Novel score function and non-dominance prioritisation method for intuitionistic fuzzy preference relations

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Abstract

Since Atanassov introduced the concept of intuitionistic fuzzy set (IFS) in 1986, much work has been done to develop decision making models with it as the information representation format. A problem that needs to be addressed in this type of decision making environment is the ranking of intuitionistic fuzzy numbers (IFNs). However, it is well known that there is no unique best approach to achieve this. A widely used approach to rank IFNs is to convert them into a representative crisp value, and perform the comparison on them. The representative crisp value developed for IFNs is known with the name of score degree.

This contribution presents a novel IFS score function which is further used to derive a fuzzy preference relation (FPR) from an intuitionistic FPR (IFPR). This result has the potential of being exploited to develop prioritisation methods for intuitionistic preferences relations based on their associated score preference relations. In particular, for IFPRs we define the intuitionistic quantifier guided dominance and intuitionistic quantifier guided non-dominance degree and propose a new prioritisation methodology for decision making problems with information represented using IFPRs. We prove that this methodology is an extension of the widely accepted methodology developed for the case of using FPRs to model information in decision making problems.

1 Introduction

Since Atanassov (1986) introduced the concept of intuitionistic fuzzy set (IFS) much work has been done to develop decision making models with it as the information representation format. A problem that needs to be addressed in this type of decision making environment is the ranking of intuitionistic fuzzy numbers (IFNs). Yager (2004) pointed out that this problem has been extensively studied for the case of FNs, and it has been concluded that there is no unique best approach. Recall that FNs are particular cases of IFNs, and therefore the same conclusion applies to the later type of numbers.

A widely used approach to rank FNs is to convert them into a representative crisp value, and perform the comparison on them (Yager, 2004). This approach is also the common one used to rank IFNs. The representative crisp values developed for IFNs are known with the names of score degree and accuracy degree. Chen and Tan (1994) developed a score function for IFSs based on the membership function and non-membership function, which was later improved by Hong and Choi (2000) with the addition of an accuracy function. Other score and accuracy functions had been proposed in (Wang et al., 2009; Lin et al., 2007; Liu and Wang, 2007; Ye, 2010).

For IFSs, we note the mathematical equivalence between IFSs and interval-valued FSs (IVFSs), and because IVFSs are special cases of interval-valued type-2 FSs, we provide a link between the development of the type reduced set (TRS) of an interval-valued type-2 FS and the score function of an IFS (Greenfield et al., 2008, 2012). This will allow us to motivate the definition of a novel IFS score function, which we prove is mathematically equivalent to the score function proposed by Chen and Tan (1994).

The novel score function proposed is used to derive a fuzzy preference relation (FPR) from an intuitionistic FPR (IFPR). This result allows the development of prioritisation methods for intuitionistic preferences relations based on their associated score preference relations. In particular, for IFPRs we define the intuitionistic quantifier guided dominance and intuitionistic quantifier guided non-dominance degree and propose a new prioritisation methodology for decision making problems with information represented using IFPRs. We prove that this methodology is an extension of the widely accepted methodology developed for the case of using FPRs to model information in decision making problems.

2 A Novel IFS Score Function

Intuitionistic fuzzy sets (IFSs) were introduced by Atanassov (1986)

***Definition* 1** (Intuitionistic fuzzy set (IFS))**.** *An intuitionistic fuzzy set (IFS) A over a universe of discourse X is given by*

$$A = \Big\{ \langle x, \mu_A(x), \nu_A(x) \rangle \, | x \in X \Big\}$$

where

$$\mu_A \colon X \to [0,1] \ , \ \ \nu_A \colon X \to [0,1]$$

and

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \quad \forall x \in X. \tag{1}$$

For each x, the numbers $\mu_A(x)$ and $\nu_A(x)$ are the degree of membership and degree of non-membership of x to A respectively.

Obviously, an IFS becomes a FS when $\mu_A(x) = 1 - \nu_A(x) \ \forall x \in X$. However, when there exists at least a value $x \in X$ such that $\mu_A(x) < 1 - \nu_A(x)$ an extra parameter has to be taken into account when working with IFSs: the hesitancy degree $\tau_A(x)$ of x to A (Montero et al., 2007)

$$\tau_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{2}$$

The hesitancy degree $\tau_A(x)$ is an indicator of the hesitation margin of the membership of element x to the IFS A. It represents the amount of lacking information in determining the membership of x to A (Burillo and Bustince, 1996; Yang and Chiclana, 2012). If the hesitation degree is zero, the reciprocal relation between membership and non-membership makes the latter one unnecessary in the formulation as it can be derived from the former.

Chen and Tan (1994) introduced a score function to evaluate the degree of suitability of an alternative with respect to a decision maker's requirement when evaluated using an intuitionistic fuzzy value.

