Perception Modelling using Type-2 Fuzzy Sets

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Publications by R. I. John on Type-2 Fuzzy Logic directly related to PhD research

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Abstract

Type-1 fuzzy logic has, for over thirty years, provided an approach for modelling uncertainty and imprecision. This methodology has been highly successful with a history of successful applications in a number of areas - particularly control. However, type-1 fuzzy systems are essentially 'crisp' in nature. This is not only paradoxical but also raises concerns for knowledge representation and inferencing. In particular type-1 fuzzy logic is flawed when representing perceptions such as colour, beauty, comfort etc. since these perceptions do not have a measurable domain.

This fundamental paradox is tackled in this research by employing a type-2 fuzzy paradigm. The investigation of the type-2 approach concludes that the uncertainty or imprecision that exists in most real problems can be more effectively modelled by a type-2 approach. The research reported in this thesis explores the properties of type-2 fuzzy sets as well as showing how useful they can be for knowledge representation and inferencing. It is shown that type-2 fuzzy sets have an important role to play in modelling perceptions. Results are given of using type-2 fuzzy sets to represent perceptions of a medical expert for shin image analysis indicating that the type-2 fuzzy paradigm is particularly helpful for perception representation.

A methodology has been developed that allows linguistic inputs to an adaptive system that implements a type-2 fuzzy system(the Adaptive Fuzzy Perception Learner (AFPL)). In this thesis, the rationale and full mathematical detail of the AFPL is presented. The approach has been applied successfully to the, so called, linguistic AND (analogous to the Boolean AND) as an aid to illustrating the methodology. Results are presented of applying the method to a real problem of classifying the acceptability of a car based on perceptions that describe certain features of the car. The AFPL is applied to this large, complex, set of data where the inputs to the network are linguistic. A detailed evaluation of the AFPL is given with recommendations for effective use of the AFPL. The results indicate that we now, truly, have an approach for learning the perceptions and relations in a type-2 fuzzy system.

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Chapter 1

Introduction

This thesis is concerned with the use of type-2 fuzzy sets for the representation of perceptions thus enabling the modelling of human perceptual categorisation by linguistic association. The modelling of perceptions and, in particular, a novel adaptive system for learning perceptions is, it is argued in this thesis, an important contribution to the development of fuzzy logic.

1.1 Vagueness and Imprecision

The problems of uncertainty, imprecision and vagueness in language have been discussed for many years. These problems have been major topics in philosophical circles with much debate, in particular, about the nature of vagueness and the ability of traditional Boolean logic to cope with concepts and perceptions that are imprecise or vague(Williamson 1998, Keefe & Smith 1997). As an introduction to this thesis, a discussion about vagueness, uncertainty and imprecision is provided. However, it is important to note that this is not a thesis on the philosophy of vagueness, but, since this work concerns itself with modelling perceptions, it is important that some of the discussion is reported. The purpose is to introduce some ideas and definitions as well as provide an overview of work carried out by others.

As early as 1923 the well known philosopher Russell discussed the notion of *Vagueness*(Russell 1923). In this early work he discusses, for example, the word 'red'. This word has no precise meaning - it is vague. The average person would clearly associate this word, for example, with the predominant colour of a Manchester United football shirt. However, as he points out, there are certain colours, as one moves through the colour spectrum, which could not perhaps unequivocally be described as red. There would be some uncertainty in associating the word red with that particular colour. There are other colours - the blue on a Leicester City football shirt for example - which are clearly not red. 'Red' is a vague term yet is one which is used every day in common language. Another example Russell uses in his discussion of vagueness is that of baldness. If a man starts with a full head of hair and the hairs are removed one by one he will eventually be bald. At what point does he become bald? There is no precise point, no particular hair that defines the move from a position of not being bald to one of baldness. Clearly baldness is an imprecise, vague concept.* He also discusses the idea that quantitative words used in science are (more or less) vague. For instance a two kilogram bag of sugar will hardly ever be exactly two kilograms. Even if it were perceived to be exactly two kilograms using the most modern measuring equipment it is unlikely to be **exactly** two kilograms since the measurement obtained is limited to the accuracy of that equipment. A two kilogram bag of sugar is in effect *about* two kilograms. All measurements have this imprecision. For the purposes of this thesis the term imprecision is used for the imprecise nature of all measurements. As Russell says *'It follows that every proposition that can be framed has a certain degree of vagueness'(Russell 1923, page 88).* In other words all propositions are vague to some degree. His argument is that the notion of Boolean AND and Boolean OR lose their real meaning when the symbols used are vague. Other philosophers (e.g. (Williamson 1998)) dispute this and argue that traditional Boolean logic can deal with these vague terms. The problem is one of ignorance. Concepts such as red and not red have crisp boundaries - we just don't know where these boundaries lie.

From the perspective of the arguments presented in this thesis whether Russell or Williamson is right does not affect the arguments in favour of type-2 fuzzy logic. Whether there is some unknown boundary between 'red' and 'not red' is almost irrelevant. For many applications, concepts and perceptions like red need to be modelled. Russells' definition of vagueness is: *a representation is vague when the relation of the representing system to the represented system is not one-one but one-many*(Russell 1923, page 89). What is meant by this? Well, if the system that is being represented can, to some degree, be related to more than one representation in another system then the representation is vague. He also points out that of course the law of excluded middle does not hold

*There are a number of puzzles considered by philosophers that are known as *sōritēs*. The baldness problem is one such example. Another is the question of a heap. At what point does a pile of grains of wheat make a heap? Is one grain a heap? Two? A thousand? There is no particular grain that makes a heap so this raises the question of whether a heap of wheat can exist? Clearly, human beings understand the notion of a heap of wheat yet traditional logic is unable to cope with this type of puzzle.

for imprecise or vague symbols but only for precise symbols[†] Black(1937) also discusses the problems of modelling vagueness. He differs from Russell in that he proposes that traditional logic can be used by representing vagueness at an appropriate level of detail and suggests that Russell's definition of vagueness confuses vagueness with generality. He discusses vagueness of terms or symbols by using borderline cases where it is unclear whether the term can be used to describe the case. When discussing scientific measurement he points out *"... the indeterminacy which is characteristic in vagueness is present also in all scientific measurement"*(Black 1937, page 429) and *"Vagueness is a feature of scientific as other discourse."*(Black 1937, page 429). An idea put forward by Black is the idea of a consistency profile or curve to enable some analysis of the ambiguity of a word or symbol. He uses three notions - a language, a situation when a user is trying to apply a symbol L to an object x and the consistency of the application of L to x. These notions are used to determine a curve that describes the consistency of application of L to x which is the number of observers who would apply the symbol L to x divided by the number of observers who would apply an alternative symbol ($\sim L$). The graph has on the vertical axis this consistency measure and on the horizontal axis the xs ranked according to the consistency. He notes that the curve will be different for different symbols. Figure 1.1 is similar to the figure in Black(1937, page 443) and shows the consistency profile for a vague symbol L and $\sim L$. Figure 1.2 shows the profile for a more precise symbol. To the fuzzy logic researcher of today these curves bear a strong resemblance to the membership functions of type-1 fuzzy sets(Zadeh 1965) but, as will be seen later, consistency profiles are different from membership functions.

The notion of 'loose concepts' is presented in a seminal article(Black 1963) by discussing at length the concept of 'tallness'. These loose concepts relate very closely to the notion of vagueness. A loose concept according to Black is where there is no sharp boundary where a concept C becomes *not* C. He gives the example of a sharply bounded concept 'short' by using the following formal definition for short: "There is a certain height, h, such that a man of that height is short, while a man of height $h + \delta$ is not short, no matter how small δ may be"(Black 1963, page 4)(clearly everyday use of short does not match this definition - short is another word that is vague). Any concept not sharply bounded he calls a loose concept (or vague concept) - short is therefore a

[†]The law of excluded middle states that if there are two sets A and its complement $\overline{A}$ then the union of A and $\overline{A}$ is the universal set X. In other words

$$A \cup \overline{A} = X$$

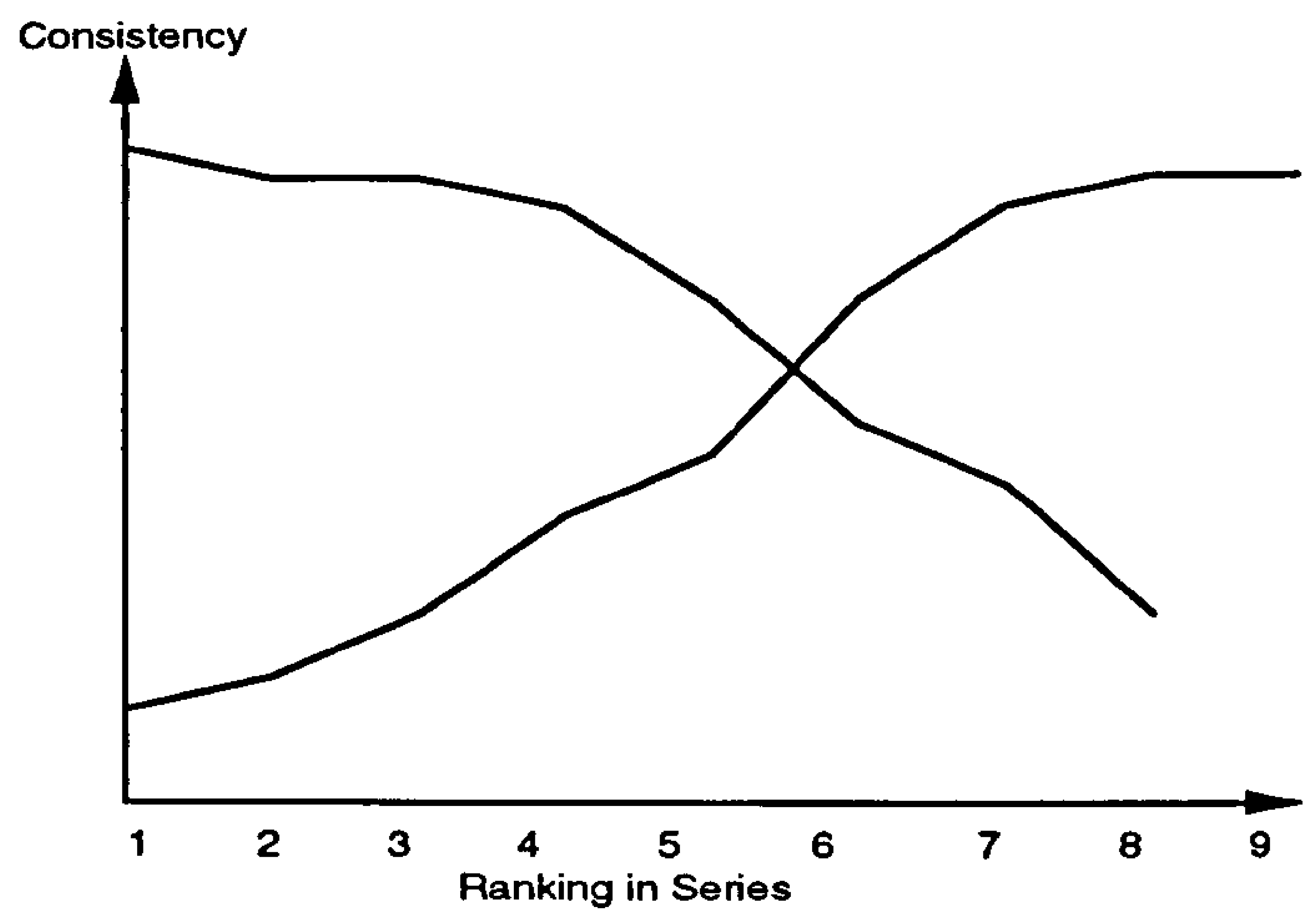

Fig. 1.1. A consistency profile for a vague symbol

loose concept. Goguen(1979) considered inexact concepts and how (type-1) fuzzy sets can be used to represent such concepts. He introduces the notion of a $J-set$ (a type of fuzzy set). J denotes the closed unit interval, the set$\{a \in Y|0 \leq a \leq 1\}$. The expression $\{a \in Y|P(a)\}$ denotes the set of all elements a of Y such that the proposition P is true of a. A $J-set$ then is a function $S : X \to J$ where J is the truth-set of S. It is interesting that Goguen briefly mentions higher type $J-sets$ ($J-sets$ which contain other $J-sets$) which he points out seem to measure the abstractness of a concept. This is a similar argument to the one made throughout this thesis about type-2 fuzzy sets.

More recently(Zadeh 1999), the modelling of perceptions has become an important topic. Consider this quote:

"...the remarkable human capability to perform a wide variety of physical and mental tasks without any measurements and any computations. Familiar examples of such tasks are parking a car; driving in heavy traffic; playing golf; understanding speech, and summarizing a story. Underlying this remarkable ability is the brain's crucial ability to manipulate perceptions - perceptions of size, distance, weight, speed, time, direction, smell, color, shape, force, likelihood, truth and intent, amongst others"

Zadeh(1999)Pages 106-107

In this quote Zadeh has highlighted that the real world is imprecise in many ways. The human being is capable of handling perceptions to carry out complex tasks that cannot

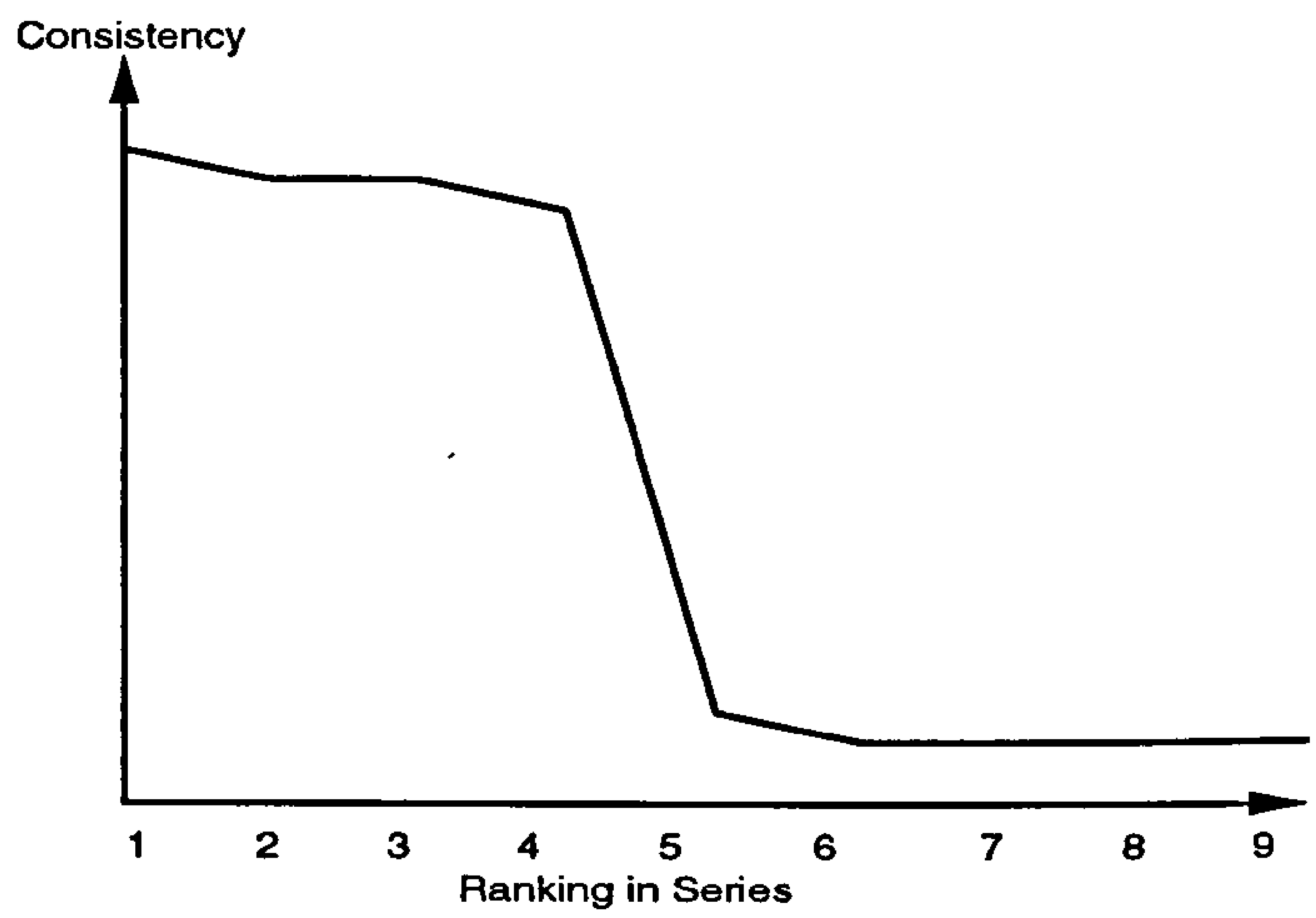

Fig. 1.2. A consistency profile for a precise symbol

successfully be modelled by traditional mathematical techniques. The central tenet of his work is that (type-1) fuzzy logic can be used to model perceptions. He develops the notion of the computational theory of perceptions where words play the role of the labels of perceptions. This paper builds on other papers (e.g. (Zadeh 1996)) where he highlights the role of fuzzy logic in Computing with Words-more of this later (Section 2.1). The computational theory of perceptions is only starting to be discussed within the fuzzy logic research community and although Zadeh's work uses many of the ideas developed by Zadeh in his earlier publications, this new theory is still not fully worked through. However, the need to model perceptions is eloquently expressed. An important point to note here is that perceptions are, by their very nature context dependent. That is, for example, the perception comfort has a different meaning for a car than, say, for a sofa.

The real world is not precise and the notions of vagueness, looseness, uncertainty, imprecision, concepts and perceptions are central to the way human beings solve problems. The discussions, briefly described here, have informed the debate about the problems of modelling notions, concepts or perceptions that are somehow vague, imprecise or uncertain. The term perceptions will be used throughout this thesis to describe concepts, ideas, notions that are (to some degree) imprecise, vague or loose.

1.2 Fuzzy Logic, Imprecision and Vagueness

Fuzzy sets(Zadeh 1965, Goguen 1967) have, over the past thirty five years, laid the basis for a successful method of modelling uncertainty, vagueness and imprecision. The use of fuzzy sets in real computer systems is extensive, particularly in consumer products and control applications.

Fuzzy logic (a logic based on fuzzy sets) is more mature than artificial neural networks with which it is often bracketed and indeed the reality is that the application of fuzzy logic is more pervasive. It is without doubt that fuzzy logic is now a mainstream technique in everyday use across the world. The number of applications is many, and growing, in a variety of areas, for example, heat exchange, warm water pressure, aircraft flight control, robot control, car speed control, power systems, nuclear reactor control, fuzzy memory devices and the fuzzy computer, control of a cement kiln, focusing of a camcorder, climate control for buildings, shower control and mobile robots(Lee 1990, Schwarz 1990, Czarnecki, John & Bennett 1995). The use of fuzzy logic is not limited to control. Successful applications, for example, have been reported in train scheduling, system modelling, computing(OMRON 1992), stock tracking on the Nikkei stock exchange(Schwarz 1990), information retrieval(Nakamura & Iwai 1982) and the scheduling of community transport(John & Bennett 1997). The fuzzy set approach to modelling is both intuitive and exciting. That this relatively simple idea can be used to model quite complex situations is extraordinary.

Zadeh(1999) presents a powerful argument for the use of fuzzy logic for manipulating perceptions. As has been discussed, his contention is that perceptions (for example, perceptions of size, safety, health and comfort) cannot be modelled by traditional mathematical techniques and that fuzzy logic is more suitable. The discussion about perception modelling is both new and exciting. As a contribution to these important developments, the position taken in this thesis is that type-2 fuzzy sets, since they have non-crisp fuzzy membership functions, can model these perceptions more effectively than type-1 fuzzy sets where the membership grades are crisp in nature.

So, the research reported in this thesis argues that, although fuzzy logic has many successful applications, there are a number of problems with the 'traditional' fuzzy logic approach which require a different set of fuzzy tools and techniques for modelling perceptions. In particular the view presented here is that fuzzy logic, as it is commonly used, is essentially **precise** in nature and that for many applications it is unable to model knowledge from an expert adequately. The contention in this thesis is that the modelling of imprecision can be enhanced by the use of type-2 fuzzy sets - providing a higher level of imprecision. Indeed, the tenet of this thesis is that the success of fuzzy logic can be built on by type-2 fuzzy sets and taken into the next generation of (type-2) fuzzy systems. The use of type-2 fuzzy sets allows for a better representation of uncertainty and imprecision in particular applications and domains. This viewpoint is presented in detail and, following on from this, a novel adaptive system has been developed to represent this type-2 uncertainty and capture the type-2 fuzzy sets and rules in a type-2 fuzzy system by the learning of perceptions.

The more imprecise or vague the data is, then type-2 fuzzy sets offer a significant improvement on type-1 fuzzy sets. Figure 1.3 shows the view taken in this work of the

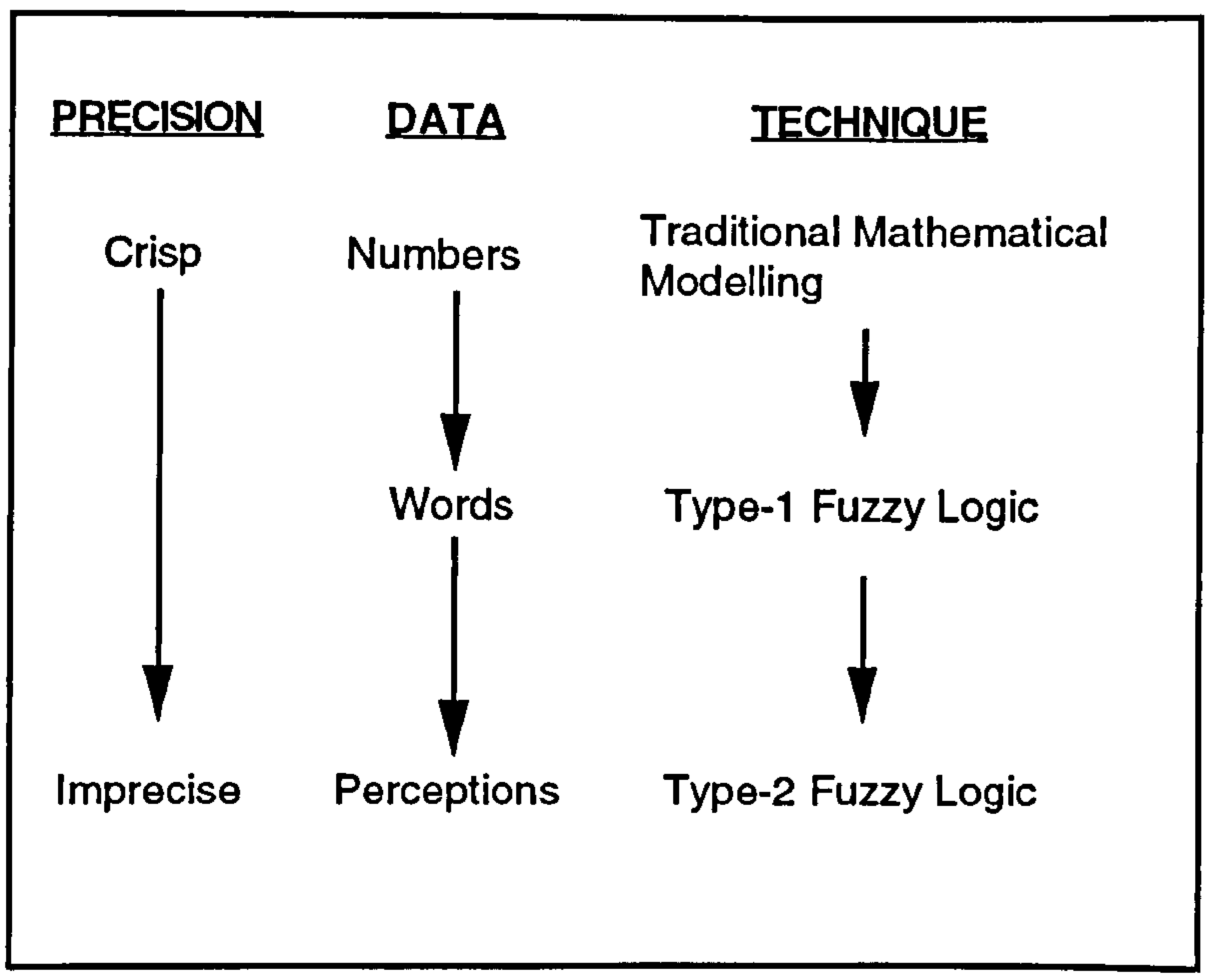

Fig. 1.3. Relationships between imprecision, data and fuzzy technique

relationships between levels of imprecision, data and technique. As the level of imprecision increases, then type-2 fuzzy logic provides a powerful paradigm for potentially tackling the problem. Problems that contain crisp, precise data do not, in reality, exist. However some problems can be tackled effectively using mathematical techniques where the assumption is that the data is precise. Other problems (for example, in control) use imprecise terminology that can often be effectively modelled using type-1 fuzzy sets.

Perceptions, it is argued here, are at a higher level of imprecision and type-2 fuzzy sets can effectively model this imprecision.

1.3 Research Hypothesis

The research hypothesis addressed in this thesis can be stated as

"Type-2 fuzzy sets have much to offer for knowledge representation and inferencing; however there is a need for some method for learning type-2 fuzzy systems. A type-2 fuzzy system that models human perceptual categorisation by linguistic association can be learnt from linguistic data that represent perceptions."

As will be seen (Chapter 3), type-2 fuzzy systems allow for modelling of perceptions that represent the way humans make decisions and use, for example, if-then rules to enhance the decision making process - this is what is meant by 'human perceptual categorisation by linguistic association'. In other words humans are capable of describing objects or situations using words (perceptions) and associating these perceptions. For example a car may be viewed as being comfortable - something that is not directly measurable. We also have a perception that a Rolls Royce is highly comfortable. We associate the perceptions together by, for example, saying that if a car is highly comfortable and very safe then it is highly acceptable. The perceptions of safety and comfort are categorised and then associated together in an if-then rule to assign a category for a perception of acceptability. In a decision-making situation, type-2 fuzzy sets offer the opportunity more so than conventional (type-1) fuzzy sets to model the use of words and phrases that, typically, experts use when solving problems. The detail of how this can be done is presented in this thesis.

However, as the argument unfolds, it will be seen that type-2 fuzzy systems have to be developed in a similar manner to type-1 fuzzy systems, thus presenting a number of problems - in particular the question of determining the type-2 fuzzy sets. The research reported here offers one solution to that particular problem. To allow for automatic development of type-2 fuzzy systems from data a novel adaptive type-2 fuzzy system has been developed. Known as the *Adaptive Fuzzy Perception Learner* this new approach is presented in detail and applied in a domain where linguistic terms need to be modelled.[‡]

[‡]It should be noted here that the author has published a number of papers on fuzzy logic and, in particular, type-2 fuzzy systems. A full list of papers and copies of the papers directly relevant to the thesis are provided in Appendix F.

1.4 Structure of the Thesis

The thesis is structured in the following way:

- Chapter 2 discusses fuzzy logic as the basis for the rest of the thesis. The (relatively) long history of fuzzy logic means that there is a huge amount of research material in the public domain. However, this Chapter considers only those components and aspects that are necessary to present the arguments for type-2 fuzzy inferencing. It describes fuzzy sets, and their role in modelling imprecision and uncertainty, as well as providing all the relevant theoretical results for type-1 fuzzy systems. The problems with type-1 fuzzy systems are highlighted. A well-known adaptive approach for type-1 fuzzy systems (ANFIS) is described as a guide to the philosophy of the novel type-2 adaptive approach which is the central result for this work. The ideas behind ANFIS are given as well as the algorithm with a brief discussion of the applications of ANFIS.

- Chapter 3 explores the literature of type-2 fuzzy sets. The properties of type-2 fuzzy sets and all the major operations are provided and some examples given to illustrate the use of these operations. The use of type-2 fuzzy sets for knowledge representation is discussed and the various approaches for inferencing with type-1 fuzzy sets using type-2 fuzzy sets are critically reviewed. A major piece of work by the author in using type-2 fuzzy sets to model the perceptions of a sports injuries consultant is described and results are presented for the use of type-2 fuzzy sets in modelling these perceptions for inputs to various unsupervised neural network paradigms. This new work highlighted, amongst other things, the need for a method for determining type-2 fuzzy sets. Type-2 fuzzy sets are just now beginning to be used more extensively in applications and an overview of these is given in this Chapter.

- Chapter 4 discusses type-2 fuzzy inferencing. The Chapter discusses an approach for inferencing with interval valued type-2 fuzzy sets, type-2 fuzzy relations are described and type-2 fuzzy systems that employ type-2 if-then rules and type-2 inferencing are explained. Finally, this Chapter provides the central philosophical approach adopted in this thesis for inferencing with type-2 fuzzy sets and how it compares with other possible approaches.

- Chapter 5 presents the novel type-2 adaptive system - the Adaptive Fuzzy Perception Learner. As well as a high level justification and description, a full mathematical exposition is given. Using an understandable example (the linguistic fuzzy

AND) the approach is shown to be highly successful in terms of using an adaptive network to learn linguistic terms. Results are presented which show how the type-2 fuzzy sets are modified by the Adaptive Fuzzy Perception Learner.

- Following on from the formal definition of the Adaptive Fuzzy Perception Learner, the application of this method is reported in Chapter 6. The problem tackled is one of determining the acceptability of a car based on perceptions describing the features of the car. Results of training the Adaptive Fuzzy Perception Learner are given showing that, for this particularly difficult problem, the network has some success.

- Chapter 7 provides the conclusion to the thesis summarising the contents of the work and discussing the opportunities for further research on the application and use of the novel Adaptive Fuzzy Perception Learner.

Chapter 2

Type-1 Fuzzy Sets, Type-1 Fuzzy Systems and ANFIS

To be able to place in context the central ideas behind the research presented in the thesis, this Chapter contains a discussion of type-1 fuzzy sets, important notions and results in type-1 fuzzy logic, some of the problems with the type-1 paradigm and the adaptive type-1 fuzzy system known as ANFIS.

Fuzzy logic has a thirty five year history. Since fuzzy sets were first introduced(Zadeh 1965), there has been huge research interest in the topic. For example, between 1970 and February 2000 the INSPEC database contained 28,697 research papers with the word fuzzy in the title or abstract and 8,160 papers in the Math.Sci.Net database(BISC email - see Appendix B). Fuzzy logic itself is well documented in numerous text books and papers (e.g. (Klir & Folger 1988, Cox 1994*a*, Cox 1994*b*, Kruse, Gebhardt & Klawonn 1994, Mendel 1995)) as well as, for example, three seminal papers by Zadeh(Zadeh 1975*a*, Zadeh 1975*b*, Zadeh 1975*c*). It is not the purpose of this chapter to purely regurgitate well-known straightforward ideas and techniques. The aim is to provide some of the basics of type-1 fuzzy sets so that the reader can contextualise the arguments made in the rest of the thesis. The central idea behind the research hypothesis (stated in the previous Chapter) is concerned with the role of type-2 fuzzy sets in modelling perceptions. Type-2 fuzzy sets are essentially a higher form of type-1 fuzzy sets. That is, they build on the ideas of type-1 fuzzy logic to, it is argued in this thesis, represent imprecision more effectively.

Before presenting the arguments for the type-2 fuzzy paradigm it is necessary to provide an overview of the basics of type-1 fuzzy logic. In this Chapter the discussion is about fuzzy sets and fuzzy logic as they are usually described in the literature. To that end the references are kept to a minimum and one reference will indicate support for the notion being described but clearly the fact that this is mainstream material means that a number of alternative references could be provided.

The rest of the Chapter is organised as follows: Section 2.1 presents a definition and discussion of fuzzy sets; Section 2.2 defines some of the operations on fuzzy sets that are important for this research and describes type-1 fuzzy systems; Section 2.3 discusses the issues faced by a fuzzy system developer and considers some of the problems that type-1 fuzzy systems present and finally Section 2.4 explores the adaptive fuzzy system known as ANFIS.

2.1 Fuzzy Sets

Fuzzy sets(Zadeh 1965) are the basis for (type-1) fuzzy logic and enable the handling of uncertainty and imprecision in real applications.* Fuzzy sets can be defined in the following way:

Definition 1 *For any fuzzy set A, the function μ_A represents the membership function for which $\mu_A(x)$ indicates the degree of membership that x, of the universal set X, belongs to set A and is, usually, expressed as a number between 0 and 1:*

$$\mu_A(x) : X \rightarrow [0,1].$$

Fuzzy sets can either be discrete or continuous. Discrete sets are written as:

$$A = \mu_1/x_1 + \mu_2/x_2 + ... + \mu_n/x_n$$

or

$$A = \sum_{i=1,n} \mu_i/x_i$$

where $x_1, x_2....x_n$ are members of the set A and $\mu_1, \mu_2, ..., \mu_n$ are their degrees of membership. A continuous fuzzy set A is written as:

$$A = \int_X \mu(x)/x.$$

*A note on terminology is required here. For the rest of this Chapter the term fuzzy set is used to mean a type-1 fuzzy set. Since the use of the terms type-1 and type-2 is not common, even amongst the research community, it was felt most appropriate here to use the term 'fuzzy set'. The history of fuzzy logic to date uses the term 'fuzzy sets' almost exclusively since, to most researchers, the notion of type-1 or type-2 does not occur naturally. The growth of interest over recent years in type-2 fuzzy sets means that this will inevitably change.

This notation can be confusing if one is new to fuzzy logic. However, for reasons of consistency with published material this thesis will use this notation throughout.
The key points to draw from this definition of a fuzzy set are:

- The members of a fuzzy set are members to some degree, known as a *membership grade* or degree of membership[†]
- A fuzzy set is fully determined by the membership function.
- The membership grade is the degree of belonging to the fuzzy set. The larger the number (in [0,1]) the more the degree of belonging[‡]
- The translation from x to $\mu_A(x)$ is known as *fuzzification*.
- A fuzzy set is either continuous or discrete. The rest of the thesis will deal almost exclusively with continuous fuzzy sets where the set is a function. All results apply for discrete sets and, indeed, software implementation of fuzzy logic often requires that these functions are discretised.
- Graphical representation of membership functions is very useful. For example, the fuzzy set 'Tall' might be represented as shown in Figure 2.1 where someone who is of height five feet has a membership grade of zero while someone who is of height seven feet is tall to degree one, with heights in between having membership grade between one and zero. The example shown is linear but, of course, it could be any function.

Fuzzy sets offer a practical way of modelling what one might refer to as 'fuzziness'. The real world can be characterised by the fact that much of it is imprecise in one form or other. For a clear exposition (important to the notion of, and argument for, type-2 sets) three ideas of 'fuzziness' can be considered important - imprecision, vagueness (linguistic uncertainty) and granularity.

[†]The terms used for the membership of a fuzzy set are varied. Throughout this thesis the term grade is used to mean the number in [0,1] that is the membership of x in the type-1 fuzzy set A. As will be seen later, grade is also used to mean the type-1 fuzzy set that represents the membership of a type-2 fuzzy set.

[‡]Goguen(1967) extends the notion of membership by the use of L-fuzzy sets where the membership is not restricted to be in [0,1]. This work essentially generalises the original work of Zadeh(1965).Formally he defines an L-fuzzy set as: An L-fuzzy set A on a set is a function $A : X \rightarrow L$. This paper presents a whole series of important results. However membership grades in fuzzy sets are now considered to be in [0,1].

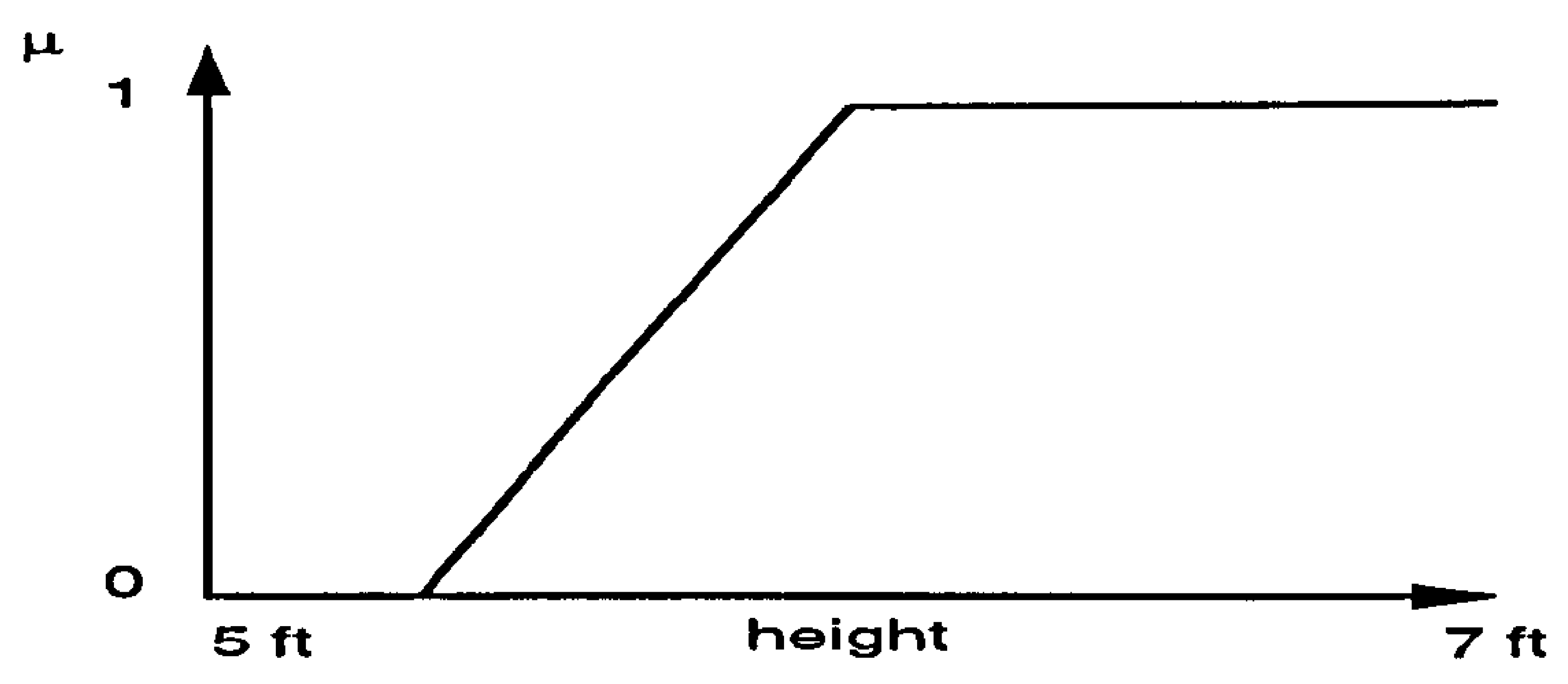

Fig. 2.1. The fuzzy set 'Tall'

Imprecision

As has already been discussed, in many physical systems measurements are never precise (a physical property can always be measured more accurately). There is imprecision inherent in measurement. Fuzzy numbers(Klir & Folger 1988, page 17) are one way of capturing this imprecision by having a fuzzy set representing a real number where the numbers in an interval near to the number are in the fuzzy set to some degree. So, for example, the fuzzy number 'About 35' might look like the fuzzy set in Figure 2.2 where the numbers closer to 35 have membership nearer unity than those that are further away from 35. The number 35.2 might belong to the fuzzy set 'About 35' to degree 0.8 whereas

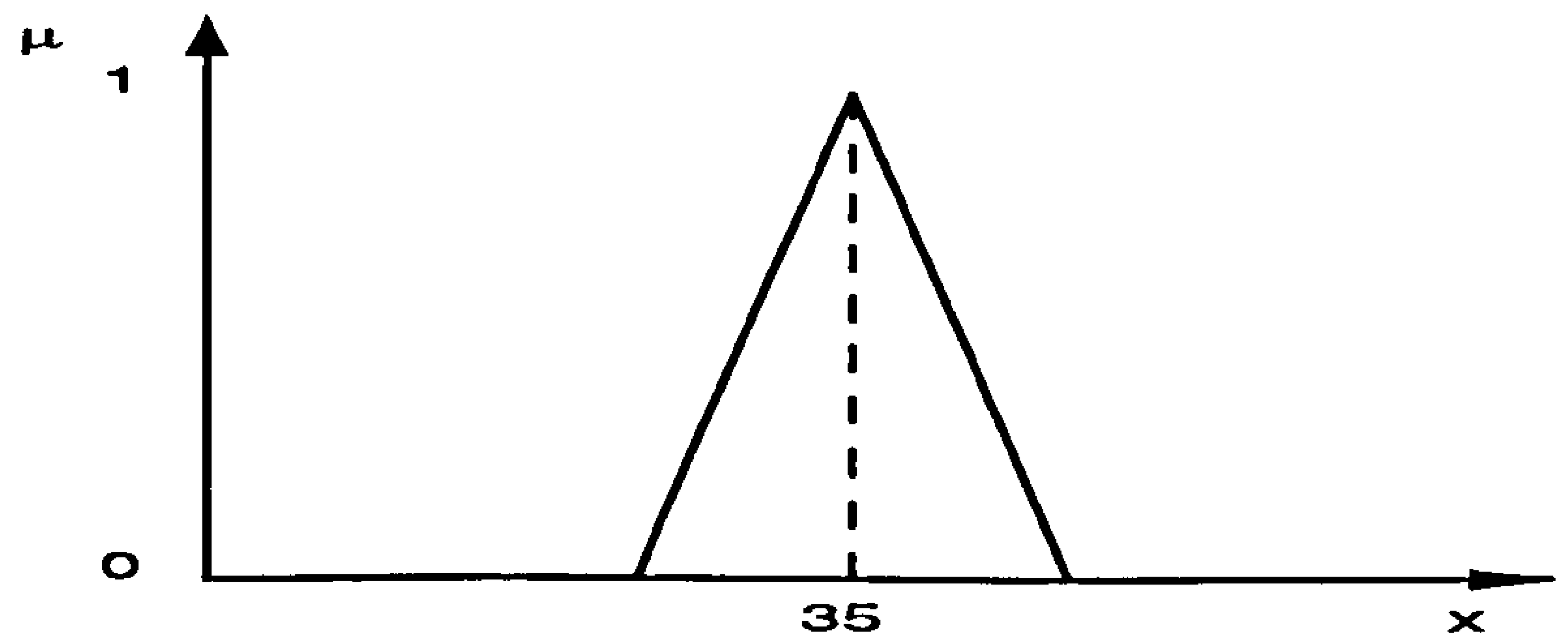

Fig. 2.2. The fuzzy number 'About 35'

35.8 might have a membership grade of 0.3 in the same fuzzy set. More formally Dubois & Prade(1980, page 26) define a fuzzy number in the following way:

Definition 2 *A fuzzy number is a convex normalised fuzzy set A of the real line* $\Re$ *such that (a)* $\exists!\ x_0 \in \Re, \mu_A(x_0) = 1$ *(*x_0 *is called the mean value of A); (b)* μ_A *is piecewise continuous.*

Fuzzy numbers are interesting but the issue of determining the fuzzy set remains.

Vagueness or Linguistic Uncertainty

Another use of fuzzy sets is where words have been used to capture imprecise notions, loose concepts or perceptions. We use words in our everyday language that we, and the intended audience, know what we want to convey but the words cannot be precisely defined. For example, where a bank is considering a loan application somebody may be assessed as a good risk in terms of being able to repay the loan. Within the particular bank this notion of a good risk is well understood. So, for example, on a scale of one to one hundred the fuzzy sets 'Good Risk' might look like the membership function in Figure 2.3. In other words it is not a black and white decision as to whether someone

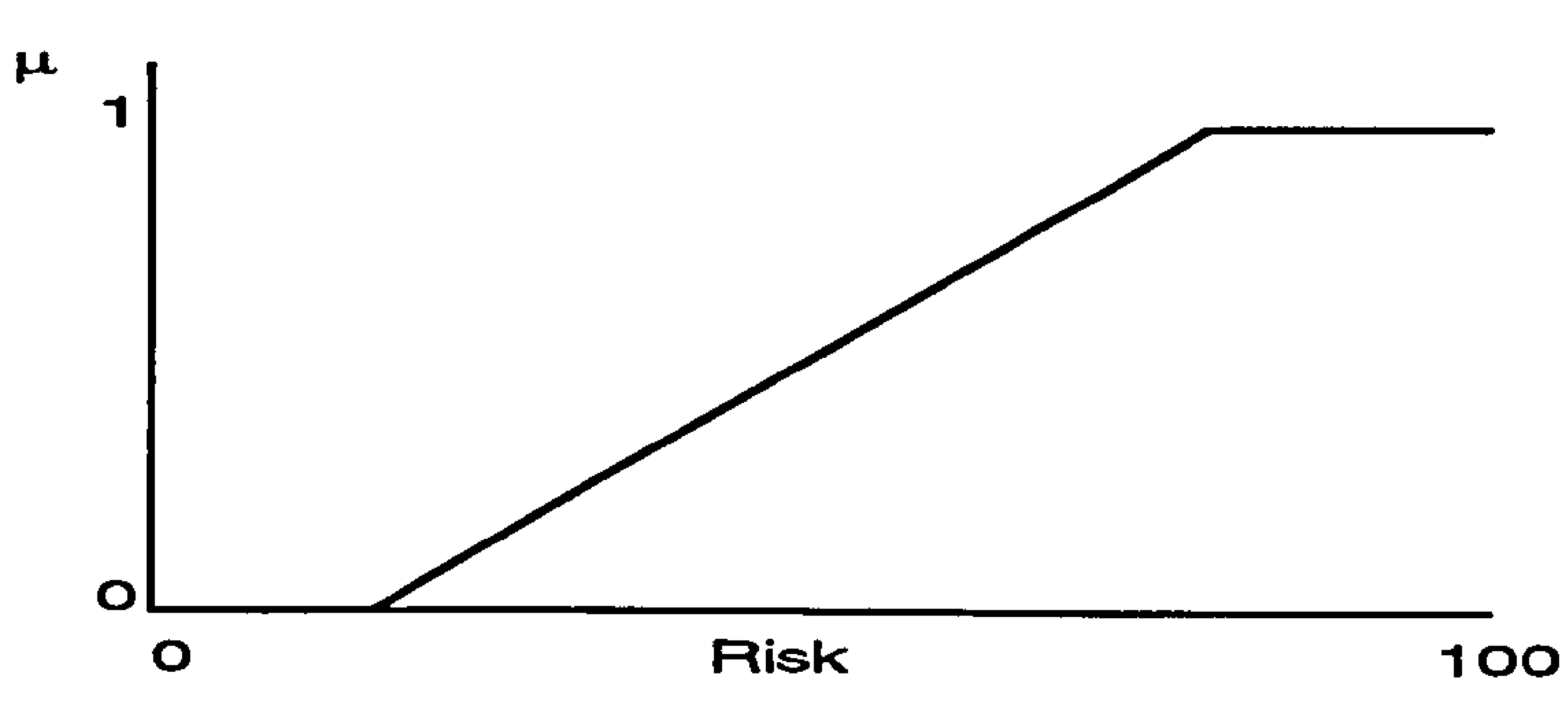

Fig. 2.3. The fuzzy set 'Good Risk'

is a good risk or not - they are a good risk *to some degree.* This use of fuzzy sets to capture the uncertainty in linguistic terms is the most common. For a particular bank someone, an expert in banking perhaps, would have to assign a number in [0,100] to that bank if the fuzzy set 'Good Risk' is to be used in a real system about the bank. Risk in a banking environment is a good example of a concept or perception that cannot be directly measured necessarily. An expert weighs up a number of factors - some directly measurable, some not (e.g. strong management of the company) - to arrive at the risk associated with the loan. It is to the modelling of these types of perceptions that type-2 fuzzy sets, it is argued in this thesis, can make a significant contribution.

Granularity

The notion of capturing and modelling linguistic uncertainty is the emphasis of this thesis; it will appear as a theme later, on a number of occasions. This linguistic uncertainty has coined a new phrase - **Computing with Words**(Zadeh 1996). In this fairly recent exposition Professor Zadeh argues that fuzzy sets capture the notion of granularity - chunks of knowledge. The reader is referred there for the detail but the central hypothesis in Zadeh's work is that words can be treated as granules of knowledge and that, by the use of fuzzy sets, these granules can be modelled. The paper presents a detailed exposition of *manipulating* granules of knowledge by fuzzy constraints and canonical forms. The argument presented in this thesis is that type-2 fuzzy sets *represent* linguistic uncertainty or granularity better than type-1 fuzzy sets and this is explored further in Chapter 3.

This Section has provided a definition of type-1 fuzzy sets and discussed some of the benefits that they bring to modelling uncertainty and imprecision. To be able to take the argument forward, the next section provides an overview of some of the (type-1) operations on type-1 fuzzy sets and how computer systems that deploy type-1 fuzzy sets can be implemented allowing the discussion of the type-2 operations to be more fruitful.

2.2 Type-1 Fuzzy Systems

This thesis is concerned with the use of fuzzy sets in computer systems since the use of fuzzy logic to enable better decision making, in any environment, invariably requires implementation on a computer. As with much of this material, there is a good deal of terminology used in the literature that can be confusing or misleading. Computer systems that employ (type-1) fuzzy sets are variously described as:

- Fuzzy Systems(Schwarz 1990, Terano, Asai & Sugeno 1992, Bezdek 1993, Kruse et al. 1994, Cox 1994*b*). This is a general term without any baggage and is the one preferred in this thesis. The IEEE Conferences and IEEE Transactions use the term Fuzzy Systems.

- Fuzzy Expert Systems. An expert system is a computer system that emulates the behaviour of an expert. They are very successful, in terms of applications but, arguably, have not lived up to their expectations. Fuzzy expert systems (e.g.(Türkşen 1996, Zadeh 1983)) are expert systems where the underlying knowledge base is fuzzy in nature. In particular expert systems usually employ if-then type rules and fuzzy expert systems will use if-then rules where the antecedents and/or consequents of these rules employ (type-1) fuzzy sets. Although fuzzy researchers do not use the term 'expert system' regularly it is the author's contention that (most) systems using fuzzy sets are indeed expert systems. They usually contain if-then rules that represent expertise in a given domain. It is suspected that the reticence in using the term is because there is an element of stigma associated with it in certain quarters.

- Fuzzy Knowledge Based Systems. Knowledge Based Systems is another term for expert systems so the term 'fuzzy knowledge based systems' appears in some literature (e.g. (Burkhardt & Bonissone 1992)).

The research reported in this thesis will use the term fuzzy system to describe a computer system that deploys fuzzy sets in if-then rules. A type-1 fuzzy system is a computer system that uses type-1 fuzzy sets in either the antecedent and/or the consequent of type-1 fuzzy if-then rules and a type-2 fuzzy system deploys type-2 fuzzy sets in either the antecedent and/or the consequent of type-2 fuzzy rules. Whichever of the above terms is used, they usually have the following features (see Figure 2.4):

- The *fuzzy sets* as defined by their membership functions. These fuzzy sets are the basis of a fuzzy system. They capture the underlying properties or knowledge in the system. For example in a control application there may be fuzzy sets that are defined for 'low pressure'. The shape of the membership functions has to be determined and the various approaches that have been used are discussed later in this Chapter (Section 2.3.2).

- The *if-then rules* that combine the fuzzy sets - in a rule set or knowledge base. There are a number of possibilities for the form these fuzzy if-then rules can take where the left hand side of the rule (the antecedent) is usually fuzzy. The right hand side (consequent) can be fuzzy or non-fuzzy. As with a conventional expert system or knowledge based system, these rules have to be acquired. The usual method is through knowledge elicitation from an expert but alternative methods are available where the rules can be gained directly from data(Hayashi, Maeda,

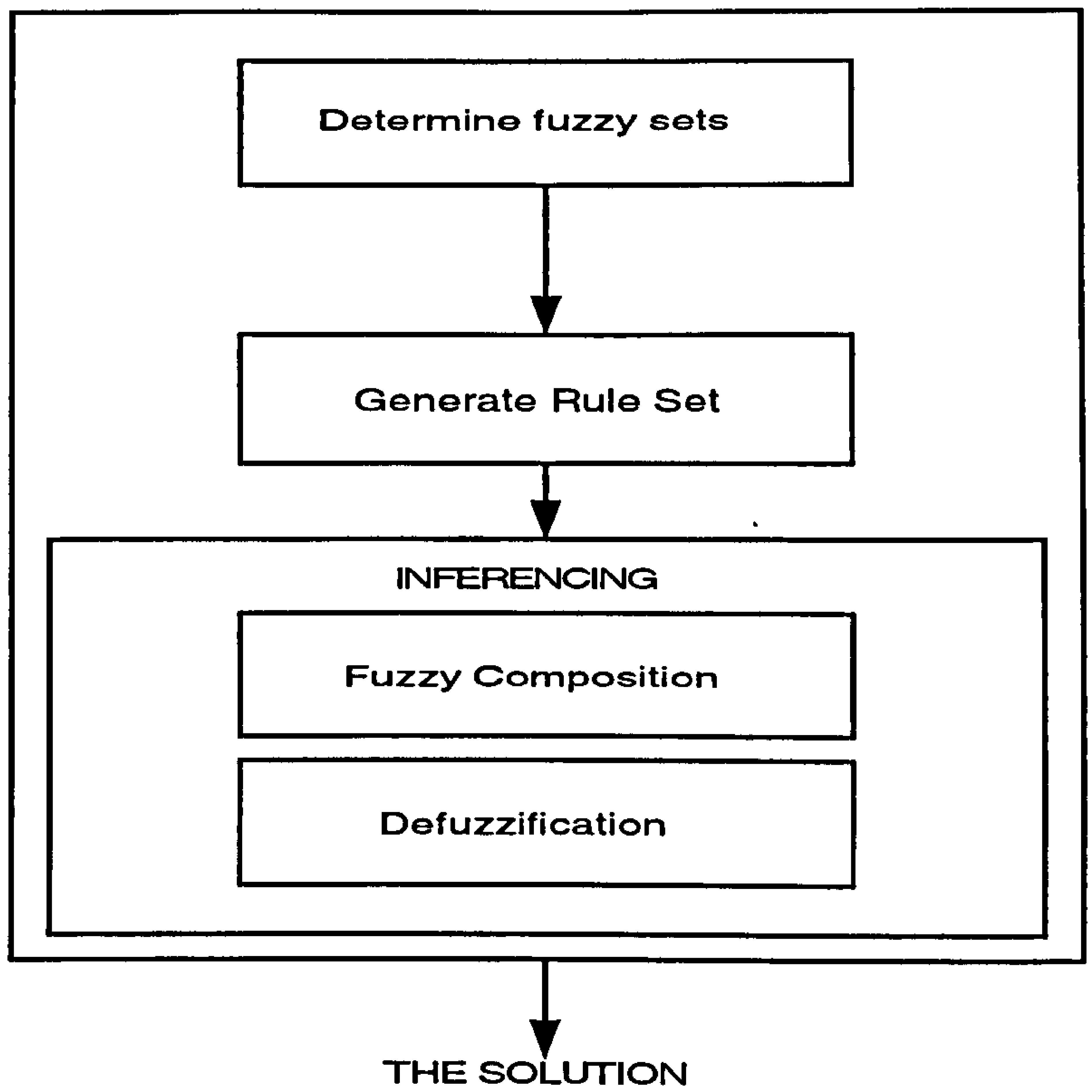

Fig. 2.4. A Fuzzy System

Bastian & Jain 1998, Shen & Chouchoulas 2000, Au & Chan 1998). An adaptive network is one such technique and the philosophy of this approach is adopted in this thesis.

- The fuzzy *composition* of the rules. Any fuzzy system that has a set of if-then rules has to combine the rules. Again, there are a number of possible approaches for combining the if-then rules. Two popular approaches are described here - the Mamdani and Takagi-Sugeno models.

- Optionally, the *defuzzification* of the solution fuzzy set. In many (most) fuzzy systems there is a requirement that the final output is a 'crisp' number. However, for certain fuzzy paradigms the output of the system is a fuzzy set. This solution set is 'defuzzified' to arrive at a number.

These different components of a fuzzy system are now discussed in more detail.

2.2.1 The Rule Set

The rules will usually be of an if-then nature. That is, more formally, a rule takes the form:

$$IF \ < antecedent > \ THEN \ < consequent > \tag{2.1}$$

For example, a fuzzy system may have rules of the form:

$$IF\ x\ is\ A\ and\ y\ is\ B\ THEN\ z\ is\ C \tag{2.2}$$

where x, y and z are from the universe of discourse X, Y and Z respectively and A,B and C are fuzzy sets. There are a number of variations available for if-then rules. For example, as will be seen later, the consequent in a Takagi-Sugeno rule is a crisp function. The essence of all the methods is that the underlying knowledge is contained in if-then rules where some component of the rule (antecedent or consequent) is fuzzy in nature. It is also worth noting that very often a fuzzy system may contain very few rules yet still be able to perform well, often exceeding the capabilities of more traditional methods.

2.2.2 Fuzzy Composition

To be able to carry out all the functions of a type-1 fuzzy system, definitions for 'AND' and 'OR' are required[§](the intersection and union respectively) for two type-1 fuzzy sets. To calculate the intersection of a pair of fuzzy sets there are a family of functions, triangular norms or *t-norms*, that meet certain requirements such as monotonicity, commutativity and associativity and the intersection of a fuzzy set with an ordinary set leads to exclusion of elements or conservation of degrees of membership. The union of two fuzzy sets employs the notion of a *t-conorm* which is commutative, associative, monotonic non-decreasing and has zero as unit element(Kruse et al. 1994) (see Appendix A for the full detail). The decision over which function to use for intersection and union is one of many a fuzzy system developer has to make. There are a wide variety of t-norms and t-conorms available to the fuzzy system developer(Kruse et al. 1994). For example, the most commonly adopted t-conorm for the union of two fuzzy sets A and B is a fuzzy set $A \cup B$ whose membership function is the maximum value of the membership function of x in A and B given by:

$$\mu_{A \cup B}(x) = max(\mu_A(x), \mu_B(x)) \tag{2.3}$$

[§]Similar operations are required for type-2 sets and these are given in Chapter 3.

In other words the membership function of the union of two fuzzy sets is gained by, for each x, taking the maximum of the membership grade of x in A and its grade in B. The intersection of two fuzzy sets A and B is often defined by the t-norm A∩B with the membership function

$$\mu_{A\cap B}(x) = min(\mu_A(x), \mu_B(x)) \tag{2.4}$$

This use of min and max often go together and are referred to as min-max. For the purposes of the rest of the thesis, the notion of t-norms and t-conorms is important since these will be required for type-2 fuzzy inferencing. So, given the fact that intersection and union of type-1 sets can be calculated, what other operations are used to combine rules and carry out inferencing in fuzzy systems? Central to fuzzy systems is the notion of *generalised modus ponens (gmp)*(Zadeh 1975*b*):

Ant1: If x is A then y is B
Ant2: x is A'
Cons: y is B'

where x and y are objects and A, A', B, B' are fuzzy sets. The fuzzy set A' is an approximation to the fuzzy set A and B' is an approximation to B. Note that the generalised modus ponens reduces to modus ponens when $A' = A$ and $B' = B$. Mizumoto and Zimmerman(1982) also describe *generalized modus tollens*:

Ant1: If x is A then y is B
Ant2: y is B'
Cons: x is A'

which, when $B' =$ *not* B and $A' =$ *not* A, reduces to modus tollens. There are various fuzzy relations that allow for the handling of fuzzy if-then rules of the form "If x is A then y is B". Mizumoto and Zimmerman(1982) review the many proposed approaches with a view to their suitability for generalised modus ponens and generalised modus tollens. This very detailed piece of theoretical comparison and the similar work by Lee(1990), although interesting in themselves, do not indicate how a fuzzy system developer might choose an implication function. So, this is another decision made by the fuzzy system developer based on 'feel' and experience.

At this stage of the thesis two of the most popular methods for inferencing with fuzzy if-then rules are introduced. The two approaches (for ease of explanation they are referred to here as the Mamdani and Takagi-Sugeno models) are the most widely deployed in real applications. An overview of both methods is now provided (Mendel(1995), in a very thorough tutorial on fuzzy logic, provides a more detailed exposition).

The Takagi-Sugeno Model

The Takagi-Sugeno model(Takagi & Sugeno 1985) has if-then rules of the form

$$\textit{IF } x \textit{ is } A \textit{ and } y \textit{ is } B \textit{ THEN } z = f(x,y) \tag{2.5}$$

where A and B are fuzzy sets but $z = f(x,y)$ is a crisp function in x, y. The antecedent could obviously be more complex with many 'AND's. The function in the consequent can be any function but in a first order Takagi-Sugeno model $f(x,y)$, typically, takes the form $f(x,y) = px + qy + r$ where p, q and r are constants. So, this type of rule has a fuzzy antecedent and crisp consequent. Figure 2.5 shows how two Takagi-Sugeno rules are combined to produce an output. The two rules are:

$$\textit{IF } x \textit{ is } A_1 \textit{ and } y \textit{ is } B_1 \textit{ THEN } z_1 = p_1 x + q_1 y + r_1$$

$$\textit{IF } x \textit{ is } A_2 \textit{ and } y \textit{ is } B_2 \textit{ THEN } z_2 = p_2 x + q_2 y + r_2$$

Given a value for x and y the membership values are found in A_1, B_1, A_2, B_2. For each rule the antecedent AND is then found by, for example, taking the minimum of the membership grades in each rule (any t-norm would suffice). This gives two values w_1 and w_2 which are the weightings for each function. A weighted average of the two functions z_1 and z_2 produces the final output

$$f = \frac{w_1 z_1 + w_2 z_2}{w_1 + w_2} \tag{2.6}$$

This model plays an important role in the (type-1) adaptive fuzzy system ANFIS, as will be seen in Section 2.4.

