modified frontier formed by the rest of the DMUs. We point out that if DMU_o is inefficient,

7

then it is positioned inside the technology and its removal in the SRDM model (2) does not affect the shape of estimated technology. Therefore, the measures of inefficient units in models (1) and (2) are identical. When $P_{ij}^- = 0$ $(i = 1,2, \dots, m; j = 1,2, \dots, n)$ the corresponding output-oriented SRDM problem for the efficient DMU$_o$ can be expressed as

$$
\begin{aligned}
&\min \quad 1 - \beta \\
&\text{s.t.} \quad \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j x_{ij} \leq x_{io}, \quad \forall i, \\
&\qquad \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j y_{rj} \geq y_{ro} + \beta P_{ro}^+, \quad \forall r, \\
&\qquad \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j = 1, \\
&\qquad \lambda_j \geq 0, \quad \forall j (j \neq o), \\
&\qquad \beta \ free.
\end{aligned}
\tag{3}
$$

Note that in model (3) the output bundle, y_{ro}, of the efficient DMU$_o$ is only scaled down by an optimal portion β^* (≤ 0) while the input bundle x_{io} preserves unaltered. When $P_{rj}^+ = 0$ $(r = 1,2, \dots, s; j = 1,2, \dots, n)$, the pertinent input-oriented SRDM model for the efficient DMU$_o$ can be formulated as

$$
\begin{aligned}
&\min \quad 1 - \beta \\
&\text{s.t.} \quad \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j x_{ij} \leq x_{io} - \beta P_{io}^-, \quad \forall i, \\
&\qquad \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j y_{rj} \geq y_{ro}, \quad \forall r, \\
&\qquad \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j = 1, \\
&\qquad \lambda_j \geq 0, \quad \forall j (j \neq o), \\
&\qquad \beta \ free.
\end{aligned}
\tag{4}
$$

where the improvement range is contingent on the input direction by defining P_{io}^-. Model (4) scales up the input bundle, x_{io}, of the efficient DMU$_o$ by an optimal portion β^* (≤ 0) while the output bundle y_{ro} preserves unchanged.

To provide a detailed view of the SRDM problem, we return to the simple numerical example given in the earlier sub-section. From Figure 1, DMUs *B*, *C*, *D* and *E* located on the efficient frontier are efficient and rated at 100% efficiency measure $(1 - \beta^* = 1)$. Put

8

differently, the discrimination problem can be observed in this performance analysis when 50% of DMUs are efficient and non-comparable. To deal with the problem, the input-oriented SRDM model (4) is applied to the efficient units in order to increase its input bundles by taking non-positive amount to the optimal β^*. For instance, consider DMU$_\text{C}$ that lies on the efficient frontier *ABCDE*. For the SRDM evaluation of DMU$_\text{C}$, we first omit this observation from the PPS. In the elimination of DMU$_\text{C}$, the piecewise segments linking DMUs *A*, *B*, *D* and *E* construct the efficiency frontier, and subsequently lead to a more restricted PPS. The resulting SRDM measure of DMU$_\text{C}$ is 1.5 (i.e., $\beta_C^* = -0.5$), implying that DMU$_\text{C}$ scales up its input bundle (-5,-2) by -0.5 to coincides with point Q whose the coordinate is $(x_{1C} - \beta_C^* P_{1C}^-, x_{2C} - \beta_C^* P_{2C}^-) = -5 - (-0.5 \times 1), -2 - (-0.5 \times 3) = (-4.5, -0.5)$. In other words, the SRDM measure of DMU$_\text{C}$ obtained from model (4) is a ratio of the length of $\overline{IQ}$ to $\overline{IC}$, i.e., $1 - \beta_C^* = \overline{IQ}/\overline{IC}$, in which $\overline{IQ}$ represents the line segment connecting the ideal point with the projection point and $\overline{IC}$ stands for the line segment connecting the ideal point with DMU$_\text{C}$. In addition, it should be also noted that the SRDM assessment of an inefficient observation such as DMU$_\text{G}$ and DMU$_\text{F}$ is not influenced by this exclusion from the technology in view of the fact that the efficient frontier constructed by the efficient units is unaltered by such an omission.

2.3. Infeasibility Trouble in the SRDM Model (Our Motivation)

The conventional super-efficiency model under VRS may suffer from the infeasibility problem (Seiford and Zhu, 1999). Given that the RDM model is based upon VRS, the SRDM model (2) may turn into infeasible for certain DMUs. Pourmahmoud et al. (2016) proved that model (2) may be infeasible if there exists at least one *i* and/or *r* for the efficient DMU$_\text{o}$ such that $P_{io}^- = 0$ and/or $P_{ro}^+ = 0$. In detail, the infeasibility problem in model (2) occurs if

a) $y_{ko} = P_{ko}^+ = 0;\quad y_{rj} < 0,\ \ j = 1,2,\dots,n, j \neq o,$

b) $x_{to} = P_{to}^- = 0;\quad x_{ij} > 0,\ \ j = 1,2,\dots,n, j \neq o,$

c) $P_{to}^- = 0;\ \ x_{to} \neq 0$ where x_{to} be outside the PPS spanned by $x_{tj}(\forall j, j \neq o)$,

d) $P_{ko}^+ = 0;\ \ y_{ko} \neq 0$ where y_{ko} be outside the PPS spanned by $y_{kj}(\forall j, j \neq o)$.