***Definition* 2** (Chen and Tan (1994) IFS Score Function)**.** *Given an IFS A over X, $A = \Big\{ \langle x, \mu_A(x), \nu_A(x) \rangle \, | x \in X \Big\}$, the following score function can be defined:*

$$S_{CT}(A) \colon X \to [-1, 1]$$

$$S_{CT}(A)(x) = \mu_A(x) - \nu_A(x) \ \forall x \in X \tag{3}$$

Note that the score function is increasing with respect to the degree of membership and decreasing with respect to the degree of non-membership, and therefore the score value associated to an alternative can be interpreted as the degree of suitability/satisfaction by x of the decision maker's requirement. Therefore, given a set of alternatives $X = \{x_1, \ldots, x_n\}$ evaluated by a decision maker against a particular criterion using an IFS A over X, the score evaluations associated to the alternatives $\{S_{CT}(A)(x_i); i = 1, \ldots, n\}$ can be used to rank them using the following ordering criterion rule:

$$x_i \prec x_j \Leftrightarrow S_{CT}(A)(x_i) < S_{CT}(A)(x_j). \tag{4}$$

Note that given an IFS A, it is always true that $\mu_A(x) \leq 1 - \nu_A(x) = \mu_A(x) + \tau_A(x)$. Since IFSs and interval-valued FSs are mathematically equivalent (Dubois et al., 2005; Deschrijver and Kerre, 2003), we can call $[\mu_A(x), \mu_A(x) + \tau_A(x)]$ an intuitionistic fuzzy number (IFN) (Yang and Chiclana, 2009; Chen, 2010), and therefore an IFS can be seen as a collection of IFNs. Interval-valued FSs are special cases of type-2 FSs (Mendel, 2001). Indeed, $\{[\mu_A(x), \mu_A(x) + \tau_A(x)] | x \in X\}$ is an interval type-2 FS over X with lower membership function (LMF) $\mu_A(x)$ and upper membership function (UMF) $\mu_A(x) + \tau_A(x)$ (Greenfield et al., 2008). The Type Reduced Set (TRS) of a type-2 FS plays an important role in the final stage of any type-2 fuzzy decision making problem (Greenfield et al., 2012). The derivation of the TRS of a type-2 FS is a challenging problem and much research has been done in this area recently (Chiclana and Zhou, 2011). For interval type-2 FS, Nie and Tan (2008) propose a computational simple, efficient, approximate method to obtain the TRS for interval type-2 FSs, which involves taking the mean of their LMF and UMF. Experimental evidence (Greenfield and Chiclana, 2011; Greenfield et al., 2009) strongly suggests that as the domain discretisation is made finer, the Nie-Tan method produces a set that defuzzifies in the same value than that of the TRS. Based on this evidence, we propose a new definition of the score function of an IFS.

Definition 3 (IFS Score Function)**.** *Given an IFS A over X, with IFNs* $A(x) = [\mu_A(x), \mu_A(x)+\tau_A(x)], x \in X$, *the following score function can be defined*

$$S_{WC}(A)\colon X \to [0,1]$$

$$S_{WC}(A)(x) = \mu_A(x) + \frac{\tau_A(x)}{2} \tag{5}$$

Note that this new score function can be re-written as follows:

$$S_{WC}(A)(x) = \frac{\mu_A(x) - \nu_A(x) + 1}{2}. \tag{6}$$

This means that $S_{WC}(A)$ and $S_{CT}(A)$ are ordering mathematically equivalent in that they will lead to the same ordering of alternatives when the ordering rule (4) is applied. However, because of its range, we will see that score function $S_{WC}(A)$ will allow the derivation of a fuzzy preference relation (FPR) from an intuitionistic fuzzy preference relation (IFPR), that we will call the score FPR (SFPR). As mentioned above, when the hesitation degree is equal to the null function, the IFS becomes a FS, and therefore an IFPR reduces to a FPR, which is equal to the SFPR as we will show in the next subsection.

3 Score FPR Associated to an IFPR

Given three alternatives x_i, x_j, x_k such that x_i is preferred to x_j and x_j to x_k, the question whether the 'degree or strength of preference' of x_i over x_j exceeds, equals, or is less than the 'degree or strength of preference' of x_j over x_k cannot be answered by the classical preference modelling (Chiclana et al., 2009). The introduction of the concept of fuzzy set as an extension of the classical concept of set when applied to a binary relation leads to the concept of a fuzzy relation. The definition of a FPR is the following one (Bezdek et al., 1978; Nurmi, 1981):

Definition 4 (Fuzzy Preference Relation (FPR))**.** *A fuzzy preference relation (FPR) P on a finite set of alternatives* $X = \{x_1, \ldots, x_n\}$ *is characterized by a membership function* $\mu_P\colon X \times X \longrightarrow [0,1]$, $\mu_P(x_i, x_j) = p_{ij}$, *verifying*

$$p_{ij} + p_{ji} = 1 \ \forall i,j \in \{1, \ldots, n\}.$$

A FPR may be conveniently denoted by the matrix $P = (p_{ij})$. The following interpretation is also usually assumed:

- $p_{ij} = 1$ indicates the maximum degree of preference for x_i over x_j.
- $p_{ij} \in]0.5, 1[$ indicates a definite preference for x_i over x_j.
- $p_{ij} = 0.5$ indicates indifference between x_i and x_j.