The Mamdani Model

The Mamdani model uses the rules as shown in Equation 2.2 where the antecedent and consequent are both fuzzy. To illustrate the Mamdani approach a two rule system is considered:

$$\textit{IF } x \textit{ is } A_1 \textit{ and } y \textit{ is } B_1 \textit{ THEN } z \textit{ is } C_1$$

$$\textit{IF } x \textit{ is } A_2 \textit{ and } y \textit{ is } B_2 \textit{ THEN } z \textit{ is } C_2$$

where $A_1, B_1, C_1, A_2, B_2, C_2$ are fuzzy sets. Figure 2.6 shows how a Mamdani model inferences with two rules. For given values of x and y the procedure is as follows (using min for AND and max for OR):

1. For the given x find the membership values in A_1 and A_2.

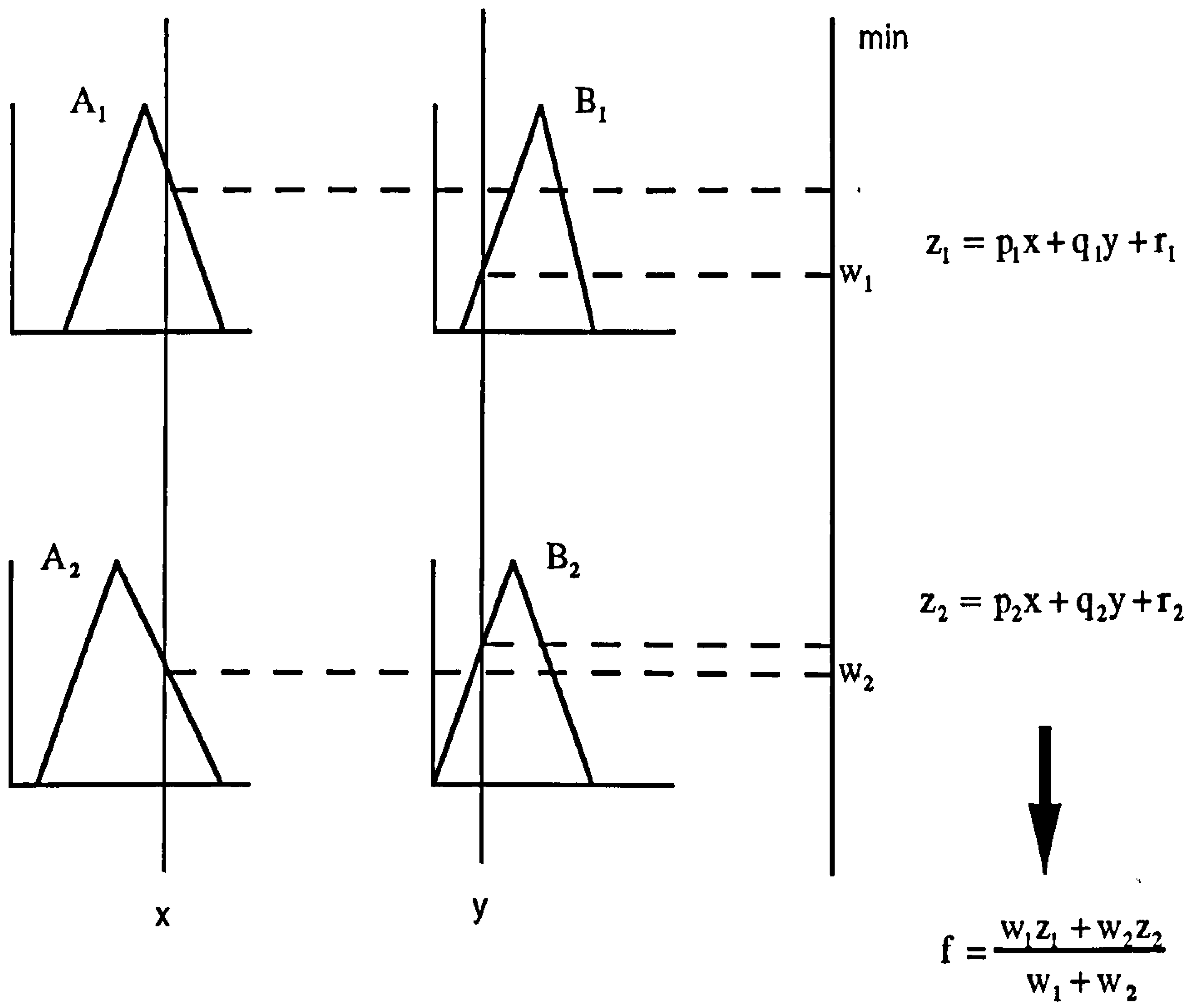

Fig. 2.5. The Takagi-Sugeno Model for two rules

2. For the given y find the membership values in B_1 and B_2.

3. For each rule take the minimum of the membership values in A_i and B_i.

4. Use this value to 'truncate' the fuzzy set $C_i (i = 1, 2)$ to produce a new set $C_i^{\prime} (i = 1, 2)$

5. For each value of z in the truncated sets take the maximum to produce the final output fuzzy set.

6. Optionally, 'defuzzify' the output set to produce a single number. This last procedure is known as *defuzzification*.

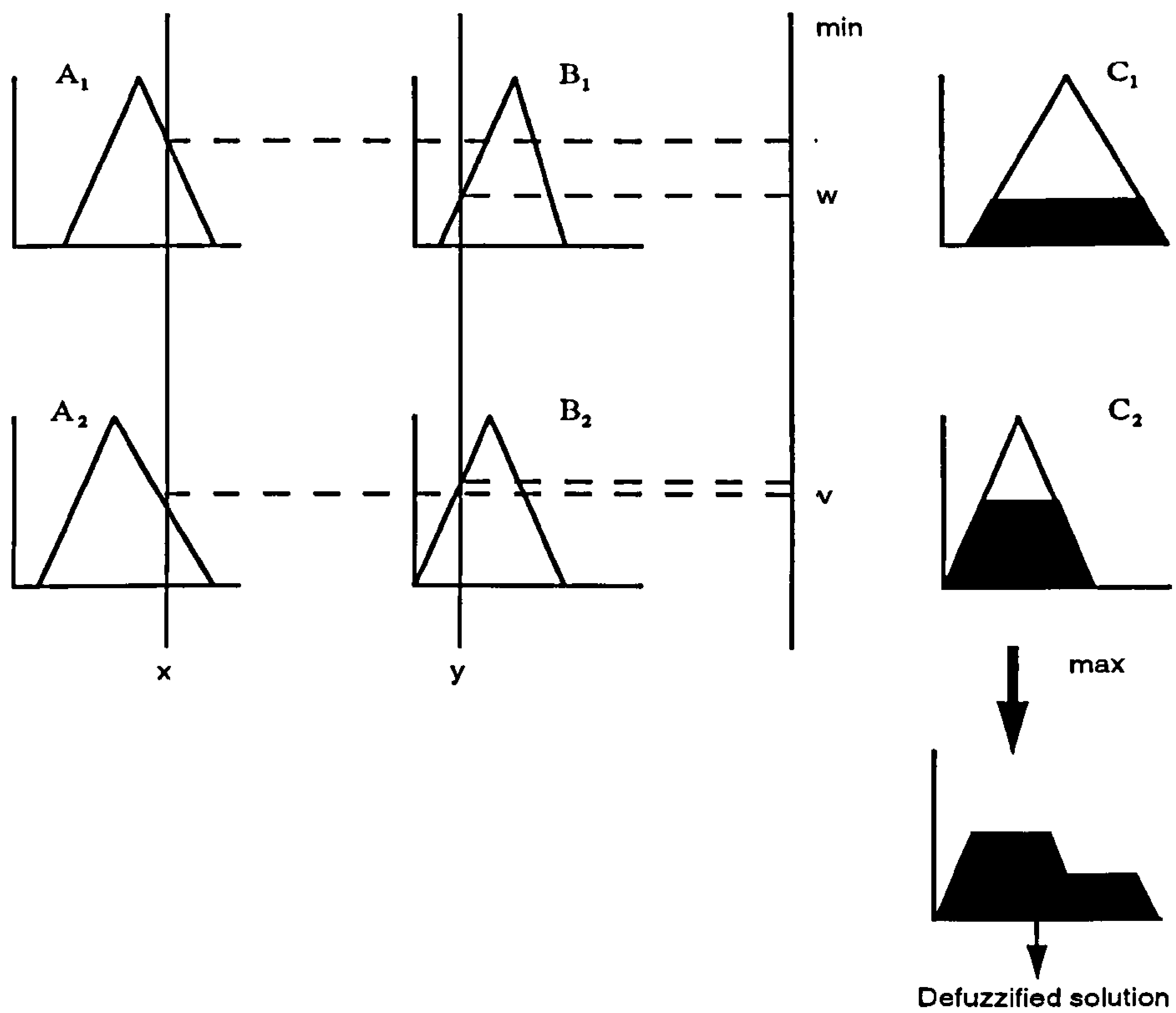

Fig. 2.6. The Mamdani Model for two rules

2.2.3 Defuzzification

Without the defuzzification phase, the final output from the inferencing is, in the Mamdani approach, a fuzzy set. For most applications (especially in control) there is a need for a 'crisp' decision. This is where defuzzification (as its name implies) reduces the fuzzy set to a single number. There are a number of defuzzification techniques available. Two particular methods (Figure 2.7 provides a schematic) will be described here using the notation used in Jang *et al*(1997):

- The Centre of Area Method.
- The Mean of Maximum Method.

The Centre of Area(COA) Method

The COA method is essentially a technique for finding a mid-point of the final output fuzzy set using a weighted average of the membership grades. Suppose there is a fuzzy set A over universe of discourse Z (the output fuzzy set here) then the Centre of Area z_{COA} is given by:

$$z_{COA} = \frac{\int_{COA} \mu_A(z) z \, dz}{\int_{COA} \mu_A(z) \, dz} \tag{2.7}$$

The Mean of Maximum(MOM) Method

The MOM method finds the average z where the membership of A is at a maximum. Where the maximum value of $\mu_A(z)$ is denoted by μ^* the Mean of Maximum z_{MOM} is given by:

$$z_{MOM} = \frac{\int_{Z'} z \, dz}{\int_{Z'} dz} \tag{2.8}$$

where $Z' = \{z \mid \mu_A(z) = \mu^*\}$

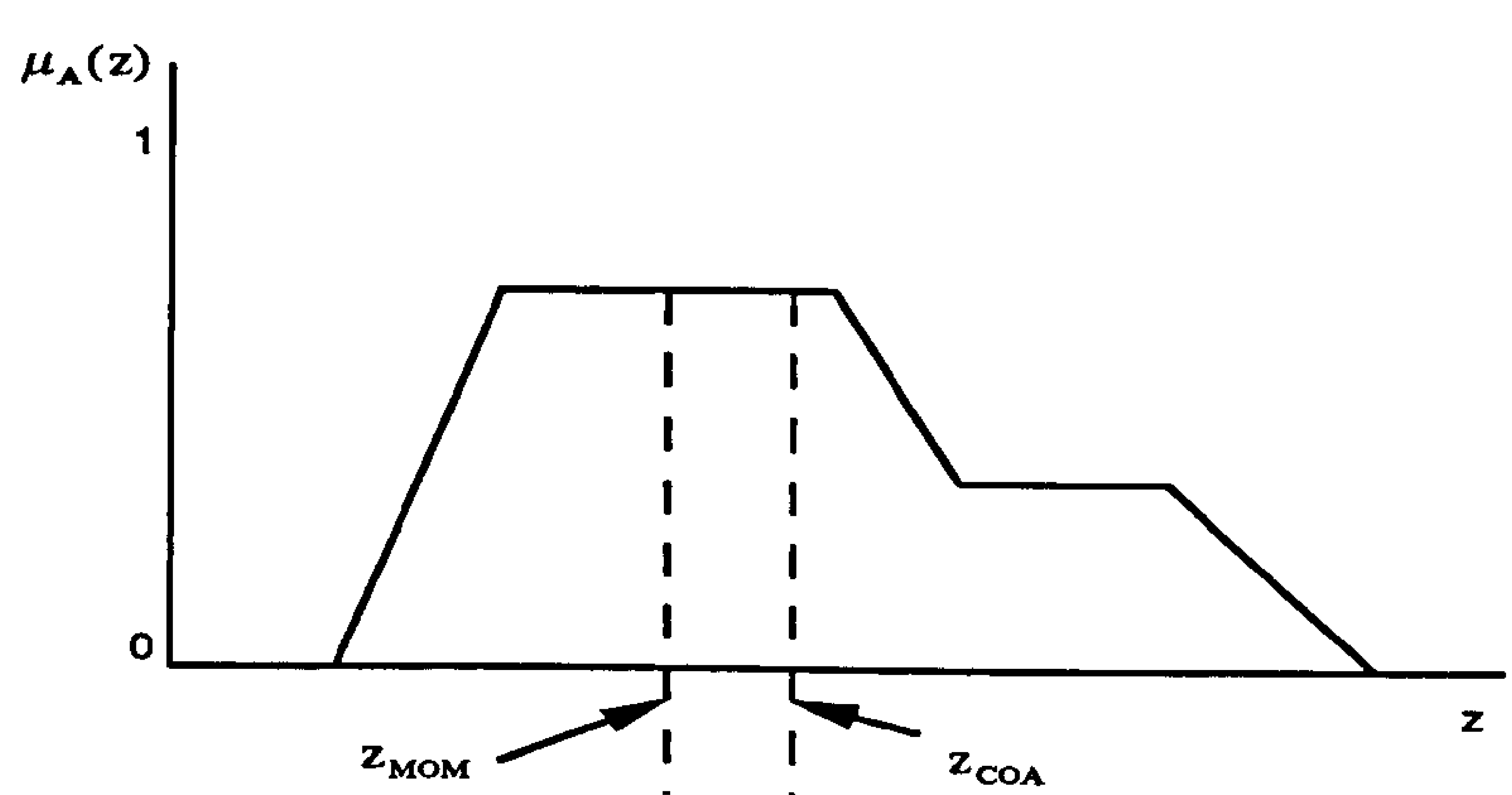

Fig. 2.7. The Centre of Area and Mean of Maximum Methods for Defuzzification of a Type-1 Fuzzy Set.

2.2.4 Summary

This section has provided a focused exposition of type-1 fuzzy sets and type-1 fuzzy systems and discussed how they are used in computer systems. In summary the following points have been made:

- Fuzzy sets have an extensive history of successful applications and fuzzy logic has been comprehensively researched during the last thirty five years.
- Fuzzy sets lay the basis for computer systems (fuzzy systems) that capture uncertainty, imprecision and vagueness - 'fuzziness'.
- The Mamdani and Takagi-Sugeno models provide examples of the capability to develop fuzzy systems with fuzzy if-then rules.
- Defuzzification is required in many cases to allow for a 'crisp' solution. Note that in the Takagi-Sugeno model there is no requirement for defuzzification and it is thus computationally more efficient which, in control applications especially, can be important. However the 'crispness' of the consequents of the rules can be counter intuitive in a real application.

As with any technique there are a number of issues that arise in the use of type-1 fuzzy logic that present problems for fuzzy system developers. The next section highlights some of these problems as a precursor to the argument for the use of type-2 fuzzy sets to model perceptions in certain domains and applications.

2.3 Some Problems with Type-1 Fuzzy Systems

Although type-1 fuzzy systems have been highly successful in a number of domains (particularly control) in capturing the uncertainty and imprecision that exists in many applications there are some problems. It is at this juncture in this thesis that a critical analysis of the type-1 fuzzy approach is necessary. In particular the problems of the nature of a type-1 membership function, determining type-1 membership functions and type-1 fuzzy inferencing are considered.

2.3.1 The Type-1 Paradox

Elkan(1993) discusses what he believes to be a paradox in (type-1) fuzzy logic. In this controversial paper he argues that fuzzy logic collapses to two-valued logic. He points out that this does not hold back development of fuzzy logic systems. This work is not concerned with this 'paradox' and is outside the scope of the thesis. The paradox discussed in this Section relates to knowledge representation in fuzzy systems.

The grade of membership for any member of a fuzzy set is a 'crisp' number in the interval [0,1]. A framework based on 'fuzziness' can be developed yet the problem is represented directly by 'non-fuzzy' numbers. As Klir and Folger(1988, page 12) point out, *" ..it may seem problematical, if not paradoxical, that a representation of fuzziness is made using membership grades that are themselves precise real numbers"*. For any type-1 application there is some translation (fuzzification) from a physical measurement or an expert's description to the membership of the fuzzy set. So, using the example of a type-1 fuzzy set 'tall' the degree of membership is a mapping from the crisp height, say 5ft 10ins., to the membership grade of 'tall' , say 0.87. Note that the height is a crisp measurement in that there is no uncertainty directly associated with it yet, even for the most accurate form of measurement, there is some uncertainty. In reality it is *about* 5ft. 10 ins., depending on the equipment being used to measure the person. The mapping to 0.87 is also crisp yet there is uncertainty in the value 0.87. For instance one hundred experts on 'tallness' would probably not all agree on the statement that *a person who is 5ft 10ins tall is tall to degree 0.87* but might agree that somebody of about *5ft 10 ins. is* quite tall. They also might agree that somebody who is 5ft 10 ins. tall has a membership in the fuzzy set tall somewhere between 0.85 and 0.90 - in the interval [0.85,0.90]. The very nature of type-1 fuzzy systems, then, is that the type-1 membership function is not capturing all the uncertainty in the knowledge representation. The paradox that fuzzy system developers fuzzify by using crisp numbers in $[0, 1]$ is worth further investigation. It is this paradox that will lay the basis for the arguments in this thesis and will be revisited on a number of appropriate occasions. In particular, an argument central to this thesis is that type-2 fuzzy sets (which help capture this (type-2) uncertainty) offer an opportunity to say that a particular person's degree of tallness is medium rather than, say, 0.62 and will allow for systems that maintain uncertainty in knowledge representation and inferencing.

Even supposing the notion of a type-1 membership function is acceptable, there is still the question of how best to determine the membership functions. The next Section considers attempts that have been made to tackle this problem.

2.3.2 Determining Type-1 Membership Functions

Where do 'they' come from? This is the title of a chapter in the book by Dubois and Prade(1980, Part IV, Chapter 1, page 255). In that chapter they summarise the problem faced by fuzzy system developers of determining the membership functions of the fuzzy sets. Others (e.g. (Lee 1990)) have also highlighted this issue. In fuzzy control problems, almost without exception, gaussian, triangular or trapezoidal membership functions are used and appear to work in many domains. In non-control applications the choice of membership function is an issue and although triangular membership functions may provide a solution, further investigations need to be carried out as to their efficacy. So, how does a fuzzy system developer design the membership functions? In general an ad hoc approach is taken although various techniques are reported in the literature.

A common approach is to conduct a knowledge acquisition exercise with one or more experts and combine the results in some way. However, attaching numbers to features of a problem is notoriously difficult and experts in a domain are unlikely to arrive at a consensus. The traditional statistical approaches(Watanabe 1993) use averages or percentages, based on informed guesses made by experts. The method for determining membership functions is constrained by the fact that numbers have to be used. A group of experts is more likely to agree that a man of 25 is middle aged to degree 'low' than agree that he is middle aged to degree, say, 0.26.

More recently the problem of determining membership functions has been tackled by researchers using some of the current 'soft computing' or 'computational intelligence' techniques such as neural networks and genetic algorithms. Takagi and Hayashi(1991) point out that fuzzy reasoning presents particular problems:

1. the lack of a definite method for determining the membership function;

2. the lack of a learning function. Fuzzy neural networks(e.g.(Carpenter, Grossberg, Markuzan, Reynolds & Rosen 1992, Simpson 1993)) and adaptive fuzzy systems(Cox 1993) overcome this problem when learning relationships between data. Fuzzy systems, like traditional knowledge based systems, do not learn.

They describe the use of Artificial Neural Networks (ANNs) to overcome these problems. The method is to investigate IF... THEN... rules by essentially using neural networks to determine the membership functions of the IF... part and then determine the THEN.... component as the output for each rule. The approach used is to take raw data (say, in a control problem), apply a conventional clustering algorithm to group the data into clusters and to use an ANN on this clustered data to determine which fuzzy sets a pattern belongs to and its membership values. This approach was applied to two applications - estimation of chemical oxygen demand density in Osaka Bay and the estimation of the roughness of a ceramic surface. Their method in both cases out performed more conventional methods. This combination of neural networks and fuzzy reasoning not only allows for automatic generation of the membership function in certain applications but the use of fuzzy IF... THEN... rules and neural networks presents a potentially powerful paradigm. Wang(1994) builds on the expertise provided by an expert and uses ANNs to 'fine tune' the membership function. His technique is applied to car preference against certain criteria. The IEEE International Conference in Seoul in 1999 had a special session on rule extraction from trained neural networks ((Fukumi & Akamatsu 1999, Imamura, Umano, Tamura & Sawada 1999, Kasabov & Woodford 1999)). Fukumi & Akamatsu(1999) present the use of an evolutionary algorithm (random optimization method (ROM)) for extracting rules from neural networks. The neural network has the number of weights reduced by ROM which generates a simple network allowing generation of rules. The method is applied to the well known benchmark iris data but the rules shown as examples in the paper are crisp - not fuzzy. Fuzzy Neural Networks have been deployed to extract fuzzy rules from data(Imamura et al. 1999). This approach uses a combination of the Kohonen Self Organising Map and back propagation to produce, in some sense, an optimal neural network that replicates a (type-1) fuzzy system. The method is applied to a very simple problem of calculating weight from height, foot size and sex as well as that of planning a kitchen. There is not enough detail for a full evaluation but the approach looks interesting. Kasabov & Woodford(1999) use Fuzzy Neural Networks that mimic a (type-1) fuzzy system and present various learning algorithms. The method shows some success with two benchmark sets of data. Other work in this area includes that of Hayashi *et al*(1992) and Gmez-Marin-Blzquez*et al*(2000).

Meredith *et al*(1992) have applied Genetic Algorithms(GAs)(Goldberg 1989) to the fine tuning of membership functions in a Fuzzy Logic Controller(FLC) for a helicopter. As measured by the movement of the helicopter, the GA based FLC out performed the FLC that did not use GAs. Karr(1991) applies GAs to the design of FLC for the cart pole problem.¶ The GA based FLC out performed the FLC designed by the engineer and interestingly the real time tuning of membership functions out performed the non-adaptive GA based FLC. Lee and Takagi(1993) also tackle the cart problem. They take a holistic approach by using GAs to design the whole system (determining the optimal number of rules as well as the membership functions). Results were encouraging but it should be noted that the problem tackled only has two input and one output variables and that further research will need to be conducted with multi-dimensional problems. GAs have also been used for development of fuzzy models(Suzuki, Furuhashi, Matsushita & Tsutsui 1999). The approach is only applied via a simulation however.

¶The cart-pole system is a 'bench mark' in control. A pole attached to a cart at a rotational joint is controlled to be kept upright by applying a force to the cart.

An evolutionary approach has been used for fuzzy modelling(Slawinski, Krone, Hammel, Wiesmann & Krause 1999). They report the results of using a hybrid evolutionary approach for learning fuzzy rules in high dimensional spaces for predicting insurance contract duration and in quality control. The results look interesting and further work will provide a more detailed presentation of the usefulness of the approach. Other methods include deformable prototypes(Bremermann 1971) and gradient search(Procyck & Mamdani 1979, Burkhardt & Bonissone 1992). More details of some of the earlier approaches are given in a working paper by the author(John 1995).

These statistical and various soft computing methods offer help in determining membership functions and have been applied with some success to certain control problems but are often very domain dependent. There is still no widely accepted approach for determining membership functions of type-1 fuzzy sets.

2.3.3 Type-1 Fuzzy Inferencing

The inferencing mechanism commonly adopted in type-1 fuzzy systems is precise and does not maintain imprecision throughout the process. Obviously, the fact that there are hundreds, if not thousands, of successful type-1 fuzzy applications means that this may not be very important in certain domains. Nevertheless there is cause for concern that a 'fuzzy' process is completely precise from fuzzification through inferencing to the final output.

The standard method for inferencing in type-1 systems is based on the generalised modus ponens(Zadeh 1975*b*), is well accepted and widely adopted. However there are a number of issues with regard to type-1 inferencing that are discussed in more detail in Chapter 3. The main points are:

- the reduction of fuzziness to a number in $[0, 1]$ means that the inferencing deals with the crisp numbers;
- the standard t-norms and t-conorms are precise in nature - there is no uncertainty attached to the inferencing process;
- there is an argument that gmp is inadequate since type-1 sets only express first order semantics(Türkşen 1995*b*), and
- type-1 implication as normally adopted does not, in many situations, reflect the inherent uncertainty in any particular situation(Hisdal 1981).

So far in this Chapter type-1 fuzzy logic has been outlined and some problems with the type-1 approach discussed. The central argument of this thesis is that type-2 fuzzy logic offers some advantages over type-1 fuzzy logic and indeed address the problems described. This thesis proposes a novel adaptive system for learning perceptions using type-2 fuzzy sets. The approach taken builds on the philosophy of ANFIS which is described in the next Section.

2.4 The Type-1 Adaptive Network Based Fuzzy Inference System (ANFIS)

In this thesis a novel 'automatic' technique for learning a type-2 fuzzy system is defined and full mathematical detail provided. For a number of reasons that will become clear, the developed method builds on the philosophy of Adaptive Network-based Fuzzy Inference Systems(Jang 1992, Jang 1993). This Section discusses the ANFIS approach for type-1 fuzzy systems since the philosophy of this approach is adopted for the novel Adaptive Fuzzy Perception Learner reported in Chapter 5.[||] The purpose of the Section is to discuss the philosophy of the ANFIS approach as well as the technical implementation so that the explanation of the Adaptive Fuzzy Perception Learner clearer.[**]

Section 2.4.1 provides a high level overview of the ANFIS approach, Section 2.4.2 provides the detail of the algorithm, Section 2.4.3 discusses some applications reported in the literature and finally Section 2.5 provides a conclusion to this Chapter.

2.4.1 ANFIS - An Overview

The ANFIS approach uses an adaptive network(Jang, Sun & Mizutani 1997) to model a type-1 fuzzy system. An adaptive network is a network of nodes and connections that, in some way, learns relationships in data, between inputs and outputs. Figure 2.8 shows a simple example of an adaptive network. These adaptive networks can take many forms. All the nodes in the network carry out a function on the inputs to the nodes. The square nodes are **adaptive nodes** and as such have parameters that are modified by the learning process. The circular nodes are **fixed nodes** and contain fixed parameters which are not modified by the learning process. The idea is to use historical data in an analogous

[||]The ANFIS approach is reported widely in the literature and there are a number of variations on the method described here. As with any adaptive system, there are many 'variations on a theme' and many options available to the system developer. The approach is described very thoroughly by Jang *et al*(1995) and is now, for example, included in the Fuzzy Logic Toolbox for Matlab^{TM}.

[**]The author, to gain an understanding of the type-1 ANFIS approach discussed here, has applied the method to a robot application(Czarnecki & John 1997). So as not to deviate from the theme of this work the details are not included but the published paper can be found in Appendix F.

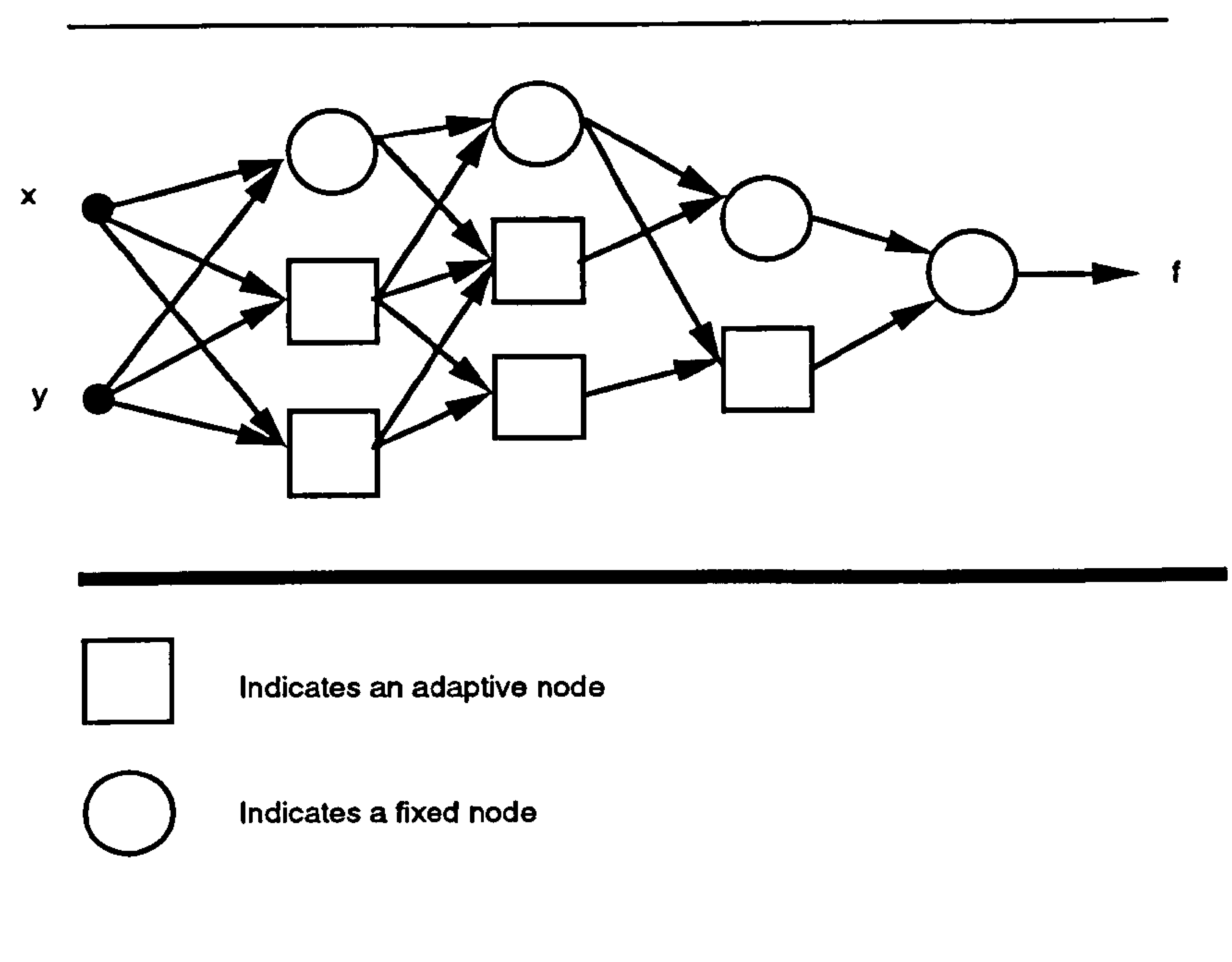

Fig. 2.8. A typical adaptive network

way to Artificial Neural Networks to learn the relationships between data represented by the inputs and outputs. In the case of Figure 2.8 there are two inputs x and y and the network learns the relationships between the inputs and the output, f. For example, In a control example (work carried out by the author of this thesis), inputs to the ANFIS were sensor readings from a robot and the outputs actions that the robot should take(Czarnecki & John 1997). The objective for the ANFIS is to learn the relationships (via a type-1 fuzzy system) between the crisp sensor readings and appropriate actions for collision avoidance. The ANFIS was successful in that for a surprisingly small amount of data (ninety eight pairs of inputs and outputs) the parameters (fifty in number) were learnt with a high success rate for decision making (ninety four percent)[††]In the case of Artificial Neural Networks, the relationships are found by learning the weights on the connections between nodes. For ANFIS there are no weights. There are parameters to be learnt within the adaptive nodes. They determine the membership functions of the fuzzy sets or the functions in the consequents of the if-then rules in the case of the Takagi-Sugeno model. Given a set of inputs and outputs, the ANFIS learns a type-1

[††]This piece of research provided the author with a good understanding of the philosophy of ANFIS as well as some implementation issues.

fuzzy system by learning the parameters of the fuzzy sets and the associated fuzzy if-then rules. Figure 2.9 provides a diagrammatic representation of the type-1 ANFIS. As can

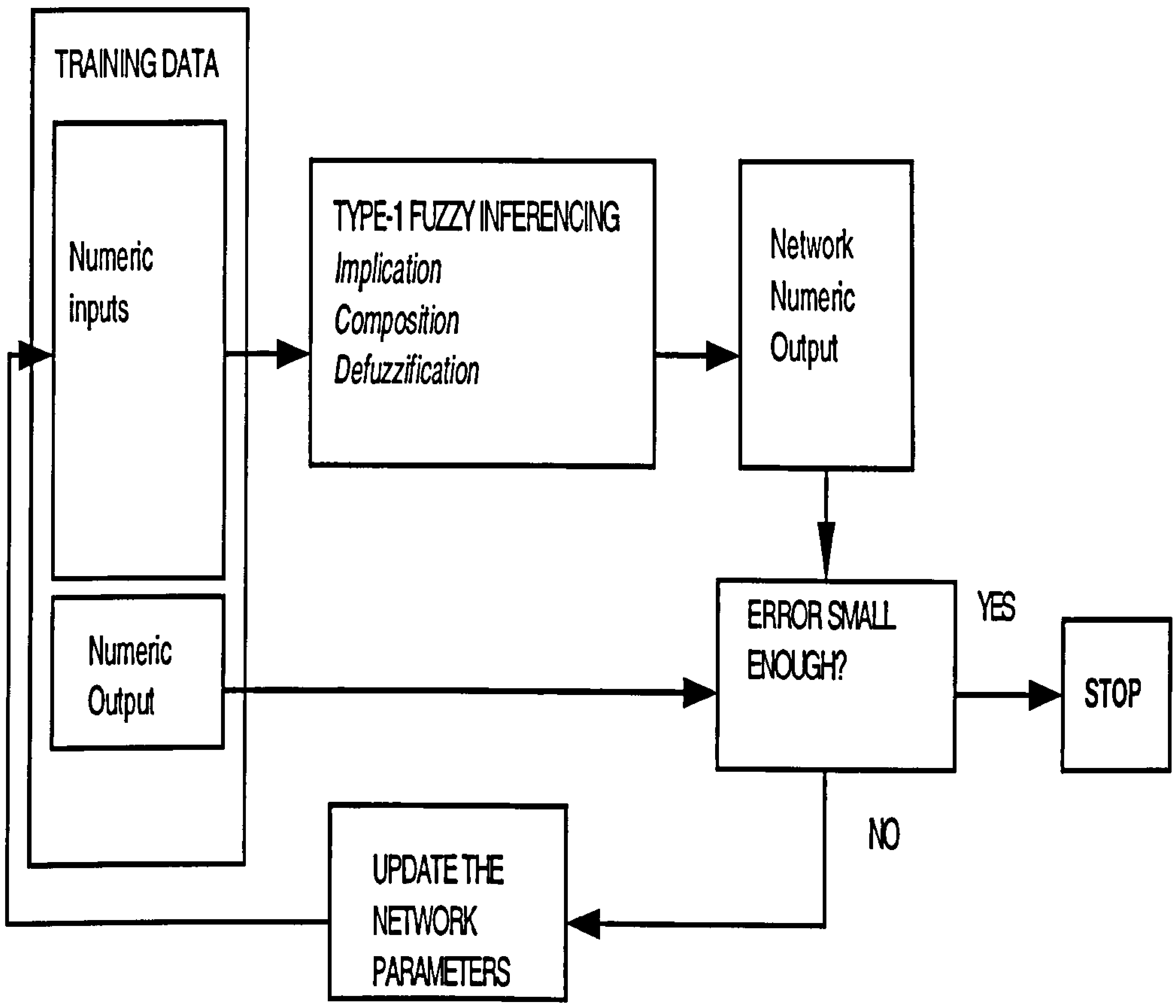

Fig. 2.9. An overview of the Type-1 ANFIS

be seen, the network takes numeric inputs. These are then processed by the network using implication, composition and defuzzification to produce a numeric output. This output is compared with the expected output and an error calculated. This error is then fed back to update all the parameters in the network (in an similar manner to the well known back propagation algorithm(Haykin 1994) for Artificial Neural Networks) until the error is acceptably small. The next Section describes the ANFIS approach for the Takagi-Sugeno model(Section 2.2.2).

2.4.2 ANFIS - The Algorithm

To illustrate the ANFIS approach and algorithms, suppose there are two inputs to an adaptive network as described (x and y) with one output f and the requirement is to build a type-1 Takagi-Sugeno system where, for the simple example shown, rules will be of the form:

$$\begin{aligned} \textit{IF } x \textit{ is } A_1 \textit{ and } y \textit{ is } B_1 \textit{ THEN } f &= p_1x + q_1y + r_1 \\ \textit{IF } x \textit{ is } A_2 \textit{ and } y \textit{ is } B_2 \textit{ THEN } f &= p_2x + q_2y + r_2 \end{aligned} \tag{2.9}$$

The ANFIS then learns the parameters that define the fuzzy sets A_1, A_2, B_1 and B_2 as well as the consequent parameters p_1, p_2, q_1, q_2, r_1 and r_2. The ANFIS for the example here is shown in Figure 2.10. The ANFIS approach consists of a forward and backward

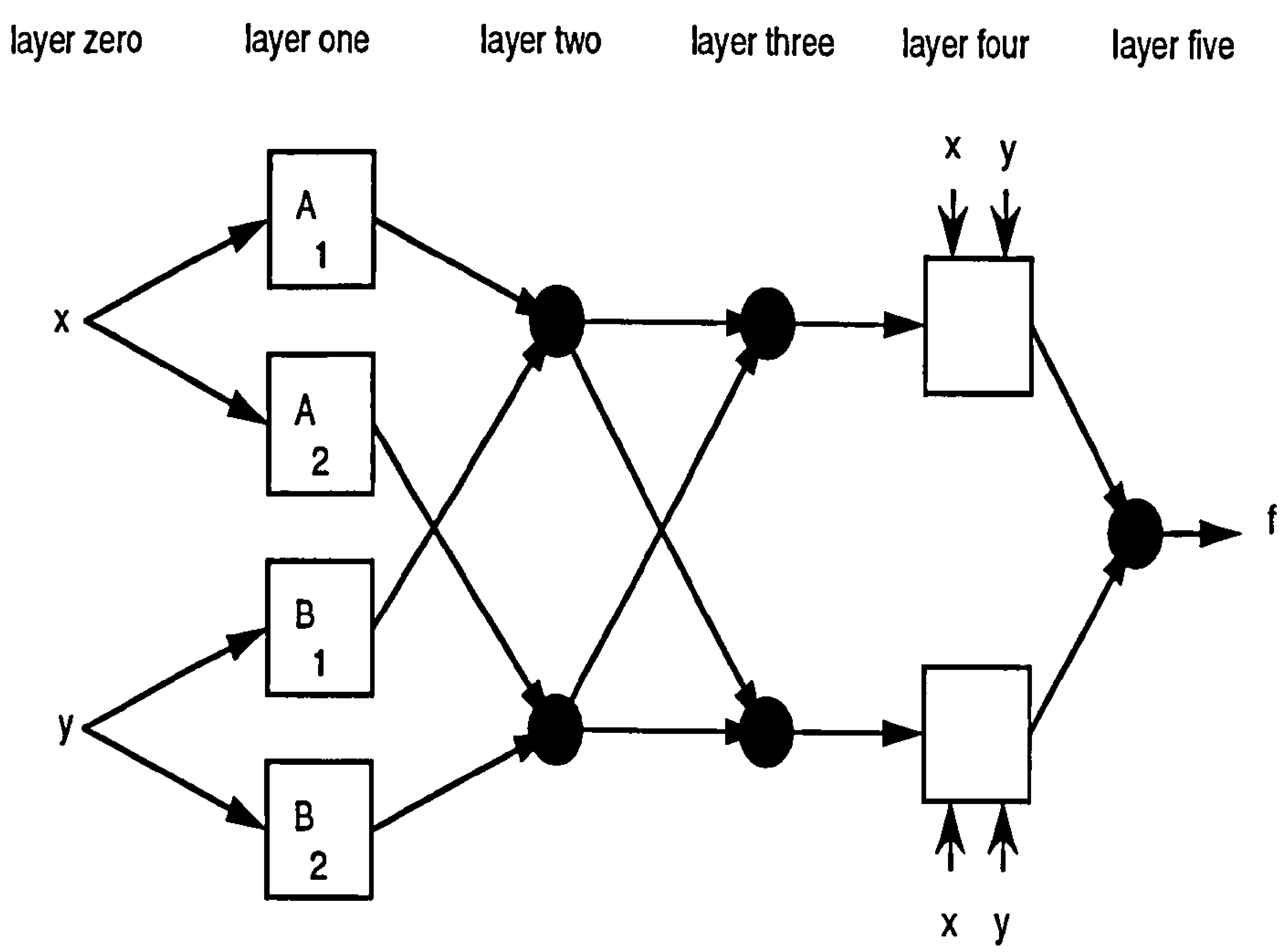

Fig. 2.10. A typical ANFIS for a two rule Takagi-Sugeno type-1 fuzzy system

pass. The forward pass propagates a pair of data items through the network to produce an output. This is compared with the expected output to produce an error which is fed back through the network in the backward pass to update the various parameters.

The Forward Pass

The purpose of the forward pass is to propagate the (in this example) two inputs through the main components of a type-1 fuzzy system (Figure 2.4). As can be seen in Figure 2.10 the ANFIS here consists of six layers. Each layer performs a function of a type-1 fuzzy system thus truly emulating the Takagi-Sugeno type-1 fuzzy system discussed in Section 2.2.2. Layer zero simply is used to present a pair of inputs x and y.

Layer one

The purpose of layer one is to carry out the fuzzification process. It performs a translation from crisp numeric input to the membership grade of a fuzzy set (A_1, A_2, B_1, B_2). So, each node in this layer is an adaptive node and has output that is the membership value for the input in the appropriate fuzzy set. For the purposes here of demonstrating the process, the membership functions are gaussian in nature of the form:

$$\begin{aligned} \mu_{A_i}(x) &= \frac{1}{1+[(\frac{x-c_i}{a_i})]^{b_i}}, i=1,2 \\ \mu_{B_i}(y) &= \frac{1}{1+[(\frac{y-c_{i+2}}{a_{i+2}})]^{b_{i+2}}}, i=1,2 \end{aligned} \quad (2.10)$$

where a_i, b_i and c_i ($i = 1 \ldots 4$) is the parameter set to be learnt for each membership function in the layer. On successful training these are the parameters that define the base type-1 fuzzy sets of the type-1 fuzzy system.

Layer two

The nodes in this layer are fixed in that they simply perform a function and do not have parameters to be learnt. The outputs from the nodes in this layer are the output of the antecedents in the fuzzy rules (Equation 2.9). Each node in this layer carries out the 'AND' in antecedent of the rules. Any appropriate t-norm (Section 2.2.2) could be used. For instance the min could be used to give:

$$w_i = min[\mu_{A_i}(x), \mu_{B_i}(y)], i = 1,2 \quad (2.11)$$

The output of each node w_i is then passed to the next layer.

Layer three

Again, the nodes in this layer are fixed in that they simply perform a function and do not have parameters to be learnt. The output of each node in the layer is simply the normalised firing strength of each rule (w_i, $i = 1, 2$).

$$\overline{w}_i = \frac{w_i}{w_1 + w_2}, i = 1, 2 \tag{2.12}$$

Layer four

Layer four contains adaptive nodes that represent the consequents of the if-then rules and uses the normalised firing strength of layer three to calculate the contribution of the consequents of the rules to f. So, the outputs are given by:

$$\overline{w}_i f_i = \overline{w}_i(p_i x + q_i y + r_i), i = 1, 2 \tag{2.13}$$

where p_i, q_i and r_i are parameters to be learnt.

Layer five

This layer has a single node which calculates the total of all the incoming signals:

$$\sum_i \overline{w}_i f_i \tag{2.14}$$

and produces the final output from the network.

What has been described is one example of a forward pass for an ANFIS implementation of the Takagi-Sugeno model. There are many types of networks for a number of various paradigms and the reader is referred to Jang *et al*(1997) for a full exposition.[‡‡] The network then is a model of a type-1 Takagi-Sugeno fuzzy system. The next section considers the method for learning the parameters in the backward pass.

The Backward Pass

There are a number of possibilities for learning the parameters in an ANFIS(Jang et al. 1997). It must be remembered that the parameters to be learnt in the Takagi-Sugeno ANFIS consist of those parameters that define the (type-1) fuzzy sets in the antecedents of the rules and the parameters that define the consequent functions. It can be shown that the output from an ANFIS network is linear in the consequent parameters and therefore these can be learnt using a least squares approach in the forward pass whilst the premise parameters are learnt using, for example, gradient descent such as back propagation. Table 2.1 reproduces a table from Jang *et al*(1997) (page 340) which summarises the

[‡‡]The use of Mamdani and Tsukamato type rules in ANFIS is more complex than the Takagi-Sugeno model and most applications seem to use the Takagi-Sugeno method.

	Forward pass	Backward pass
Premise parameters	Fixed	Gradient Descent
Consequent parameters	Least squares estimator	Fixed
Signals	Node outputs	Error signals

Table 2.1. The hybrid learning algorithm for ANFIS (Jang *et al*(1997) (page 340))

hybrid learning algorithm. Of course, this hybrid learning algorithm is efficient but is only suitable for the Takagi-Sugeno-type model.

2.4.3 ANFIS applications

The use of ANFIS for real applications is relatively new however several examples are reported in the literature. Jang(1997) reports on the use of ANFIS for printed character recognition, automobile miles per gallon prediction, non-linear system identification, channel equalization and adaptive noise cancellation. ANFIS has been used successfully in developing a course-keeping autopilot for unmanned underwater vehicles(Sutton & Crave 1997). The authors concluded that the ANFIS approach to fine tuning fuzzy if-then rules out performed a simulated annealing autopilot and a fixed rule autopilot. The control of low head hydroped plants has been carried out using ANFIS(Djukanovic, Calovic, Vesovic & Sobajic 1997). They developed a fuzzy controller using ANFIS which has been implemented on a complex high-order non-linear hydrogenerator model. Wolf(1998) reports on a number of successful applications including motor fault detection, dye concentration prediction, a dynamic biped walking robot, prediction of bursty traffic in broadband satcom systems and classification of intra-cardiac arrhythmias. As has already been mentioned, the author carried out some experimental work using ANFIS to learn a type-1 fuzzy system for mobile robot collision avoidance(Czarnecki & John 1997). This work showed that, with a relatively small amount of data, the parameters of the various membership functions could be determined with a high level of performance of the resulting system.

2.4.4 ANFIS - a summary

The ANFIS method offers a method for learning a type-1 fuzzy system from data. The approach is intuitively pleasing as well as being successful in a number of applications. It is one way of overcoming the problems of hand crafting type-1 fuzzy systems and as such offers much to the software developer. Its key features can be summarised as:

- It is an adaptive network that takes numeric input and output.

- It consists of a forward pass and a backward pass.
- The forward pass propagates the inputs through a type-1 fuzzy system to produce a numeric output.
- The backward pass uses the error between the ANFIS output and the expected output to modify the antecedent and consequent parameters in the if-then rules.
- The resulting system from training an ANFIS can be interpreted unlike an Artificial Neural Network. The rules and membership functions can be written down and examined by an expert.

2.5 Discussion

This Chapter has provided a base on which the rest of the thesis can be built. In particular:

1. The notion of a type-1 fuzzy set has been introduced as well as some of the more important operations required for implementation using if-then rules in a type-1 fuzzy system.
2. Type-1 fuzzy systems have been described (in particular the Takagi-Sugeno and Mamdani models).
3. The nature of the type-1 membership function is that the grade for any member of a fuzzy set is a 'crisp' number in [0,1]. The paradox of the 'crispness' of type-1 membership functions can be a major problem for knowledge representation since attaching numbers to imprecise perceptions removes the vagueness in the original representation. In this thesis it is argued that type-2 fuzzy sets offer a significant improvement over type-1 fuzzy sets for knowledge representation when modelling perceptions.
4. Type-1 membership grades are notoriously difficult to determine.
5. The ANFIS approach offers an automatic approach for learning a type-1 fuzzy system.

At this point it is worth re-iterating the research hypothesis stated in Section 1.3 as

"Type-2 fuzzy sets have much to offer for knowledge representation and inferencing; however there is a need for some method for learning type-2 fuzzy systems. A type-2 fuzzy system that models human perceptual categorisation by linguistic association can be learnt from linguistic data that represent perceptions."

An argument has been made that there are deficiencies with the type-1 fuzzy approach especially in capturing the 'higher level' of imprecision and vagueness contained in perceptions. The first part of the hypothesis is that type-2 fuzzy sets help with knowledge representation and inferencing. To take the discussion forward, the next Chapter reviews the various aspects of type-2 fuzzy sets and considers some of the approaches taken in the literature for their use in knowledge representation, as well as providing an understanding, by use of appropriate examples, of the effects of type-2 uncertainty.

Chapter 3

Type-2 Fuzzy Sets Model Perceptions

The previous Chapter explored some of the basics of type-1 fuzzy sets, described type-1 fuzzy systems, highlighted some of the issues and problems faced by type-1 fuzzy systems developers and described in detail the adaptive fuzzy system approach known as ANFIS. The problems associated with the use of type-1 fuzzy sets in type-1 fuzzy systems leads to consideration of alternatives. It is argued in this thesis that an effective alternative to a type-1 fuzzy paradigm is the use of type-2 fuzzy sets. To lay the basis for this argument, this Chapter provides an overview of type-2 fuzzy sets and discusses the research carried out by other workers in this area. To improve the clarity of argument, worked examples are provided using appropriate software, which has been developed by the author, that implements some of the important ideas. The rest of this Chapter provides definitions of type-2 fuzzy sets and compares them with type-1 fuzzy sets (Section 3.1); argues that type-2 fuzzy sets capture 'fuzziness' more effectively than type-1 fuzzy sets (Section 3.2); reviews the theoretical properties of type-2 fuzzy sets (Section 3.3); considers knowledge (or perception) representation with type-2 fuzzy sets (Section 3.4); describes how type-2 fuzzy sets improve on 'standard' type-1 inferencing (Section 3.5); reviews the applications of type-2 fuzzy sets in Section 3.6 and provides an overview of research by the author on modelling a medical consultant's perceptions using type-2 fuzzy sets (Section 3.7).

3.1 Type-2 Fuzzy Sets - Some Definitions

This Section provides a number of definitions of type-2 fuzzy sets. The definitions are different but complementary in that they present different ways to describe the same thing - a type-2 fuzzy set. Type-2 fuzzy sets have membership grades that are fuzzy. That is, instead of being in [0,1] the membership grades are themselves (type-1) fuzzy sets. What is important from the perspective of this thesis is that they allow for linguistic, as opposed to numeric, membership grades. Type-2 fuzzy sets were initially described by Zadeh(1974).* A particularly clear definition(Mizumoto & Tanaka 1976) of a type-2 fuzzy set is:

Definition 3 *A fuzzy set of type-2 is defined by a fuzzy membership function, the grade (that is, fuzzy grade) of which is a fuzzy set in the unit interval [0,1], rather than a point in [0,1]. A fuzzy set of type-2, A, in a set X, is the fuzzy set characterised by the fuzzy membership function μ_A as*

$$\mu_A : X \to [0,1]^{[0,1]}$$

where $\mu_A(x)$ is known as a fuzzy grade, *a fuzzy set in [0,1].*

One can see from this definition that members of type-2 fuzzy sets have membership grades that are type-1 fuzzy sets - not numbers in [0,1]. In other words, the membership grades themselves are fuzzy - not crisp. This is the strength of type-2 fuzzy sets and will be revisited and explored throughout this thesis.
Another definition(Dubois & Prade 1980, page 30) is for type-m sets which use the notion of a L-fuzzy set.

Definition 4 *A type-1 fuzzy set is an ordinary fuzzy set in X. A type-m fuzzy set ($m > 1$) in X is an L-fuzzy set whose membership values are type-$m - 1$ fuzzy sets on [0,1].*

A L-fuzzy set(Goguen 1967) is any set at least partially ordered - essentially a generalisation of the notion of a fuzzy set that we know today.† So, we can see that type-2 fuzzy set is at least partially ordered. In other words the membership grades have some ordering. From this definition a type-2 fuzzy set has membership values that are type-1 fuzzy sets, a type-3 fuzzy set has membership values that are type-2 fuzzy sets and so on. The research reported in this thesis concerns itself only with the type-2 fuzzy paradigm since:

- type-2 fuzzy sets have not been explored fully by researchers.;

*He doesn't refer to them in this paper as type-2 fuzzy sets. The use of linguistic membership grades as truth values is the essence of that work.

†A formal definition of a L-fuzzy set is as follows (Goguen 1967, page 148): An L-fuzzy set A on a set X is a function $A : X \to L$. Here L is any ordered set.

- the complexity of type-2 fuzzy sets is high and type-3 and above would be even more complex;
- it is not, at this juncture, obvious that type-3 sets and above are particularly useful.

Karnik and Mendel(1998a)[page 2] provide another, very concise, definition:

Definition 5 *A type-2 fuzzy set is characterised by a fuzzy membership function, i.e. the membership value (or membership grade) for each element of this set is a fuzzy set in [0,1], unlike a type-1 fuzzy set where the membership grade is a crisp number in [0,1].*

The characterisation in this definition of type-2 fuzzy sets uses the notion that type-1 fuzzy sets can be thought of as first order approximation to uncertainty and, therefore, type-2 fuzzy sets provide a second order approximation. Karnik and Mendel (1998a)[page 6] describe type-2 fuzzy sets by the use of a primary and secondary membership so they express the membership function as a function of 2 variables μ_1 and x where the type-2 membership function is given by

$$\mu_2(x, \mu_1) \ : X \times [0,1] \rightarrow [0,1]$$

This idea of primary and secondary membership is represented pictorially in Figure 3.1. The primary membership of x_a is μ and the secondary membership is described by the fuzzy set shown 'attached' to the primary membership. This notion of a primary and secondary membership is one that is unique to the Mendel approach to type-2 fuzzy sets. Yager(1980), in one of the earliest papers on the use of type-2 fuzzy sets in an application, provides a more formal definition of type-2 fuzzy sets:

Definition 6 *Assume X is a set of elements, then a fuzzy subset A of X is said to be of type-2 if the membership function $U_A(x)$ is a mapping from X to fuzzy sets of interval [0,1].*

This definition uses the idea of mapping from members of the given set to type-1 fuzzy sets.[‡] Finally, the father of fuzzy logic, Professor Zadeh, provides this definition(Zadeh 1975a):

Definition 7 *A fuzzy set is of type $n, n = 2, 3, \ldots$ if its membership function ranges over fuzzy sets of type $n-1$. The membership function of a fuzzy set of type-1 ranges over the interval [0,1].*

[‡]The definition, like many others, uses the term fuzzy set to describe a type-1 fuzzy set.

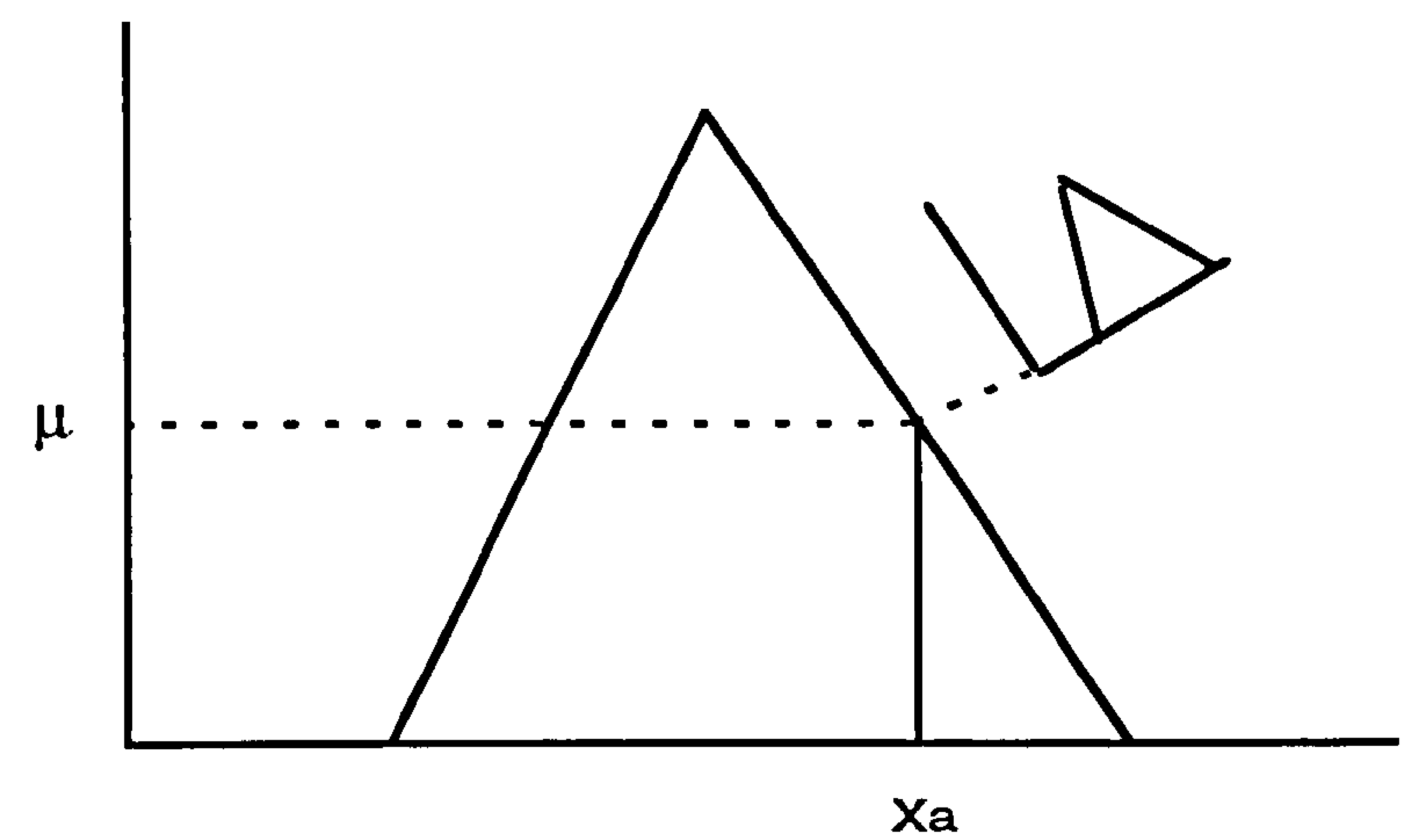

Fig. 3.1. The primary and secondary membership grades of a type-2 fuzzy set

As one might expect, this definition is concise and elegant defining type-n fuzzy sets recursively.

These definitions explain, in one way or another, that type-2 fuzzy sets are 'more fuzzy' than type-1 fuzzy sets. Type-1 fuzzy sets could be described as crisp - not fuzzy at all - since a fuzzy perception, concept or attribute is translated into a number in [0,1] for type-1 fuzzy sets. This crispness of the type-1 paradigm causes particular problems in domains where the problem is one of modelling perceptions. Type-2 fuzzy sets represent perceptions where the membership grade is itself a type-1 fuzzy set.

3.1.1 Type-1 Fuzzy Sets and Type-2 Fuzzy Sets - A Comparison

An effective way to compare type-1 fuzzy sets and type-2 fuzzy sets is by use of a simple example. Assume, for a particular application, one wishes to describe the imprecise concept of 'tallness'. One approach would be to use a type-1 fuzzy set $tall_1$. Now suppose three members of this set are considered - Michael Jordan, Danny Devito and Robert John. For the type-1 fuzzy approach one might say that Michael Jordan is $tall_1$ to degree 0.95, Danny Devito to degree 0.4 and Robert John to degree 0.6. Using the usual notation, this can be written as

$$tall_1 = 0.95/Michael\ Jordan + 0.4/Danny\ Devito + 0.6/Robert\ John.$$

A type-2 fuzzy set ($tall_2$) that models the concept of 'tallness' could be

$$tall_2 = High_1/Michael\ Jordan + Low_1/Danny\ Devito + Medium_1/Robert\ John$$

where $High_1, Low_1$ and $Medium_1$ are type-1 fuzzy sets.
Figure 3.2 shows what the sets $High_1, Low_1$ and $Medium_1$ might look like if represented graphically. As can be seen, the x axis takes values between 0 and 1, as does the y axis (μ).

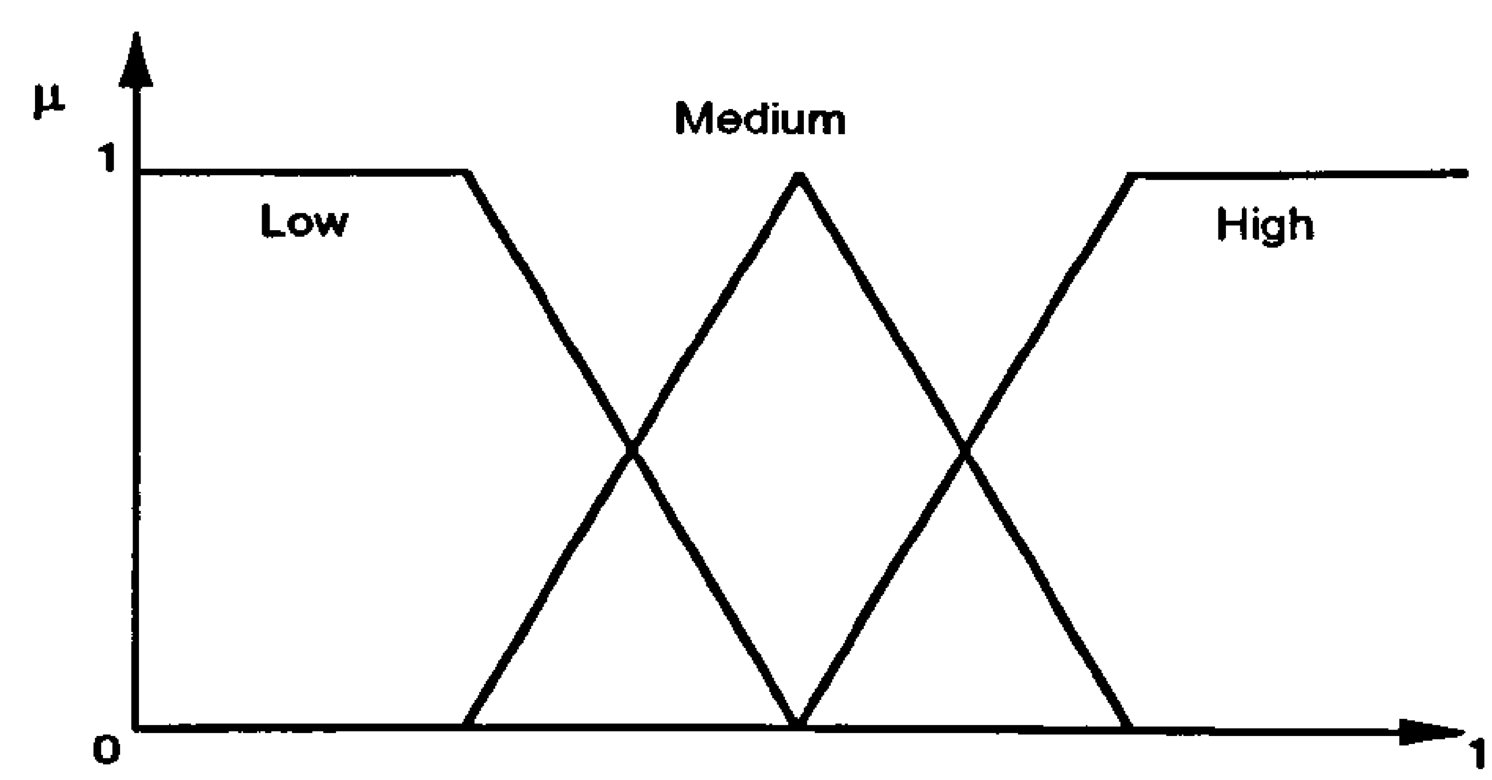

Fig. 3.2. The Fuzzy Membership Grades $High_1, Low_1$ and $Medium_1$.

This is the difference between type-1 and type-2 fuzzy sets. Type-1 sets have an x axis representing the *domain* - in this case the height of an individual. Type-2 sets employ type-1 sets as the membership grades. Therefore, these fuzzy sets of type-2 allow for the idea that a fuzzy approach does not necessarily have membership grades in [0,1] but the degree of membership for the member is itself a type-1 fuzzy set. Note here that the domain of the membership grades is [0,1] since this allows for a type-1 fuzzy set being a special case of a type-2 fuzzy set. For example Danny Devito could have membership 1/0.4 in the $tall_2$ as in Figure 3.3. As can be seen by the simple example, there is an inherent extra fuzziness offered by type-2 fuzzy sets over and above a type-1 approach. Dubois and Prade(1980) summarise type-2 fuzzy sets in the following way:

"Type-2 fuzzy sets are fuzzy sets whose grades of membership are themselves fuzzy. They are intuitively appealing because grades of membership can never be obtained precisely in practical situations."
(Dubois & Prade 1980, page 30)

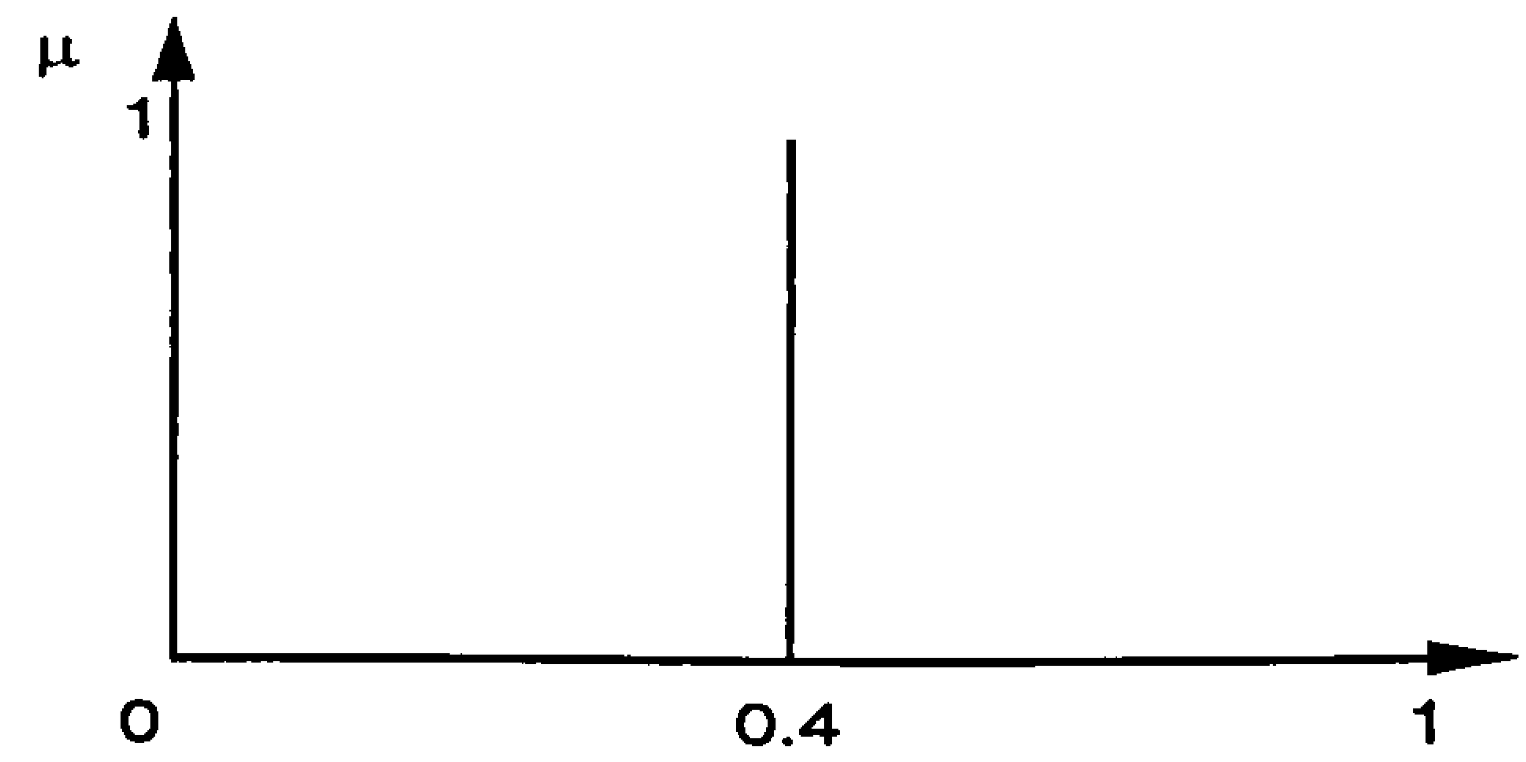

Fig. 3.3. Type-1 as a special case of Type-2

Real situations do not allow for precise numbers in [0,1]. In a control application, for instance, can it be said that a particular temperature, t, belongs to the type-1 fuzzy set hot_1 with a membership grade x (in [0,1]) precisely? No. Firstly it is highly likely that the membership could just as well be x–0.01 for example. Different experts would attach different membership grades and, indeed, the same expert might well give different values on different days! On top of this uncertainty there is always, as discussed previously in Section 1.1, some uncertainty in the measurement of t. So, an uncertain measurement is matched *precisely* to another uncertain value! Type-2 fuzzy sets on the other hand, for certain appropriate applications, allow for this uncertainty to be modelled by not using precise membership grades but type-1 fuzzy sets.