The necessary and sufficient conditions for infeasibility of model (2) for a given efficient DMU are, respectively, (*i*) a range of *zero* improvement associated with some inputs and/or some outputs, and (*ii*) the corresponding inputs (outputs) with a zero amount of improvement which are outside the PPS spanned by the inputs (outputs) of the remaining DMUs. That is,

the (*i*) necessary infeasibility condition and (*ii*) sufficient infeasibility condition simultaneously provide the circumstance that the DMU cannot be projected on the production frontier using the defined directional vectors. Mathematically, infeasibility of model (2) is caused by the non-existence of the feasible solution for the relevant linear programing model. Let us describe the infeasibility conditions of the SRDM model by means of the example depicted in Figure 1. The SRDM evaluation of DMU_E using model (4) results in the problem of infeasibility since (*i*) its x_2 improvements (difference between ideal point and observed values) is zero, i.e., $P_{2E}^- = x_{2E} - \min_j\{x_{2j}: j = A, ..., H\}$=-6-(-6)=0, which is the necessary infeasibility condition and this allows us to move on to examine the sufficient infeasibility condition, and (*ii*) DMU_E is outside the PPS spanned by x_2 of the remaining DMUs[3] which fulfills the sufficient infeasibility condition. Interestingly, model (4) for DMU_B has a feasible solution. This is because that although the necessary infeasibility condition for DMU_B is fulfilled due to a range of *zero* improvement, i.e., $P_{1B}^- = x_{1B} - \min_j\{x_{1j}: j = A, ..., H\}$=-6-(-6)=0, the sufficient infeasibility condition is not satisfied because DMU_B is still inside the PPS spanned by x_1 of DMU_A.

3. Modified SRDM Model

In this section, we propose a modified SRDM model to circumvent the infeasibility problem of SRDM in certain circumstances as well as to completely rank all the DMUs including efficient and inefficient observations in terms of their SRDM measure.

As we marked rigorously the causes and conditions of infeasibility in an earlier section, a corresponding efficient DMU under RDM is not able to get to the production frontier, formed by all the residual units, in terms of the direction of the ideal point that has the largest potential for improvement. It should be also noted that this efficient DMU that leads to infeasibility of the SRDM model has at least one output (and/or one input) with the largest (and/or smallest) amount among DMUs. In other words, the DMU under analysis has zero amount for at least one input and/or output for range of improvement as well as being outside the PPS spanned by the corresponding inputs and/or outputs of the residual DMUs. Traditionally, the typical idea to deal with the problem of infeasibility in super-efficiency DEA models, particularly under VRS assumption, is to rightly scale up the inputs (scale down the outputs) of the DMU under analysis that is unaffected in constructing the

---

[3] The removal of DMU_E alters the efficient frontier which consists of the line connecting *ABCD,* the vertical line going upward through *A* and the horizontal dashed line going from *D*.

production frontier (e.g., see Lovell and Rouse, 2003; Cook et al., 2009; Lee and Zhu, 2012). We aim at exploiting this clue to develop a modified SRDM model where the infeasible DMU moves appropriately and minimally toward the efficient frontier in both input and output directions. In this regard, we propose the following modified SRDM model to evaluate the performance of DMU_o:

$$
\begin{aligned}
\min \quad & \varphi_o = 1 - \beta + M\left(\sum_{i=1}^m s_i + \sum_{r=1}^s t_r\right) \\
\text{s.t.} \quad & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} - \beta P_{io}^- + s_i |x_i|^{max}, \quad \forall i, \\
& \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq y_{ro} + \beta P_{ro}^+ - t_r |y_r|^{max}, \quad \forall r, \\
& \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j = 1, \\
& \lambda_j \geq 0, \quad \forall j (j \neq o), \\
& s_i, t_r \geq 0 \ \forall i, r \\
& \beta \ free.
\end{aligned}
\tag{5}
$$

where M is a large positive parameter defined by a user, and $|x_i|^{max} = \max_j\{|x_{ij}| : j = 1,2,...,n$ and $yrmax = \max j y r j : j = 1,2,...,n$. To handle the problem of infeasibility of SRDM model, we add the term $s_i |x_i|^{max}$ $(i = 1,2,...,m)$ to the right hand side of the first set of constraints and at the same time subtract the term $t_r |y_r|^{max} (r = 1,2,...,s)$ from the right hand side of the second set of constraints where the sum of s_i $(i = 1,2,...,m)$ and t_r $(r = 1,2,...,s)$ is added to the objection function in the presence of a penalty term M. Therefore, the constraints are not violated anymore when there exists a range of zero improvement for DMU_o. It is necessary to note that $s_i |x_i|^{max}$ and $t_r |y_r|^{max}$ present the input saving and output surplus of DMU_o under analysis compared to the frontier that is created in the elimination of DMU_o.