We note that a FPR as above is also known as reciprocal ([0,1]-valued) preference relation. Szmidt and Kacprzyk (2002) defined the intuitionistic FPR (IFPR) as a generalisation of the concept of FPR. The adapted definition of an IFPR is the following one:

Definition 5 (Intuitionistic Fuzzy Preference Relation (IFPR))**.** *An intuitionistic fuzzy preference relation (IFPR) B on a finite set of alternatives* $X = \{x_1, \ldots, x_n\}$ *is characterised by a membership function* $\mu_B\colon X \times X \to [0,1]$ *and a non-membership function* $\nu_B\colon X \times X \to [0,1]$ *such that*

$$0 \le \mu_B(x_i, x_j) + \nu_B(x_i, x_j) \le 1 \ \ \forall (x_i, x_j) \in X \times X. \tag{7}$$

The value $\mu_B(x_i, x_j) = \mu_{ij}$ *can be interpreted as the certainty degree up to which* x_i *is preferred to* x_j*, while the value* $\nu_B(x_i, x_j) = \nu_{ij}$ *represents the certainty degree up to which* x_i *is non-preferred to* x_j*. Additionally, the following conditions are imposed:*

- $\mu_{ii} = \nu_{ii} = 0.5 \ \forall i \in \{1, \ldots, n\}$.
- $\mu_{ji} = \nu_{ij} \forall i,j \in \{1, \ldots, n\}$.

The IFPR can be conveniently represented by a matrix $B = (b_{ij})$ with $b_{ij} = (\mu_{ij}, \nu_{ij})$. When the hesitancy degree function is the null function we have that $\mu_{ij} + \nu_{ij} = 1$ $(\forall i,j)$, and therefore the IFPR $B = (b_{ij})$ is mathematically equivalent to the FPR (μ_{ij}), i.e. $B = (\mu_{ij})$. In the following, we will provide a method to derive a FPR from an IFPR $B = (b_{ij})$ via the application of the score function $S_{WC}(A)$ (5), which we call the score FPR (SFPR). We will also prove that this method in consistent with the case when the IFPR reduces to be a FPR, in which case the SFPR coincides with the original FPR.

***Theorem* 1** (Score FPR (SFPR)). *Let $B = (b_{ij})$ be an IFPR. Then $P = (p_{ij})$ where*

$$p_{ij} = S_{WC}(b_{ij})$$

is a FPR. P is called the Score FPR (SFPR) associated to the IFPR B.

Proof. We have that

$$p_{ij} + p_{ji} = S_{WC}(b_{ij}) + S_{WC}(b_{ji}) = \frac{\mu_{ij} - \nu_{ij} + 1}{2} + \frac{\mu_{ji} - \nu_{ji} + 1}{2}$$

For being $B = (b_{ij})$ an IFPR we have that $\mu_{ij} = \nu_{ji}$ and $\nu_{ij} = \mu_{ji}$, and therefore it is true that

$$p_{ij} + p_{ji} = 1 \ \forall i, j \in \{1, \ldots, n\}.$$

□

***Corollary* 1.** *Let $B = (b_{ij})$ be an IFPR and $P = (p_{ij})$ its associated SFPR. If the hesitancy degree function is the null function then $B = P$.*

The above results can be exploited to define concepts for an IFPR via the equivalent known ones in the associated SFPR. In particular, we propose a methodology to derive a priority vector for an IFPR via its corresponding SFPR based on the concept of non-dominance degree introduced by Orlovsky (1978).

4 Non-dominance Prioritisation Method for IFPRs

Yager (1996) presented a methodology to formulate linguistic expressions using ordered weighted average (OWA) operators guided by linguistic quantifiers (Zadeh, 1983). Specifically, the linguistic quantifier desired to be implemented is represented mathematically by a basic unit-monotonic (BUM) function $Q\colon [0, 1] \to [0, 1]$ such that $Q(0) = 0$, $Q(1) = 1$ and $Q(x) \geq Q(y)$ if $x \geq y$, which is used compute the OWA operator weights as follows:

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), i = 1, \ldots, n.$$

Non-decreasing relative linguistic quantifiers have been modelled in the literature with the following BMU function Q

$$Q(x) = \begin{cases} 0 & 0 \leq x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & b < x \leq 1 \end{cases}$$

$a, b \in [0, 1]$. The election of adequate values for the parameters a and b could lead to the implementation of a suitable linguistic quantifier. For example, the linguistic quantifier 'most of' can be implemented by choosing the values $(a, b) = (0.3, 0.8)$. Alternative representations for the concept of fuzzy majority can be found in the literature. For example, Yager (1996) considered the parameterized family of RIM quantifiers $Q(r) = r^a$ $(a \geq 0)$ for such representation. This family of functions guarantees (Chiclana et al., 2007) that: (i) all the experts contribute to the final aggregated value (strict monotonicity property), and (ii) associates, when $a \in [0, 1]$, higher weight values to the aggregated values with associated higher importance values (concavity property).