3.1.2 Interval-Valued Fuzzy Sets

A form of type-2 fuzzy sets, *interval-valued(i-v) fuzzy sets*, also relaxes the requirement for precise membership functions. In this case for each x, $\mu(x)$ is an interval in $[0,1]$. A more formal definition of an i-v fuzzy set is:

Definition 8 *The membership function of a type-2 i-v fuzzy set A_{i-v}, $\mu_{A_{i-v}}(x)$, is given by*

$$\mu_{A_{i-v}}(x) = [\mu^L_{A_{i-v}}(x), \mu^U_{A_{i-v}}(x)], \qquad x \in X \tag{3.1}$$

where for any x, $\mu^L_{A_{i-v}}(x)$ represents the lower end of the interval and $\mu^U_{A_{i-v}}(x)$ the upper end.

Fig. 3.4 indicates graphically what an i-v fuzzy set might look like. The use of i-v fuzzy sets for knowledge representation and inferencing is discussed in more detail in Sections

3.4 and 3.5.

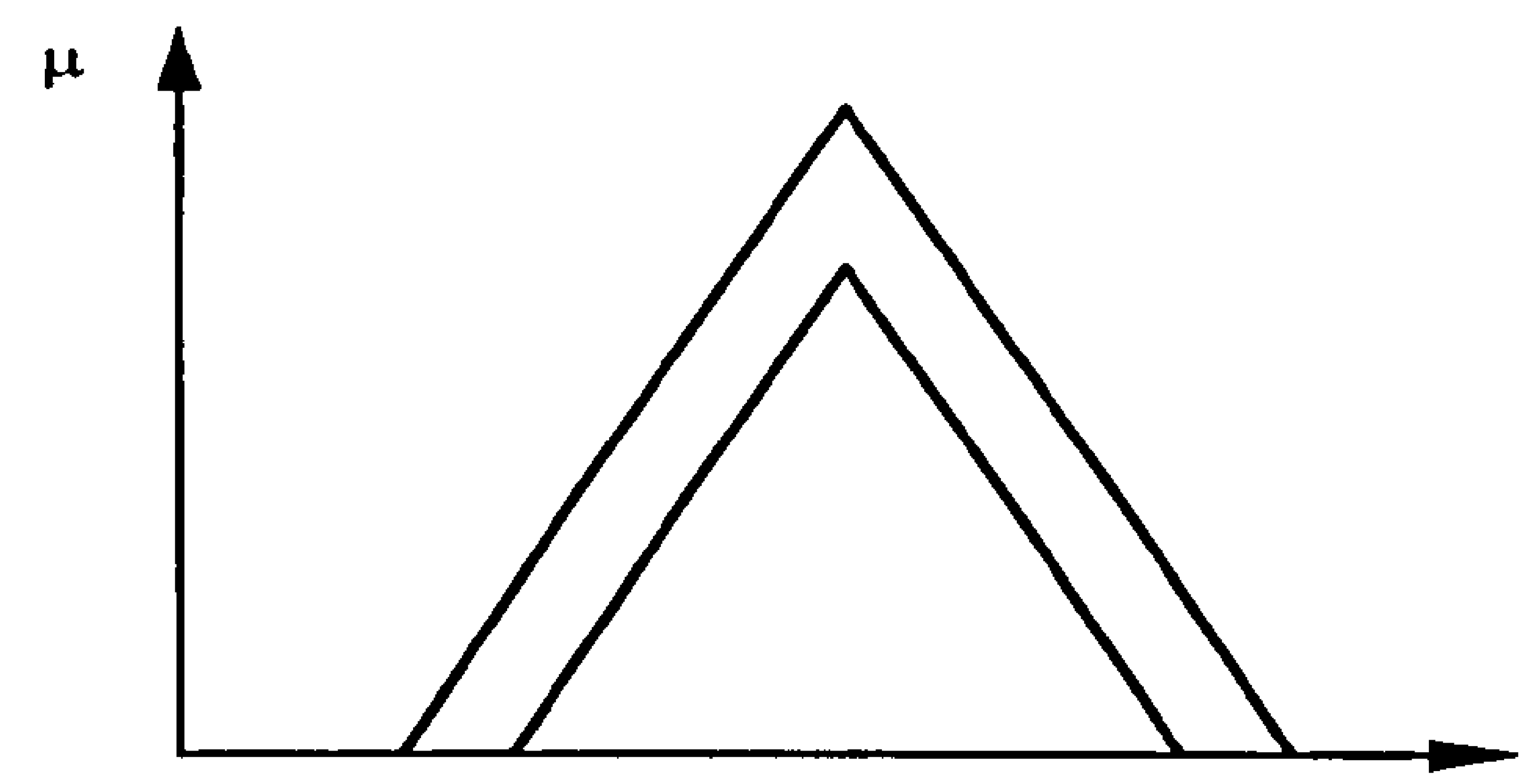

Fig. 3.4. A Typical Interval Valued Set.

This Section has provided a variety of definitions of type-2 fuzzy sets, explored their relationship with type-1 fuzzy sets and considered the particular case of i-v fuzzy sets. The next Section describes how type-2 fuzzy sets capture fuzziness and model uncertainty and imprecision.

3.2 Type-2 Fuzzy Sets Capture 'Fuzziness'

This Section now explores what type-2 fuzzy sets offer over and above type-1 fuzzy sets in more detail. Any application using type-1 fuzzy sets requires the developer to describe the membership function by numbers, in the discrete case, or by a function, where the fuzzy set has a continuous membership function. So, any system employing fuzzy sets represents the fuzziness of the particular problem using a 'non-fuzzy' (or crisp) representation. Dubois and Prade(1980, page 256) when discussing the problem of determining membership functions of type-1 fuzzy sets say *"To take into account the imprecision of membership functions, we may think of using type-2 fuzzy sets...".* As Klir & Folger(1988) point out *" ..it may seem problematical, if not paradoxical, that a representation of fuzziness is made using membership grades that are themselves precise real numbers.".* Not only is there this paradox but, indeed, type-2 fuzzy sets appear to be potentially very useful and important since the need for a 'crisp' measure of fuzziness (by a number in [0,1]) is removed and linguistic grades are allowed. Yager(1980) summarises this in the following way: *"The usefulness of fuzzy subsets of type II is that it enables us to extend membership grades to linguistic values".*

In the extensive trilogy(Zadeh 1975*a*, Zadeh 1975*c*, Zadeh 1975*b*) which covers much on type-2 fuzzy sets (amongst many fuzzy topics) Prof. Zadeh presents an argument for type-2 fuzzy sets in the following way:

On the other hand, in the case of the linguistic variable Appearance, we do not have a well defined base variable; that is, we do not know how to express the degree of beauty, say, as a function of some physical measurements

(Zadeh 1975*a*, page 205)

This is important. Type-1 fuzzy sets require a *measurable* domain, thus not allowing for fuzzy modelling of certain concepts such as beauty, comfort, safety and so on. It is this ability to capture these types of fuzziness that is the strength of type-2 fuzzy sets. Gaines (1977) in an extensive piece of work makes two interesting points about modelling such perceptions.

We are able to generate and follow arguments involving 'tallness' without having any concept of inches, centimetres or any other metric scales. To introduce the former in terms of the latter reverses the actual process of derivation and, in particular, leads to a false distinction between these concepts such as 'tallness' that have a well defined, single-parameter, physical metric, and those such as 'beautiful' which do not.

(Gaines 1977, page 51)

What is being said here is that we do not necessarily need physical measurements in our daily use of language. We know when somebody is quite tall. By the same token we know when somebody is beautiful (or indeed handsome!). In the nursing example(Lake & John 2000) nurses describe the 'stability' of a patient using terms like 'potentially unstable' - another example of a linguistic description on a non-measurable scale. The 'Dependency' of a patient could be described by a type-2 fuzzy set with membership grades (type-1 fuzzy sets) 'independent', 'becoming independent', 'dependent', 'heavily dependent' and 'totally dependent'. Dependency of a patient cannot be measured directly.

Although Gaines does not highlight type-2 fuzzy sets in his work, it is clear that since type-2 fuzzy sets allow for fuzzy grades that are not dependent on a physical measurement, they may be able to model human reasoning more readily than traditional type-1 fuzzy sets. He makes another statement that points towards consideration of type-2 fuzzy sets:

One of the key methodological questions posed to proponents of fuzzy reasoning is 'where do the numbers come from?' One appears to be simply replacing fairly simple natural language inferences with complex operations on fuzzy sets that are themselves defined in terms of fairly complex relationships between degrees of membership and physical properties.

(Gaines 1977, page 60)

He then points out that, of course, you need not use numbers and could just simply use symbolic processing. However, as he also suggests, the numerical implementation in fuzzy logic is important from a practical (and computational) point of view. Type-2 sets remove this concern to a higher level although, as will be seen later, there is a requirement to determine the fuzzy grades. This requirement is one of the driving forces behind the notion of the novel Adaptive Fuzzy Perception Learner covered in Chapter 5.

The rest of this Chapter reviews the history of type-2 fuzzy sets via consideration of their properties, their use in perception representation, and inferencing and their applications. Note that the author has published three refereed review articles on type-2 fuzzy sets(John 1998*a*, John 1998*b*, John 1998*c*)[§]

3.3 Properties of Type-2 Fuzzy Sets

Mizumoto and Tanaka(1976,1981) give a detailed analysis of the properties of type-2 fuzzy sets. Karnik and Mendel(1998) summarise these properties under both minimum and product t-norm. The results for type-2 are reproduced in Table 3.3.

The important point to absorb from this table is that for the product t-norm some of the properties for type-1 fuzzy sets do not hold for type-2. Note that recently others (e.g. (Kawaguchi & Miyakoshi 1999, Nieminen 1977)) have explored other points about the

[§]The author has applied type-2 inferencing in the domain of community transport scheduling(John 1996) so as to gain an appreciation of the effect of i-v type-2 inferencing in a real application. He has also used type-2 fuzzy sets in the pre-processing of data for submission to unsupervised neural networks in the domain of shin injuries for athletics. This collaborative work has led to a number of publications(John 1996, Innocent, Barnes & John 1997, John, Innocent & Barnes 2000, Innocent, John & Barnes 2000). A summary of this work is presented in this Chapter. More recently he has been involved in other collaborative work on the use of type-2 fuzzy sets for modelling acuity assessment in nursing using type-2 fuzzy sets(Lake & John 2000).These pieces of work have served a very useful purpose in providing an insight into the real application of type-2 fuzzy sets. However to include the detail of the community transport and nursing work here would deviate from the focus of the thesis which is an adaptive type-2 fuzzy system for learning perceptions and concepts. The papers are included in Appendix F for reference.

Set Theoretic Laws	Min.	Prod.
Reflexive	Y	Y
Anti-symmetric	Y	Y
Transitive	Y	Y
Idempotent	Y	N
Commutative	Y	Y
Associative	Y	Y
Absorption	Y	N
Distributive	Y	N
Involution	Y	Y
De Morgan's Laws	Y	N
Identity	Y	Y
Complement	Y	Y

Table 3.1. Properties of membership grades of type-2 fuzzy sets for Minimum (Min.) and Product (Prod.) (Karnik & Mendel 1998a)

theoretical properties of type-2 fuzzy sets but these pieces of work, although interesting, are not central to the development of the arguments in this thesis.

3.4 Perception Representation and Type-2 Fuzzy Sets

Section 3.2 discussed the merits of type-2 fuzzy sets in modelling 'fuzziness' by capturing imprecision that in a way that is superior to type-1 fuzzy sets. This Section investigates how type-2 fuzzy sets may be used to represent knowledge (of an expert(s)). The basic functions that enable type-2 fuzzy sets to be used in computer systems are also explained.

So that type-2 fuzzy sets can be used in a fuzzy system (in a similar manner to type-1 fuzzy sets), a method is required for computing the intersection (AND) and union (OR) of two type-2 sets. There are two different approaches reported in the literature(Zadeh 1975*a*, Hisdal 1981). To be able to discuss and compare these approaches, a common notation is adopted and introduced. Suppose there are two type-2 fuzzy sets, $\tilde{A}$ and $\tilde{B}$, in X and $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ are two *fuzzy grades* of $\tilde{A}$ and $\tilde{B}$ respectively, represented as:

$$\mu_{\tilde{A}}(x) = f(u_1)/u_1 + f(u_2)/u_2 + \ldots + f(u_n)/u_n = \sum_i f(u_i)/u_i$$

$$\mu_{\tilde{B}}(x) = g(w_1)/w_1 + g(w_2)/w_2 + \ldots + g(w_m)/w_m = \sum_j g(w_j)/w_j$$

where the functions f and g are membership functions of fuzzy grades and $\{u_i, i = 1, 2, \dots n\}$, $\{w_j, j = 1, 2, \dots m\}$ are the members of the fuzzy grades.

3.4.1 Zadeh's Union and Intersection

The most widely used definitions for union and intersection of type-2 sets are provided by Zadeh(1975*a*). His work relies on the use of the *extension principle*(Zadeh 1974). This principle lays the basis for much work on fuzzy logic.

Definition 9 (The Extension Principle) *The extension principle states that if $*$ is a binary operation in X then this operation can be applied to $\tilde{A}$ and $\tilde{B}$ by*

$$\tilde{A} * \tilde{B} = \sum_{i,j} (f(u_i) \wedge g(w_j))/(u_i * w_j)$$

where $\wedge$ is an appropriate t-norm (e.g. min).
This leads to the following definitions for union and intersection of type-2 sets.

Definition 10 (Union of Type-2 Fuzzy Sets) *The union ($\cup$) of two type-2 fuzzy sets ($\tilde{A}$, $\tilde{B}$) corresponding to $\tilde{A}$ OR $\tilde{B}$ is given by:*

$$A \cup B \Leftrightarrow \mu_{\tilde{A}\cup\tilde{B}}(x) = \mu_{\tilde{A}}(x) \sqcup \mu_{\tilde{B}}(x) \quad (3.2)$$

$$= \sum_{i,j} (f(u_i) \wedge g(w_j))/(u_i \vee w_j) \quad (3.3)$$

Definition 11 (Intersection of Type-2 Fuzzy Sets) *The intersection ($\cap$) of two type-2 fuzzy sets ($\tilde{A}$, $\tilde{B}$) corresponding to $\tilde{A}$ AND $\tilde{B}$ is given by:*

$$\tilde{A} \cap \tilde{B} \Leftrightarrow \mu_{\tilde{A}\cap\tilde{B}}(x) = \mu_{\tilde{A}}(x) \sqcap \mu_{\tilde{B}}(x) \quad (3.4)$$

$$= \sum_{i,j} (f(u_i) \wedge g(w_j))/(u_i \wedge w_j) \quad (3.5)$$

where $\vee$ is an appropriate t-conorm (e.g. max), $\sqcup$ denotes *join* and $\sqcap$ denotes *meet*(Kandel 1986). Thus the join and meet allow for us to combine type-2 fuzzy sets for the situation where it is required to 'OR' or 'AND' two type-2 fuzzy sets. Join and meet are the building blocks for type-2 fuzzy relations and type-2 fuzzy inferencing with type-2 if-then rules (Section 4.3).

The use of these definitions is best explored by an example where a comparison is made with type-1 union and type-1 intersection.

Type-1 Union and Intersection - a simple example

Suppose there are three people a, b and c who belong to the type-1 fuzzy sets *tall* and *heavy* where a is *tall* to degree 0.2 and *heavy* to degree 0.4, b is *tall* (0.7) and *heavy* (0.5) and c is *tall* (0.95) and *heavy* (0.8). The type-1 fuzzy sets representing *tall* and *heavy* are

$$tall = 0.2/a + 0.7/b + 0.95/c$$

$$heavy = 0.4/a + 0.5/b + 0.8/c.$$

So, suppose somebody who is *tall* AND *heavy* is described as being *big* then

$$big = tall \cap heavy.$$

Using min for intersection

$$big = 0.2/a + 0.5/b + 0.8/c$$

The outcome is that, for example, a is big to degree 0.2.

Join and Meet - the same example

Now consider $\tilde{tall}$ and $\tilde{heavy}$ as type-2 fuzzy sets where

$$\tilde{tall} = low/a + medium/b + high/c$$

$$\tilde{heavy} = medium/a + medium/b + high/c$$

In other words a belongs to $\tilde{tall}$ with degree *low* and to $\tilde{heavy}$ to degree *medium* The membership grades are type-1 fuzzy sets, which might be, for example:

$$low = 1.0/0.0 + 0.75/0.1 + 0.5/0.2 + 0.25/0.3$$

$$medium = 0.33/0.3 + 0.67/0.4 + 1/0.5 + 0.67/0.6 + 0.33/0.7$$

$$high = 0.25/0.7 + 0.5/0.8 + 0.75/0.9 + 1.0/1.0$$

as shown in Figure 3.5. Note that, for simplicity, the words *low*, *medium* and *high* have been used to represent the membership grades for both $\tilde{tall}$ and $\tilde{heavy}$. This is not required and, indeed, in a real application one would not expect this to be the case. As with the type-1 example, the type-2 fuzzy set $\tilde{big}$ is interpreted as $\tilde{tall}$ and $\tilde{heavy}$ ($\tilde{big} = \tilde{tall} \sqcap \tilde{heavy}$) then for person a

$$\tilde{big}_a = \tilde{tall}_a \sqcap \tilde{heavy}_a$$

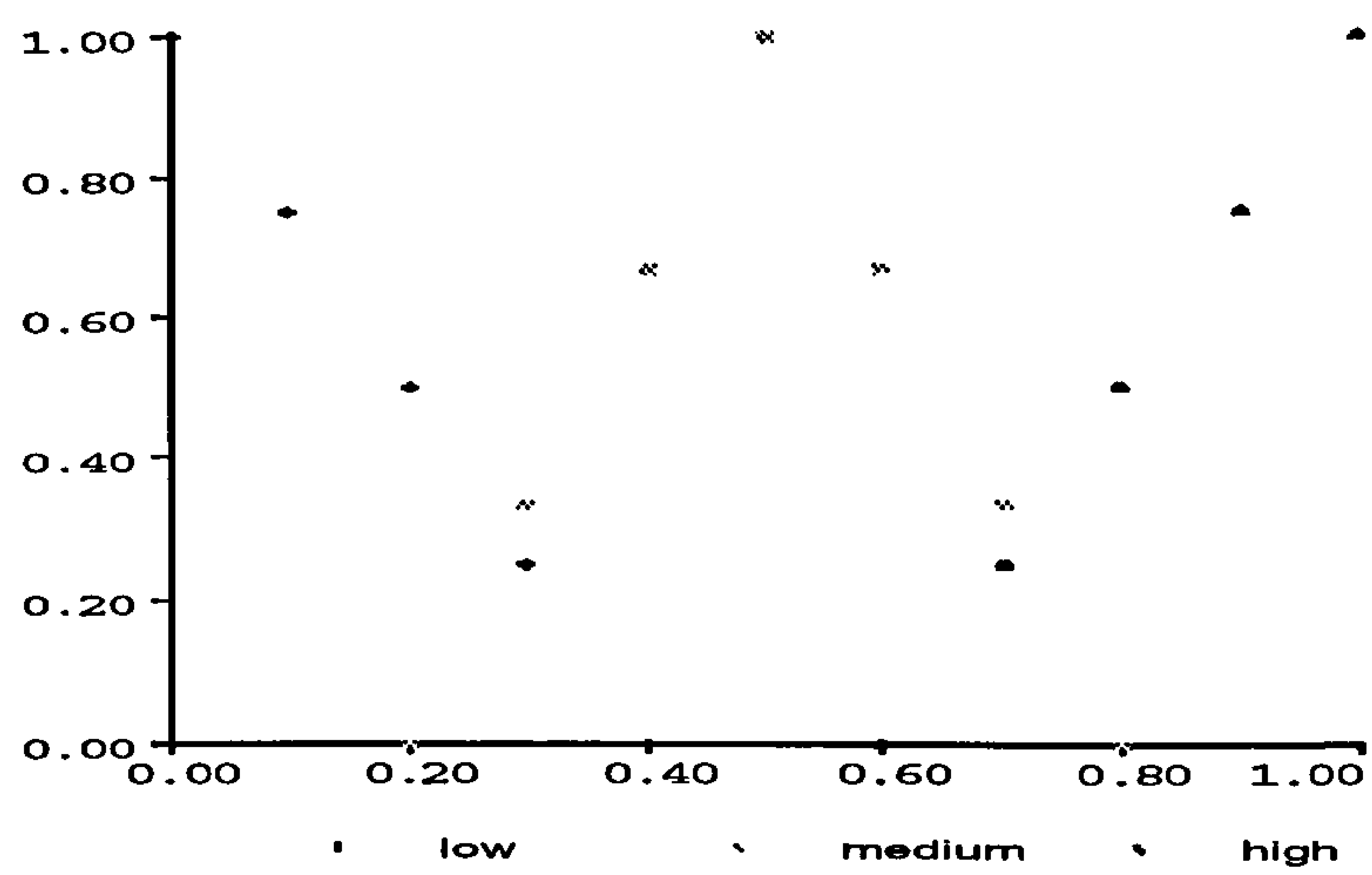

Fig. 3.5. The membership grades for the type-2 fuzzy sets $\tilde{tall}$ and $\tilde{heavy}$

which is the meet of the two fuzzy grades *low* and *medium*. Using the definition for meet (Equation 3.4)

$$
\begin{aligned}
\tilde{big}_a &= 0.33/0.0 + 0.67/0.0 + 1.0/0.0 + 0.67/0.0 + 0.33/0.0 \\
&\quad + 0.33/0.1 + 0.67/0.1 + 0.75/0.1 + 0.67/0.1 + 0.33/0.1 \\
&\quad + 0.33/0.2 + 0.5/0.2 + 0.5/0.2 + 0.5/0.2 + 0.33/0.2 \\
&\quad + 0.25/0.3 + 0.25/0.3 + 0.25/0.3 + 0.25/0.3 + 0.25/0.3 \\
&= 1.0/0.0 + 0.75/0.1 + 0.5/0.2 + 0.25/0.3 \\
&= low
\end{aligned}
$$

There are various points to be made about this comparison:

1. The type-1 representation requires the 'expert' to attach a number to each person to describe their 'tallness' or 'heaviness', whereas the type-2 approach only requires a fuzzy grade.

2. The fuzzy grades low, medium and high are the same for both tall and heavy for simplicity. They need not be.

3. The fuzzy grades low, medium and high are membership functions that still have to be determined.

4. The set 'big' for the type-1 approach is a type-1 fuzzy set whereas the type-2 approach yields a type-2 solution.

An alternative definition for meet(Zadeh 1975*a*, Zadeh 1975*b*) relies on the level-set form of the extension principle. If A is a subset of U then an α level set is a non-fuzzy set, A_α which comprises all elements of U whose grade of membership in A is greater than or equal to α. A fuzzy set A can be decomposed into its constituent level sets

$$A = \sum_\alpha \alpha A_\alpha$$

and, suppose that f is a mapping from U to V and A is a subset of U, then

$$f(A) = \sum_\alpha \alpha f(A_\alpha).$$

This leads to the following definition for the intersection of type-2 fuzzy sets

$$\mu_{\tilde{A}\cap\tilde{B}} = \sum_\alpha \alpha(\mu^\alpha_{\tilde{A}} \wedge \mu^\alpha_{\tilde{B}})$$

where $\mu^\alpha_{\tilde{A}}$ means the membership function for the α level set of $\tilde{A}$. This method produces identical results to those provided by Equations 3.4 and 3.2.

3.4.2 The Numerical Representation Approach

Hisdal(1981) proposed a different approach which uses the idea of the *numerical representation* of a fuzzy set of type-2 (Hisdal's work actually develops the idea for sets of type-N but for our purposes the type-2 case will be explained). Using Hisdal's notation let C^{t1} be a type-1 fuzzy set where

$$C^{t1} = \sum_{j=1}^{J0} c_j/v_j = \sum_{j=1}^{J0} \Pi^{t1}_C(v_j)/v_j \tag{3.6}$$

with t1 indicating 'type-1'. So a type-2 fuzzy set can be represented by specifying for each v_j, and for each value of $\Pi(v_j)$ the possibility of this $\Pi(v_j)$ i.e. $\Pi(\Pi(v_j))$ or $\Pi^{t2}(v_j)$. The set of allowed numerical values for a type-1 fuzzy set is denoted by M^{t1} and for $\Pi^{t2}(v_j)$ is denoted by M^{t2}. The numerical representation of a fuzzy set of type-2 is therefore reduced to a set of type-1 in the two dimensional space VXM^{t1}. The example given in the original paper considers the type-2 fuzzy set 'smallish' with four members, v_1, v_2, v_3 and v_4, whose grades in this set are respectively very high, high, medium and low which are represented in Table 3.2. The union and intersection for type-2 fuzzy sets are performed by carrying out these operations using the normal definitions of union and intersection for type-1 fuzzy sets. To illustrate this example Hisdal defines another type-2 fuzzy set called 'biggish' (see Table 3.3). Table 3.4 shows the union and intersection of smallish and biggish. The solution therefore is always a type-2 fuzzy set.

	$\Pi(v_j) =$				
	0.00	0.25	0.50	0.75	1.00
$v = v_1$	0	0	0	0	1
v_2	0	0	0	0.5	1
v_3	0	0.5	1	0.5	0
v_4	1	0.5	0	0	0

Table 3.2. A numerical presentation of the type-2 fuzzy set 'smallish'

	$\Pi(v_j) =$				
	0.00	0.25	0.50	0.75	1.00
$v = v_1$	1	0.5	0	0	0
v_2	0	0.5	1	0.5	0
v_3	0	0	0	0.5	1
v_4	0	0	0	0	1

Table 3.3. A numerical presentation of the type-2 fuzzy set 'biggish'

Zadeh's interpretation of the union of smallish and biggish for v_1 is

$$\mu^{v_1}_{smallish \cup biggish} = 0/0 + 0/0.25 + 0/0.75 + 1/1$$

which is clearly different from Hisdal's method. Hisdal compares the two approaches. The difference is that when using Zadeh's approach the meaning of $C = A \cap B$ is the fuzzy set 'A AND B' as if A and B had been type-1 sets. For each element of the universe set the fuzzy grade of C is obtained by using the special min operation described above (Equation 3.4) on the membership grades of A and B which leads to a fuzzy grade. For Hisdal's definition, in contrast, the fuzzy set C is described by the fuzzy set whose possibilities are $\Pi^{t2}_C(v_j) = \Pi^{t2}_A(v_j)$ AND $\Pi^{t2}_B(v_j)$. As Hisdal states(Hisdal 1981, page 400) *"..the AND connective is now between the possibilities of v_j for A and B, not between the fuzzy sets themselves"* and, as has been described, requires no new min definition. Hisdal's definition, in contrast with Zadeh's, does not require the limitation that A and B must be convex sets.

	$\prod(v_j) =$				
	0.00	0.25	0.50	0.75	1.00
$v = v_1$	1	0.5	0	0	1
v_2	0	0.5	1	0.5	1
v_3	0	0.5	1	0.5	1
v_4	1	0.5	0	0	1

(a) The union of biggish and smallish

	$\prod(v_j) =$				
	0.00	0.25	0.50	0.75	1.00
$v = v_1$	0	0	0	0	0
v_2	0	0	0	0.5	0
v_3	0	0	0	0.5	0
v_4	0	0	0	0	0

(b) The intersection of biggish and smallish

Table 3.4. The union and intersection of biggish and smallish shown using the numerical representation.

3.5 Type-2 Fuzzy Sets and Inferencing with Type-1 Fuzzy Systems

Type-2 sets play two roles in inferencing with type-1 fuzzy sets in type-1 fuzzy systems. They have been used as an improvement on inferencing with type-1 fuzzy sets(Türkşen 1986) whereby some of the uncertainty is traditionally lost by the standard inferencing methods. As has been seen (Section 2.2.2), the method for inferencing with type-1 fuzzy sets is based on the idea of *generalised modus ponens (gmp)* and is well established(Zadeh 1975*b*). However, Hisdal (1981) argues that this standard method for implication does not, in many situations, reflect the inherent uncertainty in any particular situation and that type-2 sets allow for inferencing with type-1 fuzzy sets that reflects this uncertainty. In particular the work shows that the type-1 relation for if-then often produces solutions that are incorrect (Section 3.5.1). Another hypothesis, put forward by Türkşen, is that gmp is inadequate since type-1 sets only express first order semantics and that the introduction of type-2 fuzzy sets provides increased expressive power(Türkşen 1995*b*) (Section 3.5.2).

$A \to C$	A	C
T	T	T
T	F	BLANK

Table 3.5. The truth table for a non fuzzy if-then statement as used in natural language. The if-then statement is assumed to be true.

3.5.1 Numerical Representation Inferencing

Hisdal(1981) presents a long and detailed case for a method of inferencing on type-1 fuzzy sets using type-2 relations. Hisdal's interpretation of inferencing with if-then rules relies on two ideas:

1. Mathematical logic allows for the implication to be true or false and the truth value of the consequent is specified, not inferred.

2. In (type-1) fuzzy set theory the if-then statement is assumed to be true and infers a value of y from x. This is a better representation of the if-then statement as used in natural language.

These two points lead to table 3.5.1 where the if-then statement is assumed to be true and, therefore, when the antecedent is false then the consequent is a 'don't know' state (BLANK using Hisdal's notation).

Using the notation of Hisdal, the if-then statement 'If $x = A$ Then $y = C$' is written as $C \mid A$ where A and C are fuzzy subsets of the two universes U and V respectively. The entries of the $C \mid A$ type-2 relation are given by (Hisdal(1981) Equation 6.1.2):

$$\Pi_{C|A}(\Pi(v_i) \mid u_i), \qquad i = 1, ..., I0 \quad j = 1, ..., J0 \qquad \Pi(v_j) \in M^{t1}. \tag{3.7}$$

Assuming we have the $C \mid A$ relation then we choose a particulent $P \subset U$ and deduce a set $D \subset V$. The inference operation is carried out in two steps:

1. The $C \mid A$ relation is particulated by P and this is given by

$$\Pi_{(A,C)_P}(u_i, \Pi(v_j)) = \Pi_P(u_i) \wedge \Pi_{(C|A)}(\Pi(v_j) \mid u_i). \tag{3.8}$$

2. Every row of the A, C relation is a fuzzy subset of V. The union of all the fuzzy subsets gives the deduced set D which is

$$\Pi_D(\Pi(v_j)) = \bigvee_{u_i} \Pi_{A,C}(u_i, \Pi(v_j)). \tag{3.9}$$

The $C \mid A$ and the A, C relations are, in general interval valued type-2 fuzzy sets. Before we infer on an if-then statement we are in complete ignorance. Given this set the if-then statement adds new knowledge and the solution of the $C \mid A$ relation is a restriction of this set. Hisdal applies this idea to some situations without tackling a real problem. Clearly this approach produces a *different* solution to the traditional gmp however without the application to a real problem the efficacy of this approach remains unproven.

3.5.2 Type-1 Inferencing with Interval Valued Type-2 Sets - Türkşen's Approach

Türkşen has produced a body of work using type-2 fuzzy sets when inferencing with fuzzy systems (Türkşen 1986, Türkşen 1989, Türkşen & Tian 1992, Türkşen 1993, Türkşen 1994, Türkşen 1995*a*, Türkşen 1995*b*, Türkşen 1999) and the theoretical properties of his approach including consideration of the compositional rule of inference for i-v inferencing is discussed by Yuan & Pan(1995). Türkşen argues that type-1 fuzzy sets and logics present concerns that can be tackled using type-2 fuzzy sets. In particular he states(Türkşen 1995*b*) *"uncertainty models represented by the interval-valued Type II fuzzy sets and logics have a more expressive power"*. He proposes that the combined linguistic expression of the linguistic values with the linguistic connectives should not be set arbitrarily to just one of the two fuzzy normal forms but to the interval generated by both.

His work discusses four types of knowledge representation for fuzzy systems

1. Linguistic Expression;
2. Meta-Linguistic Expression;
3. Propositional Expression;
4. Computational Expression.

1. Linguistic Expression

Türkşen describes 16 basic linguistic expressions as the basis of the representation of natural language using logic (e.g. if the temperature is high then the pressure is low). In the case of two valued logic these concepts are represented by Disjunctive and Conjunctive Normal Forms (DNF and CNF) and that for any of these 16 terms DNF=CNF. However by generalising two valued logic to fuzzy logic this does not hold and DNF$\neq$CNF(Türkşen 1986)(Türkşen 1995*a*). For type-1 fuzzy logic, as well as for two valued logic, the shortest of the forms is adopted. His central hypothesis is that since fuzzy researchers have used either DNF *or* CNF there is a loss of information. Certainly by using only one of the normal forms the type-1 fuzzy system developer is, probably in most cases unknowingly, making a decision that can affect the result of the inferencing process.

2. Meta-Linguistic Expression

This is simply a symbolic representation of linguistic expression. So implication may be represented as, for instance, X_1 *is* A_1 IMPLIES X_2 *is* A_2, $A_1 \rightarrow A_2$, NOT A_1 *or* A_2 amongst many, where X_1 and X_2 are meta-linguistic representations of the linguistic variables and A_1 and A_2 are the meta-linguistic values associated with X_1 and X_2.

3. Propositional Expressions

Here the meta-linguistic expressions are represented by normal forms. So for $A_1 \rightarrow A_2$ the DNF and CNF expressions are given by

$$\begin{aligned} DNF(A_1 \rightarrow A_2) &= (A_1 \cap A_2) \cup (\overline{A_1} \cap A_2) \cup (\overline{A_1} \cap \overline{A_2}) \\ CNF(A_1 \rightarrow A_2) &= \overline{A_1} \cup A_2 \end{aligned}$$

These normal forms are not equivalent to each other in the case of fuzzy logic. Türkşen describes *Fuzzy Normal Forms (FNFs)*(Türkşen 1995*a*). So, for example he shows that the FNFs that correspond to the usual Zadeh connectives of max-min are

$$\begin{aligned} FDNF(A_1 \rightarrow A_2) &= (A_1 \cap A_2) \cup (\overline{A_1} \cap A_2) \cup (\overline{A_1} \cap \overline{A_2}) \\ &= DNF(A_1 \rightarrow A_2) \\ FCNF(A_1 \rightarrow A_2) &= \overline{A_1} \cup A_2 \\ &= CNF(A_1 \rightarrow A_2) \end{aligned}$$

where FDNF denotes Fuzzy Disjunctive Normal Form and FCNF represents Fuzzy Conjunctive Normal Form. Note that this equivalence between the boolean normal forms and the FNFs is in propositional form only. Türkşen shows that this equality is true for all the 16 concept combinations outlined in his work.

4. Computational Expressions

This is simply the computational implementation of the propositional form. Computational expressions are formed over the elements of the fuzzy sets (i.e. the membership values) and the computational connectives. The gmp for type-1 fuzzy sets can be expressed as:

$$B' = A' o (A \rightarrow B)$$

where A, A', B and B' are fuzzy sets and o is the compositional rule of inference(Zadeh 1975*b*). In other words a solution is required to

$$A \text{ AND } (A \rightarrow B) = B.$$

The FNFs that mirror the Zadeh max-min approach for this particular situation are:

$$\begin{aligned} FDNF(A \to B) &= (A \cap B) \cup (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B}) \\ FCNF(A \to B) &= \overline{A} \cup B \end{aligned}$$

Türkşen shows that these two forms produce an interval and therefore the result of this inference is an interval valued fuzzy set [FDNF,FCNF]. Türkşen(1989) uses this approach to describe four methods for approximate reasoning using interval-valued sets. The four methods are:

1. A and A' are point valued and o is crisp;
2. A and A' are point valued and o is linguistic;
3. A and A' are interval valued and o is crisp;
4. A and A' is interval valued and o is linguistic

where 'point valued' means a type-1 set and 'crisp' o is as per Zadeh. The following describes the method(Türkşen 1989, Türkşen 1995*b*) and shows, by some simple examples, some of the effects of ***interval-valued fuzzy modus ponens.***

Türkşen's Approach Illustrated

Türkşen's approach has been implemented by the author of this thesis. Suppose the following type-1 fuzzy sets have been arbitrarily selected as examples:

$$A_1 = 0.3/a_1 \ + \ 0.5/a_2 \ + \ 1.0/a_3$$

$$B_1 = 0.4/b_1 \ + \ 0.6/b_2 \ + \ 0.8/b_3$$

$$A_2 = 0.2/a_1 \ + \ 0.2/a_2 \ + \ 0.4/a_3 \ + \ 0.5/a_4 \ + \ 0.75/a_5$$

$$B_2 = 0.2/b_1 \ + \ 1/b_2$$

$$A_3 = 0.25/a_1 \ + \ 0.66/a_2 \ + \ 0.47/a_3 \ + \ 0.95/a_4$$

$$B_3 = 0.54/b_1 \ + \ 0.64/b_2 \ + \ 0.22/b_3 \ + \ 0.83/b_4$$

The gmp is expressed as

$$B' = A \circ (A \to B)$$

where $\circ$ represents the compositional rule of inference. The rest of this Section describes the four scenarios.

A Point-Valued and ∘ Crisp

The interval valued set B' is described by the lower bound B'_L and the upper bound B'_U then the membership function for these sets using the composition based on Türkşen(1989) are:

$$b'_{jL} = \bigvee_i (a_i \wedge [(a_i \wedge b_j) \vee (a_i^c \wedge b_j) \vee (a_i^c \wedge b_j^c)]) \tag{3.10}$$

$$b'_{jU} = \bigvee_i (a_i \wedge (a_i^c \vee b_j)) \tag{3.11}$$

for all $i \in I$ and $j \in J$ where a_i and b_j are the membership values of A and B respectively.

If this is implemented using the min-max set of operators for the fuzzy sets described, the interval valued representation of $A \rightarrow B$ is given in Table 3.6(a),(b) and (c). Interval values at certain points indicating that the normal form approach captures some of the uncertainty that choosing only one of the normal forms does not. The result of applying Equations 3.10 and 3.11 are the fuzzy set

$$B'_1 = 0.5/b_1 \ + \ 0.6/b_2 \ + \ 0.8/b_3$$

$$B'_2 = 0.5/b_1 \ + \ 0.75/b_2$$

$$B'_3 = 0.54/b_1 \ + \ 0.64/b_2 \ + \ 0.47/b_3 \ + \ 0.83/b_4.$$

Note that these are not equal to the initial B_is. For the B'_i to be equal to the B_i then A has to be normalised and only those values of b_i that are greater than or equal to $\max(a_i \wedge a_i^c)$ are to be used(Türkşen 1989). It can also be seen that the B's are not interval valued sets since for distributive and conjugate pairs of operators with the pseudocomplement then $b'_{jL} = b'_{jU}$.

Now, for illustration purposes, the Yager class of union, intersection and complement is adopted(Klir & Folger 1988). Namely, for two sets A and B $min[1, (a^w + b^w)^{1/w}]$, $1 - min[1, ((1-a)^w + (1-b)^w)^{1/w}]$ and $(1 - a^w)^{1/w}$ are used for the union, intersection and complement respectively. If w is chosen to be 2 the results for $A \rightarrow B$ are given in Table 3.7(a),(b) and (c) and

$$B'_1 = 1/b_1 \ + \ 1/b_2 \ + \ 1/b_3$$

$$B'_2 = [0.73, 0.98]/b_1 \ + \ 1/b_2$$

$$B'_3 = 1/b_1 \ + \ 1/b_2 \ + \ [0.95, 1]/b_3 \ + \ 1/b_4.$$

	b_1	b_2	b_3
a_1	[0.6,0.7]	[0.6,0.7]	[0.7,0.8]
a_2	0.5	[0.5,0.6]	[0.5,0.8]
a_3	0.4	0.6	0.8

(a) Results for A_1 and B_1

	b_1	b_2
a_1	0.8	[0.8,1]
a_2	0.8	[0.8,1]
a_3	0.6	[0.6,1]
a_4	0.5	[0.5,1]
a_5	0.25	[0.75,1]

(b)Results for A_2 and B_2

	b_1	b_2	b_3	b_4
a_1	[0.54,0.75]	[0.64,0.75]	0.75	[0.75,0.83]
a_2	0.54	0.64	0.34	[0.66,0.83]
a_3	[0.53,0.54]	[0.53,0.641]	0.53	[0.53,0.83]
a_4	0.54	0.64	0.22	0.83

(c) Results for A_3 and B_3

Table 3.6. If A then B for min-max

Apart from the different numbers that arise by the nature of the different connectives the Yager class with two of the examples produces an interval set.

A Point-Valued and ∘ Linguistic

By describing ∘ as linguistic the crisp operators are replaced with linguistic ones. With the linguistic composition, the membership values of the fuzzy sets are:

$$b'_{jL} = \bigvee_i ((a_i \wedge b_j) \vee (a_i^c \wedge a_i)) \tag{3.12}$$

$$b'_{jU} = \bigvee_i ((a_i \wedge b_j) \vee (a_i^c \wedge a_i) \vee (a_i^c \wedge b_j \wedge b_j^c)) \tag{3.13}$$

for all $i \in I$ and $j \in J$.

A Interval-valued and ∘ Crisp

	b_1	b_2	b_3
a_1	[0.73,0.81]	[0.89,0.93]	1
a_2	[0.88,0.95]	1	1
a_3	[0.98,1]	1	1

(a) Results for A_1 and B_1

	b_1	b_2
a_1	[0.46,0.63]	[0.87,1]
a_2	[0.45,0.63]	[0.87,1]
a_3	[0.58,0.82]	1
a_4	[0.61,0.89]	1
a_5	[0.65,0.99]	1]

(b) Results for A_2 and B_2

	b_1	b_2	b_3	b_4
a_1	[0.78,0.85]	[0.84,0.92]	[0.52.0.70]	[0.94,1]
a_2	1	1	[0.67,0.97]	1
a_3	1	1	[0.63,0.88]	1
a_4	1	1	[0.70,1]	1

(c) Results for A_3 and B_3

Table 3.7. If A then B for Yager

With the linguistic composition the membership values of the fuzzy sets are:

$$b'_{jL} = \bigvee_i ((a_i \wedge b_j) \vee (a_i^c \wedge a_i)) \tag{3.14}$$

$$b'_{jU} = \bigvee_i ((a_i \wedge b_j) \vee (a_i^c \wedge a_i) \vee (b_j \wedge b_j^c)) \tag{3.15}$$

for all $i \in I$ and $j \in J$.

A Interval-Valued and $\circ$ *Linguistic*

This situation leads to the following definition of the membership values:

$$b'_{jL} = \bigvee_i ((a_i \wedge b_j) \vee (a_i^c \wedge a_i)) \tag{3.16}$$

$$b'_{jU} = \bigvee_i ((a_i \wedge b_j) \vee (a_i^c \wedge a_i) \vee (b_j \wedge b_j^c)) \tag{3.17}$$

for all $i \in I$ and $j \in J$. These provide the same membership functions as for where A is interval-valued and $\circ$ is crisp.

A recent work(Türkşen 1999) by Türkşen develops the theme of type-2 interval valued fuzzy reasoning by so called 'unification' of the type-1 approach. This work brings together much of his earlier work and looks at an unsupervised clustering based on the fuzzy c-means approach. He only reports on the use of the method for type-1 fuzzy inferencing.

In summary, Türkşen's argument is that traditional type-1 inferencing is incorrect in that it only, usually, uses one of the normal forms. The theoretical arguments are developed fully in his extensive publications but the practical implementation of the approach is still to be explored fully.

3.5.3 Type-1 Inferencing using Type-2 Fuzzy Sets: A Summary

The Numerical Representation approach of Hisdal works from the premise that in order not to introduce false information into an if-then statement, type-2 fuzzy sets allow for a suitable representation of the state of ignorance before applying the statement. By using type-2 fuzzy sets to represent this ignorance, Hisdal provides an alternative approach to the conventional gmp and compositional rule of inference that reflects the uncertainty. Hisdal does not undertake inference with type-2 sets as the antecedents but uses type-2 fuzzy sets to enhance the inferencing with type-1. In contrast the Türkşen approach highlights a deficiency in the traditional fuzzy inferencing approaches in that they only adopt one of the normal forms. For two valued logic this is not a problem as the normal forms are equivalent but for fuzzy logic they are not. His approach of using the normal forms to generate interval valued fuzzy sets appears to capture second order imprecision. So, it can be seen now that type-2 fuzzy sets offer the ability to model a second degree of fuzziness as well as assist in inferencing with type-1 fuzzy sets.

The next Section reviews the reported use of type-2 fuzzy sets in applications.

3.6 Applications of Type-2 Fuzzy Sets

The number of applications of type-2 fuzzy sets reported in the literature is growing. Practical applications of type-2 fuzzy sets include handling tolerances in fuzzy equations systems(Wagenknecht & Hartmann 1988), fuzzy regression models(Diamond 1990), determining membership functions(Türkşen 1991), the internet(Tolbert & Corbin 1994) and computer networks(Starks, Kreinovich & Narasimhamurthy 1994). Yager(1980), for example, uses type-2 fuzzy sets for multi-objective decision making. He describes a method that tackles the situation where the decision maker has a number of possible decisions he or she could make, based on a number of objectives. Each decision provides a solution to the problem to some degree. The decision is a function of the objectives which depends on how the objectives should be combined. There are many decision functions that could be used in any given situation and Yager proposes that there is a decision maker's ideal decision function which the potential decision functions match to some degree. He presents a simple example of a decision maker choosing between three banks based on certain criteria. Since this is not an actual application, the method is not validated on a real example and so significant conclusions cannot be drawn. He does point out that how to choose the best decision is difficult since the solution to the problem is a fuzzy set of type-2. Type-2 fuzzy sets have been used to aid decision making in Engineering(Chameau, Gunaratne & Altschaeffl 1987). The problem pertains to combination of opinions of safety of various materials in the air. They show that type-2 fuzzy sets can be used to combine linguistic descriptions of perceptions of health hazards, in particular for effective weighting of opinions. This is a simple example that isn't evaluated effectively but it does show the capability of type-2 sets to model effectively linguistic perceptions or opinions that have no base domain. Rocha(1994) extends interval valued fuzzy sets to simulate human cognitive categorisation and concept combination with the notion of evidence sets. Interval valued fuzzy sets have been used in control(Wu 1996) of mobile robots. The approach implements a new fuzzy control methodology (Fuzzy Interval Control (FIC)) for successfully navigating a miniature robot in an unknown maze without touching the walls. A comprehensive review paper(Nguyen, Kreinovich & Zuo 1997) on the application of interval computations and interval-valued degrees of belief describes various case studies, a number of which use interval valued fuzzy sets (the reader is referred there for the details).

The applications reported in the literature where type-2 fuzzy sets play a role can be categorised as either using type-2 fuzzy sets to represent knowledge or as a way of capturing the uncertainty in the inferencing process with type-1 fuzzy sets.
The strength of the type-2 paradigm is the capability to represent perceptions more effectively than the type-1 approach. It is this capability of type-2 fuzzy sets to represent perceptions that seems particularly appealing and interesting and led to some collaborative work on using type-2 fuzzy sets for perception modelling in a real application. Since type-2 fuzzy sets appear to offer the ability to model linguistic perceptions, the author tackled a problem as an initial exploration into the use of type-2 fuzzy sets for modelling perceptions. The next Section explores the work carried out by the author

on using type-2 fuzzy sets to represent perceptions by a medical consultant to help with unsupervised clustering of radiographic images.

3.7 Type-2 Sets for Pre-Processing of Neural Networks for Shin Images

This Section provides an overview of a substantial piece of work carried out by the author on using type-2 fuzzy sets as a perception representation method for pre-processing for unsupervised artificial neural networks. The full work is reported elsewhere(Innocent et al. 1997, John, Innocent & Barnes 1998, Innocent et al. 2000, John et al. 2000)¶

3.7.1 Medical Background

The major use of bone scanning in sports injuries is for the detection of an injury class called 'stress fractures'. Stress fractures are usually partial fractures or cracks which result from the application of habitual, non-violent, repetitive stress which exceeds the existing functional capacity of the bones. Unlike a normal fracture caused by trauma, a stress fracture does not show up on a plain radiograph until six weeks after the initial injury, whereas a bone scan will detect any abnormality within hours. It is important to confirm the diagnosis as soon as possible and to instigate the correct treatment, particularly in professional athletes. Usually stress fractures can be successfully treated by a period of rest, but, if missed, the fracture can become more serious, extending across the whole bone and, in extreme cases, becoming displaced and thus requiring fixation. Patients who presented with exercise-induced lower leg pain were originally bone scanned to eliminate the possibility of stress fractures. However, it soon became apparent that the bone scan appearance was not simply a matter of normal or abnormal as several other patterns began to emerge. These have been tentatively placed, by the consultant, into one of seven classes based on an interpretation of the image:

¶It is important to point out that this research was, by necessity, collaborative research delineated along the following lines:

- Mike Barnes is a consultant at Leicester General Hospital and provided the medical expertise and background to the problem.
- Peter Innocent is a Principal Lecturer in the School of Computing Sciences at De Montfort University. He provided and ran the neural network programs and did some analysis of the results.
- The author carried out the knowledge acquisition process with Mike Barnes, designed the questionnaire and interpreted it, using type-2 fuzzy sets.

1. **Normal.** A completely uniform image
2. **Athletic Normal.** Increased uptake along the entire length of both the cortices.
3. **Stress Fracture.** There is a single, very dense, localised area of uptake.
4. **Medial Tibial Syndrome (MTS)** A line of increased uptake along some part of the lower half of the posterior cortex.
5. **Focal Multiple hot spots.** These could be sub-clinical multiple stress fractures.
6. **Healing Stress Fracture Single hot spot.** Less obvious than a stress fracture.
7. **Patchy.** Non-uniform uptake that does not fit any of the above descriptions.

The classes other than the MTS and stress fractures are based on experiential interpretation of the images alone because there are no confirming clinical factors. Hence there can be difficulties distinguishing between them. Experience has shown that re-examination of these images may reclassify them into different categories due to perceptual variation. This indicates that these classes may really be in a dimension of overlapping sub and super-classes. Since there has not been a systematic analysis of what the data base of images contains, these dimensions are unknown. This has significance since the efficacy of a particular treatment may depend on correct classification. There is no precise way, when viewing a radiographic image, of determining the problem with any given shin. Diagnosis is based on imprecise data. For instance, the distinctiveness of a line on the image or whether a line is 'much longer' than its width are important in assisting with the diagnosis. The approach taken was to describe the images using a questionnaire developed in conjunction with the expert. A series of knowledge acquisition sessions was carried out by interview. After much discussion a pro forma was generated that allowed the medical domain expert to view images and record the results.
The questionnaire was piloted, some amendments made and finally all 203 images were analysed using the questionnaire. Some of the data is essentially binary in nature whilst others could be described as imprecise. For input to a neural network these fuzzy categories (or perceptions) had to be converted into a numeric format. Earlier work(Innocent, Barnes, John & Keighley 1996) adopted a simple approach of representing the various categories by essentially a type-1 fuzzy set. For example, the question about a line on an image "Where is the line located?" was translated into a fuzzy set *Location* where members of the set could have values 0(Lower), 0.25(Junction Lower Middle), 0.5(Middle), 0.75(Junction Middle Upper) and 1(Upper). In other words, an image which was described as 'Junction Middle Upper' had 0.75 as the degree of membership of the set

Location. Clearly these numbers have been chosen arbitrarily to reflect the ordering of the description of where the line is located. Given that this transformation is very crude, the results of this early work were encouraging indicating that, using this method, some relationship could be shown between the neural network output and the consultant's interpretation of the image. At the suggestion of the author, a more sophisticated approach using type-2 fuzzy sets was adopted for representing the consultant's perceptions of the images so they could be processed by the neural networks.

3.7.2 Representation of the Fuzzy Categories using Type-2 Sets

The results of the expert filling out the questionnaire were represented using type-2 fuzzy sets for the location and length of line which are particularly important features of the image. For the *Location* of line there is a type-2 fuzzy set *Location* where any image can be a member of the set to degree low, junction low middle, middle, junction middle upper, upper where these themselves are fuzzy sets. Triangular fuzzy sets were adopted to represent the various categories. This transformation of the data into type-2 fuzzy sets was also carried out for *Ratio* (ratio of length to width) which is now a type-2 fuzzy set where the grades are themselves fuzzy sets - Same, Longer, Much Longer. The sets used are shown in Figure 3.6.

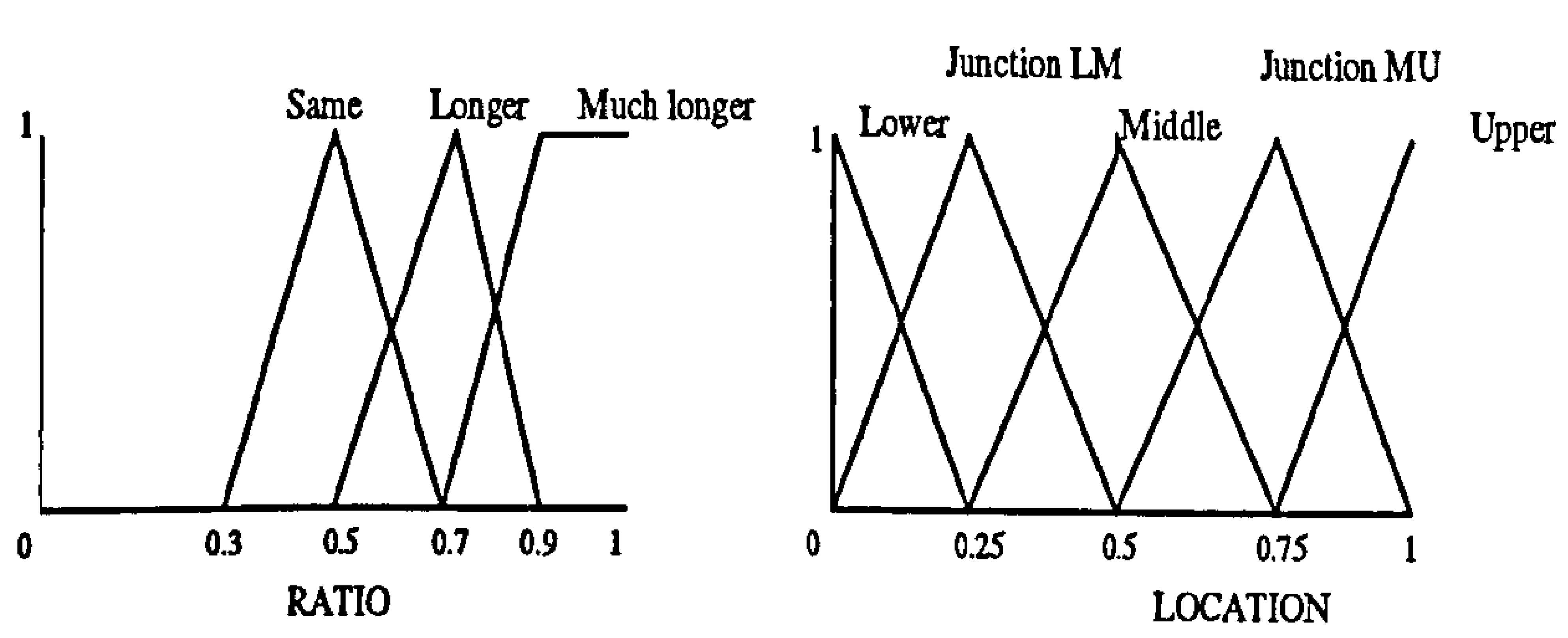

Fig. 3.6. The type-2 fuzzy sets *'Location'* and *'Ratio'*

The published work (the most detailed account being Innocent, John & Barnes(2000)) describes the neural network paradigms (FuzzyART(Carpenter, Grossberg, Marzukan, Reynolds & Rosen 1995) and FuzzyMinMax(Simpson 1993)) and the results of applying them to both a type-1 and a type-2 interpretation of the data. The nature of the data and the neural network paradigms means that presenting results in a clear well defined way is difficult. A variety of factors affect the results:

- the order of presentation of vectors for both algorithms affects the clustering;
- different control parameters produce different clusters;
- for the type-1 fuzzy sets, arbitrary numbers have been chosen to reflect the category ordering;
- for the type-2 fuzzy sets the fuzzy membership grades chosen will affect the results.

The interest here is in assisting the consultant in the diagnosis process. The nature of the classification of shin images is that the consultant is constantly learning about the processes going on, often revising the nature of the groupings and exploring potential sub clusters within clusters. The clustering algorithms and the paucity of data allow for, at best, giving the consultant some help with image classification. Although it may be possible to find an algorithm that could cluster in exactly the same way as a consultant *on a particular occasion*, the fact that experts themselves give different clustering over time indeed mirrors the algorithms in that, for a particular order of presentation and control parameter, they too provide different clustering. What is important for the shin image analysis is whether, given a small amount of data, the technique adopted can give some indication to the consultant of the type of problem with the image to a reasonable degree of accuracy. In other words, do these methods offer an insight to the consultant for those images which could not be classified originally? Further does the type-2 representation produce better results for FuzzyART or FuzzyMINMAX *given random orderings and particular control parameters*?

Two sets of experiments were carried out. Set 1 used the order of the type-1 vectors as they corresponded to the order in which the images were produced by the consultant (called the 'natural' order). The purpose of these experiments was to discover the classes of the unknown images and to compare the two neural models. Set 2 experiments concern the investigation of random input order and differences in the behaviour of the two neural models with respect to the classifications found in Set 1 when type-1 and type-2 data is used.

Single neural models

Of the original 203 images, there were 38 images the consultant could not originally classify. Using type-1 data, of the 38 unknowns, 22 were predicted with agreement from both neural models (2 as normal, 9 as healing stress fracture and 11 as MTS). These included the optimistic predictions from clusters with one class. Of the 22 predictions, 5 were not agreed with by the consultant. Of the 16 unknowns predicted without agreement, only 4 were agreed with by the consultant as being unpredictable in the existing classification scheme. The human expert was, however, able to indicate for each of these cases which classes were not possible for that image. For these unclassifiable images, it may be concluded that either the classes currently defined are not exhaustive, or the images were too uncertain in their content and could belong to any of several classes. This represents only approximately 2% of the total data base which would indicate the latter interpretation is more likely. The order of presentation of the input vectors in 10 trials was randomised and 10 separate sets of clusters produced for each of 3 values of the control parameters for both the MINMAX and FuzzyART methods. The results indicate that there is clear agreement between all the methods for all the parameters and type of data including order of presentation for areas of confusion. This shows that there can be robust inferences leading to confident conclusions about the data. Qualitatively, it appears that type-2 pre-processing and minmax clustering appears to produce least confusion in relation to consultants judgements. However, quantitative inferences are preferable and a suitable method was chosen for achieving this. The modal clustering out of 10 trials as representative of that method and parameter setting was selected. The results for type-2 data presented in random order 10 times to the ART clustering algorithm with rho=0.3 (low) show that 6 rules are involved and that there is a good overall agreement between trials on these rules.

A decision was made to ascertain if the prediction of classes of unknowns by using type-2 fuzzy sets rather than type-1 could be improved. The conclusion of this work is that the majority of single neural models usefully agrees with the expert in the classification of unknowns. The results show that there is a clear difference between type-1 and type-2 pre-processing. In particular:

- Type-1 pre-processing appears to produce better agreements than type-2, regardless of neural method and control parameter settings.
- The worst matches were for MINMAX type-1 (low and high control parameter) cases
- The best matches were for MINMAX type-2 (low control parameter), MINMAX all Types (medium control parameter) and ART all Types (medium and low control parameter).

The results indicate that type-1 produces generally more unclassifiable cases than type-2, except in the case of the ART(0.3). This is in keeping with the previous conclusion that type-1 was better than type-2 in producing agreement with the expert. This would be expected from the usual trade off between false and true positives when a decision criterion has been consistent in a neural method. Overall, it may be inferred that the accuracy of single neural models is at best 70% (MINMAX type-1, medium control parameter) and at worst, 27% (ART type-1, high control parameter). A decision was made to investigate if this accuracy could be improved by combining the predictions of the single neural models.

Combining the Models

Single neural model analysis may be considered to be equivalent to asking separate experts for a class of each image and comparing that with another independent expert. The approach taken was to combine models with a 'winner take all' approach and use majority voting to determine what may be termed a 'consensus'. This was done for all the values of the control parameters within each neural method and type of pre-processing to arrive at a consensus at that level to produce a method prediction. Then, by again using simple majority rules on the method prediction, it can be seen if particular methods and particular pre-processing are in agreement with the experts preferred classification. For example, simple majority voting rules on the predictions of each combination were applied. As with the analysis of the single neural model results, all the 38 unclassified images were processed in this way. The results were that none of the combined neural model agreements with the human expert outperforms the single neural models because the worst single neural models have been included in the combinations. However, every combination has a Kappa(Cohen 1960) value which is significant and is greater than 0.316 (significant at 5%). All the Kappa values are within a very small range possibly indicating a central tendency due to the methods of combination. Thus, a clear cut winner is not obvious and conclusions should be drawn cautiously. The best agreement is shown with ART+type-2 (60% accuracy) followed by all type-1 (60% accuracy) and MINMAX type-1 (57% accuracy). The worst combinations are for ART type-1 (54%) and MINMAX type-2 (54%). This would seem to indicate that more work needs to be done in devising suitable rules of combination. The false positive analysis shows that there is a considerable difference between the number of unclassifiable images with respect to type-1 and type-2 pre-processing within the MINMAX paradigm and overall. This is consistent with the results of the single neural model predictions from which the combinations are formed. The results do show that MINMAX produces more unclassified cases than ART and this arises from combining MINMAX with type-1 pre-processing. ART appears to be less susceptible to change in pre-processing. This may be a consequence of the larger number of small clusters generated by MINMAX compared to ART.

Discussion

The work using type-2 fuzzy sets to assist in pre-processing for unsupervised artificial neural networks was instrumental in providing the direction for the work on adaptive type-2 fuzzy systems reported in Chapters 5 & 6. For this particular application, the use of type-2 fuzzy sets to model the perceptions of the consultant showed that they have much to offer - particularly for perception representation. The ability to attach words, directly, to the features of the images proved to be particularly helpful for the medical consultant since he found that he was able to use the words that were most intuitive when describing the images. The type-2 representation of these perceptions also removed the need for arbitrary translation of words into numbers in [0,1]. The work conducted did not involve experimentation in the type of function to best represent the type-2 membership grades. They were chosen intuitively but arbitrarily. Therefore this application reinforces the importance of finding some automatic method determining the membership grades of the type-2 fuzzy sets. The thrust of the research (Chapter 5) then is how to learn these type-1 fuzzy sets (that are the type-2 fuzzy membership grades) directly from data.

3.8 Conclusions

This Chapter has considered various definitions of type-2 fuzzy sets, considered their role in modelling a second level of fuzziness, outlined their theoretical properties, type-2 knowledge representation and the use of type-2 fuzzy sets to capture the uncertainty when inferencing with type-1 fuzzy sets. It is clear that type-2 fuzzy sets offer certain advantages over type-1 fuzzy sets:

1. In perception representation an expert is likely to be more at home using linguistic grades or by representing their belief using words or intervals. This representation can be accommodated explicitly by the use of type-2 fuzzy sets where the membership grades are type-1 fuzzy sets or by the use of i-v fuzzy sets which are special instances of type-2 fuzzy sets. This knowledge representation is important since applications using type-1 fuzzy sets have usually required heuristic development of membership functions by discussion with expert(s). Type-2 sets allow for natural perception representation where the grades are type-1 fuzzy sets.

2. The uncertainty inherent in the fuzzy inferencing process can be captured in a number of ways by using i-v fuzzy sets. This is also important since it allows for an inferencing process that retains the fuzziness.

3. It is the contention here that the type-1 approach is essentially crisp in nature. Crisp numbers are translated (fuzzified) into crisp membership grades and these grades are inferenced with if-then rules using a crisp t-norm or t-conorm. Type-2 sets capture imprecision or uncertainty in the membership grades and also maintain this in the inferencing process.

4. Concepts and perceptions that have no base domain cannot be modelled effectively using type-1 fuzzy sets. Type-2 fuzzy sets do enable perceptions of beauty and comfort to be modelled. This could be considered a second level of fuzziness. It is this power of type-2 fuzzy sets that this research has explored in detail and hence led to a novel approach for automatically learning perceptions.

5. The application of type-2 fuzzy sets in both control problems and decision making is on the increase - especially using interval valued fuzzy sets - and type-2 fuzzy sets can be expected to play a more prominent role in applications generally.

6. Type-2 fuzzy sets have been used, by the author, to model the perceptions of a medical consultant. This work indicates that they have much to offer - particularly for perception representation. The ability to attach words, directly, to the features of the images is helpful as well as removing the need for arbitrary translation of words into numbers in [0,1].