To show the property of units invariance of (5), assume that the inputs x_{ij} and outputs y_{rj} are multiplied by positive α_i and μ_r, respectively. This therefore leads to $\tilde{x}_{ij} = \alpha_i x_{ij}$ $(i = 1,2,...,m; j = 1,2,...,n)$ and $\tilde{y}_{rj} = \mu_r y_{rj} (r = 1,2,...,s; j = 1,2,...,n)$, and $\tilde{P}_{io}^- = \alpha_i P_{io}^-, i = 1,2,...,m$; $\tilde{P}_{ro}^+ = \mu_r P_{ro}^+, r = 1,2,...,s$. The adjusted constraints of (5) $\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j \alpha_i x_{ij} \leq \alpha_i x_{io} - \beta \alpha_i P_{io}^- + s_i |\alpha_i x_i|^{max}$ and $\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j \mu_r y_{rj} \geq \mu_r y_{ro} + \beta \mu_r P_{ro}^+ - t_r |\mu_r y_r|^{max}$ are simply transformed to the constraints of (5). As a result, $|x_i|^{max}$ and $|y_r|^{max}$

assist in keeping model (5) unit invariant as well as making sure the feasibility of (5) as shown in the following proposition.

Proposition 1. Model (5) is always bounded and feasible.

Proof. Given the predefined parameter M, it is obvious that model (5) is bounded.

Regarding feasibility of model (5), consider two following cases:

i) Assume that $(x_o, y_o) \in T_o^s$ which implies that

$$(x_o, y_o) \in \left\{ (x, y) | x \geq \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j x_j, y \leq \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j y_j, \ \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j = 1, \ \lambda_j \geq 0 \right\}$$

Hence, there exists $0 < \beta^* < 1$ such that $(x_o - \beta^* P_o^-, y_o + \beta^* P_o^+) \in PPS$[4]. In other words, there exists $\beta^* \in (0,1)$ such that $x_{io} - \beta^* P_{io}^- \geq \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j x_{ij}$ and $y_{ro} + \beta^* P_{ro}^+ \leq \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j y_{rj}$.

Therefore, the model (2) is feasible and the optimal solutions for model (5) are $s_i^* = 0, (i = 1, 2, \dots, m)$ and $t_r^* = 0, (r = 1, 2, \dots, s)$. This shows the feasibility of model (5).

ii) Assume that $(x_o, y_o) \notin T_o^s$, which implies that

$$(x_o, y_o) \notin \left\{ (x, y) | x \geq \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j x_j, \ y \leq \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j y_j, \ \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j = 1, \ \lambda_j \geq 0 \right\}$$

This leads to $\exists i: x_{io} < \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j x_{ij}$ and/or $\exists r: y_{ro} > \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j y_{rj}$.

Models (2) and (5) are feasible if

a) $\beta^* < 0, s_i^* = 0, (i = 1, \dots, m)$ for model (5) when $\exists i: x_{io} < \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j x_{ij}$ and/or

b) $\beta^* < 0, t_r^* = 0, (r = 1, \dots, s)$ for model (5) when $\exists r: y_{ro} > \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j y_{rj}$

Otherwise, model (5) is feasible if

c) $0 < \beta^* < 1, s_i^* > 0$ when $\exists i: x_{io} < \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j x_{ij}$ and/or

d) $0 < \beta^* < 1, t_r^* > 0$ when $\exists r: y_{ro} > \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j y_{rj}$.[5]

Proposition 2. Model (2) is infeasible if and only if there exists at least one r or i such that $s_r^* > 0$ or $t_i^* > 0$ where s_r^* and t_i^* are the optimal solutions of model (5).

---

[4] This is the case that DMU$_o$ is inefficient and removing it from the PPS does not change the original technology. So, the optimal value of the RDM model is equal to the optimal value of the SRDM model (2).

[5] Under the cases (a) and (b) models (2) and (5) have a feasible solution and their optimal objective function values are identical whereas in the cases (c) and (d) model (5) is feasible and model (2) has no feasible solution.

12

Proof. (i) Assume that model (2) is infeasible. If $s_r^* = 0$ and $t_i^* = 0$ in model (5), this implies that model (2) is feasible, which is a contradiction to our earlier assumption. Therefore, some components of s_r^* or/and t_i^* are positive.