Based on this methodology, a ranking of alternatives can be obtained by applying the *quantifier guided non dominance degree, $QGNDD_i$*, to a FPR according to the following expression (Chiclana et al., 1998; Herrera and Herrera-Viedma, 2000):

$$QGNDD_i = \phi_Q\left(1 - p_{ji}^s, j = 1, \ldots, n\right), \tag{8}$$

where $p_{ji}^s = \max\left\{p_{ji}^c - p_{ij}^c, 0\right\}$ represents the degree to which x_i is strictly dominated by x_j. In our context, $QGNDD_i$, gives the degree in which each alternative is not dominated by a fuzzy majority of the remaining alternatives.

In the following we define the non-dominance concept to the case of an IFPR.

***Definition* 6** (Intuitionistic Quantifier Guided Non-dominance Degree (IQGNDD)). *Let $X = \{x_1, \ldots, x_n\}$ be a set of alternatives evaluated by a decision maker against a particular criterion using an IFPR $B = (b_{ij})$, and Q a BUM function. The intuitionistic quantifier guided non-dominance degree associated to the alternative x_i, $IQGNDD_i$, is defined as follows:*

$$IQGNDD_i = \psi_Q(1 - p_{ji}^s). \tag{9}$$

with $p_{ji}^s = \max\{p_{ji} - p_{ij}, 0\}$, $p_{ij} = S_{WC}(b_{ij})$ and ψ_Q is an OWA operator guided by the linguistic quantifier represented by the BUM function Q.

Example 1. Let

$$\mathbf{B} = \begin{pmatrix} (0.5,0.5) & (0.4,0.2) & (0.5,0.3) & (0.7,0.1) & (0.4,0.1) \\ (0.2,0.4) & (0.5,0.5) & (0.6,0.2) & (0.5,0.2) & (0.3,0.6) \\ (0.3,0.5) & (0.2,0.6) & (0.5,0.5) & (0.6,0.2) & (0.4,0.5) \\ (0.1,0.7) & (0.2,0.5) & (0.6,0.3) & (0.5,0.5) & (0.1,0.6) \\ (0.1,0.4) & (0.6,0.3) & (0.5,0.4) & (0.6,0.1) & (0.5,0.5) \end{pmatrix}$$

be an IFPR. The associated SFPR is

$$\mathbf{P} = \begin{pmatrix} 0.50 & 0.60 & 0.60 & 0.80 & 0.65 \\ 0.40 & 0.50 & 0.70 & 0.65 & 0.35 \\ 0.40 & 0.30 & 0.50 & 0.70 & 0.45 \\ 0.20 & 0.35 & 0.65 & 0.50 & 0.25 \\ 0.35 & 0.65 & 0.55 & 0.75 & 0.50 \end{pmatrix}.$$

Using the above linguistic quantifier *'most of'*, the OWA operator weighting vector is

$$W = (0.0, 0.4, 0.5, 0.1)^T.$$

The intuitionistic quantifier guided non-dominance degree associated to each one of the alternatives are

$$IQGNDD_1 = 1.00, IQGNDD_2 = 0.87, IQGNDD_3 = 0.82, IQGNDD_4 = 0.57, IQGNDD_5 = 0.97.$$

The alternatives can be ranked from best to worst according to the degree up to which an alternative is not dominated by *'most of'* the rest of alternatives:

$$x_1 \succ x_5 \succ x_2 \succ x_3 \succ x_4.$$

From corollary 1 we have that when the hesitancy degree function is the null function then $B = P$, and therefore we have $IQGNDD_i = QGNDD_i$. Therefore, it is clear that the following result is true:

***Theorem* 2.** *The IQGNDD associated to an IFPR extends the QGNDD associated to a FPR.*

5 Conclusions

A novel score function for IFSs is introduced, and applied to derive a score FPRs from IFPRs. It has been proved that when the hesitancy degree function is null the intuitionistic fuzzy relations coincide with their associated score fuzzy relations, respectively. The score function for IFSs introduced in this paper is mathematically equivalent to Chen and Tan's score function for IFS. Finally, a methodology is proposed to derive a priority vector for an IFPR via its corresponding SFPR based on the concept of non-dominance degree.

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