There is a convincing case for using type-2 fuzzy sets in certain applications. For real applications there is a need to employ type-2 fuzzy sets in systems that, for example, consist of type-2 if-then rules. The next Chapter explores the more advanced uses of type-2 fuzzy sets *directly* in decision making systems. In particular it examines using i-v sets in fuzzy systems, type-2 fuzzy relations and type-2 fuzzy logic systems that employ type-2 if-then rules.

Chapter 4

Type-2 Fuzzy Relations and Type-2 Fuzzy Inferencing

Before proceeding with this Chapter some findings and points made so far are reiterated.

- Modelling of vagueness and imprecision is a problem that has been considered by a number of philosophers as a particularly difficult problem in the confines of traditional Boolean logic.
- This thesis is concerned with the modelling of perceptions (which are imprecise and not, usually, directly measurable) by the use of type-2 fuzzy sets. Recently Zadeh(1999) has considered the role of type-1 fuzzy logic in modelling perceptions.
- Type-1 fuzzy logic has a history of successful applications, particularly in the control field. However, type-1 fuzzy systems are crisp in nature. This paradox leads to the consideration of type-2 fuzzy sets which employ fuzzy membership grades rather than numbers in [0,1].
- In this thesis, various aspects of type-2 fuzzy sets have been explored. The work to this point in the thesis indicates that type-2 fuzzy sets can model a higher level of uncertainty than a type-1 paradigm. In particular, the type-2 paradigm seems well suited to the modelling of perceptions.
- The number of applications of type-2 fuzzy sets is on the increase. The use of type-2 fuzzy sets to represent expert knowledge that can be expressed in the form of perceptions has been applied, by the author, in a medical application.

This Chapter explores how type-2 fuzzy sets can be used directly in decision making through type-2 fuzzy relations and type-2 fuzzy logic systems that employ type-2 fuzzy if-then rules and type-2 fuzzy inferencing methods. The Chapter is organised as follows: Section 4.1 presents the inferencing system using i-v type-2 fuzzy sets proposed by Gorzalczany(Gorzalczany 1987); in Section 4.2 Type-2 fuzzy relations are described in detail; Type-2 fuzzy systems that employ type-2 if-then rules and type-2 inferencing are explained in Section 4.3 and, finally, Section 4.4 provides, as part of a general discussion, the central philosophical approach adopted in this thesis for inferencing with type-2 fuzzy sets and how it compares with other possible approaches.

4.1 Inferencing with Interval-Valued Type-2 Sets - Gorzalczany's Approach

Gorzalczany proposes a method of fuzzy inferencing with i-v sets(Gorzalczany 1987, Gorzalczany 1989).* It is interesting in that it appears to be the first attempt to inference with interval-valued type-2 fuzzy sets although the later work of Türkşen (already discussed in Section 3.5.2), almost as a by product, considers inferencing with i-v sets. The following presents the main definitions and proposed inferencing algorithm.

Union of i-v sets

Suppose we have two i-v sets $\tilde{A}$ and $\tilde{B}$, then the membership function of $\tilde{A} \cup \tilde{B}$ is given by

$$\overline{\mu}_{\tilde{A}\cup\tilde{B}}(x) = [\mu^L_{\tilde{A}\cup\tilde{B}}(x), \mu^U_{\tilde{A}\cup\tilde{B}}(x)], \qquad x \in X$$

where

$$\mu^L_{\tilde{A}\cup\tilde{B}}(x) = max[\mu^L_{\tilde{A}}(x), \mu^L_{\tilde{B}}(x)]$$
$$\mu^U_{\tilde{A}\cup\tilde{B}}(x) = max[\mu^U_{\tilde{A}}(x), \mu^U_{\tilde{B}}(x)]$$

Intersection of i-v sets

Suppose we have two i-v sets $\tilde{A}$ and $\tilde{B}$ then the membership function of $\tilde{A} \cap \tilde{B}$ is given by

$$\overline{\mu}_{\tilde{A}\cap\tilde{B}}(x) = [\mu^L_{\tilde{A}\cap\tilde{B}}(x), \mu^U_{\tilde{A}\cap\tilde{B}}(x)], \qquad x \in X$$

where

$$\mu^L_{\tilde{A}\cap\tilde{B}}(x) = min[\mu^L_{\tilde{A}}(x), \mu^L_{\tilde{B}}(x)]$$

*Remember the definition of an i-v set is: The membership function of a type-2 i-v fuzzy set A_{i-v}, $\mu_{A_{i-v}}(x)$, is given by $\mu_{A_{i-v}}(x) = [\mu^L_{A_{i-v}}(x), \mu^U_{A_{i-v}}(x)]$, $\quad x \in X$ where for any x, $\mu^L_{A_{i-v}}(x)$ represents the lower end of the interval and $\mu^U_{A_{i-v}}(x)$ the upper end. See Section 3.1.2 for more discussion.

$$\mu^U_{\tilde{A}\cap\tilde{B}}(x) = min[\mu^U_{\tilde{A}}(x), \mu^U_{\tilde{B}}(x)]$$

Compatibility of i-v sets

The compatibility measure $\overline{\varphi}(\tilde{A}, \tilde{A}')$ of two i-v fuzzy sets $\tilde{A}$ and $\tilde{A}'$ is given by:

$$\overline{\varphi}(\tilde{A}, \tilde{A}') = [\varphi^L(\tilde{A}, \tilde{A}'), \varphi^U(\tilde{A}, \tilde{A}')]$$

$$\varphi^L(\tilde{A}, \tilde{A}') = min[\varphi_1(\tilde{A}, \tilde{A}'), \varphi_2(\tilde{A}, \tilde{A}')]$$

$$\varphi^U(\tilde{A}, \tilde{A}') = max[\varphi_1(\tilde{A}, \tilde{A}'), \varphi_2(\tilde{A}, \tilde{A}')]$$

where

$$\varphi_1(\tilde{A}, \tilde{A}') = \frac{max_{x\in X}\{min[\mu^L_{\tilde{A}}(x), \mu^L_{\tilde{A}'}(x)]\}}{max_{x\in X}[\mu^L_{\tilde{A}}(x)]}$$

$$\varphi_2(\tilde{A}, \tilde{A}') = \frac{max_{x\in X}\{min[\mu^U_{\tilde{A}}(x), \mu^U_{\tilde{A}'}(x)]\}}{max_{x\in X}[\mu^U_{\tilde{A}}(x)]}$$

This compatibility measure has three fundamental properties:

1. The compatibility of i-v set $\tilde{A}$ with itself is unity.
2. If the intersection between i-v sets $\tilde{A}$ and $\tilde{B}$ is empty then their compatibility is zero.
3. The compatibility of $\tilde{A}$ with $\tilde{B}$ is not necessarily equal to the compatibility of $\tilde{B}$ with $\tilde{A}$.

Consider two arbitrary i-v sets $\tilde{A}_1, \tilde{A}_2$. Let

$$\overline{\varphi}(\tilde{A}_1, \tilde{A}_2) = \overline{\varphi}_{1,2} = [\varphi^L_{1,2}, \varphi^U_{1,2}]$$

An i-v set $\Phi_{1,2}$ is given by:

$$\Phi_{1,2} = \{(y, \overline{\mu}_{\Phi_{1,2}}(y))\}, \qquad y \in Y, \qquad \overline{\mu}_{\Phi_{1,2}} = \overline{\varphi}_{1,2}$$

Gorzalczany then goes on to show the intersection of Φ as given above with any non empty i-v set $\tilde{B}$ on Y is itself an i-v set $\tilde{B}_\sim$ given by:

$$\overline{\mu}_{\tilde{B}_\sim} = [\mu^L_{\tilde{B}_\sim}(y), \mu^U_{\tilde{B}_\sim}(y)]$$

where

$$\mu^L_{\tilde{B}_\sim}(y) = min[\mu^L_\Phi \cdot \hat{\mu}^L_{\tilde{B}}, \mu^L_{\tilde{B}}(y)]$$

$$\mu^U_{\tilde{B}_{\sim}}(y) = min[\mu^U_{\Phi} \cdot \hat{\mu}^U_{\tilde{B}}, \mu^U_{\tilde{B}}(y)]$$

where $\hat{\mu}^L_{\tilde{B}} = max_{y \in Y}[\mu^L_{\tilde{B}}(y)]$ and $\hat{\mu}^U_{\tilde{B}} = max_{y \in Y}[\mu^U_{\tilde{B}}(y)]$

The inference mechanism uses these definitions. Gorzalczany describes the mechanism for both the one and multi dimensional cases. For the purposes of this thesis the one dimensional example is explained. Suppose we have an object with one input X and one output Y governed by the rules of the form:

IF $\tilde{A}_1$ THEN $\tilde{B}_1$
IF $\tilde{A}_2$ THEN $\tilde{B}_2$

.

.

IF $\tilde{A}_n$ THEN $\tilde{B}_n$

where the $\tilde{A}_i$ and $\tilde{B}_i$ are i-v sets in X and Y respectively. Suppose also that, given these rules, we have the set $\tilde{A}'$ (on X) then the inference method needs to find a $\tilde{B}'$ that corresponds to the $\tilde{A}'$. Remember here that the fuzzy sets are all interval valued. The method of inference can be represented mathematically as follows:

1. For each i determine the compatibility measure of $\tilde{A}'$ with$\tilde{A}_i$:-

$$\overline{\varphi}^{i'} = \overline{\varphi}(\tilde{A}_i, \tilde{A}'), \qquad i = 1, 2, \dots n$$

2. Using these compatibility measures calculate $\Phi^{i'}$ for $i = 1, 2, \dots n$

3. Generate $\tilde{B}' = \cup_{i=1}^{n}(\Phi^{i'} \cap \tilde{B}_i)$ as above.

This algorithm appears complicated but it actually follows a similar approach to more conventional fuzzy inferencing in that the input sets $\tilde{A}'$ are compared with the antecedents $\tilde{A}_i$ and a compatibility measure is made. For each rule a new i-v fuzzy set is generated using these compatibility measures then for each rule this new set is intersected with the $\tilde{B}_i$. The union of all these rules gives us $\tilde{B}'$. This inference method has many properties that would be expected for a suitable approach(Gorzalczany 1989). There is no practical work reported in the literature on its use. The key advantage to this approach is that the expert is able to describe the linguistic terms using intervals.

This approach, then, is of interest but it is specific to i-v sets which are only one form of type-2 fuzzy set. The rest of this Chapter considers the more general situation where we wish to use type-2 fuzzy sets *directly* for decision making. In the first instance type-2 fuzzy relations are considered.

4.2 Relations

Relations between members of (type-1 or type-2) fuzzy sets is a very useful idea for the development of many theoretical aspects of both type-1 and type-2 fuzzy logic. This Section defines crisp, type-1 and type-2 fuzzy relations. It also develops the idea of composition in a type-2 fuzzy context thus enabling the discussion of type-2 fuzzy if-then rules.

4.2.1 Crisp Relations and Type-1 Fuzzy Relations

A type-1 fuzzy relation is an extension of the crisp relation defined(Klir & Folger 1988, pages 65-66) as representing the presence or absence of association, interaction, or interconnectedness between the elements of two or more crisp sets. Type-1 fuzzy relations measure the degrees of strength of relation or interconnectedness between elements. To explore this in more detail, some definitions are provided.

Definition 12 *The Cartesian product of 2 crisp sets X and Y is given by*

$$X \times Y = \{(x, y) | x \in X \; and \; y \in Y\} \tag{4.1}$$

So, the Cartesian product is the crisp set of all ordered pairs where the first element is a member of X and the second element a member of Y.

Definition 13 *The generalised Cartesian product for a family of crisp sets $\{X_i | i = 1 \ldots n\}$ is given by*

$$X_1 \times X_2 \times \ldots \times X_n = \{(x_1, x_2, \ldots, x_n) | x_i \in X_i \; for \, all \, i = 1, \ldots n\} \tag{4.2}$$

Definition 14 *A relation in crisp sets $X_1, X_2, \ldots, X_n$ is a subset of the Cartesian product $X_1 \times X_2 \times \ldots \times X_n$ and is denoted by $R(X_1, X_2, \ldots, X_n)$. It can be defined by a characteristic function given by:*

$$\mu_R(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if and only if } (x_1, x_2, \ldots x_n) \in R \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

A crisp relation indicates that the elements of the n crisp sets are somehow related or associated with each other. One representation of a crisp relation is by an array consisting of ones and zeros indicating the value of the characteristic function. For example, suppose we have two crisp sets:

$$A = \{Michael\ Owen, Ryan\ Giggs, Andy\ Cole, Emile\ Heskey, Steve\ Walsh\}$$

and

$$B = \{Leicester\ City, Manchester\ United, Liverpool\}$$

The crisp relation *plays for* could be represented by the following array:

	Leicester City	*Manchester United*	*Liverpool*
Michael Owen	0	0	1
Ryan Giggs	0	1	0
Andy Cole	0	1	0
Emile Heskey	0	0	1
Steve Walsh	1	0	0

So, we can see the relation *plays for(Michael Owen, Liverpool)* has the value 1 indicating the relation is true. The relation *plays for(Michael Owen, Leicester City)* has the value 0 indicating that it is false. The (type-1) fuzzy relation is a simple extension of the crisp relation where, instead of the values of the characteristic functions being zero or one, the relation is a fuzzy set which has a membership function indicating the degree of belonging to the type-1 fuzzy relation. The simplest form of type-1 fuzzy relation is the binary type-1 fuzzy relation.

Definition 15 *A binary type-1 fuzzy relation is a type-1 fuzzy set defined on the crisp sets X, Y where the tuples (x, y) have varying degrees of membership in [0,1].*

The extension of this concept to n crisp sets is now given.

Definition 16 *A n-ary type-1 fuzzy relation is a type-1 fuzzy set defined on the Cartesian product of the crisp sets $X_1, X_2, \ldots, X_n$ where the n-tuples $(x_1, x_2, \ldots x_n)$ have varying degrees of membership in the relation in* $[0, 1]$.

Consider the example(Klir & Folger 1988, page 68) of two crisp sets

$$X = \{New\ York\ City, Paris\}$$

and

$$Y = \{Bejing, New\ York\ City, London\}$$

then the fuzzy relation 'very far', R, could be represented by the array

	NYC	*Paris*
Bejing	1	0.9
NYC	0	0.7
London	0.6	0.3

So, for example, the membership of the relation 'very far' of the pair $(NYC, Paris)$ would be 0.7. An important concept in relations (both crisp and fuzzy) is that of *composition.*

Definition 17 *The composition of two crisp relations $R(X,Y)$ and $S(Y,Z)$ is denoted by $P(X,Z) = R(X,Y) \circ S(Y,Z)$ and is defined as the subset P such that $(x,z) \in P$ if and only if there exists at least one $y \in Y$ such that $(x,y) \in R$ and (y,z) in S*

This composition can be interpreted(Klir & Folger 1988, page 75) as indicating a relational chain between elements of X and Z. The binary type-1 fuzzy relation is now provided.

Definition 18 *The composition of the binary type-1 fuzzy relations R and S on $X \times Y$ and $Y \times Z$ respectively is given by*

$$\mu_{R \circ S}(x,y) = sup_{y \in Y}(\mu_R(x,y) \star \mu_S(y,z)) \forall x \in X, \forall z \in Z \qquad (4.4)$$

where $\star$ is an appropriate t-norm (e.g. min, product) and sup is the least upper bound.

Consider a simple example(Klir & Folger 1988, pages 75-76). Suppose we have the (crisp) sets $X = (1,2,3,4)$, $Y = (a,b,c)$ and $Z = (\alpha, \beta)$ and two binary relations $P(X,Y)$ and $Q(Y,Z)$ given by the following matrices:

$$P(X,Y) = \begin{pmatrix} & a & b & c \\ 1 & .7 & .5 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & .4 & .3 \end{pmatrix}$$

$$Q(Y,Z) = \begin{pmatrix} & \alpha & \beta \\ a & .6 & .8 \\ b & 0 & 1 \\ c & 0 & .9 \end{pmatrix}$$

Now suppose $R(X, Z) = P(X, Y) \circ Q(Y, Z)$ then if we use max and min in Equation 4.4 then, for example:

$$\mu_R(1, \beta) = max[min(.7, .8), min(.5, 1)] \quad (4.5)$$
$$= max[.7, .5] \quad (4.6)$$
$$= .7 \quad (4.7)$$

Type-1 fuzzy composition using max-min, then, can be considered to represent the strength of the relational chain (the strength of (x, z) is its membership grade in $R \circ S$) where the strength of each chain is given by its weakest link (min) and the strength of the relation is the strength of the strongest chain (max).

4.2.2 Type-2 Fuzzy Relations

Few people have explored type-2 fuzzy relations in detail. Miyakoshi *et al*(1980) use the term *fuzzy-fuzzy relation* to describe type-2 relations and present an account of their properties and use them in a simple classification technique. Karnik and Mendel(1998*a*, Chapter 4) also present a discussion on type-2 fuzzy relations and type-2 composition. Dubois and Prade (1980, page 91) refer to type-2 fuzzy relations as *fuzzy-valued fuzzy relations* and only discuss them briefly, while Klir and Folger(1988, page 68) mention type-2 fuzzy relations without actually defining them. A definition of a type-2 fuzzy relation is as follows:

Definition 19 *A type-2 fuzzy relation is a type-2 fuzzy set defined on the Cartesian product of the crisp sets* $X_1, X_2, \ldots, X_n$ *where the tuples* $(x_1, x_2, \ldots x_n)$ *have varying degrees of membership which are type-1 fuzzy sets.*

In other words, the type-2 fuzzy relation indicates a degree of membership which is itself a type-1 fuzzy set - not a number in [0,1]. One can see why this might be referred to as a fuzzy-valued fuzzy relation(Dubois & Prade 1980).

Consider the example earlier of the type-1 fuzzy relation 'very far'. The type-2 fuzzy relation $very\_far_2$ (the 2 denoting a type-2 relation) could look like:

	NYC	*Paris*
Bejing	*veryhigh*	*veryhigh*
NYC	*zero*	*high*
London	*medium*	*low*

where $zero, low, medium, high$ and $veryhigh$ are type-1 fuzzy sets that represent our belief in the statement for example 'New York City is very far from London'. We are used to dealing with type-1 fuzzy membership grades in [0,1] and the use of type-2 fuzzy sets in this way could be considered to be describing the membership as an approximate number in [0,1]. For example, the fuzzy set $veryhigh$ might be given by:

$$\textit{very high} = 1/1.0 \ + \ 0.9/0.9 \ + \ 0.4/0.8 \ + \ 0.1/0.7$$

In this way the uncertainty in the relation is expressed by the type-1 membership grades. Our confidence that the value of the relation $very\_far_2$ for $Paris, Bejing$ is $veryhigh$ is likely to be higher than the confidence that the relation is 0.9, as in the case of the type-1 fuzzy relation.

Definition 20 (e.g. Karnik and Mendel 1998*a*) *Consider two type-2 fuzzy relations R and S on $U \times V$ and $V \times W$ respectively the type-2 composition is:*

$$\mu_{R \circ S}(u, w) = \sqcup_{v \in V}[\mu_R(u, v) \sqcap \mu_S(v, w)] \tag{4.8}$$

where $\sqcup$ and $\sqcap$ denote join and meet which are analogous to union and intersection for type-1 fuzzy sets. Join and meet are defined by:

$$\begin{aligned} A \sqcup B &= \mu_A(x) \sqcup \mu_B(x) && (4.9) \\ &= \sum_{i,j} (f(u_i) \wedge g(w_j))/(u_i \vee w_j) && (4.10) \end{aligned}$$

$$\begin{aligned} A \sqcap B &= \mu_A(x) \sqcap \mu_B(x) && (4.11) \\ &= \sum_{i,j} (f(u_i) \wedge g(w_j))/(u_i \wedge w_j) && (4.12) \end{aligned}$$

Definition 21 (e.g. Karnik and Mendel 1998*a*) *The composition of a type-2 fuzzy set $R \in U$ and a type-2 relation $S(U, V)$ is given by*

$$\mu_{R \circ S} = \sqcup_{u \in U}[\mu_R(u) \sqcap \mu_S(u, v)] \tag{4.13}$$

This last result is important as it allows for inferencing with type-2 if-then rules. Finally there is the definition of the Cartesian product for type-2 fuzzy sets(Karnik & Mendel 1998*a*, page 68).

Definition 22 *Consider two universes of discourse given by* $U = \{u_1, u_2, \ldots, u_n\}$ *and* $V = \{v_1, v_2, \ldots, v_m\}$ *and two type-2 fuzzy sets F on U and G on V then the Cartesian product of the membership grades is given by the use of the meet operation:*

$$\mu_{F \times G}(u_i, v_j) = \mu_F(u_i) \sqcap \mu_G(v_j)$$

In summary we are able to define type-2 fuzzy relations, type-2 composition and the type-2 Cartesian product. The definition of type-2 composition allows for type-2 if-then rules in an analogous way to type-1 if-then rules.

4.3 Inferencing with Type-2 Fuzzy If-Then Rules

Type-2 fuzzy sets can also be used in if-then rules directly where (some of) the fuzzy sets in the if-then rules are type-2. The theoretical exposition of type-2 fuzzy systems is relatively new. The research reported in this thesis has developed in such a way that, to enable development of the Adaptive Fuzzy Perception Learner (Chapter 5), the implication, composition and defuzzification aspects of a type-2 fuzzy system had to be developed since they were not all reported in the literature at that time. However, for this research, they were needed and the computational approach was developed by the author(John & Czarnecki 1998). Karnik and Mendel have published in the area of type-2 fuzzy logic systems(Karnik & Mendel 1998*a*, Karnik & Mendel 1998*b*). Literally at the same time as the type-2 adaptive system (AFPL) was being implemented, they were developing a detailed theoretical exposition of type-2 fuzzy inferencing and, indeed, presented their work at the same conferences (FUZZ-IEEE'98(Karnik & Mendel 1998*b*, John 1998*b*, John et al. 1998) , IEEE Systems Man and Cybernetics'98(John & Czarnecki 1998, Karnik 1998) and FUZZ-IEEE'99(John & Czarnecki 1999, Liang & Mendel 1999)). Indeed discussion with Professor Mendel at these conferences was very helpful in providing clarification of some of the ideas in this thesis[†] The approach taken to fuzzy inferencing by the author turns out to be, more or less, the same as that covered by Professor Mendel and his students. There are some differences in relation to knowledge representation which are discussed in Section 4.4.

Type-2 inferencing is carried out in a similar method to that of type-1 fuzzy systems using the operations for union (join) and intersection (meet) described in Section 3.4.1 as well as the notion of type-2 fuzzy relations (Section 4.2.2). There are some differences

[†]Professor Mendel and the author are to jointly chair a special session on type-2 fuzzy sets at NAFIPS2001 in Vancouver.

especially relating to defuzzification but inferencing with type-2 fuzzy if-then rules is summarised in the following Sections.

4.3.1 Type-2 Fuzzy If-Then Rules.

Type-2 fuzzy if-then rules (type-2 rules) are similar to type-1 fuzzy if-then rules (Section 2.2.1). For example the Mamdani equivalent (a 'pure' type-2 rule and the type of rule used for the rest of this thesis) is given by Equation 4.14

$$IF\ x\ is\ \tilde{A}\ AND\ y\ is\ \tilde{B}\ THEN\ z\ is\ \tilde{C} \tag{4.14}$$

where $\tilde{A}, \tilde{B}$ and $\tilde{C}$ are type-2 fuzzy sets. Obviously the rule could have a more complex antecedent connected by *AND*. Also the consequent of the rule could be type-1 or, indeed, crisp (like the Sugeno type-1 fuzzy model). The rule will be considered to be a type-2 fuzzy if-then rule if there are type-2 fuzzy sets in the antecedent. The difference then with the type-1 fuzzy knowledge base is that the sets are type-2 with membership grades that are type-1 fuzzy sets rather than numbers in [0,1].

4.3.2 Type-2 Inferencing

The antecedent of the rule may well have multiple type-2 fuzzy sets connected by 'AND'. These are combined using the meet operation (Equation 3.4. Given the rules are of the form in Equation 4.14 the inferencing from antecedent to consequent employs type-2 composition as given in Equation 4.13.

What does this mean from a computational viewpoint? Suppose the type-2 fuzzy sets $\tilde{A}$ and $\tilde{B}$ have discrete membership grades given by :[‡]

$$\mu_{\tilde{A}} = \sum_i \alpha_i / v_i$$

$$\mu_{\tilde{B}} = \sum_j \beta_j / w_j$$

Then the output fuzzy membership grade for $\tilde{C} : \tilde{A} \rightarrow \tilde{B}$ are given by (Zadeh 1975*c*, Equation 3.31, Page 339):

$$\mu_{\tilde{C}} = \sum_{i,j} (\alpha_i \wedge \beta_j)/(1 - v_i) \vee (v_i \wedge w_j) \tag{4.15}$$

The output of the type-2 if-then rules can then be combined using the join operation (Equation 3.2. The output of all of this is a type-2 fuzzy set which, for most applications, will require defuzzification.

[‡]This can be done without loss of generality and indeed for computational implementations there is a requirement that any continuous membership grades are discretized.

4.3.3 Type-2 Defuzzification

Type-2 defuzzification is covered extensively elsewhere(Karnik & Mendel 1998*a*) and, as with type-1 defuzzification there are a number of possible approaches (e.g. Centroid Type-Reduction, Centre of Sums Type-Reduction, Height-Type Reduction, Modified Height Type-Reduction and Center-of-Sets Reduction). These methods use similar approaches to type-1 defuzzification - they extend some of the ideas to the type-2 paradigm. All of the approaches take the same philosophy of reducing the output type-2 fuzzy set to a type-1 fuzzy set (type-reduction). This type-1 fuzzy set is then reduced, where needed, to a number by type-1 defuzzification (Section 2.2.3)

In relation to defuzzification, the Adaptive Fuzzy Perception Learner uses an approach similar to the 'height type-reduction' method reported elsewhere(Karnik & Mendel 1998*a*). Note that the timing is such that the method adopted in the AFPL was developed at more or less the same time as Karnik and Mendel. The AFPL defuzzification is reported in detail in Chapter 5.

4.4 Discussion

There are very few researchers in the world working with type-2 fuzzy sets. In the recent past the most prominent and prolific are Professor Mendel from the University of Southern California, Professor Türkşen from the University of Toronto and the author. Professor Türkşen's work has been reported in this thesis in detail and concentrates on the use of i-v sets for helping with the inferencing process. Professor Mendel and his Ph.D. students are working on the theoretical aspects of type-2 fuzzy systems which employ type-2 fuzzy if-then rules. Professor Mendel and the author tackle type-2 fuzzy sets from different perspectives[§]

The interest of this thesis relates to the use of type-2 fuzzy sets for modelling perceptions. and then the development of automatic methods for learning type-2 systems that perform this linguistic association. Karnik and Mendel on the other hand, have explored a number of issues but come from the viewpoint that type-2 fuzzy sets capture the uncertainty in type-1 fuzzy sets. In other words they use type-2 fuzzy sets as approximations

[§]Discussions at FUZZ-IEEE'98 and IEEE Systems Man and Cybernetics'98 conferences started an attempt to clarify the different approaches. The approaches are different and indeed Professor Mendel has suggested that he and the author work together to provide a clear exposition of the different philosophies. The author presented a paper 'An Adaptive Type-2 Fuzzy System for Learning Linguistic Membership Grades' at FUZZ-IEEE'99 where Professor Mendel also presented two type-2 fuzzy papers(Liang & Mendel 1999, Karnik & Mendel 1999) in the same stream. It was at this conference and in subsequent emails that the difference in the two approaches was clarified.

to type-1 fuzzy sets and have developed all the necessary theoretical underpinning in their report 'An Introduction to Type-2 Fuzzy Logic Systems'(Karnik & Mendel 1998*a*).

The application of type-2 fuzzy sets in both control and decision making is on the increase - especially using interval valued fuzzy sets - and we can expect type-2 fuzzy sets to play a more prominent role in applications generally. So, given that there is an expectation, and indeed evidence, that type-2 applications will become more important, what particular problem is of interest here? This thesis is concerned with the fact that type-2 fuzzy systems currently still have to be hand crafted in the same way as with type-1 fuzzy systems. In the case of type-2 fuzzy systems, there is the particular problem of determining the membership grades of the type-2 fuzzy sets.

It has been shown that type-2 fuzzy sets have much to offer *particularly for perception representation* since the use of words, as opposed to numbers is a powerful notion. However, how can we construct a type-2 fuzzy system? The central idea in this thesis is that a type-2 fuzzy system that supports human perceptual categorisation by linguistic association can be learnt from linguistic data using an adaptive network. The next Chapter describes the Adaptive Fuzzy Perception Learner that learns a complete type-2 fuzzy system from linguistic data.

Chapter 5

The Adaptive Fuzzy Perception Learner

This Chapter contains the detail of a novel learning methodology that implements a type-2 fuzzy system(John & Czarnecki 1998, John & Czarnecki 1999). In this thesis the research has reported on the problem of determining type-1 membership functions (in type-1 fuzzy systems) and some of the remedies contained in the literature (Section 2.3.2). There is a similar problem with type-2 fuzzy systems. The question of 'Where do 'they' come from?'(Dubois & Prade 1980) can just as well be asked about type-2 fuzzy sets as type-1. The lack of a mechanism for learning type-2 fuzzy membership grades is a problem that is tackled in the rest of this thesis. In this Chapter a novel approach for determining the membership grades of type-2 fuzzy sets which represent perceptions is presented (the AFPL) and the following Chapter presents some results of its application. This Chapter is structured as follows: Section 5.1 provides a rationale; Section 5.2 gives an overview of the approach to provide a broad understanding and in Section 5.3 a detailed mathematical exposition of the algorithm, based on a simple example, is presented.

5.1 The Rationale for the AFPL

This research has provided a detailed argument for the capability of a type-2 paradigm to represent perceptions. However the membership functions of type-2 fuzzy sets (the membership grades which are type-1 fuzzy sets) have still to be provided to enable a computer system to model what are, essentially, type-2 fuzzy relations. The only known piece of work on determining type-2 membership functions uses ANNs(Ishibuchi

& Moriaka 1995). The problem tackled is one of determining, for example, the linguistic grades of the type-2 fuzzy set 'middle age' by collecting data through a survey and using this data to train a neural network. It is only a report of some initial ideas and does not appear to have been extended (confirmation of this was made with the first author by email).

As has been seen in the previous Chapter, type-2 fuzzy systems can be developed that are similar in many ways to type-1 fuzzy systems (Section 3.5). Type-2 fuzzy systems consist of if-then rules that employ type-2 fuzzy sets in either the antecedent and/or the consequent. These type-2 fuzzy if-then rules are combined using various type-2 fuzzy operations and relations to produce an output. The output is a type-2 fuzzy set which, for most applications, will require defuzzification. All type-2 fuzzy systems (whether they be the conventional type-2 or where they use type-2 i-v fuzzy sets) require that the membership grades (usually type-1 fuzzy sets) and rules are hand crafted. That is, the type-2 fuzzy sets and type-2 fuzzy rules have to be determined. Note, however, that for a particular system, each possible grade in each type-2 fuzzy set will have to be developed individually either in discussion with an expert or via some statistical or artificial neural network technique.

For a type-1 fuzzy system, each (type-1) fuzzy set has essentially only one function to be determined. For example, suppose there is a three rule type-1 fuzzy system represented by the following rules:

$$IF\ x\ is\ \tilde{A}\ and\ y\ is\ \tilde{B}\ THEN\ z\ is\ \tilde{C}$$

$$IF\ x\ is\ \tilde{D}\ and\ y\ is\ \tilde{E}\ THEN\ z\ is\ \tilde{F}$$

$$IF\ x\ is\ \tilde{G}\ and\ y\ is\ \tilde{H}\ THEN\ z\ is\ \tilde{I}$$

where $\tilde{A} \ldots \tilde{I}$ are type-1 fuzzy sets. In this case there would be nine membership functions to be determined. Suppose that $\tilde{A} \ldots \tilde{I}$ are type-2 fuzzy sets each with only three membership grades (which are themselves type-1 fuzzy sets). In this case there would be twenty seven membership functions to be determined. There are therefore, potentially, significantly more grades to be acquired from an expert for a type-2 fuzzy system than for a comparable type-1 system.*

Each grade will usually be determined by a number of parameters. A type-2 learning system may have to learn significantly more parameters than a type-1 learning system. However it may well be the case that a type-2 fuzzy system will require less rules than an 'equivalent' (in the general sense) type-1 fuzzy system.

This problem of determining type-2 membership functions is one which could hold back the use of a type-2 approach. The type-2 paradigm is a complex one and one would expect the interpretation by a domain expert of the detailed *contents* of a type-2 fuzzy system to be difficult. From the expert's point of view, the strength of a type-2 fuzzy paradigm is for *perception representation* via type-2 fuzzy sets and type-2 fuzzy rules. The expert is able (and indeed would prefer) to represent their knowledge by use of words. However, the detail of a particular membership grade of a particular type-2 fuzzy set will be difficult to determine since it is known to be difficult for type-1 fuzzy sets (Section 2.3.2). It seems important, therefore, that some methodology is developed for learning the linguistic grades of type-2 fuzzy sets.

The research reported in this thesis has led to the adaptive learning system known as the *Adaptive Fuzzy Perception Learner (AFPL).*

- The system is *Adaptive* in that it is modified by data. The idea is based on that of an adaptive network(Jang et al. 1997) where, in this case, a type-2 fuzzy system is represented by a connected network of nodes that are either fixed or adaptive. That is, some nodes contain parameters that are adapted by the learning process (as will be seen, these parameters are those that define the membership grades of the type-2 fuzzy sets that lay the basis for the type-2 fuzzy system).
- It is *Fuzzy* in that it is built around a type-2 fuzzy paradigm. The trained network is essentially a type-2 fuzzy system with type-2 fuzzy sets and type-2 fuzzy if-then rules.

*As an observation this may possibly be one of the reasons why type-2 fuzzy systems have not been explored in the same depth as type-1 systems. It seems odd to the author that type-2 fuzzy sets were discussed as early as 1974(Zadeh 1974) yet only recently has their use started to appear more often. Many leading fuzzy researchers, as well as Professor Zadeh, have applied or discussed type-2 fuzzy sets (e.g. Yager(1980), Dubois & Prade(1980), Klir & Folger(1988)) yet still, by comparison with type-1 fuzzy logic, there is little research reported in the literature. It is a personal opinion but one expects that this is due to the emphasis that has been placed on the application of 'standard' (type-1) fuzzy logic in real applications, the lack (until the recent past) of computer power that is needed for the more complex type-2 approach and the already stated problem of the growth in the amount of membership grades to be determined for a real application.

- The term *Perception* is used deliberately to distinguish the approach from the work by others (e.g Prof. Mendel and his students) where type-2 fuzzy sets are used to approximate type-1 fuzzy sets. Although the AFPL could be modified to allow for this, the approach adopted is to have an adaptive system that learns perceptions that in the main are not directly measurable but which can be described (perhaps by experts) using words. The research here uses the notion of a type-2 fuzzy set in its truest sense - each member of the type-2 fuzzy set has a membership grade which is a type-1 fuzzy set.

- Finally, the term *Learner* is used since the perceptions are learnt from historical data. The AFPL allows for linguistic inputs (membership grades of type-2 fuzzy sets) to an adaptive type-2 fuzzy system. This model learns all the parameters necessary to model linguistic terms as well as the type-2 fuzzy sets embedded in the consequents of the if-then rules in the system.

As has been seen in Section 2.4, the (type-1) ANFIS approach(Jang 1993) has been successfully applied in a number of application areas(Jang et al. 1997). For type-1 ANFIS a type-1 fuzzy inference system is represented in an adaptive network where the parameters defining the fuzzy sets are 'learnt' from data. There are advantages to this approach.

- These networks have strong generalisation capabilities - given an input it has not seen before, a trained network will usually perform well in 'guessing' the output.

- The parameters are learnt by the training process thus reducing the amount of knowledge acquisition.

- They are known to be robust.

However, for certain applications, they have significant disadvantages. These systems take numerical input which is assumed to be 'certain'. These certain numbers are fuzzified, again in a certain manner, into a number in [0,1] representing the membership grade in a type-1 fuzzy set. Three problems can be identified with this approach:

- As has already been discussed (Section 1.1), there is almost always noise in data. For example, sensor readings have an error attached to them. In other, non-control applications the measurements used as inputs to a type-1 ANFIS will invariably be imprecise to some degree or other.

- The data for an ANFIS is almost certainly imprecise and there are situations where it would be more beneficial to use words as inputs to a learning system rather than the approach that is typically adopted of pre-processing these words by translating them into numbers. For example a property can perhaps best (from an estate agent's point of view) be described as small, medium or large. An artificial neural network or ANFIS approach that, for example, is attempting to learn the relationship between various features of a property and the sale price would by some transformation translate these linguistic descriptors directly into numbers (e.g. 0.25, 0.5, 0.75). However the words describing the property are inherently fuzzy. By attaching a number to the words this fuzziness is lost. The strength of the new approach proposed here is that this fuzziness is preserved and indeed is modelled as a type-1 fuzzy set which is the membership grade of a type-2 fuzzy set.

- Finally, not only does a type-1 ANFIS not capture the imprecision in the data it also can be considered crisp in the inferencing process. As seen in Chapter 2, type-1 fuzzy logic is not inherently uncertain in the inferencing with if-then rules.

The approach taken here in this new learning technique is to follow the basic ANFIS philosophy of modelling a fuzzy inferencing system and develop a learning paradigm for type-2 fuzzy inferencing(4.3). This approach has two significant advantages over a type-1 ANFIS.

1. The method allows for linguistic inputs. This means the data can be presented by an expert for submission to the AFPL in a manner that removes the need for a precise number. Also, where there is known uncertainty in 'precise' measurements, this uncertainty can be modelled using a type-1 fuzzy set (for example as a fuzzy number).

2. The type-2 inferencing retains the imprecision of the inputs. By using type-2 inferencing there can be type-2 rules containing type-2 fuzzy sets in both the antecedent and consequent. The rules are therefore able to capture the granularity needed for a system that is essentially linguistic in nature.

The next Section provides an overview of the method so that the mathematical exposition in Section 5.3 is more easily understood.

5.2 The Method

The methodology proposed uses an adaptive network to learn parameters that define a type-2 fuzzy inferencing system. These parameters define the membership grades of the type-2 fuzzy sets in

- the antecedents of type-2 fuzzy if-then rules, and
- the consequents of type-2 fuzzy if-then rules.

At the highest level the AFPL aims to learn the relationships between linguistic inputs (perceptions) and (crisp) outputs by a type-2 knowledge base of if-then rules, thus modelling human perceptual categorisation by linguistic association. As with any supervised learning mechanism, there is a requirement for data of an input-output nature. The AFPL needs available a set of data representing the inputs and outputs of the network. The purpose is to train the network to learn the type-2 if-then rules that map the inputs to the outputs. In other words, there are a number of training pairs consisting of inputs and outputs very similar to the type-1 ANFIS. However the AFPL differs in that there are, for each training pair, linguistic inputs and numeric outputs. The ability to allow linguistic outputs is an extension of the approach and is work for the future. Figure 5.1 shows a schematic of the proposed process.

The inputs to the network are linguistic. It is these words that are membership grades of type-2 fuzzy sets and are, as far as the network is concerned, type-1 fuzzy sets defined by a set of parameters to be learnt. One important point to note here that will be explored in Chapter 6 is that it may be there is some intuitive feel for the functions that represent the grades. For example it may be that the linguistic membership grade *low* would intuitively be an exponential membership function as in Figure 5.2. This provides an initial set of parameters that will, possibly, speed up the training of the network.

The training process consists of a forward pass and a backward pass. In the forward pass, for each training pair, the linguistic inputs are submitted to the AFPL which are then propagated through the type-2 inferencing process modelled by the AFPL. The AFPL has all the functionality of a type-2 fuzzy inferencing system(Karnik & Mendel 1998*a*) which was explained in detail in Section 4.3. The adaptive network, that is the AFPL, is essentially a representation of type-2 fuzzy rules where the links in the network represent the relationships between the type-2 fuzzy sets in the antecedents and consequents. That is, the perceptions are combined via if-then fuzzy rules, thus allowing linguistic association. A fundamental difference to a multi-layered perceptron (or, indeed, any artificial neural network) is that the interpretability of the learnt type-2

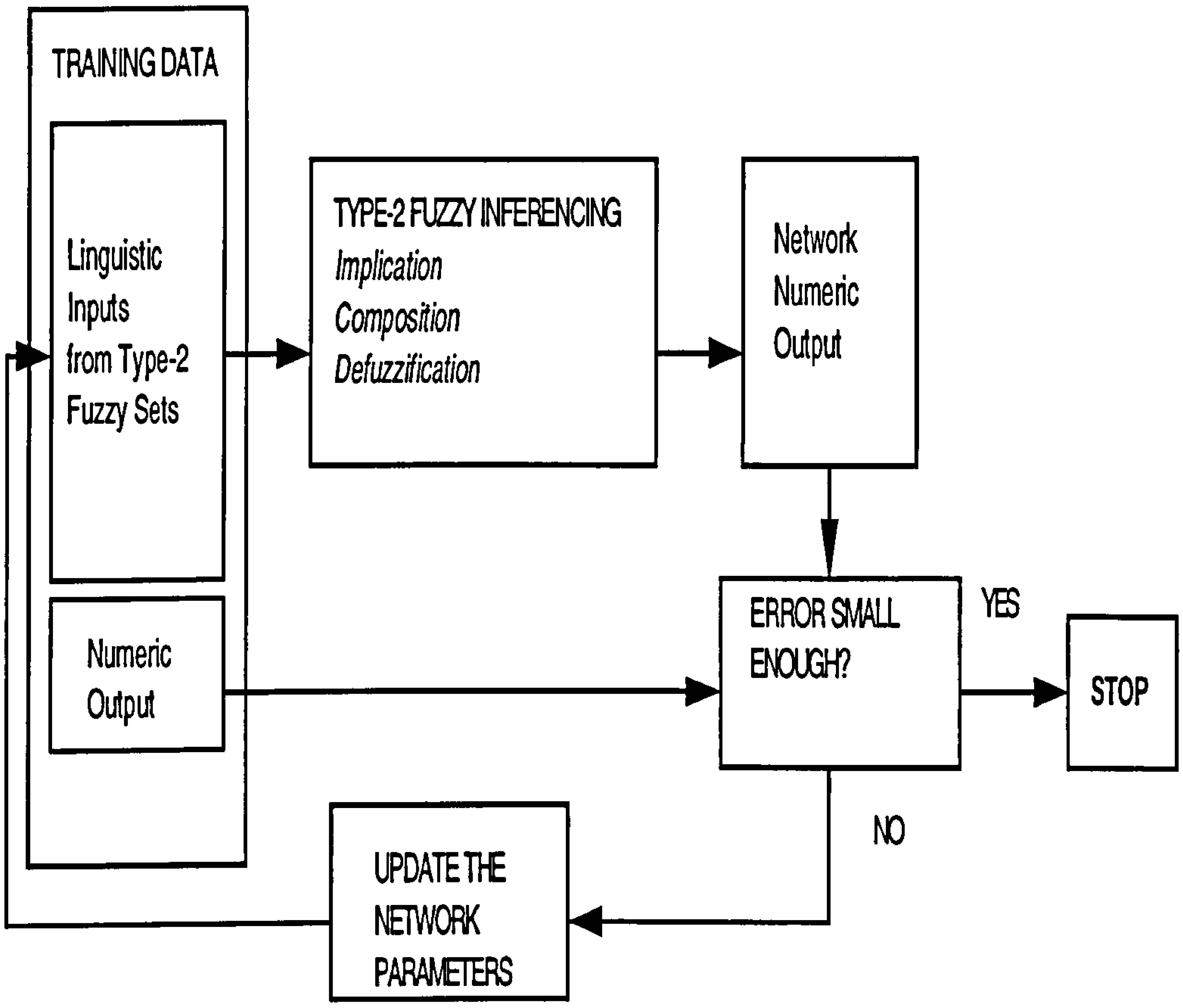

Fig. 5.1. An overview of the Adaptive Fuzzy Perception Learner approach

fuzzy system is provided and throughout the AFPL type-2 inferencing is used. The outcome of the forward pass is a numeric value (the result of type-2 defuzzification). In the backward pass, for each training pair, the output is compared with the expected numeric value to produce an error. The errors are used to update the parameters in the membership grades of the type-2 fuzzy sets, via a steepest descent algorithm. Once the error is acceptably low training stops.

So, the AFPL borrows the philosophy of the ANFIS methodology. However, the AFPL has the fundamental difference of propagating membership grades (type-1 fuzzy sets) through a type-2 fuzzy system. The only known software that implements a type-2 fuzzy system was posted on the World Wide Web by Professor Mendel after the devel-

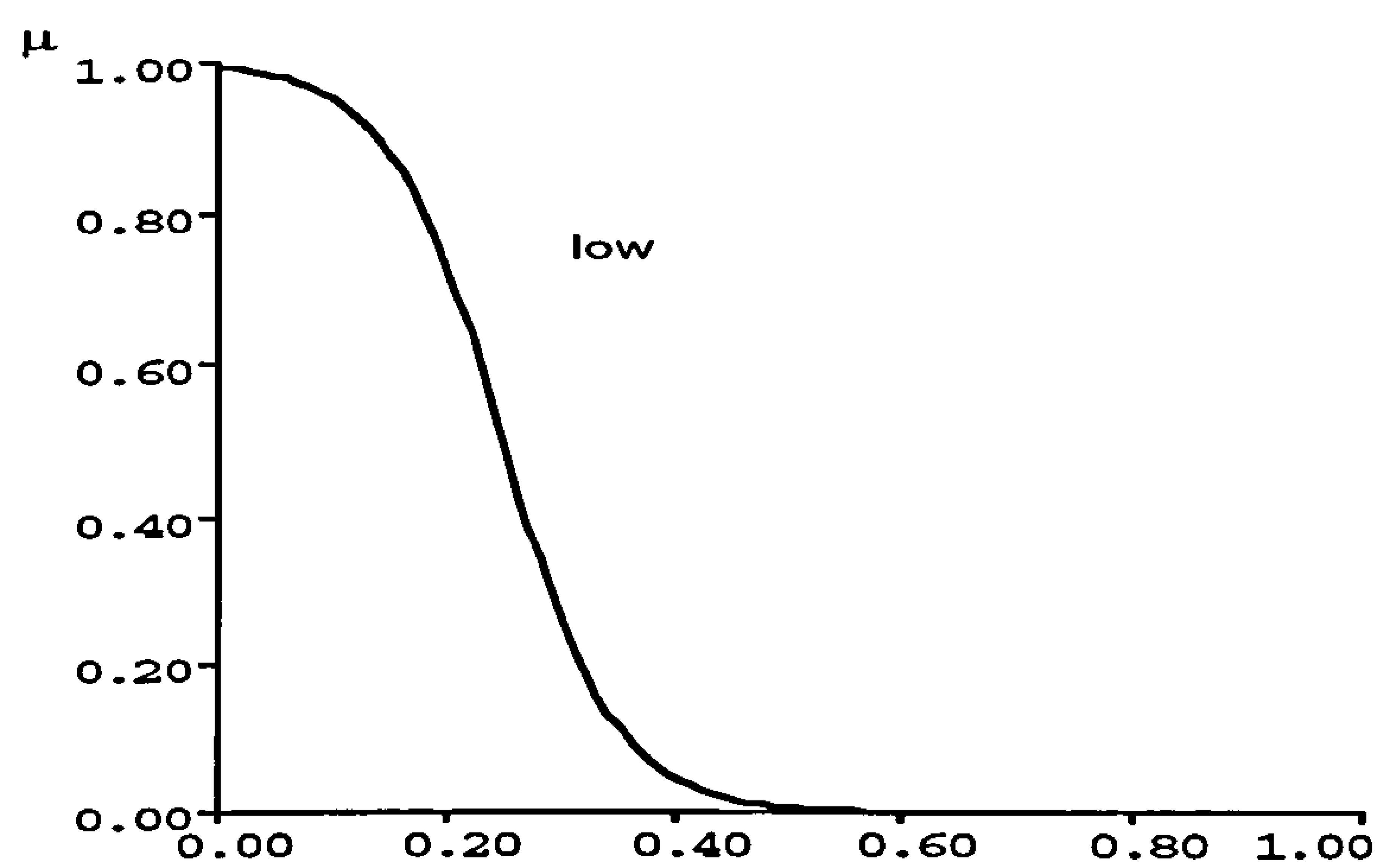

Fig. 5.2. A possible initial membership grade for *low*

opment of the AFPL (http://sipi.usc.edu/~mendel/). Since the AFPL performs all the functions of a type-2 fuzzy system, the author was among the first to write software that implements a type-2 fuzzy system.

To provide a thorough understanding of the AFPL, the next Section provides the full mathematical detail and, firstly, illustrates its workings by use of a straightforward example.

5.3 Mathematical Exposition

This Section provides the detail of the AFPL from a mathematical or algorithmic perspective. As with, for example, artificial neural networks there are a number of decisions that have to be made by the user of the AFPL which, in a real application, may affect the performance and results. This Section provides no discussion on this issue (this is included in Chapter 6) since to do so would be to confuse matters. Throughout this Section the explanation of the mathematical detail of the AFPL is illustrated by a straightforward example. The example was chosen to be simple enough to understand easily, be small so that detail of the workings could be provided and it should not detract from the main purpose of the Section which is to describe the method of the AFPL.

5.3.1 The Example

The example adopted is analogous to the Boolean 'AND'. For the Boolean case a '1' represents True and a '0' False. Table 5.1 gives the results of Boolean (B)AND. The linguistic version of this is the Linguistic AND ((L)AND). As can be seen in Table 5.2, there are two linguistic inputs 'low' and 'high' which are to be ANDed to provide numeric output.

Table 5.1. Boolean AND

x	y	(B)AND
0	0	0
1	1	1
0	1	0
1	0	0

Table 5.2. Linguistic AND

x	y	(L)AND
low	low	0.25
high	high	0.75
low	high	0.25
high	low	0.25

The numbers in the column (L)AND could be seen to represent the membership of a (type-1) fuzzy set (L)AND.[†] In other words, the numbers in the (L)AND column represent 'degree of truth'. For example, x and y both having the value high provides a value of 0.75 which could represent a high degree of truth - the membership grade in the type-1 fuzzy set (L)AND. The output that is being mapped to is a type-1 fuzzy set.[‡] It is this set of data that is used to train an AFPL[§] and is used to illustrate the mathematical exposition.

[†]This is an interesting notion and is explored in more detail in Chapter 6.

[‡]So, the approach described in this work employs numeric output for the AFPL. Clearly, in an ideal world, (type-2) linguistic output would be preferable. However the nature of the approach is that there is much to be gained from, initially, considering only type-1 output and the extension to type-2 output is further work and beyond the scope of this thesis.

[§]Note that this is extendable and indeed in Chapter 6 results are presented for problems with more than two variables. Note also that the nature of the problem is that there is no independent test or validation set. The example in the next Chapter uses an independent set of data for testing the trained AFPL as well as a validation set for monitoring the training.

5.3.2 The AFPL for (L)AND

The purpose of the AFPL in this Section is to learn the parameters that define the type-2 fuzzy sets in a type-2 fuzzy system that will map the linguistic inputs x and y to the type-1 membership value in the column (L)AND in Table 5.2. So, how is this mapping achieved? There are a number of different topologies that could have been adopted but a very simple model has been chosen to illustrate the method.[¶] Suppose there are two type-2 fuzzy sets - $\tilde{A}$ which has two linguistic membership grades $low_{\tilde{A}}$ and $high_{\tilde{A}}$ and $\tilde{B}$ ($low_{\tilde{B}}$ and $high_{\tilde{B}}$). These are combined in a type-2 fuzzy system where the relationship between $\tilde{A}$ and $\tilde{B}$ is given by if-then rules that model the (L)AND shown in Table 5.2. For the two inputs, one output model there are two fuzzy rules:

$$\begin{aligned} &IF\ x\ is\ \tilde{A}\ and\ y\ is\ \tilde{B}\ THEN\ f\ is\ \tilde{C} \\ &IF\ x\ is\ \tilde{A}\ and\ y\ is\ \tilde{B}\ THEN\ f\ is\ \tilde{D} \end{aligned} \tag{5.1}$$

where $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$ are type-2 fuzzy sets. The output of the AFPL is the result of the composition of the two rules and has target values 0.25 or 0.75, depending on the linguistic values of x and y. To develop the model, a decision has to be made about the nature of the type-2 fuzzy sets $\tilde{C}$ and $\tilde{D}$. The initial experiments deployed two membership grades for each - $low_{\tilde{C}}, high_{\tilde{C}}, low_{\tilde{D}}, high_{\tilde{D}}$ - where, for example $low_{\tilde{C}}$ is the membership grade *low* in the type-2 fuzzy set $\tilde{C}$.

Note that, for ease of understanding, the same words (labels) have been used for the membership grades of each type-2 fuzzy set. However, on completion of training, it is likely they will have different membership functions since the term *low* will have different meanings in different type-2 fuzzy sets. Figure 5.3 shows the topology of the AFPL that will be used in the discussion.

The AFPL consists of six layers:

(a) **Layer zero** feeds the linguistic terms into the AFPL. So for each row in Table 5.2 a value for x and y is passed to layer one.

(b) **Layer one** contains the type-2 fuzzy sets in the antecedents of the type-2 if-then rules and carries out a matching between the output from layer zero and the membership grades of the type-2 fuzzy sets which have connections to the x or y.

(c) **Layer two** carries out the type-2 AND on the antecedents. The existence of a connection between a node (from the previous layer) containing a type-2 fuzzy set and a node in this layer indicates that the type-2 fuzzy set concerned is to be ANDed with all other connections that come into that node.

[¶]More complex topologies are considered in the next Chapter. The purpose here is to explain the detail of the method and a simple example is the best way to achieve a coherent explanation.

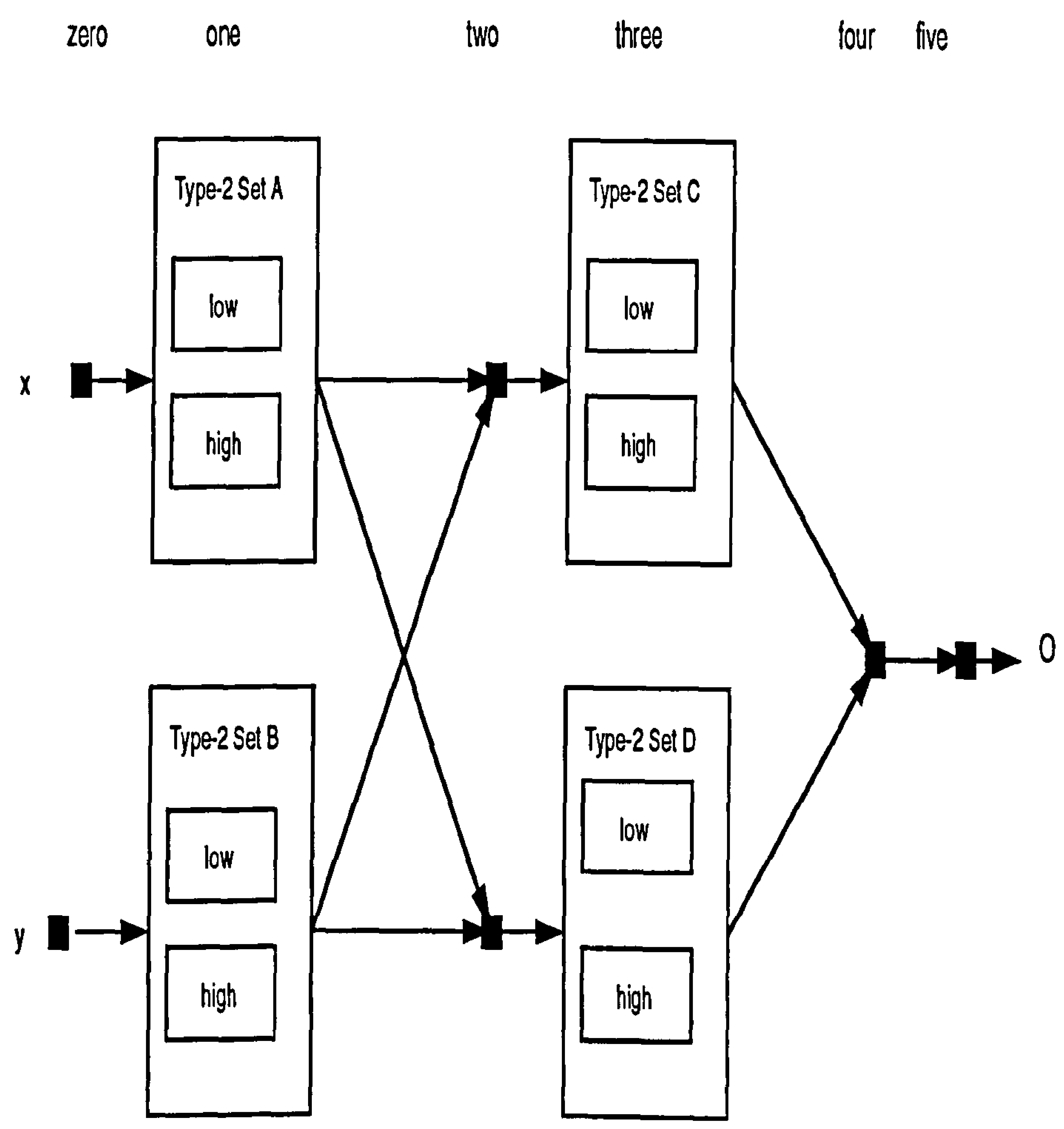

Fig. 5.3. The Adaptive Fuzzy Perception Learner

(d) Type-2 implication is performed by **layer three.** In other words, given the result of the meet of the antecedent type-2 fuzzy sets, the appropriate implication ($\Rightarrow$) is carried out.

(e) **Layer four** combines all the rules using the join operation.

(f) Finally **layer five** carries out defuzzification to enable comparison with the expected type-1 output and thus allow an 'error' to be calculated.

Another decision that has to be made is which operators to use for intersection ($\wedge$) and union ($\vee$). Throughout the chosen example the minimum will be used for intersection and maximum for union. The training of the AFPL consists of a forward pass and a backward pass.

The Forward Pass of the AFPL

This Section describes how, given two inputs, the forward pass through the AFPL operates. The purpose of the forward pass is to take the inputs and propagate them through the type-2 if-then rules. So, each layer in the AFPL performs a type-2 function in a type-2 fuzzy system (Section 4.3). Each layer of the AFPL consists of either adaptive or fixed nodes. Adaptive nodes contain parameters to be learnt and fixed nodes are non-adaptive and merely carry out some function.

Layer zero

This first layer contains the inputs to the AFPL which are the linguistic membership grades of the type-2 fuzzy sets $\tilde{A}, \tilde{B}$ (e.g. $low_{\tilde{A}}$). This layer is essentially an input one. The nodes are fixed and merely parse the inputs for submission to layer one.

Layer one

In layer one there are two nodes. The layer receives the linguistic inputs and every node in this layer is adaptive. Each node has attached to it two linguistic grades of membership of the type-2 fuzzy set which will match, from a linguistic point of view, the inputs supplied for training. There are clearly a number of possibilities for the 'shape' (function) which is the type-1 fuzzy set. For illustration purposes in order to tackle the (L)AND problem, the *low* and *high* are represented by two gaussian type-1 fuzzy sets. The grades are represented by the functions in Equations 5.2 and 5.3 below.

$$O_{i,l}(x) = \frac{1}{1 + exp(-a_{i,l}(x - c_{i,l}))} \qquad for\ a_{i,l} < 0,\ 0 \leq x \leq 1 \qquad (5.2)$$

$$O_{i,h}(x) = \frac{1}{1 + exp(-a_{i,h}(x - c_{i,h}))} \qquad for\ a_{i,h} > 0,\ 0 \leq x \leq 1 \qquad (5.3)$$

where $O_{i,l}(x), O_{i,h}(x)$ are the outputs from node $i : (i = \tilde{A}, \tilde{B})$.

There are, therefore, a total of four parameters to be learnt for each type-2 fuzzy set (each node) in layer one ($a_{i,l}$, $c_{i,l}$, $a_{i,h}$, $c_{i,h}$). For each membership grade the parameter a controls the slope at the crossover point c. Figures 5.4 and 5.5 provide two examples of different parameters.

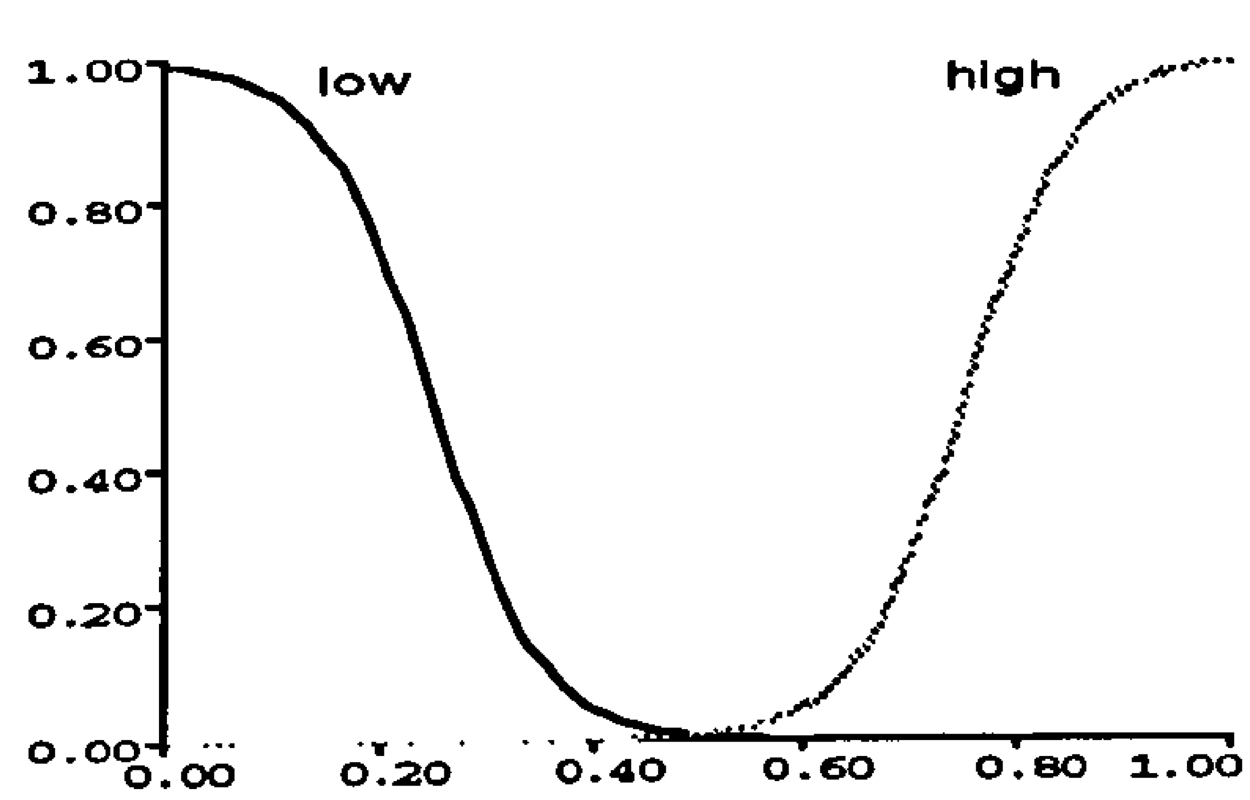

Fig. 5.4. Gaussian membership grades with $a_{i,l} = -20$, $c_{i,l} = 0.25$, $a_{i,h} = 20$, $c_{i,h} = 0.75$

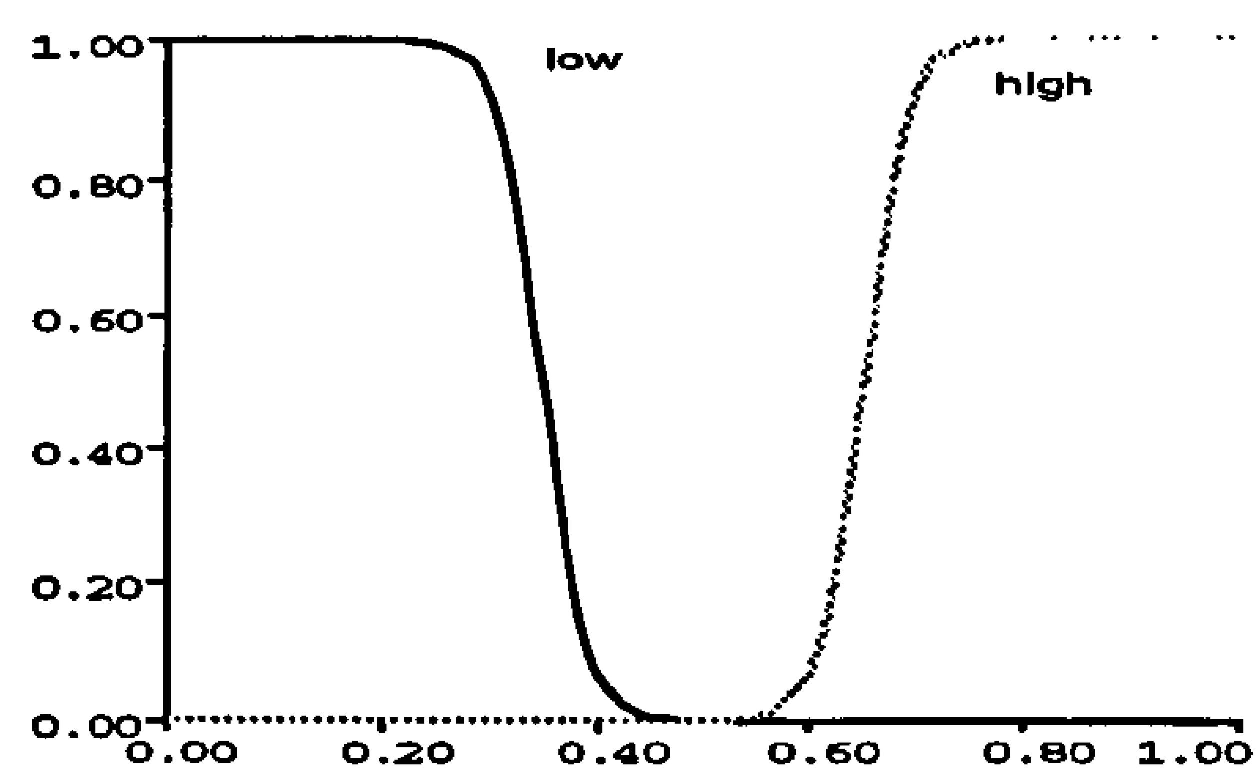

Fig. 5.5. Gaussian membership grades with $a_{i,l} = -50$, $c_{i,l} = 0.35$, $a_{i,h} = 50$, $c_{i,h} = 0.65$

As can be seen, as the absolute value of a increases so does the steepness of the slope and changing the value of c has the effect of shifting the slope along the x-axis.