(ii) Assume that some components of s_r^* and/or t_i^* are positive (i.e., $s_r^* > 0$ and/or $t_i^* > 0$), and model (2) is feasible. This shows that $s_r^* = 0$ and $t_i^* = 0$ are feasible solutions to model (5) that is a contradiction to our assumption. Therefore, from (i) and (ii) the proof completes. ∎

The value of $\bar{\varphi}_0^* = 1 - \beta^* + \sum_{r=1}^{s} s_r^* + \sum_{i=1}^{m} t_i^*$ is called *SRDM measure* of DMU$_o$ where β^*, s_r^* and t_i^* are the optimal solutions of model (5).

Note that the efficiency measure of the inefficient units resulted from models (2) and (5) is identical to the conventional RDM measure since its removal does not change the shape of the technology.

Lemma 1. Model (5) is equivalent to model (2) when model (2) is feasible.

In particular, the corresponding output- and input-oriented modified SRDM problem for the efficient DMU$_o$ can be expressed as

$$\min \quad 1 - \beta + M \sum_{r=1}^{s} t_r$$

$$\text{s.t.} \quad \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j x_{ij} \leq x_{io}, \quad \forall i,$$

$$\sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j y_{rj} \geq y_{ro} + \beta P_{ro}^+ - t_r |y_r|^{max}, \forall r,$$

$$\sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j = 1,$$

$$\lambda_j \geq 0, \quad \forall j (j \neq o),$$

$$\beta \text{ free.}$$

$$(6)$$

$$\min \quad 1 - \beta + M \sum_{i=1}^{m} s_i$$

$$\text{s.t.} \quad \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j x_{ij} \leq x_{io} - \beta P_{io}^- + s_i |x_i|^{max}, \quad \forall i,$$

$$\sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j y_{rj} \geq y_{ro}, \quad \forall r,$$

$$\sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j = 1,$$

$$\lambda_j \geq 0, \quad \forall j (j \neq o),$$

$$\beta \text{ free.}$$

$$(7)$$

13

For further perusal of the method, consider DMU_E in Figure 1 that is infeasible when applying model (4). The production frontier is $ABCDK$ when DMU_E is excluded from the technology. The distance from the line DK to the line EW is 1, in which DK is the segment of the frontier and EW can be extended to the left from E to coincide the point I (ideal point). Equivalently, solving model (7) results in $s_2^* |x_2|^{max} = 1$ where $s_2^* = 1/6$ and $|x_2|^{max} = 6$, meaning that for projecting DMU_E onto the frontier DK, its x_1 is decreased by $\beta P_{1E}^- + s_1|x_1|^{max} = 0.5(2 - (-6)) + 0(6.5) = 4$ and its x_2 is scaled up by $\beta P_{2E}^- + s_2|x_2|^{max} = 0.5(-6 - (-6)) + \frac{1}{6}(6) = 1$. In such case, DMU_D is defined as the benchmark for DMU_E since $x_{1E}^* = 2 - 4 = -2$ and $x_{2E}^* = -6 - 1 = -5$.

In addition, a *SRDM measure*, $\bar{\varphi}_0^*$, for DMU_o may not be greater than one while the super-efficiency scores for efficient units should be preferably greater than one. For example, the SRDM measure of DMU_E as a RDM efficient unit is 0.6667, i.e., $1 - \beta_C^* + \sum_{r=1}^s s_r^* + \sum_{i=1}^m t_i^* = 1 - 0.5 + 1/6 = 0.6667$ where $s_1^* = t_1^* = 0$ and $s_2^* = 1/6$. To deal with the problem, we modify the SRDM measure yielded by model (5) in terms of the concept extended by Chen (2005) and Lee and Zhu (2012). In so doing, SRDM as a directional model can be considered as input saving and/or output surplus for an efficient DMU under analysis where it moves towards the frontier in an input and output improvement direction. Let vectors $\hat{\imath}$ and $\hat{o}$ denote the input saving index and output surplus index, respectively:

$$\hat{\imath} = \begin{cases} 0 & if\ I = \emptyset \\ \dfrac{\sum_{i \in I}\left(\dfrac{1 + s_i^*}{1}\right)}{|I|} & if\ I \neq \emptyset \end{cases}$$

$$\hat{o} = \begin{cases} 0 & if\ R = \emptyset \\ \dfrac{\sum_{r \in R}\left(\dfrac{1}{1 - t_r^*}\right)}{|R|} & if\ R \neq \emptyset \end{cases}$$

where $I = \{i | s_i^* > 0\}$ and $R = \{r | t_r^* > 0\}$, and $|R|$ and $|I|$ are the cardinality of the sets R and I, respectively. Note that the vector $(\hat{\imath}, \hat{o})$ presents the distance from DMU_o to the frontier established by the remaining DMUs. In other words, $\hat{\imath}$ and $\hat{o}$ show the increase in inputs and decrease in outputs of DMU_o, respectively, so as to reach the production frontier. Consequently, the modified SRDM measure can be defined as $\bar{\varphi}_0^* = 1 - \beta^* + \hat{\imath} + \hat{o}$ where $1 - \beta^*$, $\hat{\imath}$ and $\hat{o}$ are the efficiency, the input saving index, and output surplus index, respectively. When the sets I and R are not empty, the values of input saving index and output

14

surplus index are, respectively, defined as $\frac{\sum_{i \in I}\left(\frac{1+s_i^*}{1}\right)}{|I|} = \frac{\sum_{i \in I}\left(\frac{|x_i|^{max} + s_i^* |x_i|^{max}}{|x_i|^{max}}\right)}{|I|}$ and

$\frac{\sum_{i \in I}\left(\frac{|y_r|^{max}}{|y_r|^{max} - t_r^* |y_r|^{max}}\right)}{|R|} = \frac{\sum_{i \in I}\left(\frac{1}{1 - t_r^*}\right)}{|R|}$. For DMU$_E$ in Figure 1, $\bar{\varphi}_E^* = 1.6667$ where $\beta_E^* = 0.5$,

$\hat{\imath} = 1.1667$ and $\hat{o} = 0$.

In summary, our proposed model (5) in this study yields a modified SRDM measure of DMUs even if model (2) is either feasible or infeasible. It is worth noting that in the case of feasibility of model (2) our modified SRDM measures are exactly equivalent to the original SRDM.

4. Numerical Examples

In this section, two examples are quoted to illustrate the applicability of our approach. The examples are based on the same datasets as the study of Hadi-Vencheh and Esmaeilzadeh (2013). The General Algebraic Modeling System (GAMS)[6] software is utilized to solve the proposed models.

4.1. Example 1

The first example evaluates 13 DMUs with two inputs {x_1, x_2} and three outputs {y_1, y_2, y_3} as listed in Table 1. The values of x_2, y_2 and y_3 for all the DMUs are non-positive while x_1 and y_1 values are strictly positive. Thanks to the negative value in the data set, we run the *SRDM* model (1) to calculate the efficiency score $(1 - \beta^*)$ of thirteen DMUs as presented in the 2nd column of Table 2. A higher discriminatory power is required as soon as we see five efficient DMUs {C, G, H, K, M} in the result. To rank these efficient units, the SRDM model (2) is solved as shown in the 3rd column of Table 2. However, model (2) is infeasible for DMUs G, H and M since $P_{x_1 G}^- = P_{x_2 H}^- = P_{y_1 M}^+ = P_{y_3 M}^+ = 0$. We deal with the infeasibility trouble and give the efficiency measure by employing a modified SRDM model (5) as reported in the 4th column of Table 2. As can be seen, applying model (5) to DMUs G, H and M yields 2.1306, 3.8320 and 3.7632, respectively where M as a user-defined parameter is set equal to 10^8. Note that the scores for inefficient units in the RDM, SRDM and modified SRDM models are identical. In addition, the amount of input saving for G and H is $s_1^* |x_1|^{max} = 0.0056 \times 10.8 = 0.0605$ and $s_2^* |x_2|^{max} = 0.7586 \times 2.32 = 1.7600$, respectively, and the amount of output surplus for M is $t_1^* |y_1|^{max} = 0.7762 \times 9.56 = 7.4205$ and $t_3^* |y_3|^{max} = 0.0475 \times$

---

[6] http://www.gams.com

3.79 = 0.1800. The ranking order of DMUs that is a result of the proposed super-efficiency model is reported in the 4th column of Table 2. Therefore, we have the capability of making the difference between the efficient units {C, G, H, K, M} resulting from the RDM model. That is, our method provides the ranking of the efficient DMUs as $H \succ M \succ G \succ K \succ C$.

----Insert Tables 1 and 2 Here----

4.2. Example 2

This example contains six DMUs with two inputs {x_1, x_2} and one output {y_1} listed in Table 3. In detail, x_1 and y_1 are positive for some DMUs and negative for others while x_2 is positive for all the DMUs. Applying the RDM model to DMUs yields the 100% efficiency score for DMUs D and F. The modified SRDM model (5) proposed in this study can be used to deal with the problem of infeasibility in the SRDM model (2) when evaluating DMUs D and F. Note that M as a user-defined parameter is set equal to 10^8. Therefore, DMU_F is superior to DMU_D as shown in the 5th column of Table 4. In addition, the amount of output-surplus for DMU_D is $t_1^*|y_1|^{max} = 0.1667 \times 3 = 0.5001$ and the amount of input saving for DMU_F is $s_2^*|x_2|^{max} = 1.1250 \times 12 = 13.5$.