Layer one, then, takes the output from layer zero and matches the linguistic term with the appropriate gaussian membership functions. To enable the programming of the method, the grades have to be represented within the computer in an appropriate manner. The nature of the various operations in a type-2 fuzzy system is that an appropriate method for storing the membership grades is by the use of arrays. The next consideration is how 'coarse' the arrays should be that represent the grades. The decision taken was to store each grade in an array of size ten[‖] For example, suppose the data pair under consideration is the third line in Table 5.2 - low,high,0.25 - then low for example with a = -20 and c = 0.25 (as in Figure 5.4) would be represented by the array (to 3 decimal places)

$$0.993; 0.953; 0.731; 0.269; 0.047; 0.007; 0.001; 0.000; 0.000; 0.000$$

[‖]This is one of a number of decisions to be made that are essentially heuristic. The size of array chosen will affect performance and speed. The smaller the array the more likely performance will suffer. A large array will lead to degradation of speed of training. An array of size ten provides a representation of each grade that is not too coarse as to lose a significant amount of information. To represent the grades using an array significantly larger than size ten would significantly increase training time.

Layer two

The nodes in layer two are fixed in that there are no parameters to be learnt. The purpose of this layer is to implement the AND in the antecedent of the rules (Equation 5.1). The inputs to each node are two linguistic grades which are the outputs of the nodes in layer 1. Let us denote the grade from $\tilde{A}$ to be f and from $\tilde{B}$ to be g. Then using Zadeh's definition(Zadeh 1975*c*) for intersection (Equation 3.4), the output of each node is given by:

$$O_{2,i} = \sum_{j,k} (f(u_j) \wedge g(w_k))/(u_j \wedge w_k) \quad i = 1, 2 \tag{5.4}$$

So, the output of each node in this layer is a type-1 fuzzy set which is a membership grade. For example, using row 3 in Table 5.2 the inputs into layer two are low_A and $high_B$, where using the initial membership grades in Figure 5.4:

$$low_A = 0.993; 0.953; 0.731; 0.269; 0.047; 0.007; 0.001; 0.000; 0.000; 0.000$$

$$high_B = 0.000; 0.000; 0.000; 0.000; 0.001; 0.007; 0.047; 0.269; 0.731; 0.953$$

then the output from the node will be

$$low_A \wedge high_B = 0.993; 0.953; 0.731; 0.269; 0.047; 0.007; 0.001; 0.000; 0.000; 0.000$$

which happens to be low_A.[*] Note then that the output of this layer is a type-1 fuzzy set (a membership grade). These grades are then passed to layer three.

Layer three

Layer three contains adaptive nodes which represent the consequent type-2 fuzzy sets (Equation 5.1) $\tilde{C}$ and $\tilde{D}$. In general for the AFPL there are a number of possibilities for the consequents of the type-2 fuzzy rules. These are:

- The outputs of the rules could merely be crisp functions as in the type-1 Sugeno model (Section 2.2.2). As pointed out in Chapter 2.4, this is the usual approach taken in the application of type-1 ANFIS. An example rule might be

  $$IF\ x\ is\ \tilde{A}\ and\ y\ is\ \tilde{B}\ THEN\ f = px + qy + r$$

  where $\tilde{A}$ and $\tilde{B}$ are type-2 fuzzy sets and p, q and r are constants.

[*] The result of a type-2 AND is by no means always one of the initial grades. The nature of Equation 5.4 is such that the resultant grade will usually be different. This will become clearer in Chapter 6. As an aside, there is much work to be done on the effects of the various t-norms and t-conorms in type-2 inferencing.

- The consequents could be type-1 fuzzy sets along the lines of the Mamdani model (2.2.2). So for example the rule might be

$$\textit{IF } x \textit{ is } \tilde{A} \textit{ and } y \textit{ is } \tilde{B} \textit{ THEN } f \textit{ is } C$$

where $\tilde{A}$ and $\tilde{B}$ are type-2 fuzzy sets and C is a type-1 fuzzy set.
- They could be Mamdani-style type-2 fuzzy sets (Section 4.3). As has been seen (Equation 5.1), in this example this is the approach taken.

The Sugeno model and the use of type-1 in Mamdani style are a 'watering down' of the type-2 granularity that is implicit in type-2 knowledge representation and type-2 inferencing. To retain the type-2 nature of the AFPL it was decided to make the consequents type-2 fuzzy sets where each set has two linguistic grades - labelled low and high for convenience - which are assumed to be gaussian. The equations for these are of the same form as for those in layer one, represented by Equations 5.5 and 5.6.

$$O_{i,low}(x) = \frac{1}{1+exp(-a_{i,l}(x-c_{i,l}))} \qquad for\ a_{i,l} < 0,\ 0 \le x \le 1 \qquad (5.5)$$

$$O_{i,high}(x) = \frac{1}{1+exp(-a_{i,h}(x-c_{i,h}))} \qquad for\ a_{i,h} > 0,\ 0 \le x \le 1 \qquad (5.6)$$

where i represents either $\tilde{C}$ or $\tilde{D}$ in the example and the parameters to be determined for each consequent fuzzy set are $a_{i,l}$, $c_{i,l}$, $a_{i,h}$ and $c_{i,h}$. This layer performs the implication of the if-then rules ($\Rightarrow$). Given the output from layer two, the AFPL infers using Zadeh's extension principle. For $O_{2,i} \Rightarrow C_i$ the membership grade $O_{3,i}$ (where i is $\tilde{C}$ or $\tilde{D}$) is given by:

$$O_{3,i} = \sum_{i,j} (\alpha_i \wedge \beta_j)/(1-v_i) \vee (v_i \wedge w_j) \qquad (5.7)$$

where $O_{2,i}$ and $O_{C_{i,z}}$ ($z = l$, or h) have been discretised to

$$O_{2,i} = \sum_i \alpha_i/v_i \qquad (5.8)$$

$$O_{\tilde{C}_{i,z}} = \sum_j \beta_j/w_j \qquad (5.9)$$

The output of this layer is thus a type-2 fuzzy set. Using row 3 in Table 5.2 for the first rule (given initial parameters in Figure 5.4), the output of the $\tilde{C}$ node is two membership grades shown in Figure 5.6. For the (L)AND example there are two rules and therefore two nodes in this layer. Each node has as its output a type-2 fuzzy set. These outputs are passed forward to layer four to be combined.

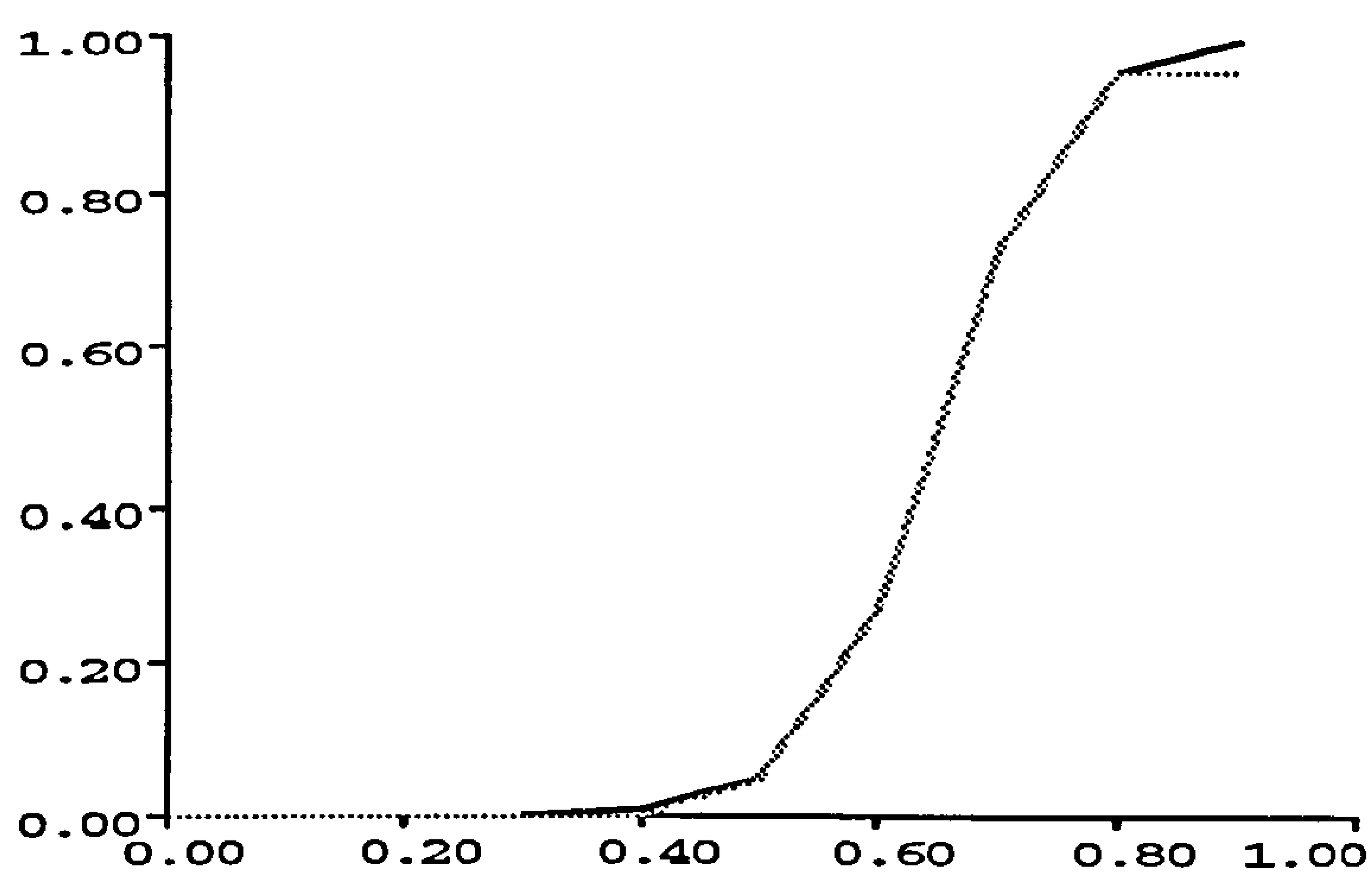

Fig. 5.6. Implication for the first rule in the (L)AND

Layer four

Layer four contains a single fixed node that composes all the outputs of the rules. This is the union (or *join* as it is known in the case of type-2 sets) of membership grades output from layer three. So the output from layer four is given by:,

$$O_4 = \sqcup_i O_{3,i} \quad i \; = \; \tilde{C}, \tilde{D} \tag{5.10}$$

The output of this layer is thus also a type-2 set containing two grades for each data pair that passes through the AFPL. A sample of the grades after 30 iterations is:

0.0000; 0.0000; 0.0000; 0.0001; 0.0009; 0.0063; 0.0449; 0.2580; 0.7198; 0.9500

0.0000; 0.0001; 0.0005; 0.0040; 0.0285; 0.8973; 0.9848; 0.9979; 0.9985; 0.9985

These are shown in Figure 5.7. So, the output of this layer is a type-2 fuzzy set. Again, we can see that the imprecision of the inputs to the AFPL is maintained throughout the inferencing process. To enable the updating of the AFPL parameters, this type-2 fuzzy set has to be defuzzified.

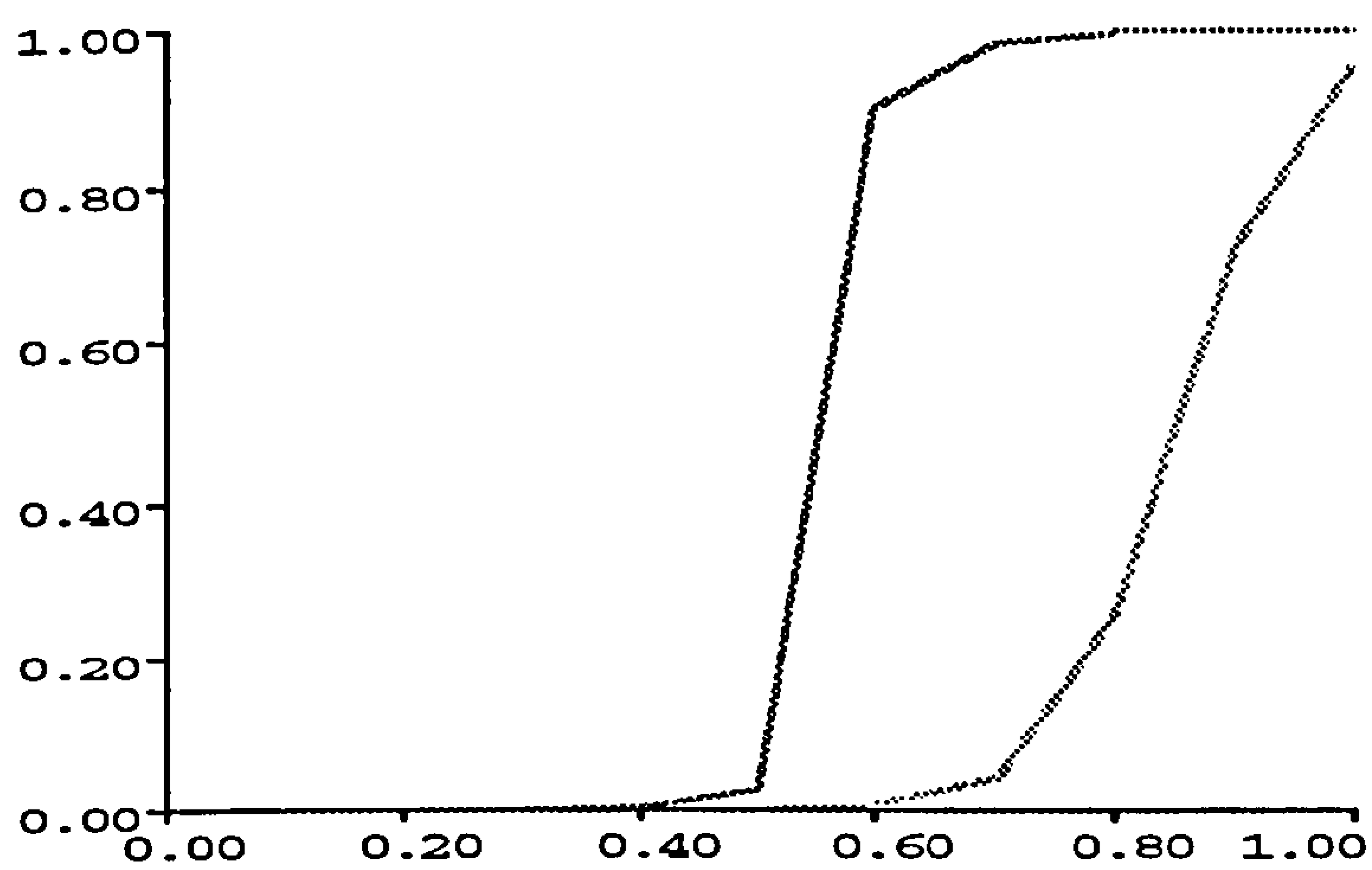

Fig. 5.7. Composition for the (L)AND

Layer five

This layer is fixed and performs the defuzzification of the output from layer four. As discussed in Section 4.3, there are a number of options available for defuzzification of type-2 fuzzy sets. The approach adopted here is to use a heuristic approach which, it transpires, is a version of ***height type-reduction*** (Karnik & Mendel 1998*a*, pages 79-80). This method reduces each membership grade to a single point which has the maximum membership. This leaves a set of maximums and these are combined using the meet (yet another choice - the min is adopted here throughout). So, suppose there are two membership grades (as above) which are the membership grades of the output type-2 fuzzy set represented by $O_{1,4}$ and $O_{2,4}$ where

$$O_{1,4} = \sum_i f(u_i)/u_i$$

$$O_{2,4} = \sum_j g(w_j)/w_j$$

then the approach takes $max_{1,4} = max_i\{f(u_i)\}$ and $max_{2,4} = max_j\{g(w_j)\}$ and the output

$$O_5 = min\{max_{1,4}, max_{2,4}\}$$

It should be noted here that type-2 fuzzy defuzzification is extensively covered in the work of Karnik and Mendel 'An Introduction to Type-2 Fuzzy Logic Systems'(Karnik & Mendel 1998*a*). A clear extension of the research in this thesis would be to explore the effect of different defuzzification methods.

The Backward Pass of the AFPL

The backward pass of the AFPL takes the output from the forward pass and compares the result from the AFPL (layer five) with the expected result. This error is fed back through the AFPL to modify the parameters in the membership grades. Unlike the backpropagation algorithm for artificial neural networks, where the weights on the connections are to be learnt and the nodes perform some function, the parameters in layers one and three are updated in light of the error in the network. The error is defined by[††]

$$E = (target - output)^2 \tag{5.11}$$

where E is the error measure, *target* is the value the AFPL *for the particular data pair under consideration* is attempting to learn (0.25 or 0.75 in the example here) and *output* is the AFPL output from layer five. The derivative of the error is given by:

$$\frac{\delta E}{output} = -2(target - output) \tag{5.12}$$

Put simply, this derivative of the error defined in Equation 5.12 is fed back to layer three, where the derivative is calculated for each grade. The derivative w.r.t. x for the gaussian is given by:

$$\frac{\delta f}{x} = \frac{ae^{(-a(x-c))}}{(1+e^{(-a(x-c))})^2} \tag{5.13}$$

where the values of a and c at that particular point in the training process are used. For each node the error measure is produced for that layer by:

$$\sum_{i=1,n} E\frac{\delta f}{x} \tag{5.14}$$

where n is the number of nodes in the layer. This is fed back through layer two to layer one and repeated for that layer. For each of the two layers, where the parameters need updating, there is now some measure of the error. The parameters are updated by using this error and the derivatives of the grades with respect to each parameter:

$$\frac{\delta f}{a} = -\frac{(-x+c)e^{(-a(x-c))}}{(1+e^{(-a(x-c))})^2} \tag{5.15}$$

[††]Again, there are a number of choices for the error. This measure is one commonly adopted in, for example, the artificial neural network community.

$$\frac{\delta f}{a} = -\frac{-ae^{(-a(x-c))}}{(1+e^{(-a(x-c))})^2} \tag{5.16}$$

Note that, for all of these equations, a value for x is required to be fed into them. For all layers, except for layer zero, a weighted average value is used, based on the membership grades. This does not make sense when considering the x value to feed into the equations for layer one since linguistic terms are coming into layer one. A heuristic approach was adopted by, for example, using a value of 0.2 for *low* and 0.8 for *high*. It was found, by experimentation, that the AFPL is not particularly sensitive to this value in that it may affect the nature of the convergence but not whether a successful solution can be found. The parameters are updated by using the step size (which is a user defined value (in [0,1])). For each parameter, the current value is updated by subtracting the step size multiplied by the derivative of the parameter normalised by the length of the parameter vector in the layer under consideration. The step size affects the performance of the AFPL dramatically as with the learning rate in an artificial neural network. Too large a step size may mean that the best solution is missed, whereas too small a step size leads to slow convergence.

5.3.3 The AFPL Network Presented Formally

The AFPL is now described more formally. There are 6 layers to the network:

(i) Layer zero. Suppose layer zero has n_0 nodes representing n_0 linguistic labels. All nodes are fixed. Each node in this layer is forward connected to at least one node in layer one but not to any other nodes in any other layer.

(ii) Layer one. This layer has n_1 adaptive nodes. Each node represents a type-2 fuzzy set $\tilde{S}_i$ with $i = 1, \ldots n_1$. Each type-2 fuzzy set S_i has j_{S_i} membership grades $\mu_{S_i,k}$ $(k = 1, \ldots j_{S_i})$.

(iii) Layer two. This layer has n_2 fixed nodes that carry out the AND in the rules. The number of nodes in this layer is equivalent to the number of rules in the type-2 fuzzy system (n_2). For each node, suppose there are k connections entering node j with membership grades $\mu_{j,i}$ $i = 1, \ldots k$ then the output for this node is

$$O_{2,j} = \sqcap_{i=1,k}\mu_{j,i}$$

The output is thus a membership grade represented by a type-1 fuzzy set. Each node is forward connected to at least one node in layer three but not to any other nodes in any other layer.

(iv) Layer three contains adaptive nodes representing the consequents of the rules. In this layer each node performs the implication. For each membership grade in the consequent type-2 fuzzy set $\tilde{C}_i$ $i = 1, \ldots n_3$ Each membership grade is

$$O_{3,i} = \sum_{i,j} (\alpha_i \wedge \beta_j)/(1 - v_i) \vee (v_i \wedge w_j)$$

where $O_{2,i}$ and the membership grade of $\tilde{C}$, $\mu_{\tilde{C}_i}$ have been discretised to

$$O_{2,i} = \sum_i \alpha_i / v_i$$

$$\mu_{\tilde{C}_i} = \sum_j \beta_j / w_j$$

Each node is forward connected to the node in layer four.

(v) Layer four contains one fixed node which combines the results of layer three by using the join

$$O_4 = \sqcup_{i=1,n_3} O_{3,i}$$

This node is forward connected to the node in layer five.

(vi) Layer five contains one fixed node which carries out defuzzification. There are a number of options available. The approach adopted in this work takes the maximum of each membership grade and then uses the intersection to find a final value.

As has been seen, the parameters in the network can be learnt by an appropriate steepest descent algorithm. To illustrate the operation of the AFPL in detail the next Section considers the results of using the AFPL in the straightforward example (L)AND.

5.3.4 Results for Linguistic AND

The AFPL was implemented in C and run on HP 10.20 Unix machines (the code was also ported to Windows and Apple Macintosh - Appendix C contains a disk with the source code for Windows Visual C++). There are a number of factors which influence the outcome of the AFPL. These are:

- the number of epochs;
- the learning rate or step size;
- the type of defuzzification and
- the values used for x in feeding back to layer zero in the learning process.

These were experimented with for the (L)AND problem. A large number of runs were conducted to see the effect of, in particular, the step size and Table 5.3 provides some of the results in particular, showing the best solution. An epoch is where all the data has been submitted to the AFPL. The number of epochs shown are both the total for the run and the epoch number at which the best (in terms of root mean square error) AFPL was found. The defuzzification method used is as already described - it takes the maximum of each membership grade in the output type-2 fuzzy set and then uses the intersection to find a final value.

Table 5.3. Linguistic AND Results

Target value	LAR1	LAR2	LAR3	LAR4
0.25	0.27	0.291	0.2902	0.2894
0.75	0.59	0.751	0.7502	0.7502
0.25	0.27	0.291	0.2902	0.2894
0.25	0.27	0.291	0.2902	0.2894
Step size	0.3	0.29	0.2908	0.29125
Total epochs	100	100	50	50
'Best' epoch	9	27	29	29
RMSE	0.0798	0.036	0.034806	0.034114

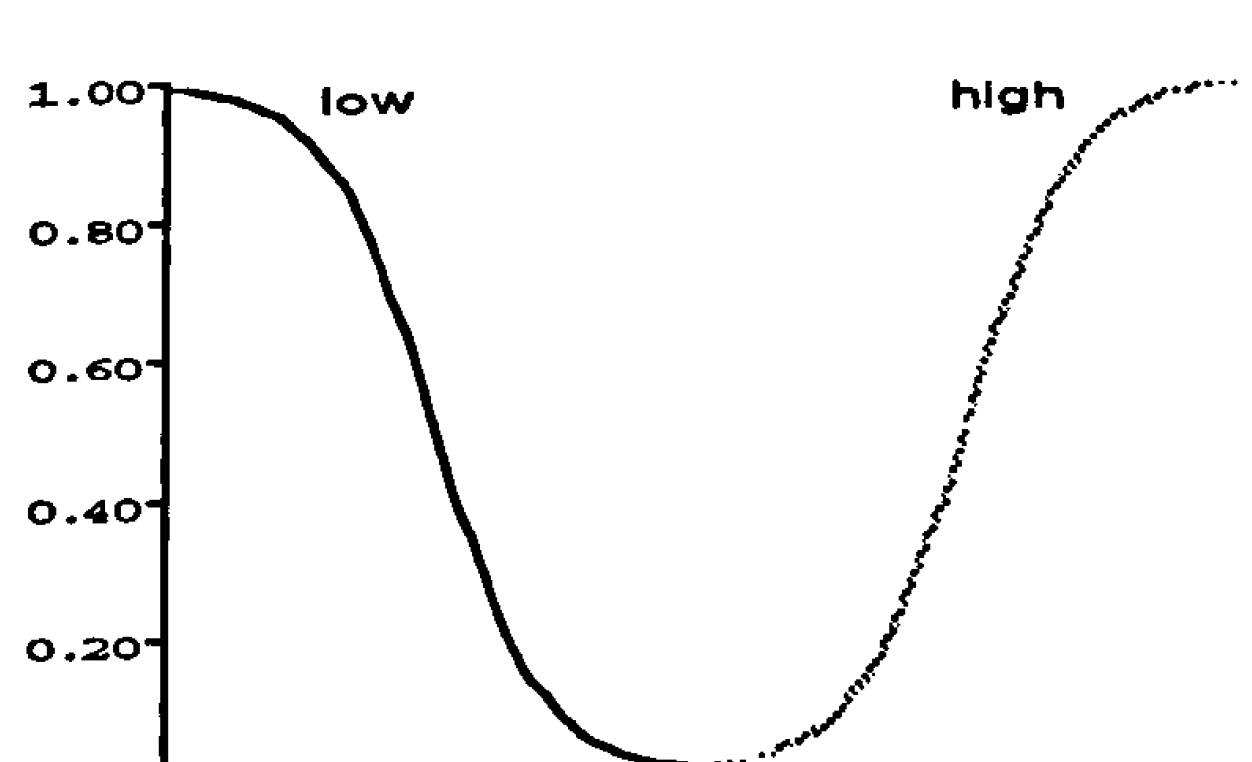

Fig. 5.8. Original grades for type-2 fuzzy sets $\tilde{A}$ and $\tilde{B}$ in (L)AND

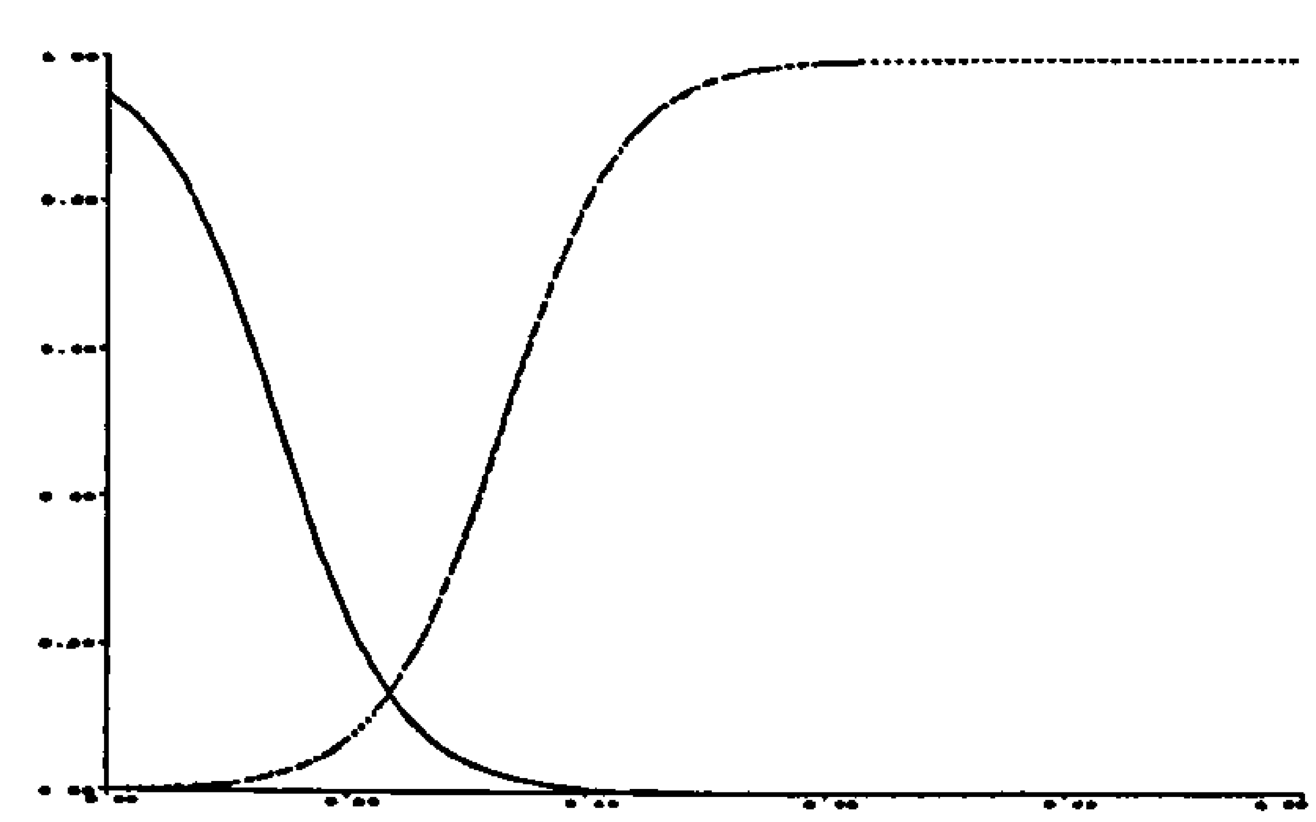

Fig. 5.9. Final grades for type-2 fuzzy sets $\tilde{A}$ and $\tilde{B}$ in (L)AND

There are some points to note about the results :

(a) The AFPL results show the same pattern in as the input. For example, where a 'low' figure of 0.25 is expected for the input, the output will produce a 'low' figure e.g. 0.29.

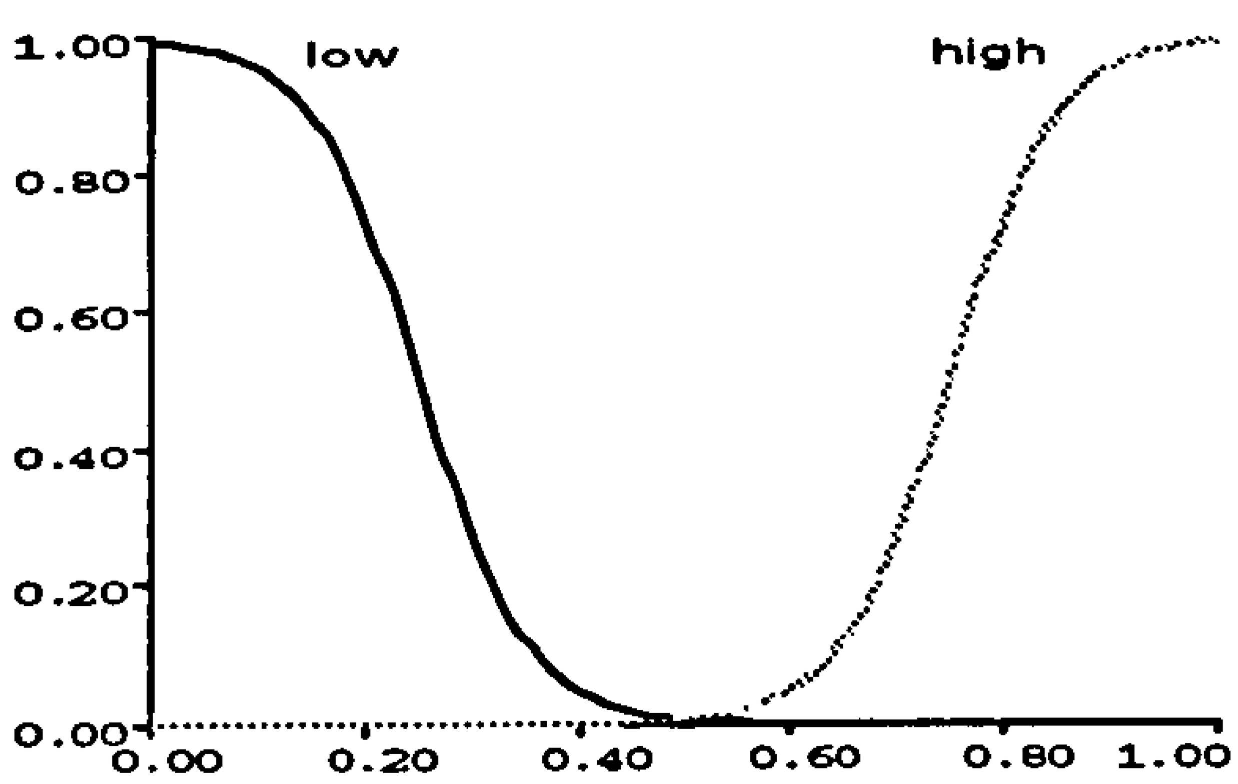

Fig. 5.10. Original grades for type-2 fuzzy sets $\tilde{C}$ and $\tilde{D}$ in (L)AND

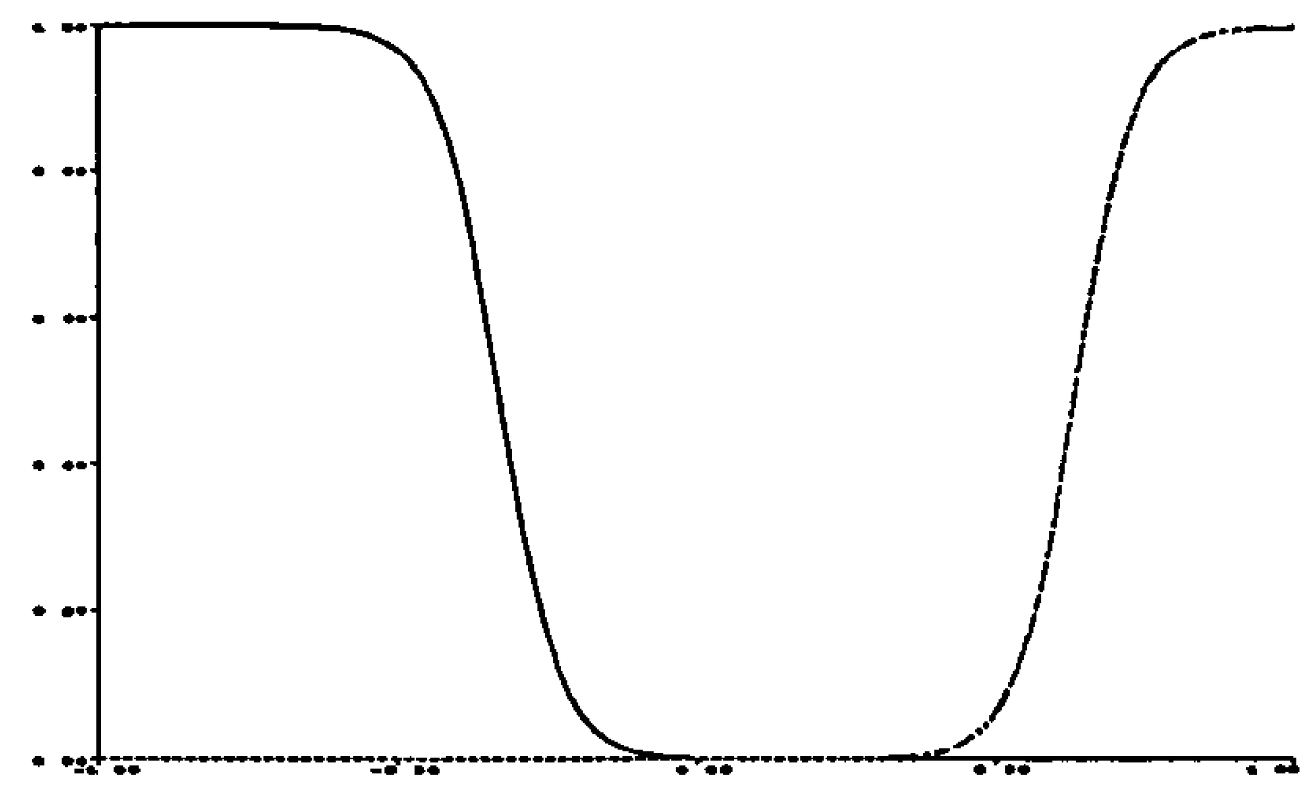

Fig. 5.11. Final grades for type-2 fuzzy sets $\tilde{C}$ and $\tilde{D}$ in (L)AND

(b) The best result in terms of least square error is found very quickly.

(c) The AFPL has to learn a total of sixteen parameters. The results are acceptable given that these sixteen parameters are estimated from only four data sets.

Figures 5.8 and 5.9 show the original membership grades for the type-2 fuzzy set $\tilde{A}$ in run LAR4 in Table 5.3. Figure 5.8 shows the original membership grades. The philosophy adopted is to take an estimate of the initial parameters that is intuitive for low and high. Chapter 6 provides some examples where the initial parameters are not intuitively selected. Figure 5.9 shows the membership grades after training for the same type-2 fuzzy set $\tilde{A}$.

A number of things can be noticed from these figures. The value of a which controls the slope hardly changes but the value of c changes significantly. Also notice that for the consequent sets $\tilde{C}$ and $\tilde{D}$ the grades take values over a different x - axis ([-1,1]). This is a very interesting feature of the results and indicates that the AFPL has, in the training process, decided to modify the parameters to accommodate the expected output to a large extent.

Although it was felt these results are encouraging, it was decided to modify the AFPL to try and improve the results. The approach taken was to add a third membership grade ('medium') to each type-2 fuzzy set in the consequents of the if-then rules (layer three). This membership grade takes the form:

$$O_{i,m} = \frac{1}{1 + \left|\frac{x - c_{i,m}}{a_{i,m}}\right|^{2b_{i,m}}} \qquad 0 \leq x \leq 1 \tag{5.17}$$

where $a_{i,m}, b_{i,m}$ and $c_{i,m}$ are three parameters to be learnt. Figure 5.12 contains an example of such a grade with the initial parameters actually used for (L)AND. Although this means the number of parameters to be learnt has increased, it was felt that the

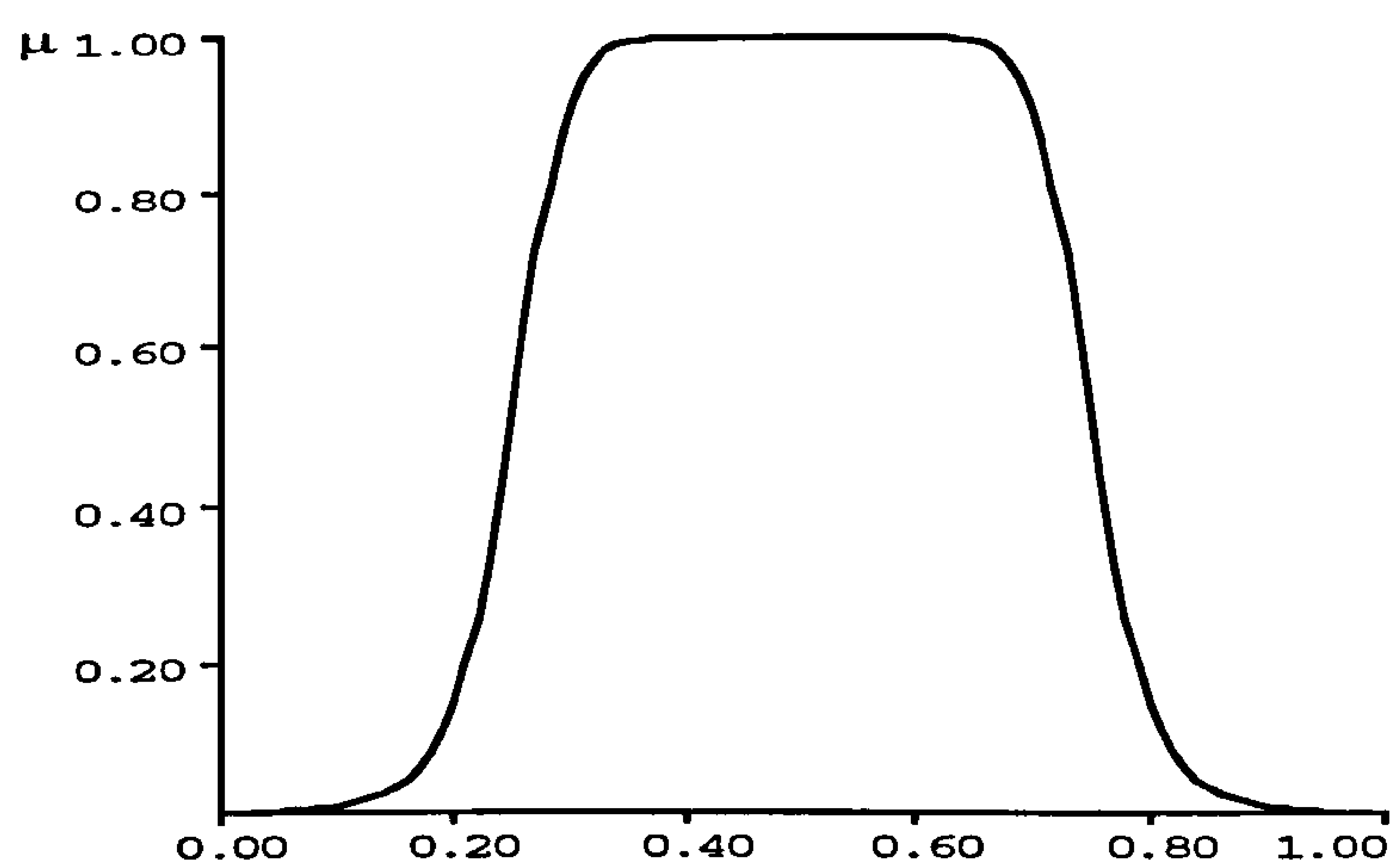

Fig. 5.12. The 'medium' membership grade for $a_{i,m} = 0.25, b_{i,m} = 5$ and $c_{i,m} = 0.5$

overlapping nature of the membership grades may lead to a smoother output surface and thus better results. The new AFPL is shown in Figure 5.13. A number of runs were carried out and a sample of some of the better results are given in Table 5.4. As can be

Table 5.4. Linguistic AND Results using medium membership grades in the consequent

Target value	LAR1M	LAR2M	LAR3M	LAR4M	LAR5M
0.25	0.235	0.250	0.239	0.251	0.250
0.75	0.777	0.776	0.754	0.787	0.750
0.25	0.263	0.267	0.256	0.257	0.250
0.25	0.235	0.250	0.239	0.251	0.251
Step size	0.269	0.2685	0.05	0.03	0.03
Total epochs	100	50	100	100	200
'Best' epoch	9	9	44	99	114
RMSE	0.0181	0.0152	0.00890	0.0187	0.000289

seen, for run LAR5M with a relatively long run (200 epochs) and low step size (0.03) the RMSE value is very low (0.000289) which is clearly excellent. The output of this LAR5M run is given in Figure 5.14. Only the epochs that show an improvement in the RMSE are shown. The initial and final parameters for this particular run are given in Table 5.5.

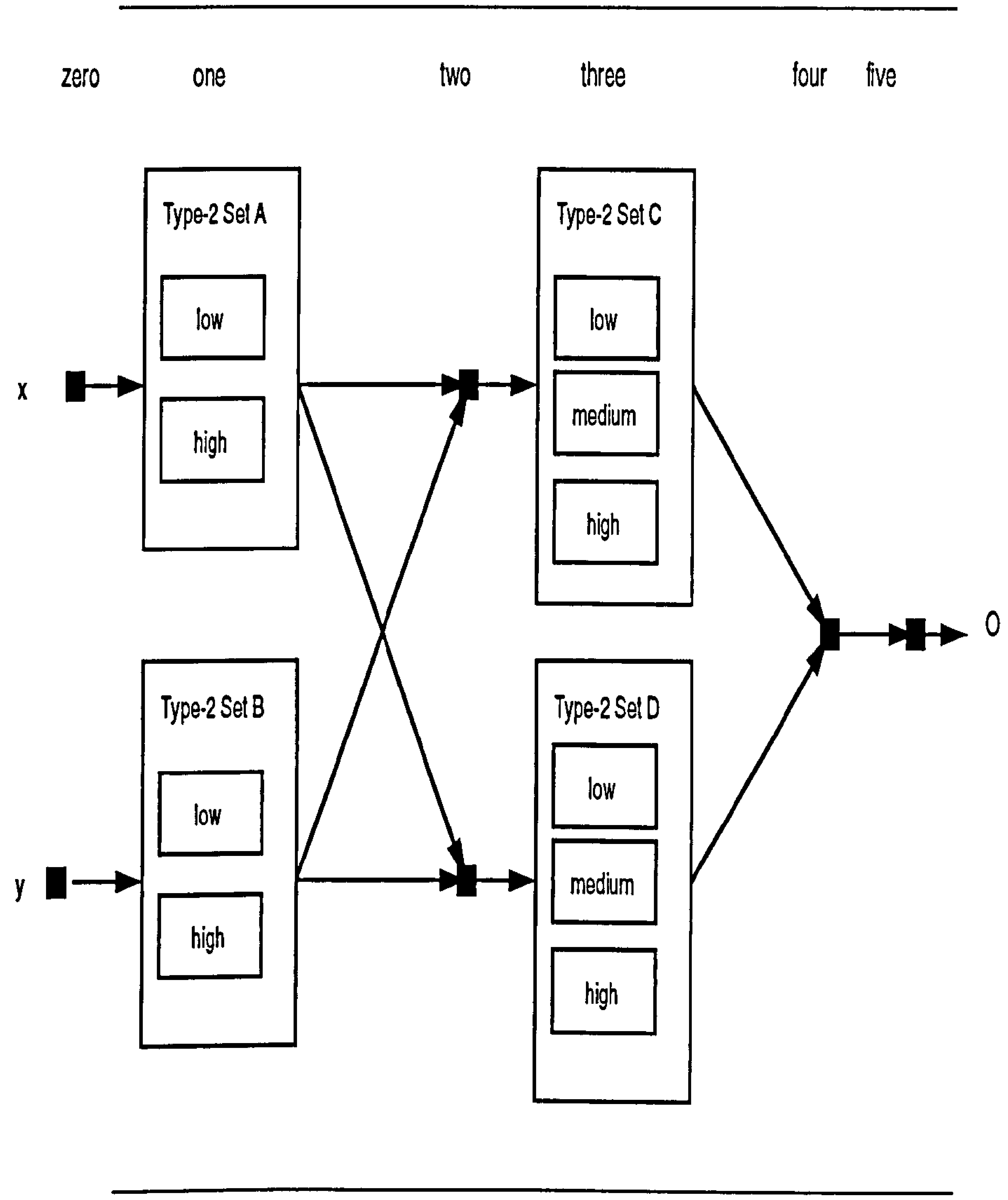

Fig. 5.13. The Adaptive Fuzzy Perception Learner with medium added into the consequent

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LINGUISTIC AND

==============

Initial stepsize = 0.03000 Error type: Root mean square error

Defuzzification: Average Maximum

epoch trn error

1 0.301201

3 0.294360

4 0.293832

5 0.293019

6 0.278680

7 0.270385

8 0.260659

iterations removed to save space ......

56 0.026088

62 0.021525

99 0.018795

101 0.023764

104 0.017049

105 0.007694

106 0.006441

107 0.004579

108 0.001889

110 0.001294

112 0.000719

114 0.000289

151 0.007247

Minimal training RMSE = 0.000289

Statistics for the training data

---

1 0.250000 0.250000

2 0.750000 0.750167

3 0.250000 0.250000

4 0.250000 0.250554

Fig. 5.14. The 'best' run for (L)AND

Table 5.5. Initial and final parameters for (L)AND where a medium membership grade is used in consequents

AFPL parameters for (L)AND		
Grade	Initial Parameters	Final Parameters
$low_{\tilde{A}}$	-20, 0.25	-20.0007256151, -0.0544010802
$high_{\tilde{A}}$	20, 0.75	20.0030202322, 0.4393052315
$low_{\tilde{B}}$	-20, 0.25	-20.0014589416, -0.0542510086
$high_{\tilde{B}}$	20, 0.75	20.0024197829, 0.4694804134
$low_{\tilde{C}}$	-20, 0.25	-20.0035878543, 0.4859625120
$medium_{\tilde{C}}$	0.25, 5, 0.5	0, 4.9921783171, 0.5055914679
$high_{\tilde{C}}$	20, 0.75	19.9995635084, 0.7535158756
$low_{\tilde{C}}$	-20, 0.25	-20.0011991390, 0.5858571255
$medium_{\tilde{C}}$	0.25, 5, 0.5	0, 4.9921783171, 0.5055914679
$high_{\tilde{C}}$	20, 0.75	19.9995635084, 0.7535158756

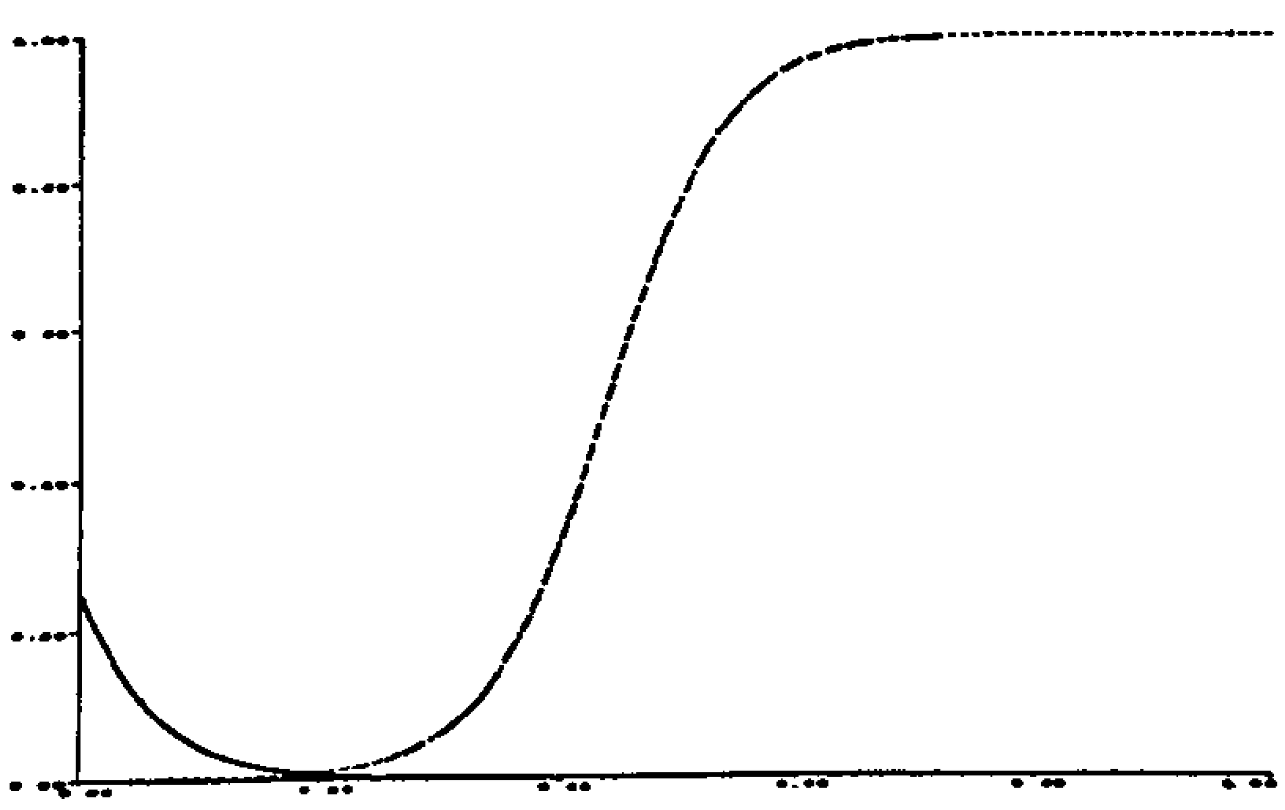

Fig. 5.15. Final grades for type-2 fuzzy set $\tilde{A}$

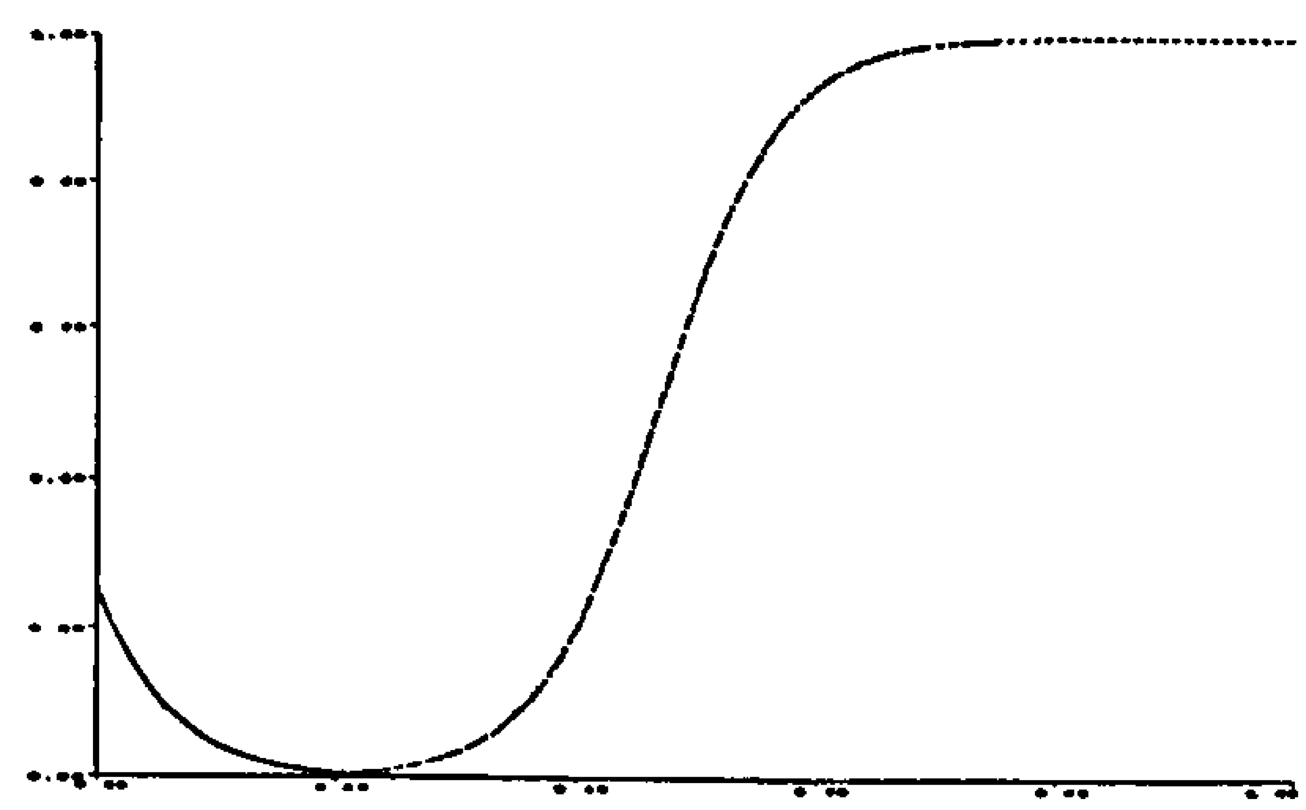

Fig. 5.16. Final grades for type-2 fuzzy set $\tilde{B}$

The final membership grades are shown in Figures 5.15, 5.16, 5.17 and 5.18. It can be seen that the a parameter in the medium grade has converged to zero. So, it appears that, for this problem, having a medium membership grade helped convergence even though it effectively disappears in the final grades.

In this Section the results for a linguistic AND ((L)AND) have been given for two, slightly different, AFPLs. What is clear from this is that the AFPL works for this relatively straightforward problem. However, these are important results. They indicate that an AFPL can be trained that learns a type-2 inferencing system for linguistic inputs to an adaptive network.

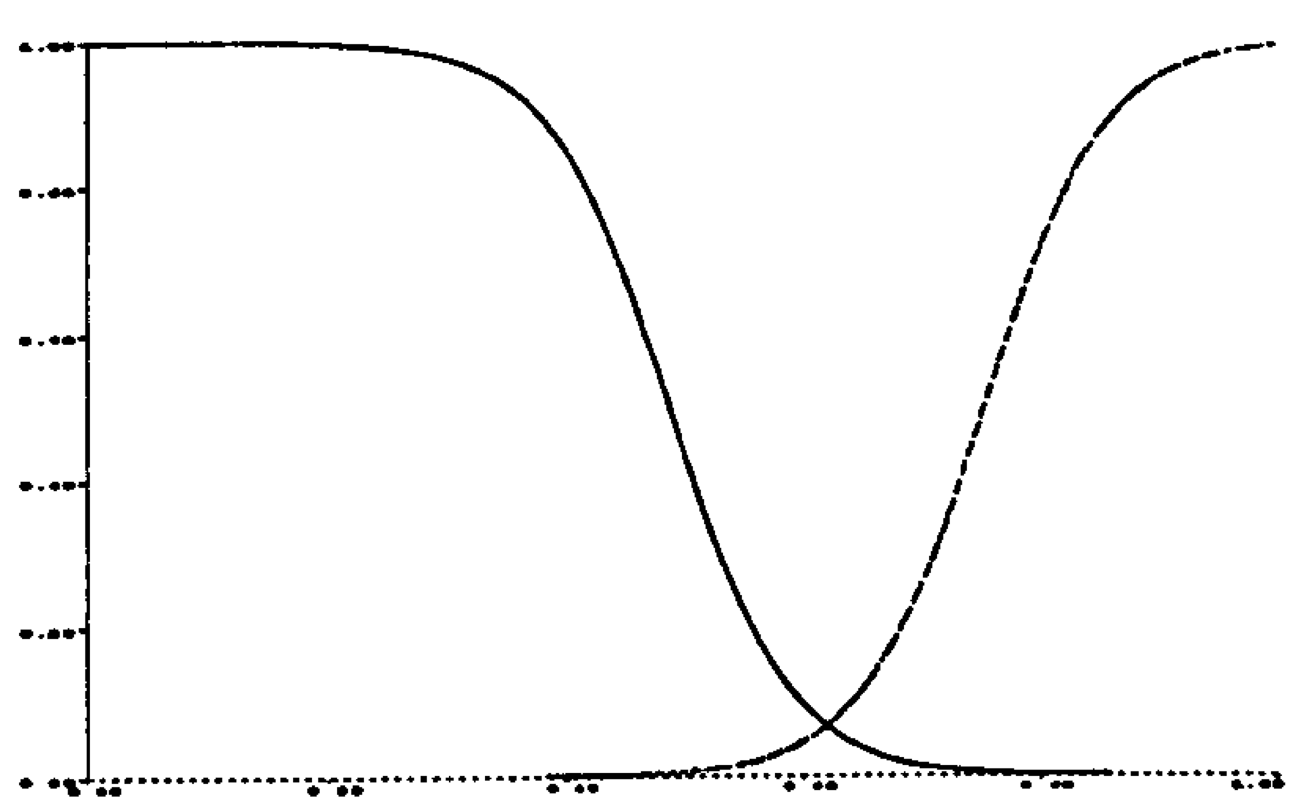

Fig. 5.17. Final grades for type-2 fuzzy set $\tilde{C}$

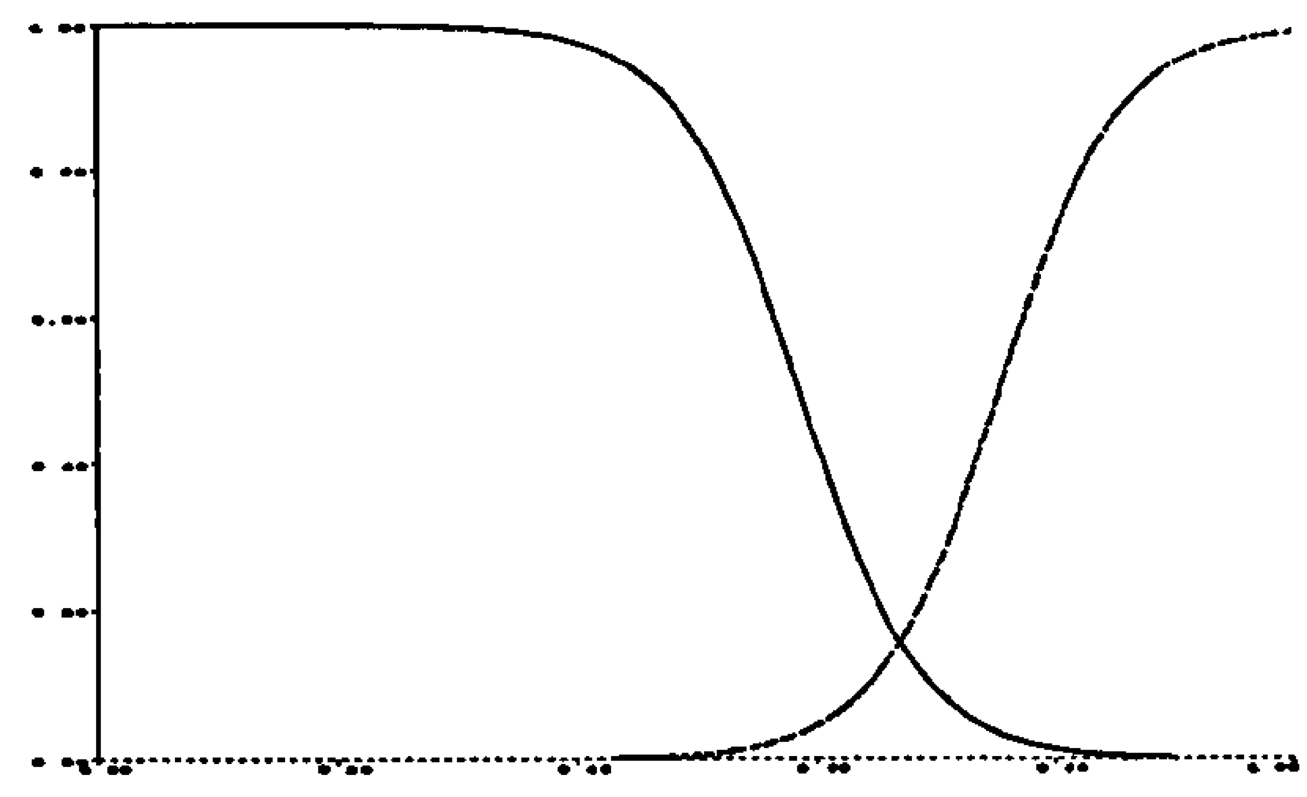

Fig. 5.18. Final grades for type-2 fuzzy set $\tilde{D}$

5.4 Adaptive Fuzzy Perception Learner - A Summary

This Chapter has described in detail a new learning algorithm - the Adaptive Fuzzy Perception Learner. The approach can be summarised in the following way:

(i) The AFPL implements, in an adaptive network, a type-2 fuzzy system where the rules have type-2 antecedents and type-2 consequents and the inferencing is type-2 in nature.

(ii) The inputs to the AFPLs are linguistic terms which are represented as type-1 fuzzy sets which are membership grades of type-2 fuzzy sets.

(iii) At this stage in the research it has been decided to keep the outputs of the AFPL as type-1 fuzzy membership grades. Future work would consider the task of allowing linguistic outputs (type-2 fuzzy sets).

(iv) The training consists of a forward pass and a backward pass. The forward pass propagates the type-2 membership grades throughout the AFPL while the backward pass uses the error in the final layer to modify the parameters in the membership grades.

(v) Each layer in the AFPL has a specific function as part of a type-2 fuzzy system.

(vi) The method was described using the (L)AND data and the AFPL trained effectively.

(vii) The algorithm modifies the membership grade parameters, in some cases considerably, from initial intuitive estimates.

(viii) The method and results on a simple example show that the technique offers an approach for learning linguistic terms.