----Insert Tables 3 and 4 Here----

5. Conclusion

The use of negative data is an interesting and challenging issue in the data envelopment analysis (DEA) literature, particularly, in real applications when observations may include negative numbers. As an example, in decentralized energy resources, the consumption of electricity may be either negative or positive regarding the heat consumption. The well-timed study of Silva Portela et al. (2004) tackled the negative data in DEA by developing the range directional model (RDM) basing on directional distance model. The common infeasibility problem in the traditional DEA super-efficiency approach can be also viewed in the super-efficiency RDM (SRDM) model when negative data occur. However, the problem of infeasibility in measuring the super-efficiency scores occurs when at least one output and/or input with a range of *zero* improvement for the evaluated DMU is outside the PPS spanned by the corresponding inputs and/or outputs of the remaining DMUs.

In this paper, we propose a modified SRDM to discriminate between efficient and inefficient DMUs as well as to differentiate between efficient DMUs when observations contain negative values. We further propose ranking procedure for DMUs based on their efficiency

scores. We define input saving index and output surplus index for an efficient DMU under analysis for SRDM in order to move towards the frontier in an input and output improvement direction.

References

Adler, N., & Yazhemsky, E. (2010). Improving discrimination in data envelopment analysis: PCA–DEA or variable reduction. *European Journal of Operational Research*, 202(1), 273-284.

Adler, N., Friedman, L., & Sinuany-Stern, Z. (2002). Review of ranking methods in the data envelopment analysis context. *European Journal of Operational Research*, 140(2), 249-265.

Ali, A.I., & Seiford, L.M. (1990). Translation invariance in data envelopment analysis, *Operations Research Letters*, 9, 403-405.

Andersen, P., & Petersen, N.C., (1993). A procedure for ranking efficient units in data envelopment analysis. *Management Science*, 39 (10), 1261–1294.

Banker, R.D., & Chang, H., (2006). The super-efficiency procedure for outlier identification, not for ranking efficient units. *European Journal of Operational Research*, 175, 1311-1320.

Banker, R.D., Charnes, A., & Cooper, W.W., (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis, *Management Science,* 30(9), 1078-1092.

Bardhan, I., Bowlin, W.F., Cooper, W.W., & Sueyoshi, T., (1996). Models for efficiency dominance in data envelopment analysis. Part I: Additive models and MED measures. *Journal of the Operations Research Society of Japan*, 39, 322–332.

Basso, A., & Funari, S. (2014). Constant and variable returns to scale DEA models for socially responsible investment funds. *European Journal of Operational Research*, 235(3), 775-783.

Chambers R., Chung Y., & Färe R., (1998). Profit, directional distance functions, and Nerlovian efficiency. *Journal of Optimization Theory and Applications*, 98, 351–364.

Chambers, R. G., Chung, Y., & Färe, R., (1996). Benefit and distance functions. *Journal of Economic Theory*, 70, 407–419.

Charnes A., Cooper W. W., & Rhodes E., (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research*, 2(6): 429-444.

Charnes, A., Haag, S., Jaska, P., & Semple, J., (1992). Sensitivity of efficiency classifications in the additive model of data envelopment analysis. *International Journal of Systems Science*, 23(5), 789-798.

Charnes, A., Rousseau, J. J., & Semple, J. H., (1996). Sensitivity and stability of efficiency classifications in data envelopment analysis. *Journal of Productivity Analysis*, 7(1), 5-18.

Chen, Y., (2005). Measuring super-efficiency in DEA in the presence of infeasibility. *European Journal of Operational Research*, 161, 545–551.

Chen, Y., & Liang, L., (2011). Super-efficiency DEA in the presence of infeasibility: One model approach. *European Journal of Operational Research*, 213, 359–360.

Chen, Y., Du, J., & Hoa, J., (2013). Super-efficiency based on a modified directional distance function. *Omega,* 41, 621-625.

Cheng, G., Zervopoulos, P., & Qian, Z., (2013). A variant of radial measure capable of dealing with negative inputs and outputs in data envelopment analysis. *European Journal of Operational Research*, 225(1), 100–105.

Cook, W. D., Tone, K., & Zhu, J., (2014). Data envelopment analysis: Prior to choosing a model. *Omega*, *44*, 1-4.

Cook, W.D., Liang, L., Zha, Y., & Zhu, J., (2009). A modified super-efficiency DEA model for infeasibility. *Journal of Operational Research Society*, 60, 276–281.

Cooper, W.W., Seiford, L., & Tone, K., (2000). *Data envelopment analysis: A comprehensive text with models, applications, references and DEA-Solver software.* Kluwer Academic Publishers: Boston, MA.

Farrell, M. J., (1957). The measurement of productive efficiency. *Journal of the Royal Statistical Society. Series A (General)*, 253-290.

Friedman, L., & Sinuany-Stern, Z., (1997). Scaling units via the canonical correlation analysis and the data envelopment analysis. *European Journal of Operational Research,* 100 (3), 629–637.

Golany, B., (1988). An interactive MOLP procedure for the extension of data envelopment analysis to effectiveness analysis. *Journal of the Operational Research Society*, 39 (8), 725–734.

Hadi-Vencheh, A., & Esmaeilzadeh, A., (2013). A new super-efficiency model in the presence of negative data. *Journal of the Operational Research Society*, 64(3), 396-401.

Hatami-Marbini, A., Tavana, M., Agrell, P. J., Hosseinzadeh Lotfi, F., & Beigi, Z. G. (2015). A common-weights DEA model for centralized resource reduction and target setting. *Computers & Industrial Engineering*, 79, 195-203.

Hinojosa, M. A., Lozano, S., Borrero, D. V., & Mármol, A. M., (2017). Ranking efficient DMUs using cooperative game theory. *Expert Systems with Applications*, 80, 273-283.

Hosseinzadeh Lotfi, F., Hatami-Marbini, A., Agrell, P. J., Aghayi, N., & Gholami, K., (2013). Allocating fixed resources and setting targets using a common-weights DEA approach. *Computers & Industrial Engineering*, 64(2), 631-640.

Jahanshahloo, G. R., Junior, H. V., Hosseinzadeh Lotfi, F., & Akbarian, D. (2007). A new DEA ranking system based on changing the reference set. *European Journal of Operational Research*, 181(1), 331-337.

Kerstens, K., & Van de Woestyne, I., (2011). Negative data in DEA: a simple proportional distance function approach. *Journal of the Operational Research Society*, 62(7), 1413-1419.

Kerstens, K., & Van de Woestyne, I., (2014). A note on a variant of radial measure capable of dealing with negative inputs and outputs in DEA. *European Journal of Operational Research*, 234(1), 341-342.

Lee H. S., & Zhu, J., (2012). Super-efficiency infeasibility and zero data in DEA. *European Journal of Operational Research,* 216, 429–433.

Lee, H.S., Chu, C.W., & Zhu, J., (2011). Super-efficiency DEA in the presence of infeasibility. *European Journal of Operational Research,* 212, 141–147.

Li, Y., Xie, J., Wang, M., & Liang, L. (2016). Super efficiency evaluation using a common platform on a cooperative game. *European Journal of Operational Research*, 255(3), 884-892.

Lovell, C.A.K., (1995). Measuring the macroeconomic performance of the Taiwanese economy. *International Journal of Production Economics*, 39, 165-178.

Lovell, C.A.K., & Pastor, J.T., (1995), Units invariant and translation invariant DEA models. *Operations Research Letters*, 18, 3, 147-151.

Lovell, C.A.K., & Rouse, A.P.B., (2003). Equivalent standard DEA models to provide super efficiency scores. *Journal of the Operational Research Society*, 54 (1), 101–108.

Pastor, J. T., & Aparicio, J., (2015). *Translation invariance in data envelopment analysis. In data envelopment analysis* (pp. 245-268). Springer US.

Pastor, J. T., & Ruiz, J. L., (2007). *Variables with negative values in DEA. In modeling data irregularities and structural complexities in data envelopment analysis* (pp. 63-84). Springer US.

Pastor, J.T., (1996). Translation invariance in data envelopment analysis: A generalization. *Annals of Operations Research*, 66, 93-102.

Pourmahmoud, J., Hatami-Marbini, A., & Babazadeh, E., (2016). A comment on a new super-efficiency model in the presence of negative data. *Journal of the Operational Research Society,* 67, 530-534.

Rousseau, J. J., & Semple, J. H., (1995). Radii of classification preservation in data envelopment analysis: A case study of 'Program Follow-Through'. *Journal of the Operational Research Society,* 943-957.

Seiford, L.M., & Zhu, J., (1999). Infeasibility of super-efficiency data envelopment analysis models. *INFOR: Information Systems and Operational Research*, 37 (May), 174–187.

Sexton, T.R., Silkman, R.H., & Hogan, A.J., (1986). Data envelopment analysis: Critique and extensions. In: Silkman, R.H. (Ed.), *Measuring efficiency*: *An assessment of data envelopment analysis*. Jossey-Bass, San Francisco, CA (pp. 73– 105).

Silva Portela, M.C.A., Thanassoulis, E., & Simpson, G., (2004). A directional distance approach to deal with negative data in DEA: An application to bank branches. *Journal of Operational Research Society*, 55 (10): 1111-1121.

Thrall, R.M., (1996). Duality, classification and slacks in DEA. *Annals of Operations Research,* 66, 109–138.

Torgersen, A.M., Forsund, F.R., & Kittelsen, S.A.C., (1996). Slack-adjusted efficiency measures and ranking of efficient units. *Journal of Productivity Analysis,* 7, 379–398.

Zhu, J. (1999). Infeasibility of super-efficiency data envelopment analysis models. *INFOR: Information Systems and Operational Research*, 37(2), 174-187.