The research hypothesis of the thesis was stated in Chapter 1 as:

"Type-2 fuzzy sets have much to offer for knowledge representation and inferencing; however there is a need for some method for learning type-2 fuzzy systems. **A type-2 fuzzy system that models human perceptual categorisation by linguistic association can be learnt from linguistic data that represent perceptions."**

The AFPL appear to satisfy the second aspect of the hypothesis. However, to really test the efficacy of the method, the next Chapter explores the application of the AFPL to a particular problem where perceptions play a central role in decision making.

Chapter 6

The Adaptive Fuzzy Perception Learner and the Association of Perceptions of a Car with its Acceptability

The previous Chapter described in detail the algorithm for a novel type-2 adaptive fuzzy system - the AFPL. As part of that explanation, a straightforward example was used to illustrate the methodology. This example showed that the AFPL could learn the so called Linguistic AND (Table 5.2). This Chapter explores the use of the AFPL in more detail, by considering its application to a large, complex, set of data where the inputs to the network are linguistic and where there are more than two inputs. The purpose of the Chapter is to investigate the use of the AFPL on 'real' data. In order to evaluate the research hypothesis, it is clearly of benefit if the AFPL can learn a type-2 fuzzy system on real data. The Chapter is structured as follows: Section 6.1 describes the data in detail; the topology of the AFPL for this application is described in Section 6.2; Section 6.3 provides the detail of the results of the application of the AFPL to the chosen set of data and finally Section 6.4 provides a review of the Chapter, exploring the implications for the AFPL in real applications.

6.1 The Data

The analysis of the efficacy of the AFPL can only be deemed worthwhile if the data chosen for the experiments has integrity.

6.1.1 The Criteria for the Data

To be able to effectively and satisfactorily evaluate the AFPL there is a requirement for data that meets certain criteria:

1. **Linguistic.** Clearly there is a requirement for the data to be essentially linguistic in nature representing perceptions. In particular, the AFPL requires that there are linguistic inputs to the network.

2. **Sourced.** The data should be from a reliable source.

3. **Representative.** The data should be valid in that it should be representative. All learning algorithms (e.g. artificial neural networks and ANFIS) require that the data used for training is representative of the problem in that there is a wide variety of examples and the data is not biased.

6.1.2 The Data in Detail and How it Meets the Criteria

The data set chosen for the experiments is the Car Evaluation Database which is available at the Carnegie Mellon University Data Repository on the World Wide Web(Blake, Keogh & Merz 1998). The problem is one of determining the acceptability of a car, based on, primarily, linguistic characteristics, which can be considered to be perceptions that describe the car.

Linguistic Criteria

As a test bed for the AFPL, this data is very appropriate since the perceptions that describe each car can be considered as type-2 fuzzy sets and the terms that describe the perceptions for a particular instance are membership grades of the type-2 sets. Therefore the data meets the Linguistic criteria.

Sourced Criteria

The hierarchical decision model, from which this dataset is derived, was originally developed for the demonstration of DEX(Bohanec & Rajkovic 1988) - software that combines multi-attribute decision making and expert systems. The data comes from a published, reliable source and thus meets the Sourced criteria for the data.

The Detail of the Car Evaluation Data

The cars are evaluated according to concepts that are mapped hierarchically into a decision tree. This decision tree (or criteria tree in the original work) is shown in Figure 6.1. The meaning of the different concepts is contained in Table 6.1. The concepts in upper case indicate (what the original work describes as) aggregate criteria in that they aggregate the other lower case descriptors as shown in Figure 6.1.

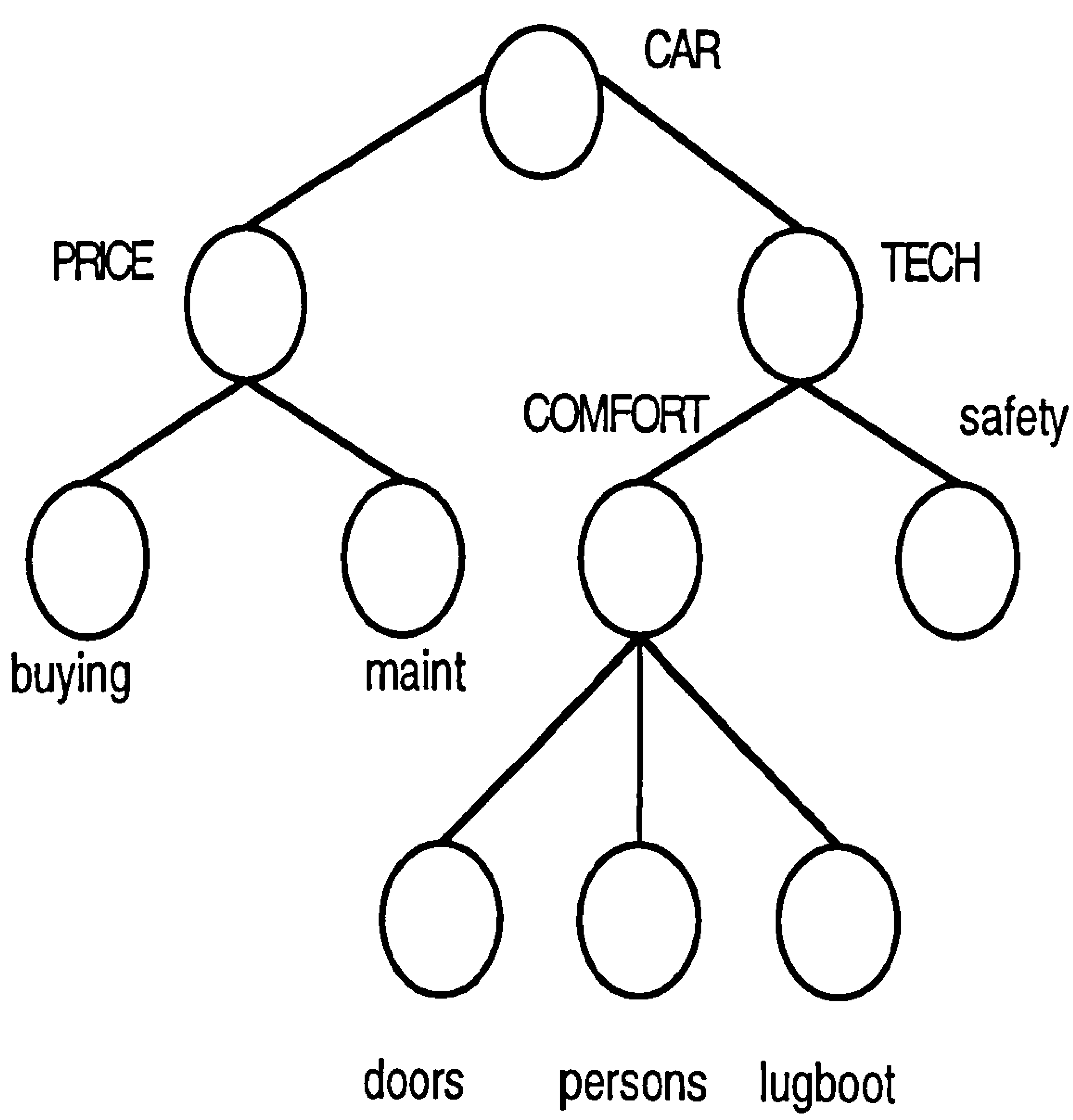

Fig. 6.1. The Car Concept Structure (Blake, Keogh & Merz 1998)

As can be seen (Figure 6.1 and Table 6.1) there are four aggregate concepts - CAR, PRICE, TECH and COMFORT. The overall target concept (CAR) describes the acceptability of the car. It is this perception that the AFPL will have as the output of the network. The CAR concept is an aggregate one which is directly related to two other aggregate concepts - PRICE and TECH. One could say that CAR is the 'father' of the concepts PRICE and TECH. PRICE is an aggregation of the buying price (buying) and

Concept	Description
CAR	car acceptability
PRICE	overall price
buying	buying price
maint	price of the maintenance
TECH	technical characteristics
COMFORT	comfort
doors	number of doors
persons	capacity in terms of persons to carry
lugboot	the size of luggage boot
safety	estimated safety of the car

Table 6.1. The Car Concepts and their Descriptions

the maintenance price (maint) whereas TECH is an aggregation of COMFORT (itself an aggregated concept) and safety. COMFORT is an aggregation of the number of doors (doors), the capacity in terms of persons (persons) and the size of the luggage boot (lugboot). The Car Evaluation Database itself contains 1728 examples of cars with the structural information removed. The data set does not contain any knowledge about the structure in Figure 6.1. It directly relates CAR to the six input attributes: buying, maint, doors, persons, lugboot, safety. In other words, the six input attributes are in a flat record with the acceptability of the car. The different attributes can take a number of different values and these are shown in Table 6.2. The full data set is contained in Appendix D. Table 6.3 shows some example cars taken from the original database.

Attribute	Attribute Values			
buying	vhigh	high	med	low
maintenance	vhigh	high	med	low
doors	2	3	4	5-more
persons	2	4	more	
lugboot	small	med	big	
safety	low	med	high	

Table 6.2. The Car Attributes

The acceptability of the car (CAR) takes one of four possible values: unacceptable (unacc), acceptable (acc), good (good) or very good (vgood). These are not equally represented in the database. Table 6.4 provides a detailed breakdown for the full data

buying	maint	doors	persons	lugboot	safety	CAR
low	high	2	more	big	low	unacc
low	high	2	more	big	med	acc
low	high	2	more	big	high	vgood
low	high	3	2	small	low	unacc

Table 6.3. Some Example Cars taken from the Car Evaluation Database.

set. As can be seen, there are considerably more instances of unacceptable than any

Class	Instances	Percentage
unacc	1210	70.02
acc	384	22.22
good	69	3.99
vgood	65	3.76
Total	1728	100

Table 6.4. The Car Classes

other class. This therefore does not look like a representative sample of data and does not meet the Representative criteria. However, as was stated at the beginning of the Chapter, a good test of the AFPL is its ability to learn a type-2 fuzzy system from data that is 'real'. Real data for classification problems will invariably not equally represent all classes. The data has the added merit that it is in the public domain (Bohanec & Rajkovic 1988) and so it was decided to proceed. This data set lays the basis for the testing of the AFPL. It is partitioned for training, testing and validation and, later, a subset of the data is used to provide a different challenge that is 'fairer' to the AFPL.

The research reported in this Chapter is interested primarily in testing the efficacy of our approach with linguistic terms so the two 'crisp' values (doors and persons) were removed from the database. Also the algorithm requires a crisp output so the CAR concept was translated into numbers between zero and one - unacceptable was changed to 0.2, acceptable (0.4), good (0.6) and very good (0.8). These can be viewed as the degree of membership in a type-1 fuzzy set CAR. Figure 6.2 shows the concept diagram amended to reflect the removal of crisp features. Table 6.5 shows the original cars from Table 6.3 modified for submission to the AFPL. The fully modified data is found in Appendix E.

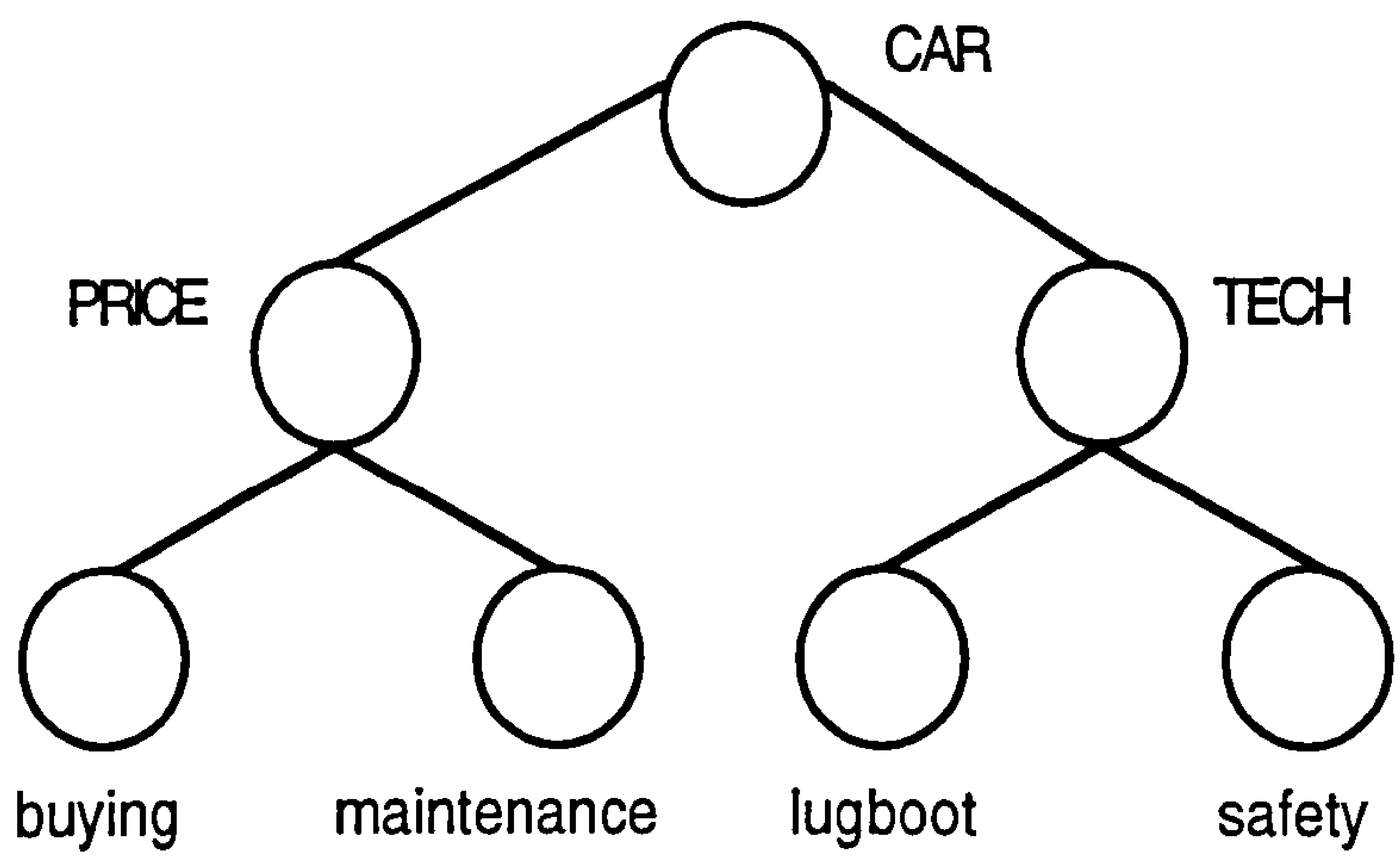

Fig. 6.2. The Car Concepts for the modified data

buying	maint	lugboot	safety	CAR
low	high	big	low	0.2
low	high	big	med	0.4
low	high	big	high	0.8
low	high	small	low	0.2

Table 6.5. Some Example Cars taken from the Data Set used for Training

Summary of the Car Data

The data then can be summarised in the following way:

(a) The data has been taken from a reliable source and consists of 1,728 cars.

(b) In the raw database each car has six attributes which can take a variety of possible linguistic values.

(c) Each car is described as either unacceptable, acceptable, good or very good.

(d) Crisp values (accounting for two of the attributes) have been removed.

(e) The data set meets two of the three criteria for a useful set of data. The third criteria - that the data be representative is not ideally met but since the data can be considered to be 'real' it was decided to proceed.

The adaptive system that is AFPL was, therefore, set a difficult problem in that the number of inputs has been reduced from the original database and the data is 'biased' since one particular classification accounts for 70% of the data. The next Section discusses the structure of AFPL for the Car Evaluation Database.

6.2 The Adaptive Fuzzy Perception Learner for Car Evaluation

The first thing to decide when using the AFPL (as with any adaptive network) is the topology, or structure, of the network. Figure 6.3 shows the chosen schematic of the AFPL for this particular problem. The structure mimics that of Figure 6.2. In other words the AFPL has been deliberately structured to represent the linguistic association as provided by an expert. The perceptions buying and maintenance are combined to produce the aggregated concept PRICE. The aggregated concept COMFORT is achieved by combining lugboot and safety and the two aggregated concepts are combined to produce the target concept CAR. The previous Chapter described the algorithm in detail. To remind the reader it is worth describing, at a high level, how this particular network operates. This particular set of data, in its original form, is 'ordered' in that similar cars are often placed together in the original database. The data could have been re-ordered but random selection of cars for submission to the AFPL was deemed to be the best approach so that the order of submission of car to the network does not have any impact on the learning. For a particular car, selected at random, the attributes are submitted to layer zero. These are linguistic terms. Layer one contains the membership grades to be learnt for the type-2 fuzzy sets buying, maintenance, lugboot and safety. The grades 'med' are described by a bell shaped function containing three parameters (Equation 5.17), and the other membership grades by gaussian membership functions represented by two parameters (big, high and vhigh are as in Equations 5.3, and low and small are as in Equation 5.2). These were given 'sensible' initial values as in Table 6.6. They are represented graphically in Figure 6.4. Layer two carries out the intersection in the antecedents of the rules. There are a choice of possible t-norms here but the min operator has been chosen.* Layer three carries out the implication. A decision had to be made here about the possible values the grades could take for the consequent type-2 fuzzy sets and low and high have been chosen as two gaussian membership functions with parameters

*The development of any adaptive network requires many decisions to be made about topology etc. The AFPL is no exception. The point is made again that the efficacy of the approach is being tested here. This new technique will be open to much more experimentation in future work.

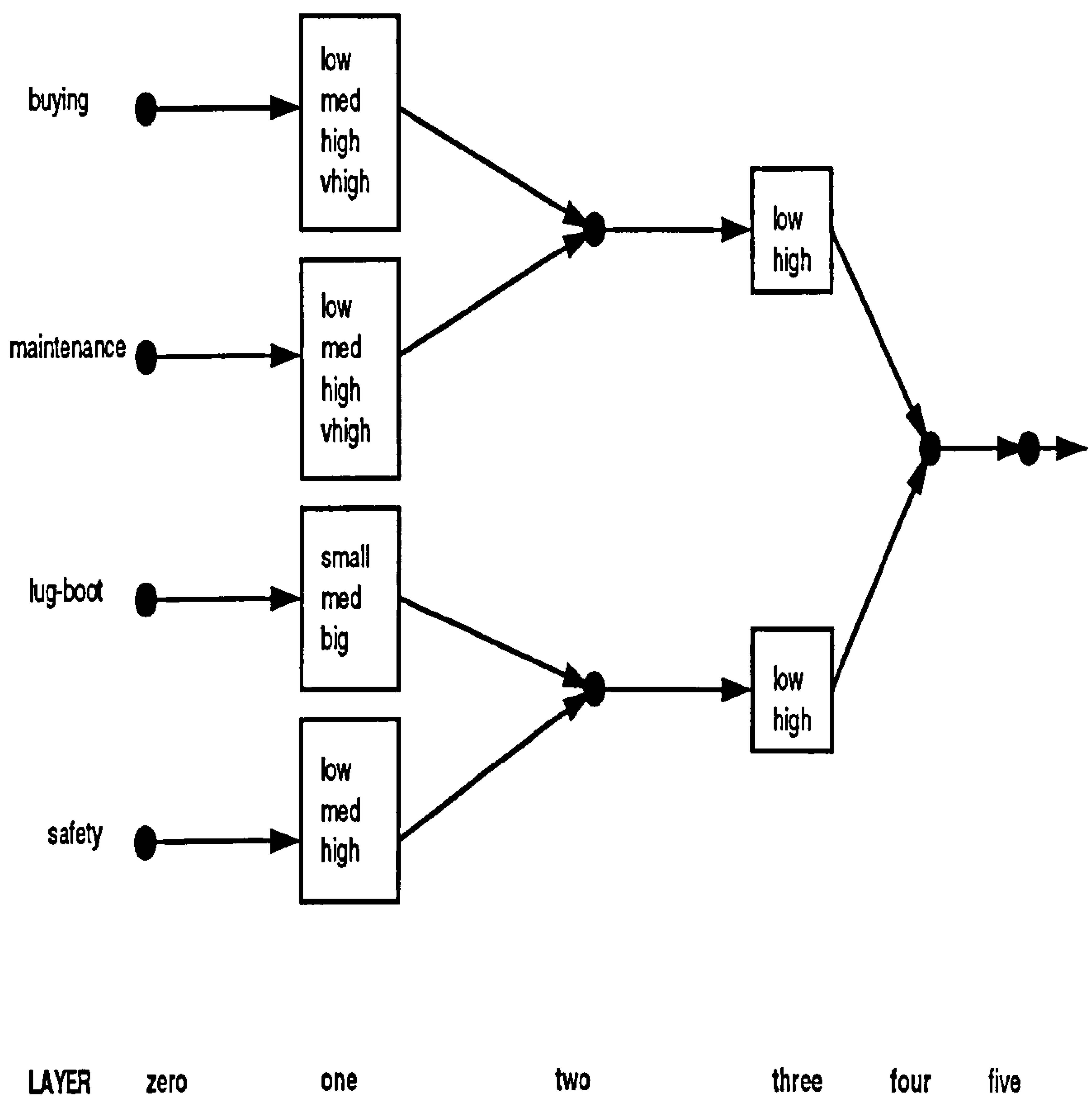

Fig. 6.3. AFPL for the Car Evaluation problem

to be learnt with initial values as in Table 6.6. The composition of, in this example, the two rules is carried out by layer four. Defuzzification is carried out in layer five. Another decision has to be made here - which method of type-2 defuzzification should be adopted? At the time of development of the AFPL no published approaches to defuzzify the type-2 output from a type-2 fuzzy system were available. The method adopted was heuristic in nature (as are in fact all defuzzification approaches). The output of the AFPL is a type-2 fuzzy set and as such consists of a set of membership grades which are type-1 fuzzy sets held as discrete arrays. The defuzzification method employed was to take the maximum of the type-1 membership values thus reducing each type-1 membership grade to a single number in [0,1]. These values were then averaged to produce the final output

Grade	a	b	c
low	-20		0.25
small	-20		0.25
med	0.25	5	0.5
big	20		0.75
high	20		0.75
vhigh	20		0.75

Table 6.6. Initial Values for the Membership Grade Parameters in the Car Evaluation Database

of the network.

6.3 Results

This section reports on the results achieved of applying the AFPL described to the data already discussed. Initial runs used this data as it appeared on the World Wide Web(Blake et al. 1998). It should be noted here that many experiments were carried out over a period of time. The algorithm requires a choice of stepsize for training and this has an impact on the number of epochs to successfully train an AFPL as well as whether it finds the globally optimal solution.

For easy comparison the results are presented in a consistent style which needs some explanation. The initial runs were for a specific number of epochs and the results are given for the epoch with the smallest least mean square error (LMSE). For these initial runs the data was split randomly into two sets of data - one for training and one for testing the final trained type-2 network. The training data consisted of 1555 cars with their breakdown shown in Table 6.7. The test set, chosen at random from the full

Class	Instances
unacc	1091
acc	341
good	65
vgood	58
Total	1555

Table 6.7. The Car Classes for Initial Training Set

data set, for assessing the success of the final network consisted of 172 cars and their

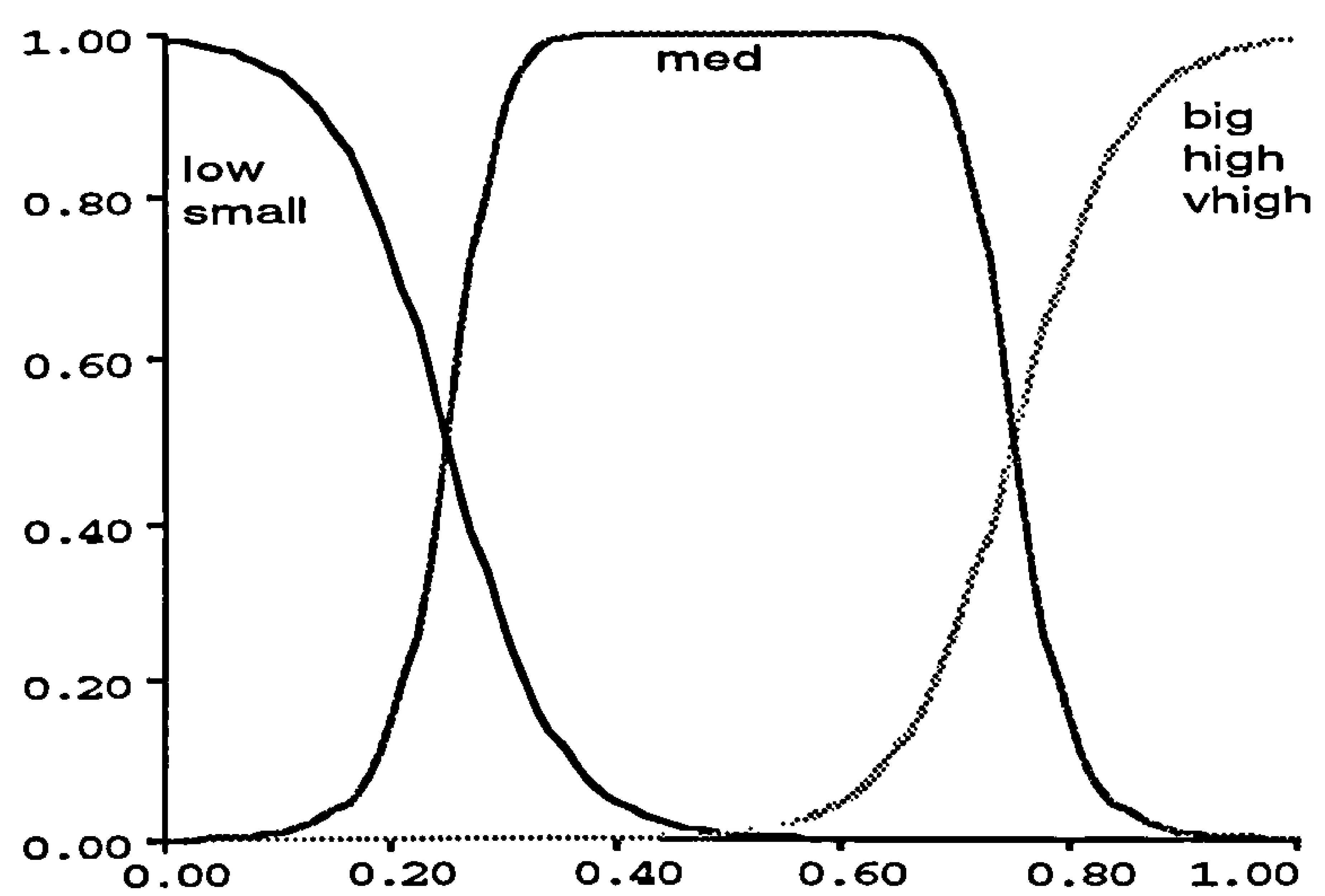

Fig. 6.4. A Graphical Representation for the Initial Values of the Membership Grade Parameters in the CAR AFPL

Class	Instances
unacc	123
acc	36
good	8
vgood	5
Total	172

Table 6.8. The Car Classes for Initial Test Set

breakdown is shown in Table 6.8. The output from the network is the type-1 membership value 0.2 (acceptable), 0.4 (unacceptable), 0.6 (good) and 0.8 (very good). The rule for deciding whether the car has been classified correctly is that the classification a car is given is the value of that classification that is nearest to the actual value. For each trained network the following is provided:

- The number of epochs. In the initial runs this is preset. For later runs this is the epoch number at which the validation set has minimum RMSE (see later for a definition of the validation set). The RMSE is the usual square root of average square error for the set of data under consideration.

- The RMSE for the training set is given for each run.
- The number of cars classified correctly and incorrectly and the percentage of the total is shown in brackets.

The first run that is shown is given in Table 6.9. In this initial run of the network

Number of epochs				5000
Root Mean Square Error				0.1549
	Unacceptable	Acceptable	Good	Very Good
Correct	94 (76)	15 (42)	2 (25)	0 (0)
Wrong	29 (24)	21 (58)	6 (75)	5 (100)
Total Correct				111 (65)

Table 6.9. Run 1 - Results for the test data

the performance indicates that it can classify the (predominant) class of unacceptable cars correctly 76% of the time but for the 5 very good cars in the test set it incorrectly classified all of them. Overall, the performance can be said to be 65% in that it classifies correctly 65% of the cars in the test set. This is one of the early runs and there is no fine tuning.

To reduce the swamping of the data by the unacceptable category, a smaller sample of the unacceptable cars was taken giving as many unacceptable cars as the other three classes combined (to make all classes equal would have been a massive reduction in data and was not deemed appropriate. As well as this amendment to the data, a change in strategy (along similar lines to many artificial neural networks approaches) was adopted where the notion of a validation set of data was introduced. The remaining data set was therefore partitioned into three distinct sets of data - training, validation and test. The training set is the set of data used to train the AFPL and modify the parameters of the type-2 fuzzy system. The validation set is used at each epoch to test the behaviour of the network on data; it is not using for training purposes. After each epoch the current state of the network is assessed by 'passing' the validation set through the AFPL in forward pass mode only to produce a value for RMSE. This is monitored and it is this validation set that provides the value for the minimum RMSE. The test data is used to evaluate a 'trained' network. Note that the bias in the data towards the unacceptable class still exists but to a lesser degree. Interestingly, when training on all the data bar 172 'test' cases the performance of the networks was very similar, indicating that, by using a subset of the data, the problem has not disappeared completely; to do much more pruning would invalidate the whole process.

The composition of the training and test data sets is shown in Table 6.10 and some typical results shown in Table 6.11. As can be seen, these results are very similar to

	Unacceptable	Acceptable	Good	Very Good	All
Training	323	233	49	36	641
Test	237	70	17	13	337

Table 6.10. The break down of the training and test sets for the Car Evaluation data where there is a validation set

Number of epochs				50000
Root Mean Square Error				0.1909
	Unacceptable	Acceptable	Good	Very Good
Correct	207 (87)	20 (29)	0 (0)	0 (0)
Wrong	30 (13)	50 (71)	17 (100)	13 (100)
Total Correct				227 (67)

Table 6.11. An example set of results where a validation set is used

the initial run except that it is able to get the unacceptable class correct 87% of the time and unable to classify good or very good at all. Although encouraging, they are not entirely satisfactory. Why might this be? Firstly two crisp parameters have been removed from the data. One option would be to include them in the AFPL. However the test of the network is that it can use only linguistic inputs. To include crisp values would mean that interpretation of the efficacy of the approach would become muddled. Another cause for degradation in performance is obviously the unrepresentative nature of the data being swamped by the unacceptable class.

To gain a feel for how successful the AFPL is with this data a comparison was made using the well known ID3 algorithm(Quinlan 1986). The software MLC++ (http://www.sgi.com/Technology/mlc) was deployed using the data as described above. The system produced variations in classification. For the unacceptable class it classified with a 70% success rate, acceptable(88%), good(52%) and very good(90%) with an overall success rate of 71%. This is quite different to the AFPL with a similar overall success rate. As with many comparisons it is difficult to draw conclusions. The purpose of the work here is to see if the AFPL is able to learn a type-2 fuzzy system, not whether it is better than any other approach since there are too many parameters that can vary in the AFPL and any other technique to make a comparison meaningful.

It was decided at this stage to see if the network was able to differentiate between the unacceptable and the acceptable classes. To achieve this, three new data sets were created that only had those classes in them. The training set and test set are given in Table 6.12 and an example run in Table 6.13. Note that the RMSE is significantly

	Unacceptable	Acceptable	Good	Very Good	All
Training	720	233	0	0	953
Test	237	70	0	0	307

Table 6.12. The break down of the training and test sets for the Car Evaluation data for only unacceptable and acceptable classes

Number of epochs		5000
Root Mean Square Error		0.0972
	Unacceptable	Acceptable
Correct	168 (71)	41 (59)
Wrong	69 (29)	29 (41)
Total Correct		209 (68)

Table 6.13. An example set of results where a validation set is used

improved and that, although the overall percentage is similar, there is a more 'even' distribution.

For a more detailed analysis of one of the more successful runs, results are given for a particular run to look at the errors in classification and the effects on the type-2 membership grades. The results of the run are shown in Table 6.14. This is the best run with a stepsize of 0.05. Notice, firstly, that this has a RMSE of 0.0942 after 10,000 iterations. It learns the acceptable class as successfully as the unacceptable class even though there are considerably more unacceptable cars. The final parameters for this run

Number of epochs		10000
Root Mean Square Error		0.0942
	Unacceptable	Acceptable
Correct	167 (70)	49 (70)
Wrong	70 (30)	21 (30)
Total Correct		216 (70)

Table 6.14. An example set of results with a stepsize of 0.05

are shown in Table 6.15. An interesting phenomenon is that some parameters change very little and some change considerably with the crossover point being modified much more than the slope. notice also that the parameters for $medium_{safety}$ do not change at all. The final membership grades are represented graphically in Figures 6.5, 6.6, 6.7,

Table 6.15. Initial and final parameters for a run with the car data set

AFPL parameters for a run with the car data		
Grade	Initial Parameters	Final Parameters
low_{buying}	-20, 0.25	-20.0094669116, 0.9258546604
$medium_{buying}$	0.25, 5, 0.5	0.0320011769, 4.9997567174, 0.2716213560
$high_{buying}$	20, 0.75	20.0047790712, 0.3563176837
$vhigh_{buying}$	20, 0.75	19.9993240563 0.8932147431
$low_{maintenance}$	-20, 0.25	-20.0099265860, 0.9404244215
$medium_{maintenance}$	0.25, 5, 0.5	0.0326471645, 4.9997904917, 0.2699721176
$high_{maintenance}$	20, 0.75	20.0051484423, 0.3399655243
$vhigh_{maintenance}$	20, 0.75	19.9987522157 0.8499214070
$low_{lugboot}$	-20, 0.25	-20.0106225134, 0.9611964283
$medium_{lugboot}$	0.25, 5, 0.5	0.0331402825, 4.9997254402, 0.2730772739
$high_{lugboot}$	20, 0.75	20.0043323760, 0.3766479707
low_{safety}	-20, 0.25	-19.9090322065, -0.0685114076
$medium_{safety}$	0.25, 5, 0.5	0.2500000000, 5.0000000000, 0.5000000000
$high_{safety}$	20, 0.75	19.3941845435, 0.5486190056
low_{PRICE}	-20, 0.25	-20.0068167711, 1.4422131013
$high_{PRICE}$	20, 0.75	20.0059131544, 0.9704274386
$low_{COMFORT}$	-20, 0.25	-26.2469328486, 0.9134841247
$high_{COMFORT}$	20, 0.75	20.0058529728, 0.9677633650

6.8, 6.9 and 6.10. As can be seen the various parameters are altered considerably, and differently within each type-2 fuzzy set. Figure 6.11 shows a scatter diagram of the errors for the test set for the particular run. The test file is ordered in that the unacceptables appear before the acceptables so one can see that it is more successful with the unacceptable class, as expected. A sample of some of the results are shown in Table 6.16 as a cross section of successful and unsuccessful classifications. Consider two particular cases. Car number 75 in the test set has the descriptors high for buying and maintenance, med for lugboot and high for safety and is described as unacceptable. The AFPL performs poorly for this car. However, in contrast, car number 19 has vhigh for buying, high for maintenance, med for lugboot and low for safety. This car is described as unacceptable yet the AFPL performs very well. The observation can be made that the low for safety has perhaps made this car unacceptable more clearly and the AFPL is struggling between the vhigh and high on buying in conjunction with this difference

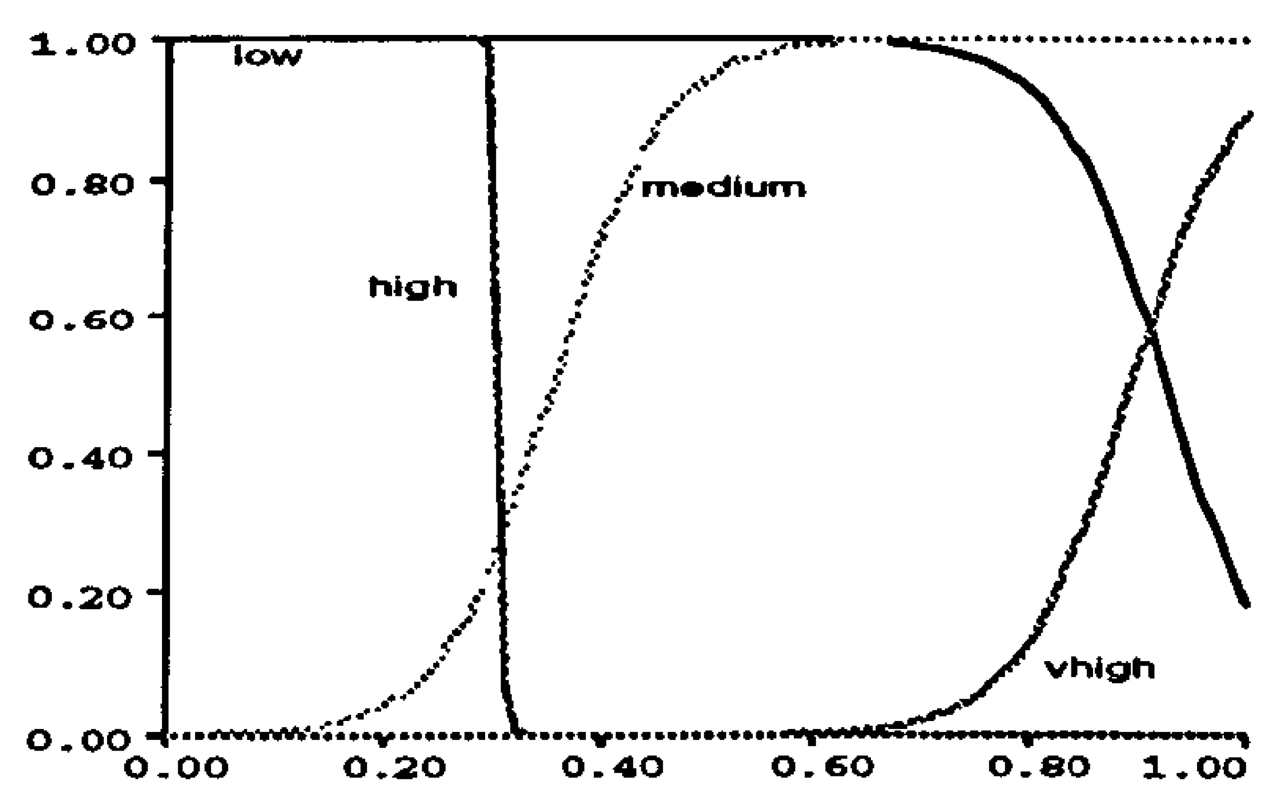

Fig. 6.5. Final grades for buying

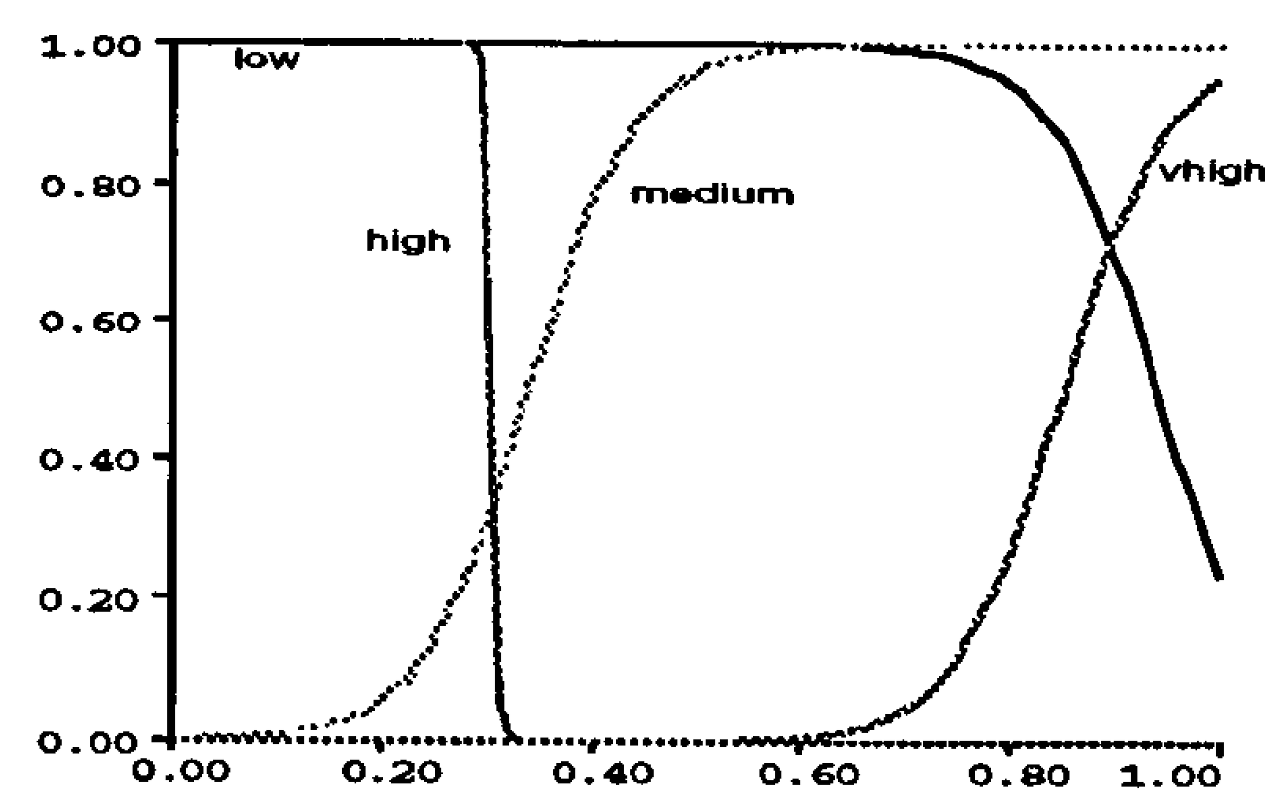

Fig. 6.6. Final grades maintenance

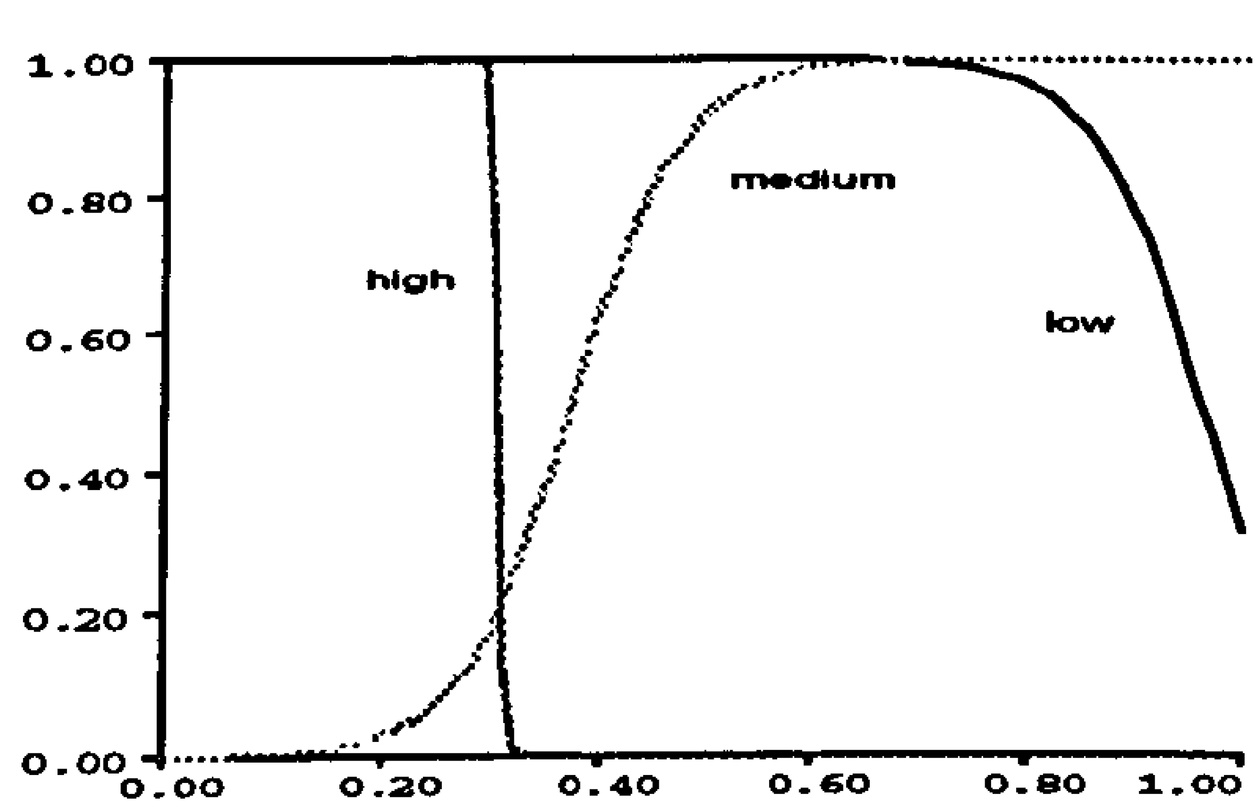

Fig. 6.7. Final grades for lugboot

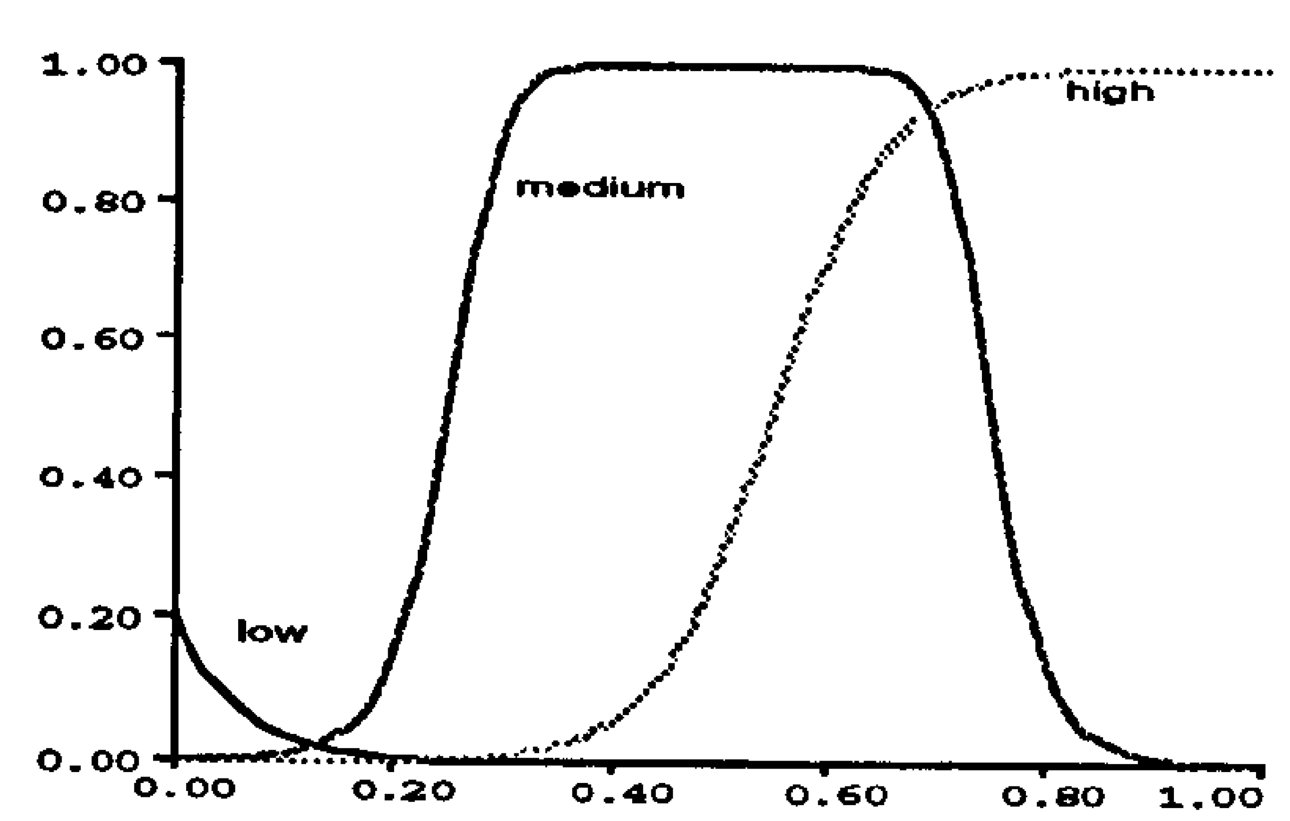

Fig. 6.8. Final grades for safety

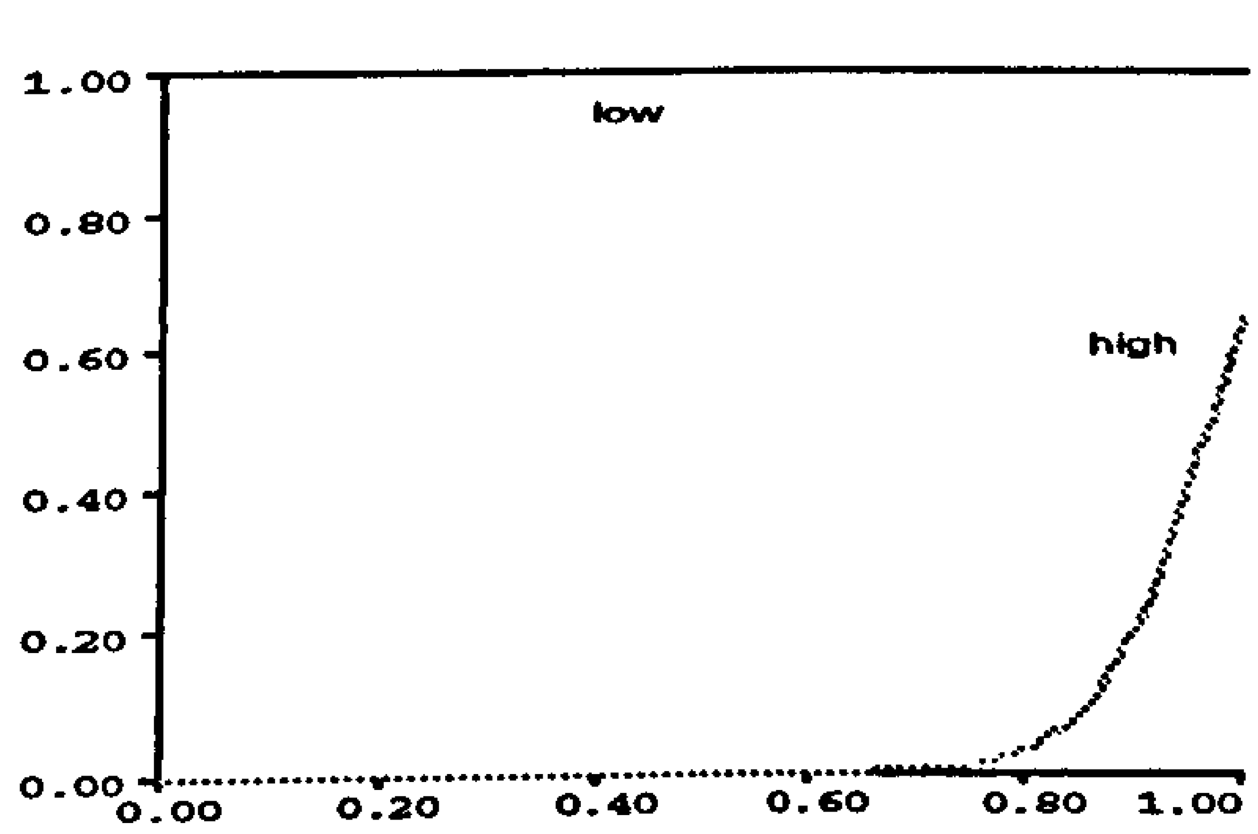

Fig. 6.9. Final grades for PRICE

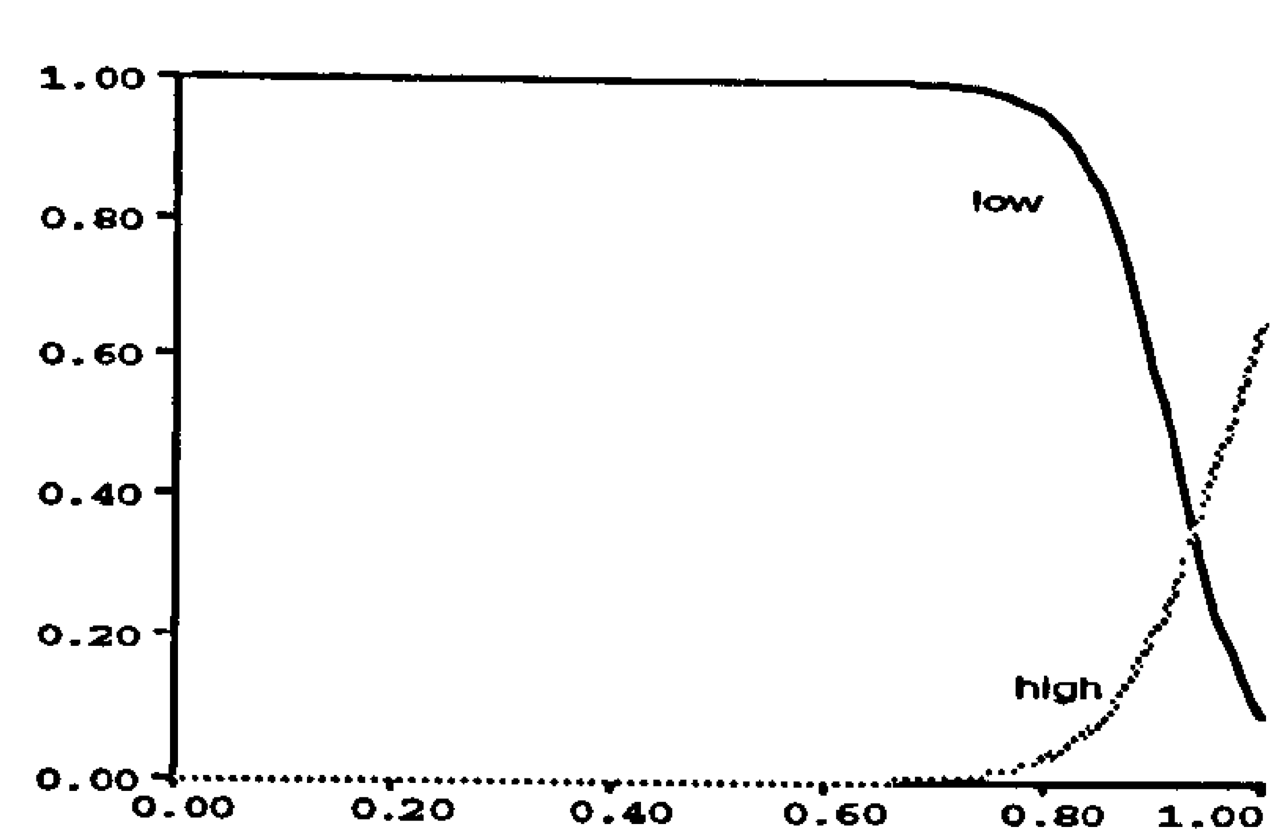

Fig. 6.10. Final grades for COMFORT

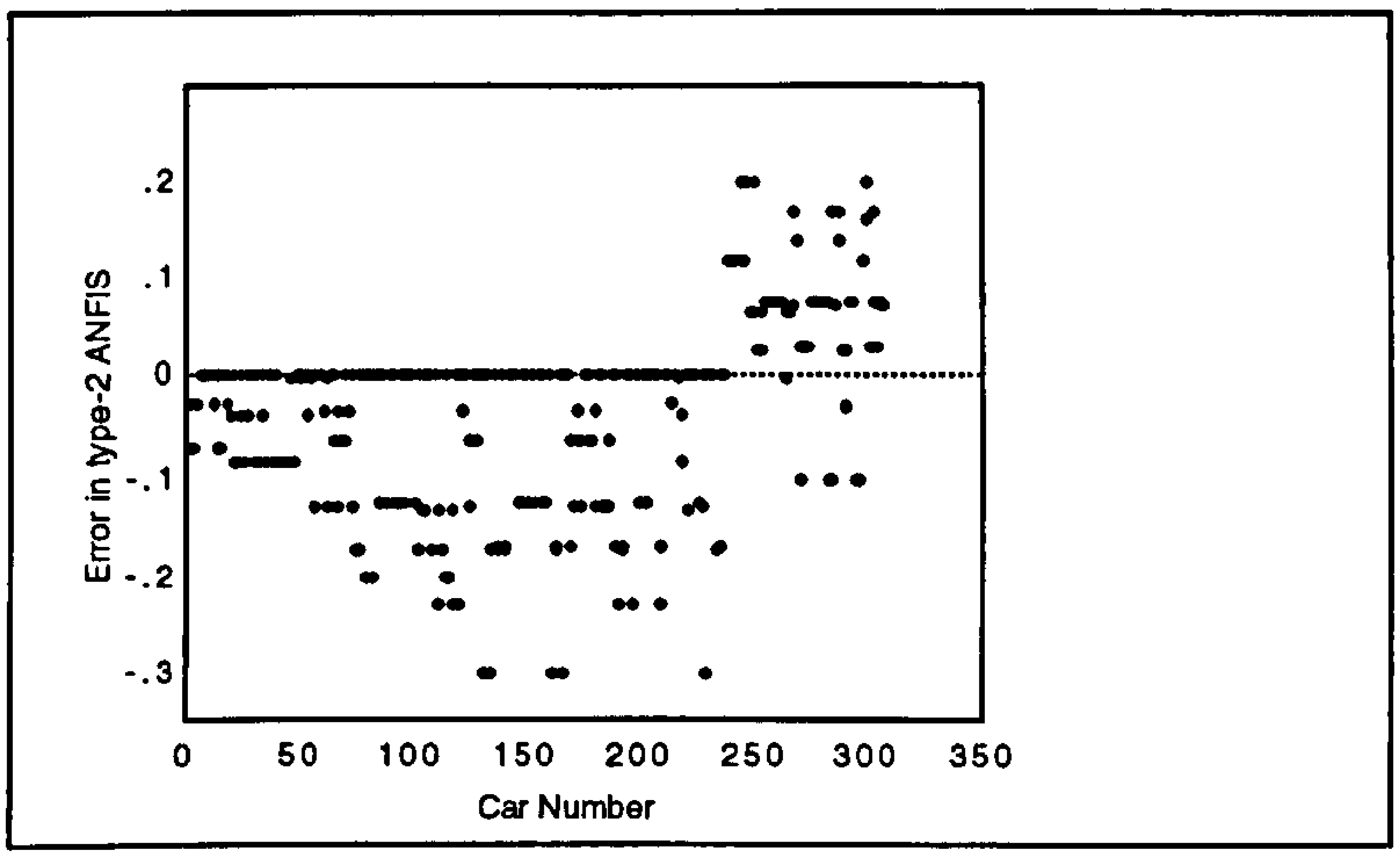

Fig. 6.11. A scatter plot of the test set errors for a particular run of the AFPL on the car data

on safety.

To test how susceptible the solution was to step sizes, length of run and the random processing of the data, an investigation was made of some of the better runs in more detail. The runs are shown in Figure 6.12.Some Receiver Operating Characteristic(ROC) curves are shown in Figure 6.13. ROC curves are well known for analysing systems that carry out some form of classification(Swets, Pickett, Whitehead, Getty, Schnir, Swets & Freeman 1979, Woods & Bowyer 1997). The ROC curve plots false positive against true positive. In other words, in this case, the horizontal axis (false positive (FP)) is for those cars that the AFPL decided were acceptable when they were unacceptable and the vertical axis (true positive (TP)) is the proportion of those cars correctly classified as acceptable. Each run will generate a ROC curve, in this case, by changing the point at which the boundary is chosen for the decision on which class to place the output of the AFPL. A ROC curve for random guessing would be a straight line at 45 degrees. A good ROC shows a steep curve to the top of the graph and then a horizontal line. The ROC curve is an indicator of the success or otherwise of the approach. In Figure 6.13 we see that the ROC curves shown are 'good' in the sense that the AFPL is better than a random guess. What is more interesting is that the curves are very similar for each run. This indicates that there is some stability in the approach.

Table 6.16. A cross section of some of the results for the test set

Car	Expected	Actual
4	0.200000	0.228071
5	0.200000	0.228071
6	0.200000	0.199561
7	0.200000	0.199994
8	0.200000	0.199561
18	0.200000	0.228071
19	0.200000	0.199994
20	0.200000	0.238833
21	0.200000	0.282122
22	0.200000	0.199994
72	0.200000	0.232454
73	0.200000	0.199994
74	0.200000	0.328753
75	0.200000	0.372753
250	0.400000	0.200354
251	0.400000	0.333358
252	0.400000	0.372753
253	0.400000	0.333358
254	0.400000	0.372753
264	0.400000	0.400259

6.4 Discussion

In the previous Chapter the AFPL was shown to successfully learn the linguistic AND. This Chapter reports on its use for a far more complex set of data on car classification based on linguistic descriptors of perceptions of the car. The results for this set of data are mixed but what is clearly shown is that the AFPL offers an opportunity to capture the linguistic association between the words that describe the cars and the classification of the same. There are a number of factors that influence the success of the AFPL not least of which are the stepsize, the number of epochs, the structure of the network and the many decisions made in the development of the AFPL described at various points in the thesis.

One can conclude from this and the previous Chapter that the AFPL does work. As a novel approach to learning linguistic relationships, the AFPL offers much to the system developer. The next Chapter summarises the thesis, draws conclusions about the efficacy of the type-2 approach and the AFPL and discusses future research possibilities for this work.

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INITIAL PARAMETERS

-20.000000 0.250000 0.250000 5.000000 0.500000
20.000000 0.750000 20.000000 0.750000 -20.000000
0.250000 0.250000 5.000000 0.500000 20.000000
0.750000 20.000000 0.750000 -20.000000 0.250000
0.250000 5.000000 0.500000 20.000000 0.750000
-20.000000 0.250000 0.250000 5.000000 0.500000
20.000000 0.750000 -20.000000 0.250000 20.000000
0.750000 -20.000000 0.250000 20.000000 0.750000
Number of epochs: 10000
Using random selection of training patterns
Initial stepsize = 0.050
Minimal training RMSE = 0.096587
Classification rates
Unacceptable Acceptable
Right 168 (71) 47 (67)
Wrong 69 (29) 23 (33)
Total Number right 215 (70)
Final parameters
-20.0065154141 0.8230527383 0.0324231217 4.9997508511 0.2716702455
20.0046802601 0.3607982111 19.9993303042 0.9128125600 -20.0067543877
0.8321642320 0.0328068456 4.9997882475 0.2699907869 20.0050496351
0.3442851429 19.9993191156 0.8793819628 -20.0074934004 0.8585291566
0.0335778117 4.9997192587 0.2731300411 20.0042673486 0.3797331674
-19.9084204375 -0.0696335677 0.2500000000 5.0000000000 0.5000000000
19.4668261899 0.7865440857 -20.0066805850 1.4363310572 20.0057111783
0.9657588344 -20.0129494122 1.3554775848 20.0056914114 0.9640482449

Fig. 6.12. Run A

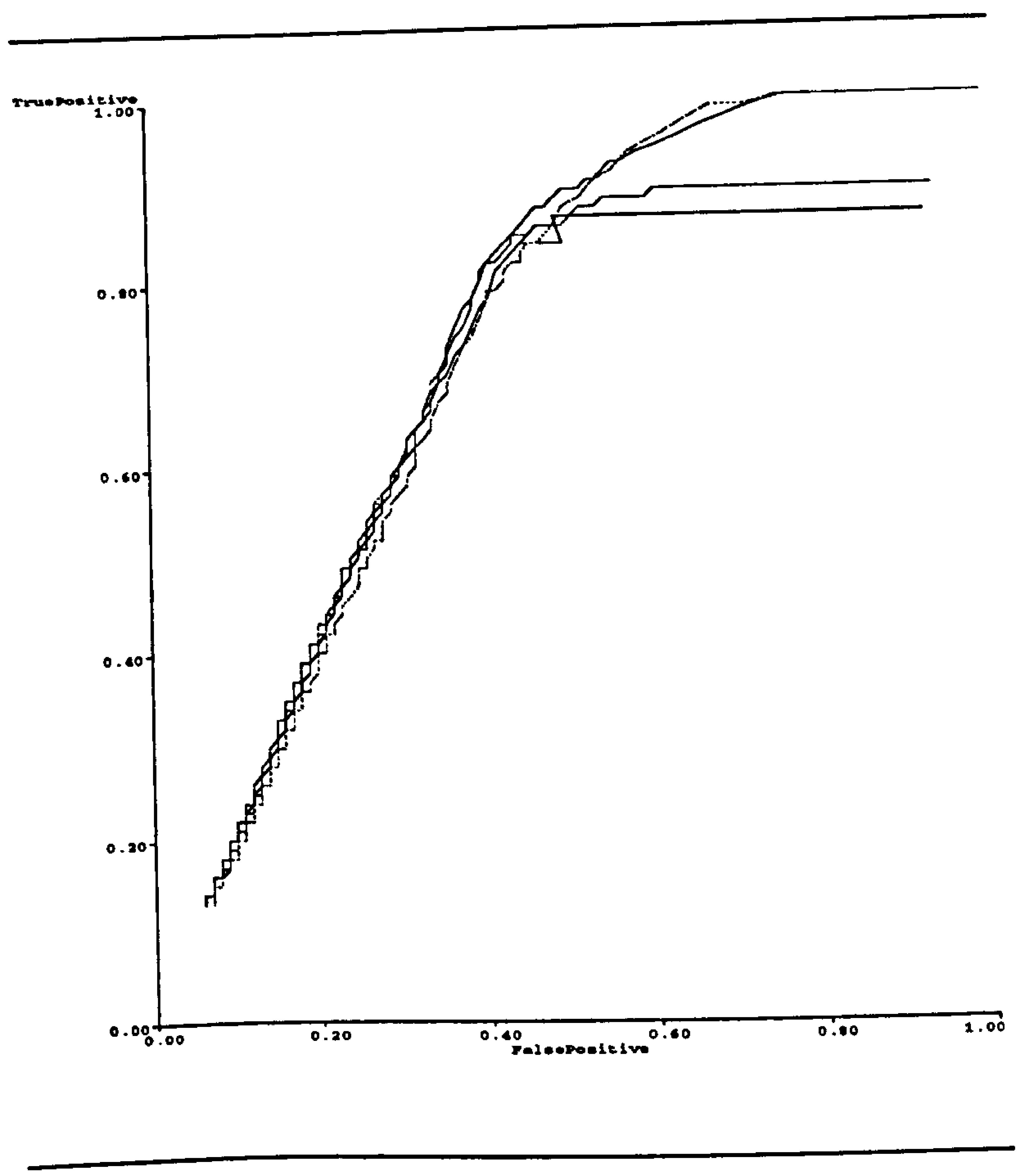

Fig. 6.13. Sample ROC curves for the AFPL

Chapter 7

Conclusion and Discussion

This Chapter concludes the thesis by summarising the main points and results and presents a discussion about further work for the AFPL. The research hypothesis in the thesis has been stated in Chapter 1 as:

> *"Type-2 fuzzy sets have much to offer for knowledge representation and inferencing; however there is a need for some method for learning type-2 fuzzy systems. A type-2 fuzzy system that models human perceptual categorisation by linguistic association can be learnt from linguistic data that represent perceptions."*

The structure of this Chapter follows the thread of argument that concerns itself directly with this hypothesis. The central tenets supporting the research hypothesis are:

(i) *Type-1 Fuzzy Systems Are Not Fuzzy.* Type-1 fuzzy systems have a long history of successful applications, mainly in control. However, the research in this thesis presents an argument that type-1 fuzzy systems are crisp and do not therefore model perceptions well.