Zhu, J., (1996). Robustness of the efficient DMUs in data envelopment analysis. *European Journal of Operational Research,* 90 (3), 451–460.

Zhu, J., (2001). Super-efficiency and DEA sensitivity analysis. *European Journal of Operational Research*, 129(2), 443-455.
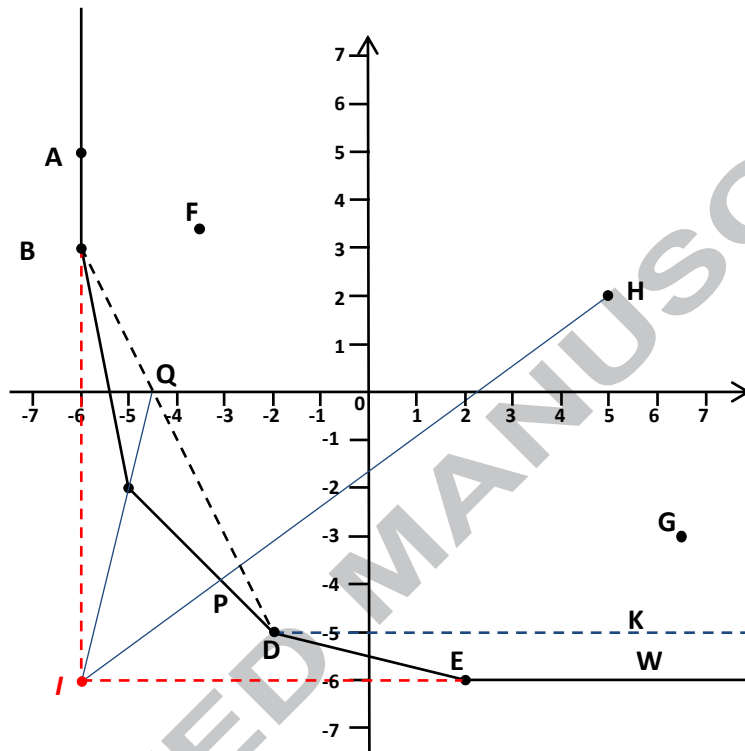
Figure 1. Example with two inputs.

Table 1. Input-output data for Example 1.

DMUs	x_1	x_2	y_1	y_2	y_3
A	1.03	-0.05	0.56	-0.09	-0.44
B	1.75	-0.17	0.74	-0.24	-0.31
C	1.44	-0.56	1.37	-0.35	-0.21
D	10.8	-0.22	5.61	-0.98	-3.79
E	1.3	-0.07	0.49	-1.08	-0.34
F	1.98	-0.1	1.61	-0.44	-0.34
G	0.97	-0.17	0.82	-0.08	-0.43
H	9.82	-2.32	5.61	-1.42	-1.94
I	1.59	0	0.52	0.00	-0.37
J	5.96	-0.15	2.14	-0.52	-0.18
K	1.29	-0.11	0.57	0.00	-0.24
L	2.38	-0.25	0.57	-0.67	-0.43
M	10.3	-0.16	9.56	-0.58	0.00

Table 2. Results of RDM, SRDM and the modified SRDM models.

DMU	RDM	SRDM	φ_0^*	Rank
A	0.9649	0.9649	0.9649	8
B	0.9181	0.9181	0.9181	10
C	1	1.2377	1.2377	5
D	0.7352	0.7352	0.7352	13
E	0.9243	0.9243	0.9243	9
F	0.9708	0.9708	0.9708	7
G	**1**	**Infeasible**	2.1306	3
H	**1**	**Infeasible**	3.8320	1
I	0.9945	0.9945	0.9945	6
J	0.8596	0.8596	0.8596	11
K	1	1.9375	1.9375	4
L	0.8448	0.8448	0.8448	12
M	**1**	**Infeasible**	3.7632	2

22

Table 3. Input-output data for Example 2.

DMUs	x_1	x_2	y_1
A	-2	12	-0.1
B	-2	8	0.1
C	1	5	2
D	6	4	3
E	6	6	2.5
F	-0.5	2.5	1

Table 4. Results of RDM, SRDM and the modified SRDM models.

DMUs	RDM	SRDM	φ_0^*	Rank
A	0.9355	0.9355	0.9355	5
B	1.7273	1.7273	1.7273	3
C	1.2800	1.2800	1.2800	4
D	**1**	**Infeasible**	3.5333	2
E	0.7619	0.7619	0.7619	6
F	**1**	**Infeasible**	6.4583	1

23

Highlights

- We determine the source of infeasibility problems of RDM super-efficiency (RDMS).

- We propose a modified RDMS model in the presence of negative data.

- We enhance discriminatory power of the RDM model by differentiating the efficient DMUs.

- The new approach is illustrated through two numerical examples.

24