(ii) *Type-2 Fuzzy Systems Are More Fuzzy Than Type-1 Fuzzy Systems.* In this thesis it is argued that type-2 fuzzy sets capture a higher level of imprecision than type-1 fuzzy sets.

(iii) *Type-2 Fuzzy Sets Help With Perception Representation.* This is the first part of the hypothesis and is central to the work contained in this thesis. The point is that type-2 fuzzy sets allow an expert to describe perceptions using words, removing the need for a translation into a number in [0,1].

(iv) *Type-2 Membership Grades Have To Be Determined.* The membership grades of a type-2 fuzzy set are type-1 fuzzy sets and some approach has to be adopted for their formation.

(v) *The Adaptive Fuzzy Perception Learner Offers An Effective Method For Learning Perceptions.* The research reported here has described a method for full determination of a type-2 fuzzy system, given some linguistic data, thus removing the need to 'hand craft' the membership grades of the type-2 fuzzy sets.

7.1 Type-1 Fuzzy Systems are not Fuzzy

Type-1 fuzzy logic(Zadeh 1965, Goguen 1967) has over thirty years of successful applications, primarily in the control field. Chapter 2 presented a detailed discussion of type-1 fuzzy sets and how they are used to measure and model imprecision, vagueness (linguistic uncertainty) and granularity. Type-1 fuzzy sets are primarily deployed in type-1 fuzzy systems that contain type-1 fuzzy if-then rules. These systems have a history of exhibiting a strong capability to model imprecision and uncertainty. A type-1 fuzzy system consists of:

- Type-1 Fuzzy Sets. These are the basis for the fuzzy system and are used to model some of the underlying knowledge in the system.

- Fuzzy If-Then Rules. The rules in a type-1 fuzzy system typically take the form:

  $$IF\ x\ is\ A\ and\ y\ is\ B\ THEN\ z\ is\ C$$

  where x, y and z are from the universe of discourse X, Y and Z respectively and A,B and C are type-1 fuzzy sets. There are variations on this type of rule including, for example, the Takagi-Sugeno rule (Equation 2.5).

- Fuzzy Composition. The rules in a type-1 fuzzy system have to be combined in some way. There are a number of approaches available. A basic requirement is the ability to carry out an 'AND' and an 'OR' and there are a number of functions available known as t-norms and t-conorms that provide this functionality.

- Defuzzification. The output of a type-1 fuzzy system is a type-1 fuzzy set. For most applications some 'crisp' decision is required so this fuzzy set has to be reduced to a number. This process is known as defuzzification for which there are a number of approaches available (Section 2.2.3).

It is argued in this thesis that the traditional type-1 fuzzy system is not fuzzy. The membership functions of the base type-1 fuzzy sets are precise functions, or numbers in [0,1]. The composition of the type-1 fuzzy rules is a precise process and the usual definitions for 'AND' and 'OR' are flawed in that only one of the possible Normal Forms is deployed(Türkşen 1995*a*). Section 2.3.1 presented a detailed discussion of this type-1 paradox. As well as this paradox, there is the question of "Where do 'they' come from?"(Dubois and Prade(1980, Part IV, Chapter 1, page 255)). This is the question that (should) tax type-1 fuzzy system developers. All type-1 fuzzy systems require membership functions to be determined. Section 2.3.2 discusses this problem in some detail. The conclusion is that, although there are a number of possible approaches for determining membership functions of type-1 fuzzy sets, they are essentially domain dependent and there is no well accepted method.

In summary, it is argued in this thesis that type-1 fuzzy systems are essentially crisp. In particular, the fuzzification process in a type-1 fuzzy system translates, what is assumed to be, crisp input into a number in [0,1]. This perceived crisp input will almost certainly have some imprecision in reality. Nothing can be measured precisely. In fact the inputs to all type-1 fuzzy systems are imprecise, even though the opposite is usually assumed. This fuzzification which translates a number into a membership grade in [0,1] also assumes that the membership function of the type-1 fuzzy set is also precise. So there are two assumptions inherent in a type-1 fuzzy system:

- the inputs to the system are precise;
- the membership functions of the type-1 fuzzy sets can be found exactly.

This paradox that underpins all (traditional type-1) fuzzy systems is addressed in this thesis by the use of type-2 fuzzy sets.

As well as not being fuzzy, type-1 fuzzy systems are not good for modelling and representing perceptions that are not directly measurable. For example, comfort is a perception that, as human beings, we are perfectly capable of describing yet cannot be measured either directly or indirectly. Type-1 fuzzy systems require a domain that can be fuzzified. It is argued in this thesis that type-2 fuzzy systems offer the capability for modelling perceptions as well as being more fuzzy than type-1 fuzzy systems.

7.2 Type-2 Fuzzy Systems Are More Fuzzy Than Type-1 Fuzzy Systems

Five definitions of type-2 fuzzy sets were given in Chapter 3. Perhaps the most elegant is(Zadeh 1975*a*):

A fuzzy set is of type $n, n = 2, 3, \ldots$ if its membership function ranges over fuzzy sets of type $n-1$. The membership function of a fuzzy set of type-1 ranges over the interval [0,1].

In other words, a type-2 fuzzy set has membership grades which are type-1 fuzzy sets. They have fuzzy membership functions. The key strength of type-2 fuzzy sets is that they allow for linguistic membership grades not requiring a number in [0,1], but maintaining any fuzziness in the initial linguistic description. Below, Figure 1.3 is reproduced to summarise the relationships between imprecision, data and fuzziness. The ability of

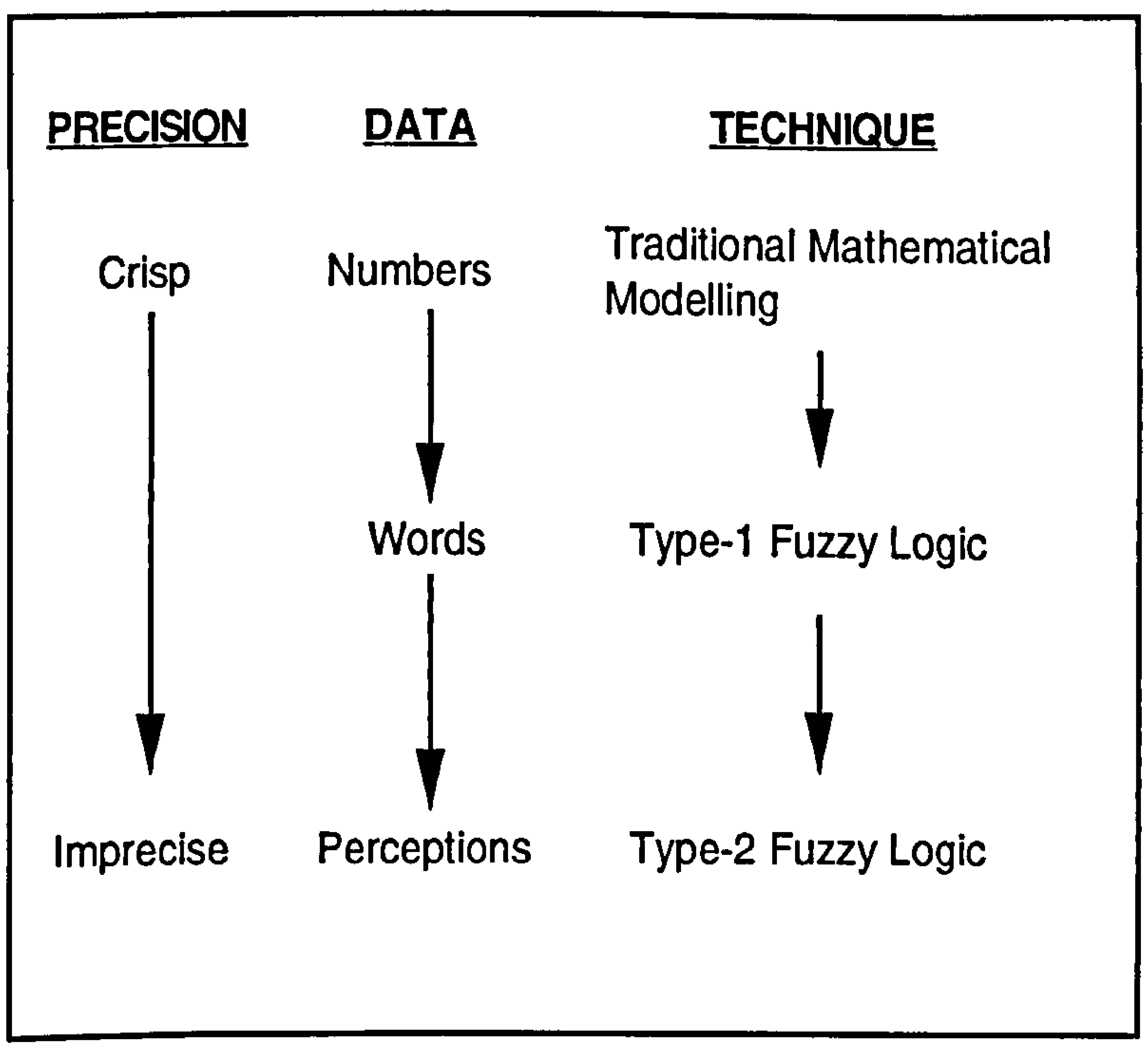

Fig. 7.1. Relationships between imprecision, data and fuzzy technique

type-2 fuzzy sets to have fuzzy membership grades allows for a higher level of imprecision to be modelled than with the type-1 paradigm. Chapter 3 explores the extra fuzziness inherent in the type-2 paradigm. The key points can be summarised in the following way:

- Perceptions can be modelled using type-2 fuzzy sets - in particular where there is no base domain.
- The fuzzy inferencing process is enhanced by i-v fuzzy sets (a special case of a type-2 fuzzy set).
- The reporting of type-2 fuzzy applications is on the increase.

The author has successfully used type-2 fuzzy sets for representing perceptions of a medical consultant to enable the analysis of shin images using artificial neural networks(John et al. 1998, John et al. 2000). This work showed that the expert could describe the location and type of shin problem using words and that these could be represented as type-1 fuzzy sets which are membership grades of type-2 fuzzy sets.

Chapter 4 explored the literature on type-2 fuzzy relations(Definition 19) and type-2 inferencing with type-2 fuzzy if-then rules. That Chapter laid the basis for the fuzzy inferencing that uses type-2 fuzzy if-then rules such as:

$$IF\ x\ is\ \tilde{A}\ AND\ y\ is\ \tilde{B}\ THEN\ \ z\ is\ \tilde{C}$$

where $\tilde{A}, \tilde{B}$ and $\tilde{C}$ are type-2 fuzzy sets. These rules are at the heart of a type-2 fuzzy system. As with a type-1 fuzzy system type-2 fuzzy composition is required (Equation 4.13). The output of the type-2 if-then rules is combined using the join operation (Equation 3.2). The output of this inferencing is a type-2 fuzzy set which, for most applications, will require defuzzification. There are a number of possibilities for type-2 defuzzification(Karnik & Mendel 1998*a*) which reduces a type-2 fuzzy set (the output of a type-2 fuzzy system) to a single number. Type-2 fuzzy relations and type-2 fuzzy inferencing are at the core of the Adaptive Fuzzy Perception Learner which learns a type-2 fuzzy system from linguistic data.

7.3 Type-2 Membership Grades Have To Be Determined

All type-2 fuzzy systems (whether they be the conventional type-2 or where they use type-2 i-v fuzzy sets) require that the membership grades (usually type-1 fuzzy sets) and rules are hand crafted. That is, the type-2 fuzzy sets and type-2 fuzzy rules have to be determined. Note, however, that for a particular system each possible grade in each type-2 fuzzy set will have to be developed individually either in discussion with an expert or via some statistical or artificial neural network technique. This could be a massive task and will not assist in the wide acceptance of type-2 fuzzy systems for real applications. The type-2 paradigm is complex. From the expert's point of view, the strength of a type-2 fuzzy paradigm is for *perception representation* via type-2 fuzzy sets and type-2 fuzzy rules. The expert is able to represent their knowledge by use of words. However, the detail of a particular membership grade of a particular type-2 fuzzy set will be difficult to determine since it is known to be difficult for type-1 fuzzy sets (Section 2.3.2). It seems important, therefore, that some methodology is developed for learning the linguistic grades of type-2 fuzzy sets.

7.4 The Adaptive Fuzzy Perception Learner Offers An Effective Method For Learning Perceptions

The AFPL is the major result of this research. The mathematical detail and its application to a theoretical problem is provided in Chapter 5. The use of the AFPL on a real problem - for modelling perceptions relating to the acceptability of a car - is explored in detail in Chapter 6. The AFPL uses linguistic data to learn the contents of a type-2 fuzzy system. In particular, there is no requirement for the expert to transform perceptions into numbers since the words (that represent the perceptions) are the inputs to the system. The inferencing process is also type-2 and thus maintains the imprecision inherent in the perception representation in the inputs. It can be summarised in the following way:

- The AFPL is an adaptive network which models a type-2 fuzzy system using type-2 if-then rules and type-2 fuzzy inferencing (thus maintaining uncertainty in the inferencing process).
- The inputs to the AFPL are linguistic terms which are considered to be membership grades of type-2 fuzzy sets.

- The network learns the parameters of the membership grades of the type-2 fuzzy sets via a forward pass and a backward pass.

The (L)AND problem (reproduced in Figure 7.1) was effectively modelled by the AFPL.

Table 7.1. Linguistic AND

x	y	(L)AND
low	low	0.25
high	high	0.75
low	high	0.25
high	low	0.25

In this case the inputs to the AFPL are the words in the x and y columns with the output contained in the column headed (L)AND which is considered to be a type-1 fuzzy set. The AFPL learns this problem very effectively. The best run (LAR5M) had 200 epochs with a very low RMSE of 0.000289 with near perfect output. The use of the AFPL to tackle this problem shows that an AFPL can be trained that learns a type-2 inferencing system for linguistic inputs to an adaptive network.

Chapter 6 explores the use of the AFPL to model linguistic associations between perceptions that relate to the acceptability of a car. The data used is the Car Evaluation Database from the Carnegie Mellon University Data Repository(Blake et al. 1998). It contains 1728 cars that fall into four classes - acceptable, unacceptable, good and very good. The 'split' is contained in Table 7.2. As can be seen, 70% of the cars fall into one

Class	Instances	Percentage
unacc	1210	70.02
acc	384	22.22
good	69	3.99
vgood	65	3.76
Total	1728	100

Table 7.2. The Car Classes

class. This data represents a particularly difficult challenge for the AFPL, partly because of this swamping by one class in the data. Each car is described by six factors - buying (the buying price), maint (the price of the maintenance), doors (the number of doors), persons (the capacity in terms of persons to carry), lugboot(the size of the luggage boot) and safety (the estimated safety of the car). Four of these can be considered to be perceptions (buying, maint, lugboot and safety) represented by words - the other two are precise and (for 'purity') have been removed. It is these perceptions that are the type-2 fuzzy sets in the type-2 fuzzy system modelled by the AFPL shown in Figure 7.2.

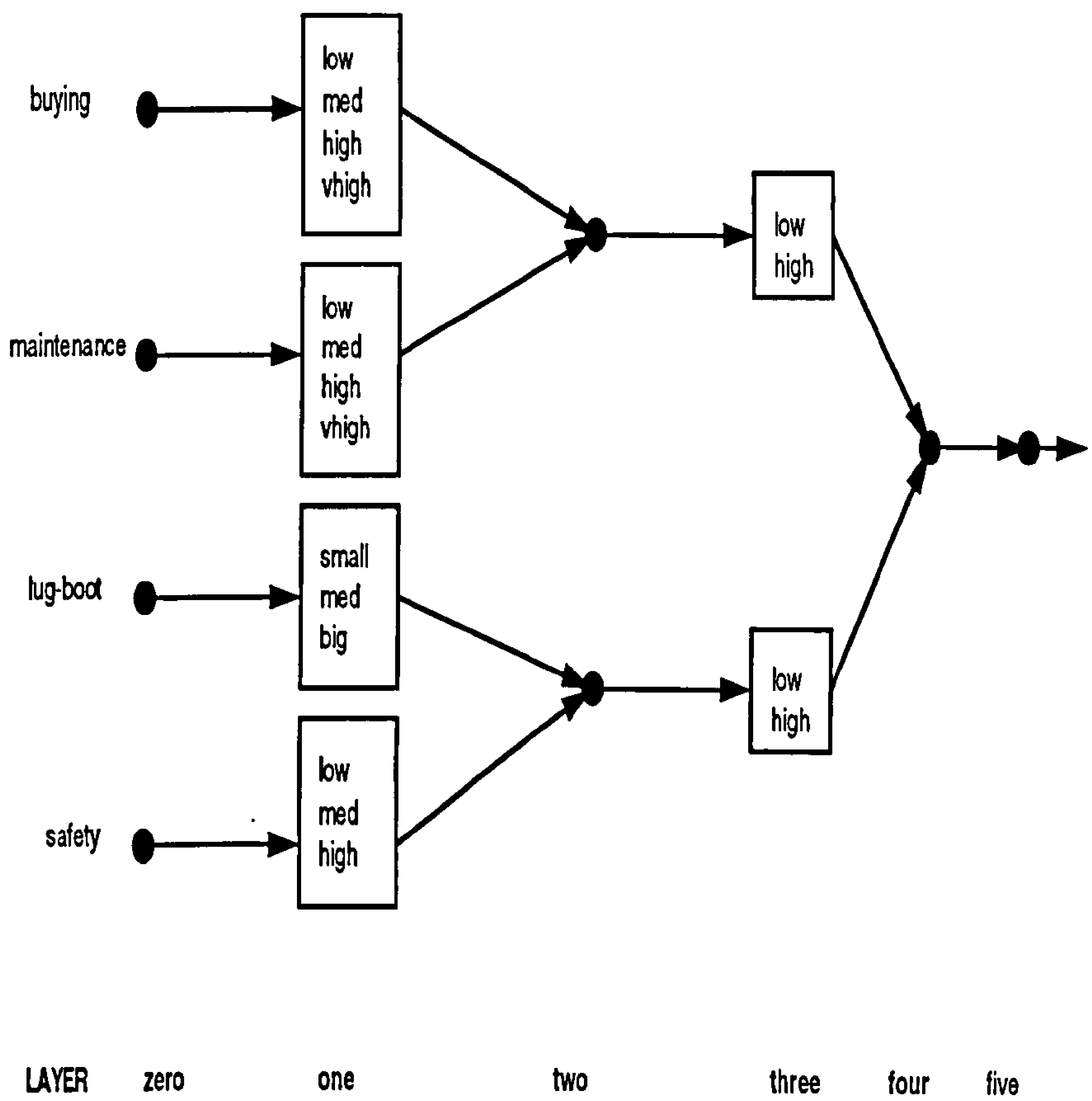

Fig. 7.2. AFPL for the Car Evaluation problem

Early results showed that the AFPL struggles with all the data since the data is swamped by one class and the precise factors have been removed from the problem.Two of the classes were removed to see whether the AFPL can differentiate between acceptable and unacceptable. The remaining data set was partitioned into three distinct sets of data - training, validation and test. The training set is the set of data used to train the AFPL and modify the parameters of the type-2 fuzzy system. The validation set is used at each epoch to test the behaviour of the network during training and the test data used to assess the efficacy of the trained AFPL. The best results are shown in

Table 7.3, indicating correct classification 70% of the time. ROC curves for a number

Number of epochs		10000
Root Mean Square Error		0.0942
	Unacceptable	Acceptable
Correct	167 (70)	49 (70)
Wrong	70 (30)	21 (30)
Total Correct		216 (70)

Table 7.3. The best network for the Car AFPL

of runs show that the AFPL is stable in that similar ROC curves are produced for a variety of different parameters. The results of applying the AFPL to this particular problem indicate that given linguistic input and crisp output (type-1 membership) the AFPL learns the membership grades of the type-2 fuzzy sets that represent perceptions. There is nothing about the problem chosen that would make one believe that the AFPL could not be used on other problems where type-2 fuzzy sets can be used to represent perceptions.

The AFPL offers, for the first time, the ability to learn a type-2 fuzzy system where the inputs are perceptions. This has implications for problems that employ perceptions on non measurable domains. For example, it is clear that many medical applications would be particularly suitable for the AFPL approach (indeed current work already alluded to on modelling nursing intuition supports this view). Another suitable problem could be, for example, psychometric testing for job applications where the qualities of an applicant may well best be described using words that represent perceptions. This type-2 paradigm offers opportunities to tackle problems (outside those conventionally tackled by type-1 fuzzy logic) where human decision making lays at the centre of the process. Another important aspect of this work is that users of type-2 fuzzy systems will find them more intuitive than other approaches since, by their very nature, they allow for representations that are linguistic, not numeric.

7.5 Further Work

This thesis has presented a novel adaptive type-2 fuzzy system but in so doing has raised some issues for further research that are common to most adaptive networks.

- The AFPL employs a number of operators at various points. In particular, there are a variety of t-norms and t-conorms that are possible which will almost certainly impact the performance of the AFPL for a given application. Further research could carry out a detailed study of the impact of various operators in a variety of applications with particular consideration to the performance in classification and stability of the learned type-2 fuzzy systems.

- The learning algorithm adopted is essentially gradient descent and for a relatively large system is slow. It is also known to find local minima. Further research would consider the alternatives available such as the use of simulated annealing and genetic algorithms which may avoid the problem of local minima and speed up the process of learning.

- The defuzzification method requires further investigation and comparisons could be made between the variety on offer. A research project could investigate whether particular applications are best suited to a particular defuzzification technique by conducting a series of experiments and comparing performances.

- As with any adaptive system a network topology has to be selected. Future work could consider the effects of the topology on the solution to a particular problem. Another project could explore whether genetic algorithms can be used to produce an optimal network topology.

- As has already been alluded to (Section 5.1) there is work to be done on the number of rules and the 'trade off' with the number of membership grades in each type-2 fuzzy set. In other words how can the optimum number of rules be determined for a given type-2 fuzzy system and a given set of data. Again a GA approach is worth consideration for this particular problem.

- A study is required into the scalability of the AFPL. That is, as the number of inputs increases the training time will increase but, by how much? This would require an investigation of applications where the number of inputs are different.

- The AFPL, currently, only works with numeric outputs. A true AFPL would allow for linguistic outputs which would be membership grades of type-2 fuzzy sets. This is a difficult problem in that an adaptive learning algorithm requires some comparison with expected and actual output. The approach could be to find some 'distance' measure between the expected type-2 fuzzy set and the type-2 fuzzy set produced by the AFPL (prior to defuzzification) and to use this measure to inform the parameter modification algorithm. Indeed, this would remove the need for defuzzification.

7.6 Summary

This thesis has concerned itself with type-2 fuzzy sets for knowledge representation - in particular the representation of perceptions. The research indicates that type-2 fuzzy sets have much to offer for decision making and there is almost certain to be an explosion in the application of type-2 fuzzy sets to problems that require this higher level of imprecision. The argument throughout has been that type-1 fuzzy systems are not fuzzy. Type-2 fuzzy systems are more fuzzy by allowing for linguistic membership grades. Results of using type-2 fuzzy sets to represent perceptions of a medical consultant showed early promise but highlighted the need to learn the membership grades of the type-2 fuzzy sets. The major contribution made by the work reported here is the Adaptive Fuzzy Perception Learner that is an adaptive network that learns the content of a type-2 fuzzy system representing linguistic association between perceptions. The AFPL has been applied to two problems. The results indicate that the approach has much to offer when tackling problems containing perceptions.

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APPENDICES

Appendix A

Some Fuzzy Definitions

A.1 Fuzzy Union

As described in the text, for a function u to qualify for fuzzy union of two fuzzy sets A and B the following must be true:

u *is commutative* if $u(a,b) = u(b,a)$

u *is associative* if $u(u(a,b),c) = u(a,u(b,c))$

u *is monotonic* if $a \leq a'$ and $b \leq b'$ then $u(a,b) \leq u(a',b')$

and $u(0,0) = 0; u(0,1) = u(1,0) = u(1,1) = 1$ i.e. has zero as unit element and behaves as with crisp sets.

A.2 Fuzzy Intersection

As described in the text, for a function i to qualify for fuzzy intersection of two fuzzy sets A and B the following must be true:

i must be *commutative, associative* and *monotonic* as above

and $i(1,1) = 1; i(0,1) = i(1,0) = i(0,0) = 0$ i.e. behaves as the classic intersection.

Appendix B

An email from the Berkeley Initiative for Soft Computing - Prof. L.A. Zadeh

Subject: updated statistics
Date: Sat, 12 Feb 2000 22:44:08 +0100 (MET)
From: "Michelle T. Lin" ¡michlin@cs.berkeley.edu¿
To: Multiple recipients of list ¡fuzzy-mail@dbai.tuwien.ac.at¿

****
Berkeley Initiative in Soft Computing (BISC)
****

To: BISC Group From: L. A. Zadeh ¡zadeh@cs.berkeley.edu¿ Re: Updated statistics

Dear Members of the BISC Group:

At my request, Ann Jensen, Head of the Mathematics, Statistics and Astronomy Library, has come up with the following statistics regarding the number of papers listed in the INSPEC database (Engineering/Science) and Math.Sci.Net database (Mathematics) which contain the words "fuzzy," "fuzzy control," or "soft computing" in title. The data for 1999 are not complete.

INSPEC/fuzzy Math.Sci.Net/fuzzy
1970-1980 566 453
1980-1990 2,361 2,476
1990-2000 21,162 5,017

1970-2000 24,089 7,946

INSPEC/fuzzy control Math.Sci.Net/fuzzy control
1970-1980 38 0
1980-1990 214 39
1990-2000 4,356 175

1970-2000 4,608 214

INSPEC/soft computing Math.Sci.Net/soft computing
1970-1980 0 0
1980-1990 0 0
1990-2000 1,962 5

1970-2000 1,962 5

Warm regards to all,

Lotfi

cc. Ann Jensen

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Appendix C

The Source Code for the Type-2 ANFIS

Appendix D

The Original Car Evaluation Database

1. Title: Car Evaluation Database

2. Sources:
   (a) Creator: Marko Bohanec
   (b) Donors: Marko Bohanec (marko.bohanec@ijs.si)
   Blaz Zupan (blaz.zupan@ijs.si)
   (c) Date: June, 1997

3. Past Usage:

The hierarchical decision model, from which this dataset is derived, was first presented in

M. Bohanec and V. Rajkovic: Knowledge acquisition and explanation for multi-attribute decision making. In 8th Intl Workshop on Expert Systems and their Applications, Avignon, France. pages 59-78, 1988.

Within machine-learning, this dataset was used for the evaluation of HINT (Hierarchy INduction Tool), which was proved to be able to completely reconstruct the original hierarchical model. This, together with a comparison with C4.5, is presented in

B. Zupan, M. Bohanec, I. Bratko, J. Demsar: Machine learning by function decomposition. ICML-97, Nashville, TN. 1997 (to appear)

4. Relevant Information Paragraph:

Car Evaluation Database was derived from a simple hierarchical decision model originally developed for the demonstration of DEX (M. Bohanec, V. Rajkovic: Expert system for decision making. Sistemica 1(1), pp. 145-157, 1990.). The model evaluates cars according to the following concept structure:

CAR	car acceptability
. PRICE	overall price
. . buying	buying price
. . maint	price of the maintenance
. TECH	technical characteristics
. . COMFORT	comfort
. . . doors	number of doors
. . . persons	capacity in terms of persons to carry
. . . lug_boot	the size of luggage boot
. . safety	estimated safety of the car

Input attributes are printed in lowercase. Besides the target concept (CAR), the model includes three intermediate concepts: PRICE, TECH, COMFORT. Every concept is in the original model related to its lower level descendants by a set of examples (for these examples sets see http://www-ai.ijs.si/BlazZupan/car.html).

The Car Evaluation Database contains examples with the structural information removed, i.e., directly relates CAR to the six input attributes: buying, maint, doors, persons, lug_boot, safety.

Because of known underlying concept structure, this database may be particularly useful for testing constructive induction and structure discovery methods.

5. Number of Instances: 1728
(instances completely cover the attribute space)

6. Number of Attributes: 6

7. Attribute Values:

buying	v-high, high, med, low
maint	v-high, high, med, low
doors	2, 3, 4, 5-more
persons	2, 4, more
lug_boot	small, med, big
safety	low, med, high

8. Missing Attribute Values: none

9. Class Distribution (number of instances per class)

class	N	N[%]
unacc	1210	(70.023 %)
acc	384	(22.222 %)
good	69	(3.993 %)
v-good	65	(3.762 %)

names file (C4.5 format) for car evaluation domain

| class values

unacc, acc, good, vgood

| attributes

buying: vhigh, high, med, low.
maint: vhigh, high, med, low.
doors: 2, 3, 4, 5more.
persons: 2, 4, more.

lug_boot: small, med, big.
safety: low, med, high.

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Appendix E

The Car Evaluation Database Modified for the Type-2 ANFIS

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Appendix F

Publications by R. I. John in Fuzzy Logic and Fuzzy Sets and Copies of Published Papers by R.I. John Directly Relevant to the Thesis

John, R.I., Innocent,P.R. and Barnes, M.R., (2000) "Neuro-Fuzzy Clustering of Radiographic Tibia Image Data using Type-2 Fuzzy Sets". Information Sciences, 125/1-4, pp 203–220.

John, R.I. and Mooney, G.J., (2000) "Fuzzy User Modelling and the World Wide Web". To appear in Knowledge and Information Systems.

Lake, S. and John, R.I., (2000) "Patient Assessment in Nursing Care using Fuzzy Logic" Proceedings Nursing Informatics Conference, pp 468–475

John, R.I., (2000) "Fuzzy Sets and Knowledge Representation" in Fuzzy Systems in Medicine, Studies in Fuzziness and Soft Computing, Physica-Verlag, pp 78–79

Innocent, P.R. , John, R.I. and Barnes, M.R., (2000) "Neuro-fuzzy models of radiographic image classification" in Fuzzy Systems in Medicine, Studies in Fuzziness and

Soft Computing, Physica-Verlag, pp 361–393

John, R.I., (1999) "Type-2 Fuzzy Sets", Expert Update, 2(2)

John, R.I. and Czarnecki, C.(1999) "An Adaptive Type-2 Fuzzy System for Learning Linguistic Membership Grades". Proc. 8th Intl. Conf. on Fuzzy Systems FUZZ-IEEE'99, pp 1552–1556.

John, R.I. and Czarnecki, C., (1998) "A Type 2 Adaptive Fuzzy Inferencing System", Proceedings IEEE Systems, Man and Cybernetics, pp 2068–2073.

John, R.I., (1998) "Type 2 Fuzzy Sets: An Appraisal of Theory and Applications", International Journal of Uncertainty, Fuzziness and Knowledge Based Systems, 6(6), pp 563–576.

John, R.I., (1998) "Type 2 Fuzzy Sets for Knowledge Representation and Inferencing", Proc. 7th Intl. Conf. on Fuzzy Systems FUZZIEEE'98, pp 1003–1008.

John, R.I. and Innocent, P.R. and Barnes, M.R., (1998) "Type 2 Fuzzy Sets and Neuro-Fuzzy Clustering of Radiographic Tibia Images", Proc. 7th Intl. Conf. on Fuzzy Systems FUZZ-IEEE'98, pp 1373–1376.

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John R I and Bennett S C, (1997) "The Use of Fuzzy Sets for Resource Allocation in an Advance Request Vehicle Brokerage System - a Case Study", Journal of the Operational Research Society, 48, pp 117–123.

John, R.I., (1996) "Type 2 Fuzzy Sets for Community Transport Scheduling", Proceedings of of the Fourth European Congress on Intelligent Techniques and Soft Computing - EUFIT96, 2, 1369–1372.

John R I, (1996) "Representing Knowledge with Type 2 Fuzzy Sets", Proceedings of Knowledge Transfer '96, Ed. A. Behrooz, pp 82-89

Innocent, P and Barnes, M and Keighley J and John R, (1996) "Neuro-fuzzy clustering of shin images", Proceedings of 2nd International Conference on Neural Networks and Expert Systems in Medicine and Healthcare (NNESMED96), Ed. Emmanuel C. Ifeachor, pp 14-21

John R I and Bennett S C, (1996) "Fuzzy Sets and Community Transport", in Fuzzy Logic, edited by Prof. J F Baldwin, pp 245-253, John Wiley: Chichester, Sussex. ISBN 0471 96281 3.

Bennett S C and John R I, (1995) "Fuzzy Inferencing Applied to Vehicle Assignment in Community Transport", International Symposium on Fuzzy Logic '95, Zurich, Switzerland, May 1995.

John R I and Bennett S C, (1995) "Fuzzy Sets and Community Transport", Applied Decision Technologies Conference, Brunel University, April 1995.

John R I, (1995) "Fuzzy Inferencing Systems", De Montfort University School of Computing Sciences Working Paper No. 64

Bennett S C and John R I, (1994) "The Application of Fuzzy Set Theory to Vehicle Brokerage", 27th ISATA, Dedicated Conference on Mechatronics, Aachen, Germany, November 1994.

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Vol. 6, No. 6 (1998) 563–576

TYPE 2 FUZZY SETS: AN APPRAISAL OF THEORY AND APPLICATIONS

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This paper provides a guide and tutorial to type 2 fuzzy sets. Type 2 fuzzy sets allow for linguistic grades of membership thus assisting in knowledge representation. They also offer improvement on inferencing with type 1 sets. The various approaches to knowledge representation and inferencing are discussed, with worked examples, and some of the applications of type 2 sets are reported.

Keywords: Type 2 Fuzzy Sets, Knowledge Representation, Inferencing, Linguistic Grades

1. Introduction

This paper has been written to provide a guide to *type 2 fuzzy sets*, review the research in this area, discuss the various issues surrounding their use and provide a 'tutorial' on type 2 sets.

Any application using fuzzy sets requires the developer to describe the membership function by numbers, in the discrete case, or by a function where the fuzzy set has a continuous membership function. So, any system employing fuzzy sets represents the fuzziness of the particular problem using a 'non-fuzzy' representation. As Klir and Folger[1] point out, *" ..it may seem problematical, if not paradoxical, that a representation of fuzziness is made using membership grades that are themselves precise real numbers."* As well as this paradox there are two particular problems that are faced by fuzzy systems developers:

- **Determining membership functions.**

  One of the major problems faced by any fuzzy system developer is that of determining the membership function[2]. Attaching numbers to features of a problem is notoriously difficult and experts in a domain are unlikely to arrive at a consensus. The traditional statistical approaches[3] use averages or percentages based on informed guesses made by experts. Adaptive approaches to determining membership functions, such as neural networks[4,5] and genetic algorithms[6,7] provide an alternative. These methods offer help in determining membership functions and have been applied successfully to control problems but they do not utilise fully the knowledge available from the expert.

- **Inferencing methods.**

  Inferencing with type 1 sets presents another choice for the fuzzy system developer (described in detail in Section 4). The method of combining type 1 sets using for example 'AND', 'OR' or for implication in an if-then rule traditionally takes one particular form. However there are two forms, that are equivalent for boolean inferencing, that should be adopted. If these two forms are used, an upper and lower bound are determined for the result of inferencing with fuzzy sets.

Zadeh[8] introduced the idea of type 2 fuzzy sets that help with both these problems (researchers differ in their terminology with some calling them Type II). The rest of this paper is structured as follows: Section 2 defines type 2 sets, Section 3 considers the different approaches for combining them with Union and Intersection, Section 4 describes inferencing with type 2 sets and Section 5 discusses some of the applications of type 2 sets.

2. Definition Of Type 2 Fuzzy Sets

Type 2 sets that allow linguistic membership grades were initially defined by Zadeh[8]. However a particularly clear definition of a type 2 fuzzy set is provided by Mizumoto and Tanaka[9] - *"A fuzzy set of type 2 is defined by a fuzzy membership function, the grade (that is, fuzzy grade) of which is a fuzzy set in the unit interval [0,1], rather than a point in [0,1]"*. A fuzzy set of type 2, A, in a set X, is the fuzzy set characterised by the fuzzy membership function μ_A as

$$\mu_A : X \rightarrow [0,1]^{[0,1]} \tag{1}$$

where $\mu_A(x)$ is known as a *fuzzy grade*, a fuzzy set in [0,1]. As an example, suppose we wish to use the fuzzy set 'tall'. Using type 1 fuzzy sets we might say that Michael Jordan is tall to degree 0.95, Danny Devito to degree 0.4 and Robert John to degree 0.6. This can be written as

$$tall = 0.95/MichaelJordan + 0.4/DannyDevito + 0.6/RobertJohn.$$

A type 2 interpretation of this set could be

$$tall = High/MichaelJordan + Low/DannyDevito + Medium/RobertJohn$$

where High, Low and Medium are themselves fuzzy sets.

Figure 1 shows what the sets High, Low and Medium might look like if represented graphically. As can be seen, the base axis takes values between 0 and 1 as does the vertical axis (μ). Type 1 sets have a base axis representing the *domain*

- in this case the height of an individual. Type 2 sets employ type 1 sets as the membership grades and these grades are usually on the domain [0,1]. Therefore, these fuzzy sets of type 2 allow for the idea that the members of a fuzzy set do not necessarily have membership grades in [0,1] but the degree of membership for the member is itself a fuzzy set. As Yager[10] points out *"The usefulness of fuzzy subsets of type II is that it enables us to extend membership grades to linguistic values"*.

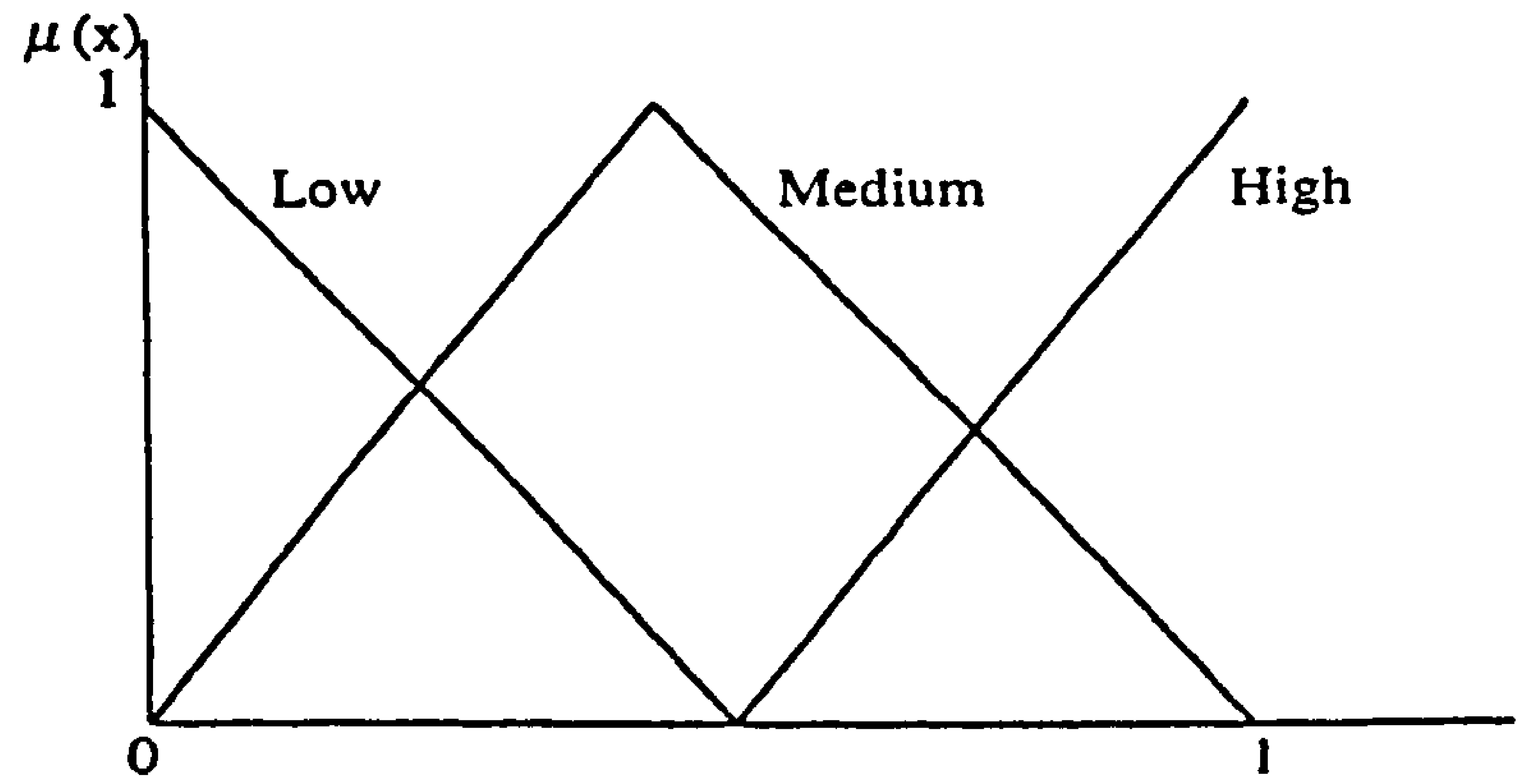

Fig. 1. The Fuzzy Sets High, Low and Medium.

A form of type 2 sets, *interval valued fuzzy sets*, also relaxes the requirement for precise membership functions. In this case for each x, $\mu(x)$ is an interval in $[0,1]$. Nguyen *et al*[11] argue that the use of intervals is necessary to to describe an expert's degree of belief. Gehrke *et al*[12] state *"Many people believe that assigning an exact number to an expert's opinion is too restrictive, and that the assignment of an interval of values is more realistic"*. This type of set also plays an important role in inferencing with fuzzy sets as will be seen in Section 4.

The theoretical properties of type 2 sets are described extensively by Mizumoto and Tanaka[9,13] and the reader is referred there for the details.

3. The Union and Intersection of Type 2 Fuzzy Sets

To enable the use of type 2 sets in a computer system that uses if-then rules a method is required for computing the intersection and union of two type 2 sets. There are two different approaches reported in the literature. Using the notation provided by Mizumoto and Tanaka[9] we have type 2 sets, A and B, in X and $\mu_A(x)$ and $\mu_B(x)$ are two *fuzzy grades* of A and B respectively, represented as:

$$\mu_A(x) = f(u_1)/u_1 + f(u_2)/u_2 + \dots + f(u_n)/u_n = \sum_i f(u_i)/u_i$$

$$\mu_B(x) = g(w_1)/w_1 + g(w_2)/w_2 + ... + g(w_m)/w_m = \sum_j g(w_j)/w_j$$

where the functions f and g are membership functions of fuzzy grades.

3.1. *Union and Intersection using the Extension Principle*

The most widely used definitions for union and intersection of type 2 sets are provided by Zadeh[14]. His work relies on the use of the *extension principle*[8]. This states that if $*$ is a binary operation in X then this operation can be applied to A and B by

$$A * B = \sum_{i,j}(f(u_i) \wedge g(w_j))/(u_i * w_j)$$

This leads to the following definitions for union and intersection of type 2 sets.
Union

$$A \cup B \Leftrightarrow \mu_{A\cup B}(x) = \mu_A(x) \sqcup \mu_B(x) \quad (2)$$
$$= \sum_{i,j}(f(u_i) \wedge g(w_j))/(u_i \vee w_j) \quad (3)$$

Intersection

$$A \cap B \Leftrightarrow \mu_{A\cap B}(x) = \mu_A(x) \sqcap \mu_B(x) \quad (4)$$
$$= \sum_{i,j}(f(u_i) \wedge g(w_j))/(u_i \wedge w_j) \quad (5)$$

where $\sqcup$ denotes *join* and $\sqcap$ denotes *meet*[15]. Of particular interest are the implications these definitions have for handling fuzzy sets. This is best illustrated by using a simple example.

Consider the type 1 situation first. Suppose there are three people a,b and c where a is tall to degree 0.2 and heavy to degree 0.4, b is tall (0.7) and heavy (0.5) and c is tall (0.95) and heavy (0.8). The type 1 sets representing tall and heavy are

$$tall = 0.2/a + 0.7/b + 0.95/c$$

$$heavy = 0.4/a + 0.5/b + 0.8/c.$$

So, suppose we describe somebody who is tall and heavy as being big then we would arrive at

$$big = tall \cap heavy.$$

Using min for intersection

$$big = 0.2/a + 0.5/b + 0.8/c.$$

The outcome is that, for example, a is big to degree 0.2. Now consider tall and heavy as type 2 sets where

$$tall = low/a + medium/b + high/c$$

$$heavy = medium/a + medium/b + high/c$$

where for example

$$low = 1.0/0.0 + 0.75/0.1 + 0.5/0.2 + 0.25/0.3$$

$$medium = 0.33/0.3 + 0.67/0.4 + 1/0.5 + 0.67/0.6 + 0.33/0.7$$

$$high = 0.25/0.7 + 0.5/0.8 + 0.75/0.9 + 1.0/1.0$$

As with the type 1 example, big is interpreted as tall and heavy ($big = tall \cap heavy$) then for person a

$$big_a = tall_a \sqcap heavy_a$$

which is the meet of the two fuzzy grades low and medium. Using the definition for meet described in (8) above we have that

$$\begin{aligned} big_a &= 0.33/0.0 + 0.67/0.0 + 1.0/0.0 + 0.67/0.0 + 0.33/0.0 \\ &\quad + 0.33/0.1 + 0.67/0.1 + 0.75/0.1 + 0.67/0.1 + 0.33/0.1 \\ &\quad + 0.33/0.2 + 0.5/0.2 + 0.5/0.2 + 0.5/0.2 + 0.5/0.2 + 0.33/0.2 \\ &\quad + 0.25/0.3 + 0.25/0.3 + 0.25/0.3 + 0.25/0.30.25/0.3 \\ &= 1.0/0.0 + 0.75/0.1 + 0.5/0.2 + 0.25/0.3 \\ &= low \end{aligned}$$

There are various points to be made about this comparison:

1. The type 1 representation requires the 'expert' to attach a number to each person to describe their 'tallness' or 'heaviness' whereas the type 2 approach only requires a fuzzy grade.
2. The fuzzy grades low, medium and high are the same for both tall and heavy for simplicity purposes. They need not be.
3. The fuzzy grades low, medium and high are membership functions that still have to be determined.
4. The set 'big' for the type 1 approach is a type 1 set whereas the type 2 approach yields a type 2 solution.

An alternative definition for intersection is provided by Zadeh[14,16]. This definition relies on the level-set form of the extension principle. If A is a subset of U then an α level set is a non-fuzzy set, A_α which comprises all elements of U whose grade of membership in A is greater than or equal to α . A fuzzy set A can be decomposed into its constituent level sets

$$A = \sum_{\alpha} \alpha A_\alpha$$

and, suppose that f is a mapping from U to V and A is a subset of U, then

$$f(A) = \sum_{\alpha} \alpha f(A_\alpha).$$

This leads to the following definition for the intersection of type 2 sets

$$\mu_{A \cap B} = \sum_{\alpha} \alpha(\mu_A^\alpha \wedge \mu_B^\alpha)$$

where μ_A^α means the membership function for the α level set of A. This method produces identical results to those provided by Eqns. 3 and 5.

3.2. *The Numerical Representation Approach*

Hisdal[17] proposes a different approach which uses the idea of the *Numerical Representation* of a fuzzy set of type 2 (Hisdal's work actually develops the idea for sets of type N but for our purposes the type 2 case will be explained). Using Hisdal's notation we let C^{t1} be a type 1 fuzzy set where

$$C^{t1} = \sum_{j=1}^{J0} c_j/v_j = \sum_{j=1}^{J0} \Pi_C^{t1}(v_j)/v_j \qquad (6)$$

with t1 indicating 'type 1'. So a type 2 set can be represented by specifying for each v_j, and for each value of $\Pi(v_j)$ the possibility of this $\Pi(v_j)$ i.e. $\Pi(\Pi(v_j))$ or $\Pi^{t2}(v_j)$. The set of allowed numerical values for a type 1 set is denoted by M^{t1} and for $\Pi^{t2}(v_j)$ is denoted by M^{t2}. The numerical representation of a fuzzy set of type 2 is therefore reduced to a set of type 1 in the two dimensional space VXM^{t1}. The example given in the original paper considers the type 2 fuzzy set 'smallish' with four members v_1, v_2, v_3, v_4 whose grades in this set are respectively very high, high, medium and low which are represented in table 1. The union and intersection for type 2 sets are performed by carrying out these operations using the normal definitions of union and intersection for type 1 sets. To illustrate this example Hisdal defines another type 2 set called 'biggish' (see table 2). Table 3(a) and 3(b) show the union and intersection of smallish and biggish. The solution therefore is always a type 2 set.

If Zadeh's interpretation of the union is used (Eqn. 3) the union of smallish and biggish for v_1 is

$$\mu^{v_1}_{smallish \cup biggish} = 0/0 + 0/0.25 + 0/0.75 + 1/1$$

which is clearly different from Hisdal's method. Hisdal compares the two approaches. The difference is that using Zadeh's approach the meaning of $C = A \cap B$

Table 1. A numerical presentation of the type 2 fuzzy set 'smallish'

	$\prod(v_j) =$				
	0.00	0.25	0.50	0.75	1.00
$v = v_1$	0	0	0	0	1
v_2	0	0	0	0.5	1
v_3	0	0.5	1	0.5	0
v_4	1	0.5	0	0	0

Table 2. A numerical presentation of the type 2 fuzzy set 'biggish'

	$\prod(v_j) =$				
	0.00	0.25	0.50	0.75	1.00
$v = v_1$	1	0.5	0	0	0
v_2	0	0.5	1	0.5	0
v_3	0	0	0	0.5	1
v_4	0	0	0	0	1

is the fuzzy set 'A AND B' as if A and B had been type 1 sets. For each element of the universe set the fuzzy grade of C is obtained by using the special min operation described above (Eqn. 4) on the membership grades of A and B which leads to a fuzzy grade. For Hisdal's definition, in contrast, the fuzzy set C is described by the fuzzy set whose possibilities are $\Pi_C^{t2}(v_j) = \Pi_A^{t2}(v_j)$ AND $\Pi_B^{t2}(v_j)$. As Hisdal states *"..the AND connective is now between the possibilities of v_j for A and B, not between the fuzzy sets themselves"* and, as has been described, requires no new min definition. Hisdal's definition, in contrast with Zadeh's, does not require the limitation that A and B must be convex sets.

4. Inferencing and Type 2 Fuzzy Sets

This section considers two of the approaches that have been adopted for inferencing on type 1 sets using type 2 sets (Gorzalczany[18] reports on a fuzzy inference method for type 2 sets). The standard method for inferencing with type 1 sets is based on the idea of *generalised modus ponens (gmp)* and is well established[16]. Hisdal[17] argues that this standard method for implication does, in many situations, not reflect the inherent uncertainty in any particular situation and that type 2 sets allow for inferencing with type 1 sets that reflects this uncertainty. In particular the work shows that the type 1 relation for if-then often produces solutions that are incorrect. Another hypothesis, put forward by Türkşen, is that gmp is inadequate since type 1 sets only express first order semantics and that the introduction of type 2 sets provides increased expressive power[19].

4.1. *Numerical Representation Inferencing*

Hisdal[17] presents a long and detailed case for a method of inferencing on type 1 sets using type 2 relations. Hisdal's interpretation of inferencing with if-then rules relies on two ideas:

Table 3. The union and intersection of biggish and smallish shown using the numerical representation.

	$\prod(v_j) =$ 0.00	0.25	0.50	0.75	1.00
$v = v_1$	1	0.5	0	0	1
v_2	0	0.5	1	0.5	1
v_3	0	0.5	1	0.5	1
v_4	1	0.5	0	0	1

(a) The union of biggish and smallish

	$\prod(v_j) =$ 0.00	0.25	0.50	0.75	1.00
$v = v_1$	0	0	0	0	0
v_2	0	0	0	0.5	0
v_3	0	0	0	0.5	0
v_4	0	0	0	0	0

(b) The intersection of biggish and smallish

Table 4. The truth table for a non fuzzy if-then statement as used in natural language. The if-then statement is assumed to be true.

$A \to C$	A	C
T	T	T
T	F	BLANK

1. Mathematical logic allows for the implication to be true or false and the truth value of the consequent is specified, not inferred.

2. In fuzzy set theory the if-then statement is assumed to be true and infers a value of y from x. This is a better representation of the if-then statement as used in natural language.

These two points lead to table 4 where the if-then statement is assumed to be true and that, therefore, when the antecedent is false then the consequent is a 'don't know' state (BLANK using Hisdal's notation).

The if-then statement 'If $x = A$ Then $y = C$' is written as $C \mid A$ where A and C are fuzzy subsets of the two universes U and V respectively. The entries of the $C \mid A$ type 2 relation are given by (Hisdal[17] eqn. 6.1.2):

$$\Pi_{C|A}(\Pi(v_i) \mid u_i), \qquad i = 1, ..., I0 \qquad j = 1, ..., J0 \qquad \Pi(v_j) \in M^{t1}. \tag{7}$$

Assuming we have the $C \mid A$ relation then we choose a particulent $P \subset U$ and deduce a set $D \subset V$. The inference operation is carried out in two steps:

1. The $C \mid A$ relation is particulated by P and this is given by

$$\Pi_{(A,C)_P}(u_i, \Pi(v_j)) = \Pi_P(u_i) \wedge \Pi_{(C|A)}(\Pi(v_j) \mid u_i). \tag{8}$$

2. Every row of the A, C relation is a fuzzy subset of V. The union of all the fuzzy subsets gives the deduced set D which is

$$\Pi_D(\Pi(v_j)) = \bigvee_{u_i} \Pi_{A,C}(u_i, \Pi(v_j)). \tag{9}$$

The $C \mid A$ and the A, C relations are, in general interval valued type 2 sets. Before we infer on an if-then statement we are in complete ignorance. Given this set the if-then statement adds new knowledge and the solution of the $C \mid A$ relation is a restriction of this set. Hisdal applies this idea to some situations without tackling a real problem. Clearly this approach produces a *different* solution to the traditional gmp however without the application to a real problem the efficacy of this approach is unproven.

4.2. *Inferencing with Interval Valued Type 2 Fuzzy Sets*

Türkşen has produced a body of work relating to the using type 2 sets when inferencing with fuzzy systems [19,20,21,22,23,24,25]. He argues that type 1 fuzzy sets and logics present concerns that can be tackled using type 2 sets. In particular he states[19] "*uncertainty models represented by the interval-valued Type II fuzzy sets and logics have a more expressive power*". He proposes that the combined linguistic expression of the linguistic values with the linguistic connectives should not be set arbitrarily to just one of the two fuzzy normal forms but to the interval generated by both.

4.2.1. *Knowledge Representation*

His work discusses four types of knowledge representation for fuzzy systems

1. Linguistic Expression;
2. Meta-Linguistic Expression;
3. Propositional Expression;
4. Computational Expression.

Linguistic Expression

He describes 16 basic linguistic expressions as the basis of the representation of natural language using logic (e.g. if the temperature is high then the pressure is low). In the case of two valued logic these concepts are represented by Disjunctive and Conjunctive Normal Forms (DNF and CNF) and that for any of these 16 terms DNF=CNF. However by generalising two valued logic to fuzzy logic this does not hold and DNF≠CNF[24,21]. For type 1 fuzzy logic, as well as for two valued logic, the shortest of the forms is adopted. The central hypothesis is that since fuzzy researchers have used either DNF *or* CNF there is a loss of information.

Meta-Linguistic Expression

This is simply a symbolic representation of linguistic expression. So implication may be represented as, for instance, $X_1 is A_1$ IMPLIES $X_2 is A_2$, $A_1 \rightarrow A_2$, NOT $A_1 or A_2$ amongst many, where X_1 and X_2 are meta-linguistic representations of the linguistic variables and A_1 and A_2 are the meta-linguistic values associated with X_1 and X_2.

Propositional Expressions

Here the meta-linguistic expressions are represented by normal forms. So for $A_1 \rightarrow A_2$ the DNF and CNF expressions are given by

$$\begin{aligned} DNF(A_1 \rightarrow A_2) &= (A_1 \cap A_2) \cup (\overline{A_1} \cap A_2) \cup (\overline{A_1} \cap \overline{A_2}) \\ CNF(A_1 \rightarrow A_2) &= \overline{A_1} \cup A_2 \end{aligned}$$

These normal forms are not equivalent in the case of fuzzy logic. Türkşen describes *fuzzy normal forms (FNFs)*[21]. So, for example he shows that the FNFs that correspond to the usual Zadeh connectives of max-min are

$$\begin{aligned} FDNF(A_1 \rightarrow A_2) &= (A_1 \cap A_2) \cup (\overline{A_1} \cap A_2) \cup (\overline{A_1} \cap \overline{A_2}) \\ &= DNF(A_1 \rightarrow A_2) \\ FCNF(A_1 \rightarrow A_2) &= \overline{A_1} \cup A_2 \\ &= CNF(A_1 \rightarrow A_2) \end{aligned}$$

Note that this equivalence between the boolean normal forms and the fuzzy normal forms is in propositional form only. Türkşen shows that this equality is true for all the 16 concept combinations outlined in his work.

Computational Expressions

This is simply the computational implementation of the propositional form. Computational expressions are formed over the elements of the fuzzy sets (i.e. the membership values) and the computational connectives.

4.2.2. *Fuzzy Inference*

The gmp for type 1 can be expressed as:

$$B' = A' o (A \rightarrow B)$$

where A, A', B and B' are fuzzy sets and o is the compositional rule of inference[16]. In other words we need a solution to

$$A \text{ AND } (A \rightarrow B) = B.$$

The fuzzy normal forms that mirror the Zadeh max-min approach have already been described above and for this particular situation are:

$$\begin{aligned} FDNF(A \rightarrow B) &= (A \cap B) \cup (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B}) \\ FCNF(A \rightarrow B) &= \overline{A} \cup B \end{aligned}$$

The result of this inference is an interval valued fuzzy set [FDNF,FCNF]. Türkşen[20] uses this approach to describe four methods for approximate reasoning using interval-valued sets. The four methods are:

1. A and A' are point valued and o is crisp;
2. A and A' are point valued and o is linguistic;
3. A and A' are interval valued and o is crisp;
4. A and A' is interval valued and o is linguistic

where 'point valued' means a type 1 set and 'crisp' o is as per Zadeh. He presents the results of using FDNF and FCNF to represent the interval valued fuzzy sets that arise from these four approaches. He then uses the simplest approach (as in 1 above) to tackle the problem of production planning in a paint factory and concludes that the results are 'good' and 'robust' and better represent the way a manager would describe the problem (linguistically). What does need investigating is whether the other approaches offer better solutions.

4.3. *Type 2 Inferencing: A Summary*

The Numerical Representation approach of Hisdal works from the premise that so as not to introduce false information into an if-then statement type 2 sets allow for a suitable representation of the state of ignorance before applying the statement. By using type 2 sets to represent this ignorance Hisdal provides an alternative approach to the conventional gmp and compositional rule of inference that reflects the uncertainty. Hisdal does not inference with type 2 sets as the antecedents but uses type 2 sets to enhance the inferencing with type 1.

In contrast the Türkşen approach highlights a deficiency in the traditional fuzzy inferencing approaches in that they only adopt one of the normal forms. For two valued logic this is not a problem as the normal forms are equivalent but for fuzzy logic they are not. His approach of using the normal forms to generate interval valued fuzzy sets appears to capture second order imprecision.

5. Applications of Type 2 Fuzzy Sets

The number of applications of type 2 sets reported in the literature is growing. Practical applications of type 2 sets include handling tolerances in fuzzy equations systems[26], fuzzy regression models[27], determining membership functions[28], community transport scheduling[29], the internet[30] and computer networks[31]. Yager [10] uses type 2 sets for multiobjective decision making. He describes a method that tackles the situation where the decision maker has a number of possible decisions he or she could make based on a number of objectives. Each decision provides a solution to the problem to some degree. The decision is a function of the objectives which depends on the how the objectives should be combined. There are many decision functions that could be used in any given situation and Yager proposes that there is a decison makers ideal decision function which the potential decision functions match to some degree. He presents a simple example of a decison maker choosing between three banks based on certain criteria. Since this is not an actual application the method is not validated on a real example and so significant conclusions cannot be drawn. He does point out that how to choose the best decision is difficult since the solution to the problem is a fuzzy set of type 2. Type 2 sets have been used to assist in the pre-processing of data for use with neural networks[34]. This research reports the results of the analysis of bone scans from stress related injuries to the tibia of athletes. Neural network based clustering techniques are used to assist the consultant in classifying the images. For this particular problem the consultant's interpretation of the image lends itself to representation using type 2 fuzzy sets. The results of the approach indicate that the use of neural clustering using a type 2 representation can improve the classification of shin images. Rocha[32] extends interval valued fuzzy sets to simulate human cognitive categorisation and concept combination with the notion of evidence sets. As an example of using interval valued fuzzy sets in control Wu[33] reports on their use for the control of mobile robots. He implements a new fuzzy control methodology that he calls fuzzy interval control (FIC). He implemented the FIC for successfully navigating a miniature robot in an unknown maze without touching the walls. Nguyen[11], in a comprehensive review paper on the application of interval computations and interval-valued degrees of belief describes a number of case studies, a number of which use interval valued fuzzy sets (the reader is referred there for the details).

So, there are a number of applications reported in the literature where type 2 sets play a role. They can be categorised as either using type 2 sets to represent knowledge in some way or as a way of capturing the unceratinty in the inferencng process with type 1 sets.

6. Conclusion

This paper has described type 2 sets and discussed various issues surrounding their use in real systems for knowledge representation and inferencing. Type 2 sets offer advantages over type 1 sets. Firstly, in knowledge representation an expert is likely to be more at home using linguistic grades or by representing their belief using intervals. Secondly, the uncertainty inherent in the fuzzy inferencing process can be captured in a number of ways by using interval valued fuzzy sets. The applications of type 2 sets in both control and decision making is on the increase - especially using interval valued fuzzy sets - and we can expect type 2 sets to play a more prominent role in applications generally.

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Neuro-fuzzy clustering of radiographic tibia image data using type 2 fuzzy sets

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Abstract

This paper presents the results of using type 2 fuzzy sets to assist in the pre-processing of data for use with neuro-fuzzy clustering for classification of sports injuries in the lower leg. This research is concerned with the analysis of bone scans from stress related injuries to the tibia. Of particular interest is whether neural network based clustering techniques can help the consultant in classifying the images. The work was motivated by the situation where there is a relatively small amount of relevant data and difficulties are faced by consultants in classifying the various types of injuries. For this particular problem the consultant's interpretation of the image lends itself to representation using type 2 fuzzy sets. This research sets out to address whether, with fuzzy neuro-clustering techniques some insights may be provided to the consultant that they can use along with their experience and knowledge. The results of this approach indicate that the use of neural clustering using a type 2 representation can improve the classification of shin images. © 2000 Elsevier Science Inc. All rights reserved.

Keywords: Fuzzy sets; Type 2 sets; Neural networks; Clustering, image analysis; Tibia; Stress fractures

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1. Introduction

This paper presents the results of using type 2 fuzzy sets to aid the classification of sports related injuries to the tibia. The most common site of stress fractures in athletes [12,8,14] is the tibia. Clinicians have built up significant expertise in recognising and classifying these injuries and have accumulated evidence to support them such as bone scans. A database of over 200 bone scan images has been established over the past 15 years and this has not been formally analysed with respect to the type or class of injury. This research is interested in neural network based clustering techniques for discovering useful clusters of these images.

The consultant faces a difficult task on an every day basis of classifying the images because the order and rate of presentation of injuries and associated symptoms varies. They are also relatively infrequent. It would not be possible with the paucity and imprecision of the data and the state of knowledge of the consultant to provide an 'automatic' classification. This research sets out to address whether, with type 2 fuzzy sets and fuzzy neuro-clustering techniques, some insights may be provided to the consultant that they can use along with their experience and knowledge.

The rest of the paper is structured as follows. Section 2 of this paper discusses the medical background to the problem, Section 3 discusses the preprocessing adopted for the tibia image data, Section 4 describes how type 2 sets have been used to represent the images, Section 5 discusses the neural network paradigms used in this research, Section 6 presents the results and finally in Section 7 the conclusions are presented.

2. Medical background

Bone scanning in sports injuries is used for the detection of an injury class called 'stress fractures'. Stress fractures are usually partial fractures or cracks which result from the application of habitual, non-violent, repetitive stress which exceeds the existing functional capacity of the bones. Patients who presented with exercise-induced lower leg pain were originally bone scanned in order to eliminate the possibility of stress fractures. However, it soon became apparent that the bone scan appearance was not simply a matter of normal or abnormal as several other patterns began to emerge (see Fig. 1 for an example image). These have been tentatively placed into one of seven classes based on an interpretation of the image:

1. *Normal.* The image is completely uniform.
2. *Athletic normal.* Lines of uniform increased uptake along the entire length of shin.
3. *Stress fracture.* There is a single, dense, localised area of uptake.

Fig. 1. A typical shin image.

4. *Medial tibial syndrome (MTS).* There is a line of increased uptake along some part of the lower half of the posterior cortex.
5. *Focal multiple spots.* These could be sub-clinical multiple stress fractures.
6. *Healing stress fracture.* Less obvious than a stress fracture.
7. *Patchy.* Non-uniform uptake.

The classes other than the MTS and stress fractures are based on experiential interpretation of the images alone because there are no confirming clinical factors. Hence there can be difficulties distinguishing between them. Experience has shown that re-examination of these images may reclassify them into different categories due to perceptual variation. This indicates that these classes may really be in a dimension of overlapping sub and super-classes. Since there has not been a systematic analysis of what the database of images contains, these dimensions are unknown. This has significance since the efficacy of a particular treatment may depend on correct classification.

3. Pre-processing of radiographic tibia images

From the discussion in Section 2, it is clear that there is no precise way, when viewing a radiographic image, of determining the problem with any given shin. Diagnosis is based on imprecise data. For instance, the distinctiveness of a line on the image or whether a line is 'much longer' than its width are important in assisting with the diagnosis. For this research, the interest was in seeing if the consultants' *description* of the image could be used to help diagnose patients. A series of knowledge acquisition sessions was carried out by interview. It became clear during the sessions that no precise rules described the relationship between the consultants' view of the images and the diagnosis. The approach taken, after much discussion, was to extract from the consultant those features that distinguish images and their possible values. A questionnaire was developed that, after prototyping and refining, the consultant used to describe all 203 images. A sample of these completed questionnaires was double checked by an independent expert. Some of the data contained in the questionnaire is essentially binary in nature whilst others could be described as imprecise or fuzzy. For input to a neural network these fuzzy categories had to be converted into a numeric format.

Earlier work [11] adopted a simple approach of representing the various categories by essentially a type 1 fuzzy set. For example the question about a line on an image 'Where is the line located?' was translated into a fuzzy set *location* with values 0(lower), 0.25(junction lower middle), 0.5(middle), 0.75(junction middle upper) and 1(upper). In other words an image which was described as 'junction middle upper' had 0.75 as the degree of membership of the set *location*. These numbers were chosen arbitrarily to reflect the ordering of the description of where the line is located. Even though this transformation is very crude the results of this early work were encouraging indicating that by using this method some relationship could be shown between the output from the questionnaire and the consultants' interpretation of the image. The next section describes how a more sophisticated approach using type 2 fuzzy sets was adopted for pre-processing.

4. Type 2 fuzzy pre-processing

As described in Section 3 the initial approach to representing fuzzy categories adopted a traditional fuzzy set approach (type 1) of assigning numeric grades to represent the membership values. This approach is crude in that it reduces the consultants' imprecise terminology to a number in [0,1]. This reduction of a linguistic term to a number could be seen as inappropriate given that the categories cannot be directly measured but can only be described by using linguistic terms. An alternative approach is the use of type 2 fuzzy sets which should more naturally represent the description offered by the consultant.

4.1. Type 2 fuzzy sets

Type 2 sets [1] that allow linguistic membership grades were initially described by Zadeh [26]. A clear definition of a type 2 fuzzy set is '*A fuzzy set of type 2 is defined by a fuzzy membership function, the grade (that is, fuzzy grade) of which is a fuzzy set in the unit interval* [0,1], *rather than a point in* [0,1]' [13]. A fuzzy set of type 2, A, in a set X, is the fuzzy set characterised by the fuzzy membership function μ_A as

$$\mu_A : X \to [0,1]^{[0,1]}, \tag{1}$$

where $\mu_A(x)$ is known as a *fuzzy grade*, a fuzzy set in [0,1]. Suppose we wish to use the fuzzy set 'tall' then with type 1 fuzzy sets we might say that Michael Jordan is tall to degree 0.95, Danny Devito to degree 0.4 and Robert John to degree 0.6. This can be written as

$$\text{tall} = 0.95/\text{Michael Jordan} + 0.4/\text{Danny Devito} + 0.6/\text{Robert John}.$$

A type 2 interpretation of this set could be

$$\text{tall} = \text{High}/\text{Michael Jordan} + \text{Low}/\text{Danny Devito} + \text{Medium}/\text{Robert John},$$

where High, Low and Medium are themselves fuzzy sets. These fuzzy sets of type 2, therefore, allow for the idea that the members of a fuzzy set do not necessarily have membership grades in [0,1] but the degree of membership for the member is itself a fuzzy set. As Yager [25] points out "The usefulness of fuzzy subsets of type II is that it enables us to extend membership grades to linguistic values". Turksen has written extensively on the use of type 2 sets for inferencing with if-then type rules [16,17,19–23]. The use of type 2 sets for decision making was described by Yager [25] in a very simple example of a company making a decision about a bank with which to place their business. Other applications of type 2 sets include handling tolerances in fuzzy equations systems [24], fuzzy regression models [7] and determining membership functions [18].

The next section discusses the use of type 2 sets to assist in the pre-processing of data for submission to two neural network paradigms for the problem of tibia image classification.

[1] Researchers sometimes describe them as type II sets. Throughout type 2 will be used and type 1 will refer to 'traditional' fuzzy sets.

4.2. Representation of the fuzzy categories using type 2 sets

The results of the consultant filling out the questionnaire have been represented using type 2 sets for the location and length of line. These are particularly important features of the image. For the *location* of line described in Section 3 there is a fuzzy set *location* where any image can be a member of the set to some degree (low, junction low middle, middle, junction middle upper, upper) where these grades themselves are fuzzy sets. Triangular fuzzy sets have been adopted to represent the various categories. This transformation of the data into type 2 sets was also carried out for *ratio* (ratio of length to width) which is now a type 2 set where the grades are themselves fuzzy sets – same, longer, much longer. The sets used are shown in Fig. 2. So the number of inputs for the networks were increased by eight (three numbers for each set replacing the one in the early work [11]). Each membership grade of location and ratio was represented for input by 3 numbers representing the triangular shapes.

The next section provides the main features of the neural network paradigms selected for the clustering process.

5. The neural network paradigms

There are many neural network approaches to clustering. The algorithms chosen for this problem were FuzzyART and the FuzzyMINMAX as they both offer approaches for clustering with fuzzy data.

5.1. FuzzyART

The adaptive resonance theory (ART) network paradigm is well known. The algorithm is reported in detail by various authors (e.g., [1–4,9]). The ART was

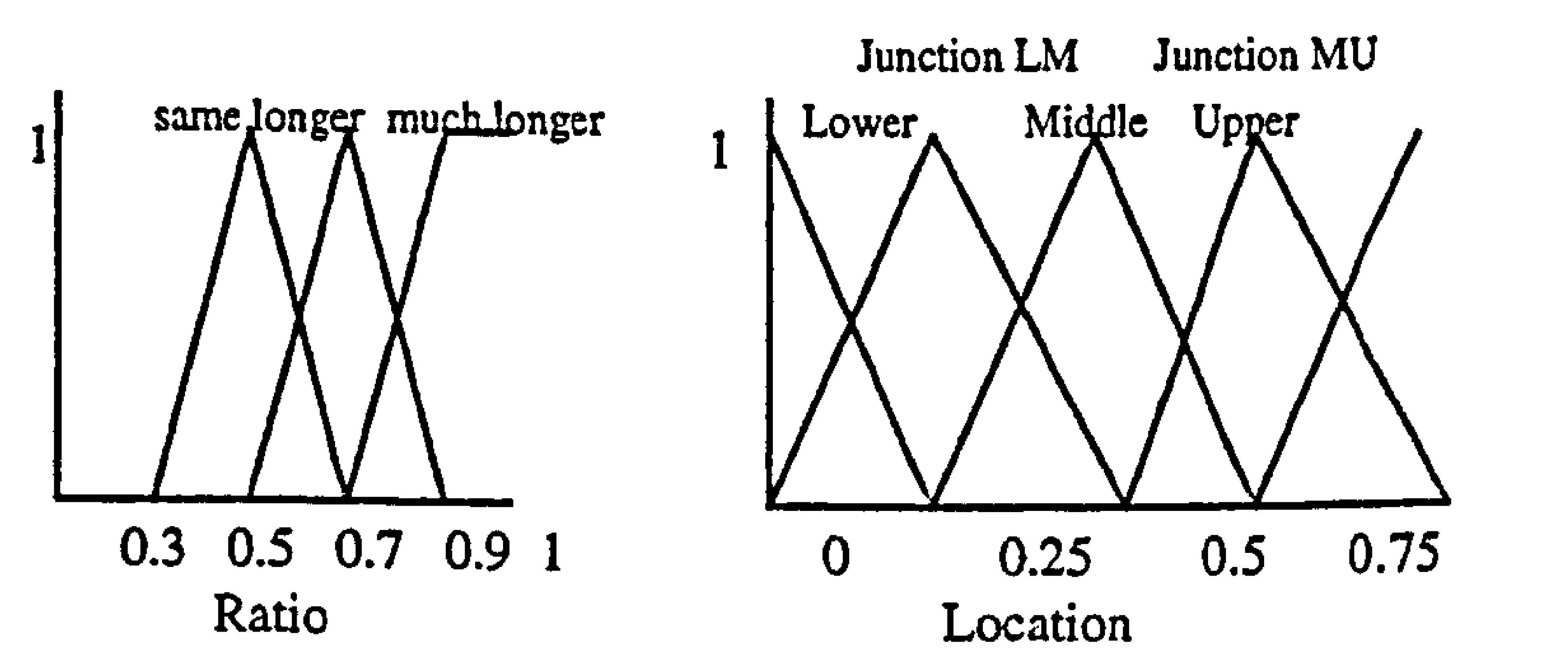

Fig. 2. The type 2 fuzzy sets location and ratio.

developed to model a massively parallel architecture for a self-organising neural pattern recognition architecture. The major feature of ART, proposed by Stephen Grossberg and Gail Carpenter, is that it has solved the stability–plasticity dilemma. That is the basic problem that neural networks have of not being able to learn new information on top of old. Trying to add a new training vector to an already trained network may have the catastrophic side-effect of destroying all the previous learning by interfering with the weight values. ART is able to switch modes between plastic (the learning state where the internal parameters can be modified) and stable (a fixed classification set).

ART networks map feature vectors in the input into the input layer where, after processing, it is mapped into a single node corresponding to a cluster identifier. If a node cannot be mapped into, a new node is established. In this way the ART network is 'plastic' though stable. This is a big advantage over fixed topology networks such as the Kohonen and multi layered feed forward nets. With the ART1 approach, the number of clusters in the output layer depends solely on the vigilance parameter and the variety in the input. The number of clusters differs as a function of the vigilance parameter. Of interest for this research is how the discovered clusters relate (if at all) to the classes defined by the medical consultants.

For this research the data is inherently fuzzy in nature so the modified ART algorithm FuzzyART [5] has been adopted. A combination of 2 FuzzyART networks (ARTa and ARTb) can be produced which are connected by a map to enable clusters to be mapped into each other (FuzzyARTMAP).

5.2. Fuzzy MINMAX clustering

An alternative to FuzzyART which has been used in this research is the MINMAX network proposed by Simpson in [15] for clustering. The network differs from FuzzyART networks in that the processing involves hyperbox (a region in n dimensional pattern space) expansion and contraction to avoid overlapping classes. This is done with respect to a control parameter, θ, which determines the maximum size of a hyperbox. It is proposed that by combining 2 FuzzyMINMAX networks connected by a mapping vector a better understanding of the clusters would be found by exploring the internal representations being made by the networks.

The detail of the algorithm is presented by Simpson [15]. The choice of FuzzyART or FuzzyMINMAX networks for analysis of data depends on the results of an investigation to determine the relative merits of each. Apart from simple comparisons relating to, for example, noise susceptibility, it was important to understand the way in which each method was to be used for discovering new information.

If, as is often the case, external information can be exploited and applied to the vectors in order to form clusters, it is then possible to see how effective an uninformed clustering method can be in discovering these clusters. Furthermore, it is possible to then explore the vector space in those regions where clusters have been discovered which do not correspond to those expected by exploiting outside knowledge. Empirical investigations were carried out to determine the characteristics of each method [9]. These showed that both methods are highly dependent on the order of presentation of the input data and on their respective control parameters. Consequently, in this research the order of presentation of the input vectors was randomised and 10 separate clusterings for each of 3 values of the control parameters for both the MINMAX and ART methods were produced. Each of these 60 clusterings was examined with respect to the consultants classification of the images and 3 of the random clusters for each control value and each method were selected. Finally, for every cluster containing a consultants 'unknown' image vector, an ARTmap and a MINMAX-map were trained and used to predict the possible class of the unknown image. The consultant was then asked to comment on the neural classification. These results are reviewed in the following section.

6. Results

The nature of the data and the neural network paradigms means that presenting results in a clear well defined way is difficult. A variety of factors affect the results:

- the order of presentation of vectors for both algorithms affects the clustering;
- different control parameters produce different clusters;
- for the type 1 sets, arbitrary numbers have been chosen to reflect the category ordering;
- for the type 2 sets the fuzzy membership grades chosen will affect the results.

The interest here is in assisting the consultant in the diagnosis process. The nature of the classification of shin images is that the consultant is constantly learning about the processes going on, often revising the nature of the groupings and exploring potential sub clusters within clusters. The clustering algorithms and the paucity of data allow for, at best, giving the consultant some help with image classification. Although it may be possible to find an algorithm that could cluster in exactly the same way as a consultant *on a particular occasion* the fact that experts' themselves give different clustering at different times indeed mirrors the algorithms in that for a particular order of presentation and control parameter they too provide different clustering. What is important for the shin image analysis is whether, given a small amount of data, the technique adopted can give some indication to the consultant of the type of problem with the image to a reasonable degree of accuracy. In other words do these methods offer an insight to the consultant for those images which could not be classified originally? Further does the type 2 representation produce better results for FuzzyART or FuzzyMINMAX *given random orderings and particular control parameters*?

Two sets of experiments were carried out. Set 1 used the order of the type 1 vectors as they corresponded to the order in which the images were produced by the consultant (called the 'natural' order). The purpose of these experiments was to discover the classes of the unknown images and to compare the two neural models. Innocent et al. [10] report the detail of the results here we present a summary for comparison purposes. Set 2 experiments are the main focus of this paper and concern the investigation of random input order and differences in the behaviour of the two neural models with respect to the classifications found in set 1 when type 1 and type 2 data is used.

6.1. Single neural models

Of the original 203 images, there were 38 images the consultant could not originally classify. Using type 1 data, of the 38 unknowns, 22 were predicted with agreement from both neural models (2 as normal, 9 as healing stress fracture and 11 as MTS). These included the optimistic predictions from clusters with one class. Of the 22 predictions, 5 were not agreed with by the consultant. Of the 16 unknowns predicted without agreement, only 4 were agreed with by the consultant as being unpredictable in the existing classification scheme. The human expert was, however, able to indicate for each of these cases which classes were not possible for that image. For these unclassifiable images, we may conclude that either the classes currently defined are not exhaustive, or the images were too uncertain in their content and could belong to any of several classes. This represents only approximately 2% of the total data base which would indicate the latter interpretation is more likely. The order of presentation of the input vectors in 10 trials was randomised and 10 separate sets of clusters produced for each of 3 values of the control parameters for both the MINMAX and FuzzyART methods. Table 1 shows how the clustering related to the consultants categories for all the variables considered. These were constructed by considering the composition of each cluster in terms of expert categories for all methods and parameters. For example, if a cluster contains vectors of images which the consultant said contained stress fractures (s) and healing stress fractures (h), then the cluster is assigned as a 'confusion' cluster between those categories (rule 23 in Table 1). There are 31 combinations of the 5 categories to consider in the table; each is shown as a separate rule. In the table, 'n' means normal, 'p' means patchy, 'f' means focal, 'm' means mts(medial tibial syndrome), 'h' means healing stress, 's' means stress and 'ART' means FuzzyART.

The main feature of these results is that there is clear agreement between all the methods for all the parameters and type of data including order of

Table 1
Cumulative confusion class frequencies over all random trials

Rule no.	Class					Type 1 data						Type 2 data					
						ART: rho value			MINMAX: theta value			ART: rho value			MINMAX: theta value		
	n	h	m	s	p	0.2	0.25	0.30	0.65	0.75	0.85	0.2	0.25	0.30	0.65	0.75	0.85
0	x	x	x	x	x	0	0	0	0	0	0	0	0	0	0	0	0
1	x	x	x	x		0	0	0	0	0	0	0	0	0	0	0	0
2	x	x	x	x	1	0	0	0	0	4	2	0	0	0	0	0	0
3	x	x	x		x	0	0	0	0	0	0	0	0	0	0	0	0
4	x		x	x	x	0	0	0	0	0	0	0	0	0	0	0	0
5	x	x		x	x	0	0	0	0	0	0	0	0	0	0	0	0
6		x	x		x	0	0	0	0	0	0	0	1	0	0	0	0
7		x	x	x		8	9	7	1	3	7	6	7	8	0	0	9
8	x	x		x		3	0	2	0	0	0	3	6	5	0	0	0
9	x		x	x		0	0	0	0	0	0	0	0	0	0	0	0
10	x	x	x			1	0	0	0	0	0	0	0	0	0	0	0
11	x		x		x	0	0	0	0	0	0	0	0	0	0	0	0
12	x	x			x	0	0	0	0	0	0	0	0	0	0	0	0
13		x		x	x	0	0	0	0	0	0	1	1	0	0	0	0
14			x	x	x	1	1	0	0	1	4	4	1	2	4	1	10
15	x				x	0	0	0	0	0	0	0	0	0	0	0	0
16			x		x	12	12	12	9	9	6	5	13	8	6	9	0
17		x	x			1	2	0	4	13	0	1	0	0	1	9	0
18			x	x		10	5	6	8	2	9	7	9	8	5	2	3
19	x		x			1	0	0	0	2	5	0	0	0	0	0	0
20		x			x	1	1	0	0	0	0	2	0	0	0	0	0
21				x	x	1	1	0	0	0	0	0	1	0	0	0	0
22	x				x	0	0	0	0	0	0	1	0	0	0	0	0
23		x		x		13	14	14	21	14	16	15	13	12	24	14	25
24	x	x				1	1	2	0	0	0	1	1	0	0	0	0

Table 1 (Continued)

Rule no.	Class					Type 1 data						Type 2 data					
						ART: rho value			MINMAX: theta value			ART: rho value			MINMAX: theta value		
	n	h	m	s	p	0.2	0.25	0.30	0.65	0.75	0.85	0.2	0.25	0.30	0.65	0.75	0.85
25	x			x		0	0	0	0	0	0	0	0	0	0	0	0
26				x		11	19	30	24	12	0	9	16	23	32	14	10
27					x	7	7	13	11	8	0	1	6	13	12	10	9
28		x				9	18	24	46	30	19	12	15	22	39	36	24
29			x			12	11	19	22	15	6	6	6	9	20	26	6
30	x					14	15	16	20	18	15	15	12	14	20	20	20

presentation for areas of confusion. This shows that there can be robust inferences leading to confident conclusions about the data. Qualitatively, it appears that type 2 pre-processing and MINMAX clustering appears to produce least confusion in relation to consultants judgements. However, quantitative inferences are preferable and a suitable method has been chosen for achieving this. The modal clustering out of 10 trials as representative of that method and parameter setting was selected. Table 2 shows the results for type 2 data presented in random order (each row) 10 times to the ART clustering algorithm with rho = 0.3(low). The entries are frequencies with which clusters are found for that rule. Since, at this stage,the interest is in confusion and not exact matching to an experts opinion we eliminate rules for single categories (rules 26–30 inclusive). Table 2 shows that 6 rules are involved overall and that there is a good overall agreement between trials on these rules.

We wanted to discover if we could improve the prediction of classes of unknowns by using type 2 fuzzy sets rather than type 1. Table 3 shows the performance of each type of pre-processing and neural model where 'experts f' means the frequency of occurrence in the data base of the given class as classified by the human expert. A single class preference has been asked of the human expert for the unknown and, from the earlier experiment, the expert was also allowed to make predictions of what was not possible for the unknown, i.e., elimination of possibilities. The impossibilities are used in a further analysis to show where the networks are performing particularly poorly. In this table, the contingency coefficient, C [16], is a measure of how much the method is associated with the expert for categorisation over all the 5 categories. When the value of C exceeds 0.65, the degree of association can be considered to be significant at the 5% level. The coefficient of agreement Kappa [6], is a measure of how much the outcome of the methods agree with the outcome of the experts categorisation. Kappa differs from C in that it is a true reflection of the proportion of agreement for each category and does not include how well the expert agrees with the mis-classifications by the models. If the z-score related to Kappa is greater than 1.96 (2 standard deviations from the mean), then Kappa is significant at the 5% level. It may then be inferred that the neural model and the human expert are in agreement to a degree greater than may reasonably be expected by chance in 1 in 20 cases. In Table 3 all the values of Kappa are significant except that for ART(0.2). Thus, we may conclude that the majority of single neural models are usefully agreeing with the expert in the classification of unknowns. These results show that that there is a clear difference between type 1 and type 2 pre-processing. In particular:

- type 1 pre-processing appears to produce better agreements than type 2 regardless of neural method and control parameter settings;
- the worst matches were for MINMAX type 1 (low and high control parameter) cases;

Table 2
Confusion class frequencies for 10 random trials using type 2 data

Trial	Reference number of the confusion rule																										Total
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	
1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	3
2	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	3	0	0	5
3	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	4
4	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	1	0	1	0	0	0	0	1	0	0	5
5	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	4
6	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	2	0	2	0	0	0	0	2	0	0	7
7	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	3
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	2	0	0	5
9	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2
10	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	2	0	1	0	0	0	0	0	0	0	5
Total	0	0	0	0	0	0	0	8	5	0	0	0	0	0	2	0	8	0	8	0	0	0	0	12	0	0	

Table 3
Proportion of agreement for all unknown images

Class	Experts frequency	MINMAX						ART					
		0.65		0.75		0.85		0.2		0.25		0.3	
		Type 1	Type 2	Type 1	Type 2	Type 1	Type 2	Type 1	Type 2	Type 1	Type 2	Type 1	Type 2
m	15	0.60	0.73	0.80	0.73	0.67	0.80	0.40	0.80	0.80	0.80	0.80	0.67
n	0	0	0	0	0	0	0	0	0	0	0	0	0
p	2	1	0	0.50	0	0.50	0	0	0	0	0.50	0.50	0
s	2	0.50	0	0.50	0	0	0	0	1	0.50	0.50	0	0.50
h	15	0.47	0.33	0.60	0.53	0.33	0.40	0.20	0.33	0.73	0.40	0.47	0.53
Contingency coefficient (C)		0.73	0.57	0.71	0.66	0.72	0.62	0.61	0.56	0.64	0.64	0.70	0.61
Kappa		0.42	0.24	0.54	0.35	0.28	0.30	0.02	0.36	0.52	0.39	0.41	0.37
z-score of kappa		4.40	2.06	4.78	2.93	2.85	2.56	0.24	3.19	3.85	3.39	3.60	3.21

- the best matches were for MINMAX type 2 (low control parameter). MINMAX all types (medium control parameter) and ART all types (medium and low control parameter).

The results indicate that type 1 produces generally more unclassifiable cases than type 2 except in the case of the ART(0.3). This is in keeping with the previous conclusion from Table 3 that type 1 was better than type 2 in producing agreement with the expert. This would be expected from the usual trade off between false and true positives when a decision criterion has been consistent in a neural method. Overall, it may be concluded that the accuracy of single neural models is at best 70% (MINMAX type 1, medium control parameter) and at worst, 27% (ART type 1, high control parameter). We wish to investigate if we could improve this accuracy by combining the predictions of the single neural models.

6.2. Combining the models

Single neural model analysis may be considered to be equivalent to asking separate experts for a class of each image and comparing that with another independent expert. Single neural model analysis may be considered to be equivalent to asking separate experts for a class of each image and comparing that with another independent expert. Better classifications can result by combining the results of several neural models in the sense that more agreement can be found with an independent expert. There are a variety of approaches to how this may be achieved. The simplest of these is a 'winner take all' approach and use majority voting to determine what may be termed a 'consensus'. We do this for all the values of the control parameters within each neural method and type of pre-processing to arrive at a consensus at that level to produce a method prediction. The approach taken here is to combine models with a 'winner take all' approach and use majority voting to determine what may be termed a 'consensus'. We do this for all the values of the control parameters within each neural method and type of pre-processing to arrive at a consensus at that level to produce a method prediction. Then, by again using simple majority rules of consensus on the method prediction, we can see if particular methods and particular pre-processing are in agreement with the experts preferred classification. For example, we apply simple majority voting rules on the predictions of each combination. As with the analysis of the single neural model results, we process the results for all the 38 unclassified images in this way. The results in Table 4 show the proportion of agreement with the human experts preferred classifications for the 34 possible predictions. None of the combined neural model agreements with the human expert outperforms the single neural models because the worst single neural models have been included in the combinations. However, every combination has a Kappa value which is significant and is greater than 0.316 (significant at 5%). All the Kappa values

Table 4
Combined model proportions for the unknown images

Class	Experts frequency	mmm-type 1	mmm-type 2	ART-type 1	ART-type2	All-mmm	All-ART	ART-mmm +ART	All type 1	All type 2
M	15	0.733	0.8	0.8	0.8	0.733	0.733	0.733	0.8	0.8
n	0	0	0	0	0	0	0	0	0	0
p	2	0.5	0	0	0	0	0	0	0	0
s	2	0	0	0	1	0	0	0	0	0.5
h	15	0.467	0.467	0.4	0.467	0.467	0.533	0.6	0.533	0.4
Contingency coefficient		0.739	0.611	0.657	0.655	0.638	0.627	0.641	0.699	0.593
Agreement coefficient (kappa)		0.40	0.34	0.316	0.45	0.35	0.39	0.38	0.43	0.36
z-score of kappa		3.86	2.79	2.73	3.87	3.31	3.27	3.18	4.04	3.10

are within a very small range possibly indicating a central tendency due to the methods of combination. Thus, a clear cut winner is not obvious and conclusions should be drawn cautiously. The best agreement is shown with ART + type 2 (60% accuracy) followed by all type 1 (60% accuracy) and MINMAX type 1 (57% accuracy). The worst combinations are for ART type 1 (54%) and MINMAX type 2 (54%). This would seem to indicate that more work needs to be done in devising suitable rules of combination. The false positive analysis shows that there is a considerable difference between the number of unclassifiable images with respect to type 1 and type 2 pre-processing within the MINMAX paradigm and overall. This is consistent with the results of the single neural model predictions from which the combinations are formed. The results do show that MINMAX produces more unclassified cases than ART and this arises from combining MINMAX with type 1 pre-processing. ART appears to be less susceptible to change in pre-processing. This may be a consequence of the larger number of small clusters generated by MINMAX compared to ART.

7. Conclusions

The problem of classification of images of the tibia is fraught with difficulties. There is a small amount of historical data much of which is imprecise. In particular the images themselves are difficult to classify and, indeed, the classifications change with time. This paper presents some results of applying neuro-fuzzy clustering techniques to the problem where some of the input data is represented by type 2 fuzzy sets. The main conclusion is that the use of neural clustering for the classification of shin images can be improved using type 2 pre-processing.

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An Adaptive Type-2 Fuzzy System For Learning Linguistic Membership Grades

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Abstract

Type 2 fuzzy sets allow for linguistic grades of membership. A type-2 fuzzy inferencing systems uses type-2 fuzzy sets to represent uncertainty in both the representation and inferencing. However, as with type-1 fuzzy systems there is till an issue with regard to the design of the appropriate membership functions. This paper presents a novel type-2 adaptive system for learning the membership grades of type-2 fuzzy sets. The paper reports on some results obtained from a type-2 system developed for car evaluation. The results lead us to believe that this approach offers the capability to allow linguistic descriptors to be learnt by an adaptive network.
Keywords: Type 2 Fuzzy Sets, Adaptive Fuzzy Systems.

1. Introduction

For a number of years artificial neural networks[3] have successfully allowed for the 'learning' of highly non linear relationships within data. However they are 'black box' in that they provide no explanation of their reasoning. Also they usually require the data to be numeric, where for many problem domains a linguistic representation is more natural. Lin and Lu[10] report some work on training a neural network with linguistic terms but this approach is not widely used and the resulting system is still not easily interpreted. The ANFIS[5] approach is an adaptive type-1 fuzzy system that learns, from numeric data, the membership functions and rules in a fuzzy inferencing system. An advantage of this approach compared with artificial neural networks is that the final system allows the developer to interrogate the network and gain some interpretability. A major drawback for a particular class of problem is that it is unable to handle linguistic inputs to the network. The novel type-2 learning approach detailed here allows for both linguistic inputs and interpretability. Previous work[6] has discussed the philosophy behind the approach. The work reported here builds on this with an exposition of the methodology and the results of applying the approach in a particular domain. The format of the papers is as follows: Section 2 describes type-2 fuzzy inferencing; Section 3 describes the type-2 ANFIS and some results from some data for car evaluation and section 4 provides a conclusion.

2. Type-2 Fuzzy Sets and Inferencing

This section briefly outlines type-2 fuzzy sets and how they may be used with if-then rules[1]. Type-2 sets[12] allow linguistic membership grades. Mizumoto and Tanaka[11] provide the following definition of a type 2 fuzzy set:

> *"A fuzzy set of type-2 is defined by a fuzzy membership function, the grade (that is, fuzzy grade) of which is a fuzzy set in the unit interval [0,1], rather than a point in [0,1]".*

A fuzzy set of type-2, A, in a set X, is the fuzzy set characterised by the fuzzy membership function μ_A as $\mu_A : X \rightarrow [0,1]^{[0,1]}$ where $\mu_A(x)$ is known as a *fuzzy grade*, a fuzzy set in [0,1]. So type-2 fuzzy sets are different from the 'conventional' (type-1) fuzzy sets in that the membership grade of a member of the set is not represented by a number but by a type-1 fuzzy set. Most type-1 fuzzy systems rely on some physical measurement (e.g. height, temperature) to then be 'fuzzified' into a number in [0,1]. However we do not necessarily use physical measurements in our daily use of language. Also, any application using fuzzy sets requires the developer to describe the membership function by numbers, in the discrete case, or by a function, where the fuzzy set has a continuous membership function. So, any system employing fuzzy sets represents the fuzziness of the particular problem using a 'non-fuzzy' representation. As Klir and Folger[9] point out, " ..*it may seem problematical, if not paradoxical, that a representation of fuzziness is made using membership grades that are themselves precise real numbers.*" Gaines[2] in an extensive piece of work makes

[1] The theoretical underpinning to type-2 fuzzy logic systems is covered extensively elsewhere by Karnik and Mendel [8]

an interesting point. So as to not lose the impact it is quoted directly:

> *We are able to generate and follow arguments involving 'tallness' without having any concept of inches, centimetres or any other metric scales. To introduce the former in terms of the latter reverses the actual process of derivation and, in particular, leads to a false distinction between these concepts such as 'tallness' that have a well-defined, single-parameter, physical metric, and those such as 'beautiful' which do not.*
> [2, page 51]

This is important. What is being said here is that we do not necessarily need physical measurements in our daily use of language. We know when somebody is quite tall. By the same token we know when somebody is beautiful (or indeed handsome!) which is not physically measurable. Although Gaines does not highlight type-2 sets in his work it is clear that since type-2 sets allow for fuzzy grades that are not dependent on a physical measurement they may be able to model human reasoning more readily than traditional type-1 sets. It is problems which exhibit these characteristics that this research addresses.

Type-2 fuzzy sets can be inferenced with if-then rules in the same manner as with type-1 sets. To illustrate, we provide the definitions for union and intersection. Suppose we have type 2 sets, A and B, in X and with two *fuzzy grades* of A and B respectively, represented using the usual notation by $\mu_A(x) = \sum_i f(u_i)/u_i$ and $\mu_B(x) = \sum_j g(w_j)/w_j$ where the functions f and g represent membership functions of fuzzy grades. The union and intersection of type-2 fuzzy sets deploy the extension principle[12]. This leads to the following definitions[13].

Union of Type-2 Fuzzy Sets

$$\mu_{A\cup B}(x) = \mu_A(x) \sqcup \mu_B(x) \quad (1)$$

$$= \sum_{i,j}(f(u_i) \wedge g(w_j))/(u_i \vee w_j) \quad (2)$$

Intersection of Type-2 Fuzzy Sets

$$\mu_{A\cap B}(x) = \mu_A(x) \sqcap \mu_B(x) \quad (3)$$

$$= \sum_{i,j}(f(u_i) \wedge g(w_j))/(u_i \wedge w_j) \quad (4)$$

where $\sqcup$ denotes *join* and $\sqcap$ denotes *meet*[7].

Now let us suppose we have a collection (knowledge base) of if-then type-2 rules where the i'th rule is of the form

IF x_1 *is* A_1^i *and* x_2 *is* $A_2^i \ldots x_n$ *is* A_n^i THEN y *is* B^i

where $A_1^i, A_2^i, \ldots A_n^i$ and B^i are type-2 fuzzy sets. A type-2 knowledge base can be inferenced with in the same way as with type-1 fuzzy systems. We need to be able to carry out intersection, implication, composition and defuzzification. The intersection is required to combine the antecedents of each rule. This is done using the meet definition provided above (Eqn. 3). The output of this will be a type-2 fuzzy set. Implication can be defined in a similar way to a type-1 system. Suppose we have two membership grades ($\mu_A(x)$ and $\mu_B(x)$) and we wish to performs $A \to B$ then the result is

$$\sum_{j,k}(f_i(u_j) \wedge g_i(w_k))/(1 - u_j) \vee (u_j \wedge w_k). \quad (5)$$

There are a number of options for defuzzification of a type-2 fuzzy system. These are discussed in detail by Karnik and Mendel[8] and we will outline the approach we have adopted in the next section.

3. An Adaptive Type-2 System

The previous section briefly described the way type-2 fuzzy sets can be inferenced with where they are in a knowledge based system using if-then rules. This section describes an adaptive type-2 fuzzy system that implements this approach in a method that mirrors the type-1 ANFIS. It is applied in a problem where all the inputs to the system are linguistic terms. Previous work [6] outlined the general philosophy and structure of a type-2 adaptive fuzzy system. Essentially they are an extension of the (type-1) ANFIS approach[5] which has been successfully applied in a number of application areas[4]. For ANFIS a typical fuzzy inference system is represented in an adaptive network. The parameters defining the fuzzy sets are 'learnt' from data. The resulting network will be able to, given a particular set of numeric inputs, 'guess' the output. The advantages of this approach are:

- these networks have strong generalisation capabilities - given an input it has not seen before a trained network will usually perform well in guessing the output;
- the fact that the parameters are learnt by the algorithm removes the need for the knowledge acquisition exercise from an expert(s);
- they are known to be robust.

However they have significant disadvantages. These systems take numerical input which is assumed to be 'certain'. However, real world applications often have uncertain or fuzzy data where for example an expert would describe a feature of a problem more intuitively using linguistic rather than numeric values. It is this particular shortcoming that our approach attempts to resolve. We take the basic ANFIS philosphy and extend it to allow linguistic inputs and type-2 inferencing.

3.1 The problem

To test the ideas discussed in [6] we use the problem of classifying cars based on a number of linguistic (and numeric) descriptors. The car evaluation database (http://www.ics.uci/m̃learn/MLRepository.html/) was derived from a simple hierarchical decision model[1]. The model evaluates cars according to the the concept structure shown in Figure 1. This simple model relates the factors that influence the acceptability of a car into a heirarchy with three intermediate factors - price, comfort and technical.

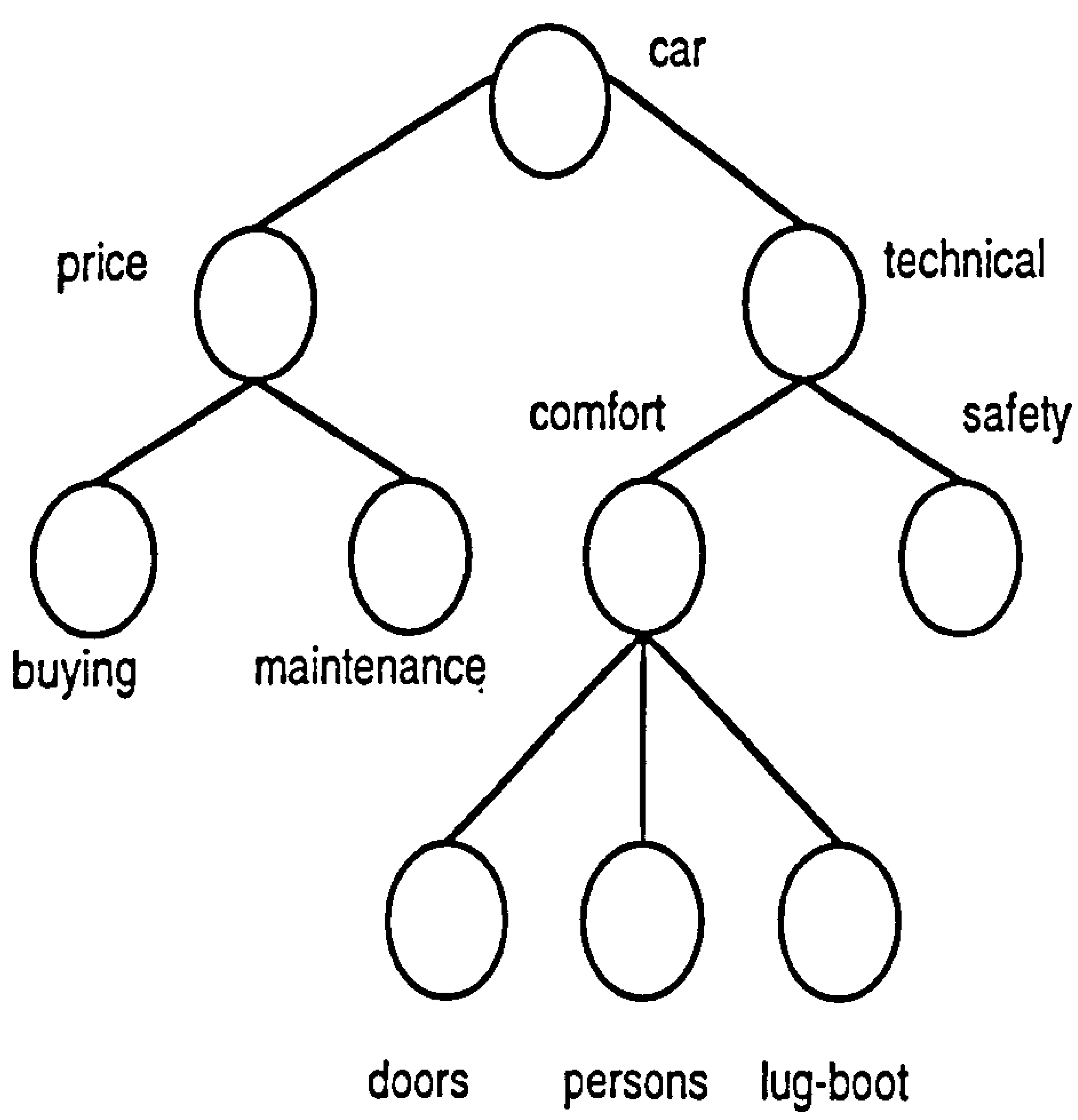

Figure 1. Concept Structure for the Car Evaluation Problem

The car evaluation database contains 1728 examples with the concept structure removed. In other words the six input attributes are in a flat record with the acceptabilty of the car.The different attributes can take a number of different values: buying (vhigh, high, med, low), maintenance (vhigh, high, med, low),doors (2, 3, 4, 5-more),persons (2, 4, more), lug-boot (small, med, big) and safety (low, med, high). The acceptability of the car takes one of four possible values (unacc (70%), acc (22%), good (4%), vgood (4%)). We were only interested in testing the efficacy of our approach with linguistic terms so the two 'crisp' values (doors and persons) were removed from the database. In the initial experiments we were setting the adaptive system a difficult problem in that we have reduced the number of inputs and the data is 'biased' since one particular classification accounts for 70 % of the data. Later experiments considered the problem of differentiating between acceptable and unacceptable.

3.2 The Type-2 ANFIS for Car Evaluation

A type-2 ANFIS[6] is an adaptive network that simulates a type-2 inferencing system with type-2 if-then rules. An adaptive network is a network structure consisting of connected nodes where the links are directional. The nodes are processing units that perform a function and can be adaptive in that the result of the processing is dependent on the parameters selected for the node. Not all nodes are necessarily adaptive. An adaptive network has a 'learning' algorithm where the parameters in the nodes are learnt by training from data (supervised learning). The type-2 ANFIS consist of five layers. Layer zero contains adaptive nodes, each of which is associated with linguistic grades for type-2 fuzzy sets. The parameters to be learnt are those that define the grades. The next layer contains fixed nodes that perform the 'AND' in the rules using the intersection (Equation 3). Layer two contains adaptive nodes that represent the grades for the type-2 sets in the consequents of the rules and they carry out the impication (Equation 5). Layer three contains a single fixed node that composes all the outputs of the rules using the join(Equation 1). Finally layer four contains a single node that performs the defuzzification. The output of the network is compared with the expected output and a learning algorith is used to determine the parameters within the network. Now we put this idea in context by describing the application of the methodology. Figure 2 shows a schematic of the type-2 ANFIS developed for this particular problem. The topology mimics the concept structure in Figure 1. The process of learning consists of a forward pass and backward pass. In the forward pass a car, selected at random, is submitted to layer zero. These are the linguistic attributes describing buying, maintenance, lug-boot and safety. Layer one contains the type-2 fuzzy sets and the appropriate membership grades to be learnt. The grades 'med' are represented by a bell shaped function containing three parameters and the others by gaussian membership functions represented by two parameters. These are initialised prior to training and we chose parameters that were sensible in that they represented typical shapes that one might expect. Depending on the particular input values, layer two carries out the intersection in the antecedents of the rules. There are a choice of possible t-norms here but we have chosen to use the min operator since this is the one often adopted in fuzzy systems. Layer three carries out the implication. A decision had to be made here about the possible values the grades could take and we have chosen low and high as two gaussian membership functions with parameters to be learnt. This is an area where further research could be carried out to see the effect of introducing more grades in the consequent type-2 sets. The compostion of, in this example, the two rules is acrried out by layer four. Defuzzification is carried out in layer five. As has already been reported there a number

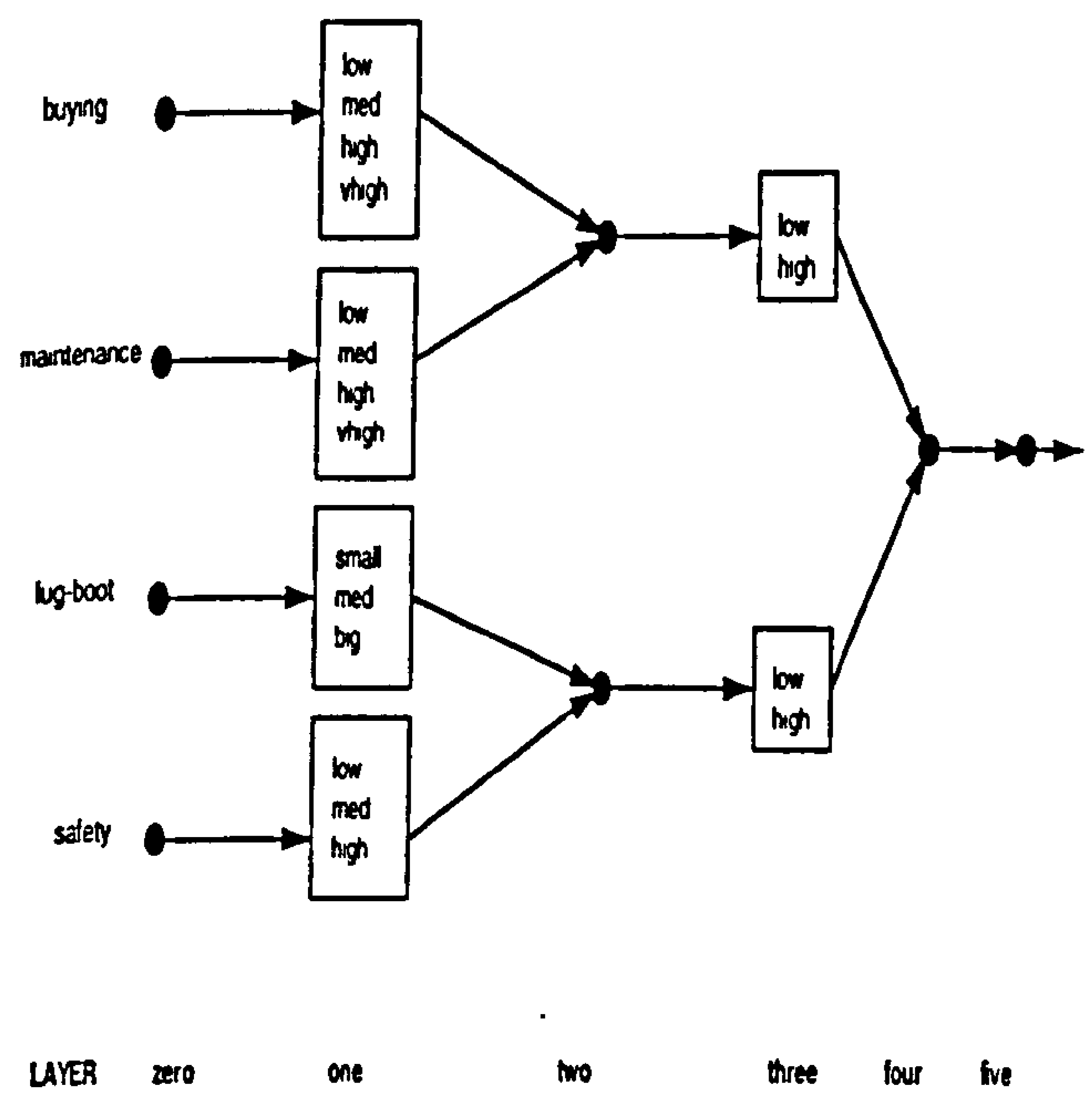

Figure 2. Type-2 ANFIS for the Car Evaluation problem

of possible type-2 defuzzification techniques. All the currently reported methods[8] use some form of type-reduction. That is the type-2 output of the inferencing process is reduced to a type-1 set. This is then defuzzified in the normal way. We have adopted the height type-reduction[8, pp. 79-80]. The various classifications have been represented as numeric numbers in [0,1]. Upon defuzzification the output value is compared to the expected value for the car's classification. This produces an eror which is fed back through the network to update the parameters in a very similar way to the back propagation algorith[3] for artificial neural networks. This involves minimising the error by feeding it back through the network and updating the parameters. Space does not allow for the detail to be reported here and this will be included in further work.

3.3 Results

This section reports on some results obtained from applying the approach described to the data already discussed. To reduce the bias of the data towards the unacceptable category some data was ignored and the remaining data was split into three sets - training, validation and test. The validation set was used at each epoch to test the behaviour of the network on data it is not using for training purposes. The test data is used to evaluate a 'trained' network. Note that the bias still exists but to a lesser degree (it is interesting to note that when training on all the data bar 172 'test' cases the performance of the networks was very similar indicating that by using a subset of the data the problem has not disappeared - to undertake more pruning would invalidate the whole process). In a similar way to artificial neural networks there are numerous heuristic decisions that will affect the performance of the network, and remains an area of research to be undertaken.

Table 1. The break down of the training and test sets for the Car Evaluation data

	Unacc	Acc	Good	V Good	All
Train	323	233	49	36	641
Test	237	70	17	13	337

The make up of the training and test data sets is shown in Table 1. In order to evaluate the succes of the network it was decided to determine a classification rate for each category. A correct classification is obtained if the output from the network falls within 0.1 of the expected classification. As would be expected the networks performed well for the unacceptable category since these still represented the majority of the data in all three sets. As the length of training time increases the performance on the other classifications improves to the detriment of the unacceptable classification. The best networks for the unacceptable category produced a correct classification for 83% of the test cases. However only 20% of the good and acceptable category were classified correctly with none of the very good. The 'best' network overall at the time of writing classified unacceptable(63%), acceptable(63%), good(15%) and very good (13%). Note that the networks took a very long time to train as there is a large amount of numerical processing being carried out. It was not uncommon to let the networks train for a day or more. As an example 500 full passes through the training set on a Macintosh Performa 5200 took nearly three hours. Performance has improved on porting to Unix workstations (HP9000 with HPUnix 10.20).

Further experiments were carried out to see whether the network was able to classify acceptable and unacceptable only. There were 953 training cases (720 acceptable, 233 unacceptable), and 307 test cases (237 acceptable, 70 unacceptable). For example after 5000 training epochs the root mean square error was 0.097 and it achieved a claasification rate of 72% for the acceptable and 60% for the unacceptable. Further work will be carried out to see if improvements can be made to the performance of these networks and comparisons drawn with ANFIS and artificial neural networks. These early experiments are encouraging. There are issues about the data and design of the network that should be reiterated:

- the input variables have been reduced to only allow for linguistic values. One would expect the performance to improve were the 'crisp' input values included;
- there is a significant bias in the data towards the unacceptable class;
- there are many heuristic decision that have been made that will certainly affect the performance of the network.

The work does indicate however that we have a new and novel method for learning membership grades of type-2 fuzzy sets that appears to offer significant advantages over artificial neural networks and type-1 ANFIS in that the final system offers interpretability and linguistic descriptions can be learnt directly from data.

4. Conclusion

This paper describes the empirical investigation of the use of a novel type-2 adaptive system that allows linguistic inputs to an adaptive network. The performance of the network on a particular problem is currently unsatisfactory yet given the nature of the data and the possible improvements that could be made we are encouraged to believe that adaptive type-2 fuzzy sytsems offer, for certain applications, the ability to learn relationships between known inputs and expected outputs. Future work will examine how to improve the results for this data as well as investigating other data sets. It is hoped that comparisons will be able to made dierectly with artificial neural networks and type-1 ANFIS approaches.

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Patient Assessment in Nursing Care using Fuzzy Logic

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Abstract

Expert nurses concurrently assess the patient need for nursing care in several domains of concern to nursing. It is suggested that the process takes into account the overall context of patient need in relation to at least five domains. The degree of need within each domain is prioritised, reflecting the need for nursing intervention. The result of this assessment may be summarised in a single sentence. This is an intuitive process in that the expert nurse uses inexact or imprecise information to make judgements based on nursing knowledge and practice wisdom. Fuzzy Logic models imprecision and intuition in a decision making environment via the use of fuzzy sets. This paper argues that fuzzy logic has much to offer in the assessment of patients for nursing care. Type-2 fuzzy sets are explained and some examples given whereby these sets are used to model the decision making process described.

1. Introduction

The clinical reality of nursing requires nurses to make decisions arising from an ongoing holistic assessment of the patient need for nursing care, based on an extensive range of knowledge [1][2]. Decision making by nurses has been studied from several points of view [3][4][5], which are more often discussed in relation to one aspect of nursing intervention [6][7]. It is suggested that nurses concurrently assess patient need for nursing care in several domains of concern to nursing before deciding where the primary focus of nursing attention should be directed. This paper suggests a method, based on fuzzy logic, for modelling such nurse decision making. The method goes some way to recognising the complexity of context and degree of acuity of the nurse patient interaction. Fuzzy logic [8] is an approach for modelling imprecision that has been widely used in many domains for decision making. It is our contention that fuzzy logic provides the ability to model the inherent complexity (granularity) of linguistic terms and, indeed, provides *Computing with Words* [9]. Central to our approach is the use of Type-2 fuzzy sets [10] to model combined domains of nursing assessment and the degrees of patient need within the domains, enabling demonstration of the holistic nature of this assessment. The paper is structured as follows: Section 2 discusses the central concepts in that paper from a nursing perspective; Section 3 reviews Fuzzy Logic and Type-2 Fuzzy Sets; Section 4 provides some illustration by example of our approach and Section 5 discusses the main findings of this work to date.

2. A Framework for Patient Assessment in Nursing Care

A simple, but comprehensive, framework for nursing assessment in concurrent domains may be found in Watson where the holistic overview of nursing is defined as the provision of a supportive, protective, and corrective mental, physical, sociocultural, and spiritual environment [11]. Holistic nursing assessment takes the physical, mental, sociocultural and environmental domains of patient need into account, as well as the need for clinical intervention. For the purposes of this paper it is proposed to simplify the framework for assessment to five domains.

As nursing is often carried out in conjunction with medical diagnosis and treatment, a primary focus of nursing concern is the physical/medical condition or diagnosis of the person. Therefore the initial domain of assessment is named 'Physical/Medical Condition'. A domain named 'Complicating Factors', which may affect the initial condition, is then also considered and taken into account. The physical capability or 'Dependency' domain of the individual is always assessed, as also is their ability to understand and co-operate with suggested interventions. In this latter domain the support available to the person from their family and environment will also affect the amount and type of nursing intervention that will be provided. This domain has been summarised as the 'Psycho-Social' domain. Last but not least, the requirement for the more obvious array of nursing clinical interventions, as titrated to patient need and condition, are combined in the 'Clinical Intervention' domain. It is suggested that these five domains provide the context of the patient need for nursing care and intervention.

Within each domain, a degree, or priority, of need is determined by nursing assessment before any required intervention is applied. To demonstrate the application of the framework, an arbitrary numerical value from 1 to 5 has been assigned to the degrees within each domain. For the Physical/Medical Condition domain 5 would equate to 'critically unwell', 4 to 'unstable', 3 to 'potentially unstable', 2 to 'becoming stable', and 1 to 'stable'. (In the 'Dependency' domain, the 5 would equate to 'totally dependent', through to 1 equating to 'independent'.)

To demonstrate the translation of such a framework into Fuzzy Logic, a comparison is drawn between two assessments of a hypothetical patient requiring elective surgery. The assessments are made over a period of four-to five days during the peri-operative and post-operative recovery, and a summary of two separate daily assessments made.

The patient is a frail 48yr old, admitted for relief of symptoms of cancer by the insertion of a celestin tube under anaesthetic to enable parenteral feeding. On return from theatre, his post-operative Physical/Medical condition is assessed as 'potentially unstable' (3), requiring routine regular observations. Complicating Factors of nausea and pain/comfort are also assessed as 'potentially unstable' (3), also requiring regular observation. His requirements for Clinical Intervention are assessed as 'complex' (4), as antibiotics, intravenous analgesia and anti-emetics are administered as necessary. His Dependency needs are assessed as 'heavily dependent' (4), both due to his generally frail physical condition, plus the necessity for added assistance in the post-operative recovery phase. His need for emotional/educational support in the Psycho-Social domain is assessed as 'moderate' (3), somewhat eased by post-operative sedation. This totals a score of 17 out of a possible 25, denoting a 'more acutely unwell' patient in the context of surgery on a person with cancer and secondary spread, who requires frequent nursing assessment, clinical intervention and physical support in the post-operative period. A summary might be: "Frail, requiring frequent intervention for relief of symptoms."

On the fourth post-operative day, he is noted to have had a miserable night with nausea, and his wife is noted to be tearful. The assessment for this day could be summarised in a single sentence as follows: "Oral intake good, requested bedsponge, emotionally and physically dependent." The nursing understanding underlying this summary is that his Physical condition is noted to be 'becoming stable' (2), requiring some observation. The Complicating Factors of nausea and pain remain 'potentially unstable' (3), requiring regular assessment. The Clinical Intervention domain is stable, little intervention is needed (1), and oral medication suffices. However, after the rough night, some regained independence regresses and the patient requests a bedsponge remaining 'heavily dependent' (4). Considerable emotional and educational support is required from the nurse for both the patient and his wife, and this domain is assessed as 'complex' (4). This totals a score of 14 out of a possible 25, denoting a moderately unwell patient, requiring considerable emotional and physical (rather than clinical), support.

Both the domains and the degree of need within each domain are imprecise, using inexact words and phrases to define or explain the nursing assessment. Such imprecision summarises or granulates the complexity of patient need into recognisable fragments of knowledge. Yet it is suggested that the context for each patient, and the weighting given the degree of need within each domain supports the relevance of the overall summary even if translated into a score. The single sentence used to summarise each nursing assessment is underpinned by nursing understanding of the degree of need for the patient in each domain. The key areas of concern to nursing are noted in the summary, but each assessment includes all domains. Notice that the scoring system requires a translation from nursing knowledge and intuition into numeric scores. The final 'score' for the patient is difficult to interpret. The score appears crisp and linear. There could be a substantial difference in reality between a score of 13 and 14 for a patient which is not reflected by this approach. The key difference in the two examples is the weighting given by nursing assessment to the different domains. The patient 'scores' 17 immediately post-operatively due to the operation and its aftermath. He 'scores' 14 after four days of nursing care due to the physical and emotional frailty attendant on his illness. The next section describes some of the basics of fuzzy logic which allows for capturing imprecision, uncertainty and vagueness in linguistic descriptions of nursing assessment.

3. Fuzzy Logic and Type-2 Fuzzy Sets

Fuzzy Logic and Type-1 Fuzzy Sets:

Fuzzy sets [8] lay the basis for fuzzy logic. To enable an understanding and appreciation of the proposed approach some background to fuzzy sets and fuzzy logic is required. The notion of a fuzzy set is very simple and straightforward. A fuzzy set allows for modelling imprecision, uncertainty or vagueness. Consider the example of describing a man as tall. In general we all understand what we mean by this concept, but it is not precise. For example a man who is 6ft 5ins would be considered to be tall by almost everyone. A man who is 5ft 6ins would not be tall. However at what point does the concept tall become true - 6ft? Does that make somebody who is 5ft 11 1/2 ins not tall? In reality men are tall *to some degree*. It is this imprecision that fuzzy sets can capture. In this example there would be a fuzzy set `tall` that would be over the domain `height` in the range say 5ft to 7ft. A member of the set `tall` would be a member to some degree represented by a number between zero and one. So we could have a fuzzy set `tall` where Danny Devito is a member to degree 0.4, Robert John to degree 0.73 and Michael Jordan to degree 0.91. The fuzzy set `tall` is shown in Figure 1. The line in this figure is known as its' membership function. A fuzzy set is fully defined by its' membership function. From now on, for reasons that will become clear we use the term Type-1 fuzzy set to describe these fuzzy sets.

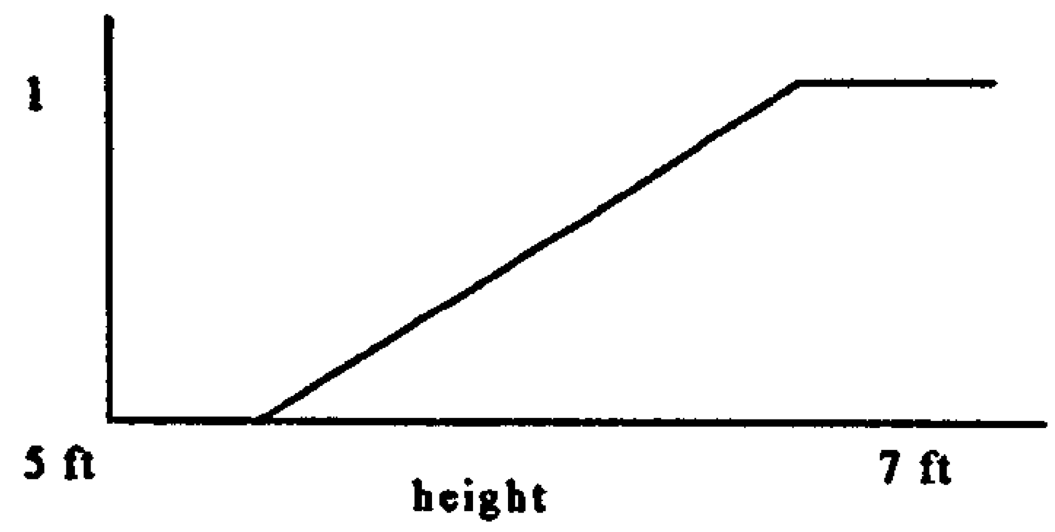

Figure 1: The Fuzzy Set `tall`

Fuzzy logic [10] is a mathematical theory that combines these Type-1 fuzzy sets by, for example AND or via the use of fuzzy if-then rules. It has a thirty year history and much successful application especially in the control field [12] [13]. However there are two particular problems with the Type-1 approach in the domain of nursing.

.1. The Nature of a Type-1 Membership Function

The grade of membership for any member of a Type-1 fuzzy set is a 'crisp' number in the interval [0,1]. A framework based on 'fuzziness' could be developed yet the problem is represented directly by 'non-fuzzy' numbers. As Klir and Folger [14] point out, "..it may seem problematical, if not paradoxical, that a representation of fuzziness is made using membership grades that are themselves precise real numbers". For any Type-1 application there is some translation (fuzzification) from a measurement or an expert's description to the membership of the fuzzy set. So, if we use the example of a Type-1 fuzzy set tall the degree of membership is a mapping from the crisp height, say 5ft 10ins, to the membership grade, say 0.87. Note that the height is a crisp measurement in that there is no uncertainty directly associated with it yet, even for the most accurate form of measurement, there is some uncertainty. The mapping to 0.87 is also crisp yet there is uncertainty in the value 0.87. The very nature of Type-1 fuzzy systems, then, is that the Type-1 membership function is not capturing all the uncertainty in the knowledge representation. Obviously, the fact that there are hundreds, if not thousands, of successful Type-1 applications means that this may not be very important in certain domains. By the same token, however, the paradox that fuzzy system developers fuzzify by using crisp numbers is worth further consideration especially for decision making problems like nursing. Even supposing the notion of a Type-1 membership function is acceptable there is still the question of how best to determine the membership functions. One of the major problems faced by any fuzzy system developer is that of determining the membership function [15]. In control problems, almost without exception, triangular or trapezoidal membership functions are used and appear to work in many domains. In non-control applications the choice of membership function is an issue and although triangular membership functions may provide a solution, further investigations need to be carried out as to their efficacy.

.2. Many Concepts in Nursing Can Not be Directly Measured

Gaines [16] in an extensive piece of work makes an interesting point – "We are able to generate and follow arguments involving 'tallness' without having any concept of inches, centimetres or any other metric scales. To introduce the former in terms of the latter reverses the actual process of derivation and, in particular, leads to a false distinction between these concepts such as 'tallness' that have a well defined, single-parameter, physical metric, and those such as 'beautiful' which do not." This is important. What is being said here is that we do not necessarily need physical measurements in our daily use of language. We know when somebody is quite tall. By the same token we know when somebody is beautiful (or indeed handsome!). For nursing assessment many of the words are not measurable. For example when a patient's physical condition is described as 'stable' we are not able to measure directly the physical condition or measure its stability.

Type-2 Fuzzy Sets:

Type-2 sets that allow linguistic, as opposed to numeric, membership grades were initially described by Zadeh [10]. Instead of attaching numbers between zero and one the membership grades are themselves Type-1 fuzzy sets. The most effective way to compare Type-1 and Type-2 fuzzy sets is by an example. Suppose we wish to use the fuzzy set tall then with Type-1 fuzzy sets we might say that Michael Jordan is tall to degree 0.95, Danny Devito to degree 0.4 and Robert John to degree 0.6.

A Type-1 representation can be written as

tall = 0.95/Michael Jordan + 0.4/Danny Devito +0.6/Robert John.

Whereas a Type-2 interpretation of this set could be

tall = High/Michael Jordan + Low/Danny Devito +Medium/Robert John

where High, Low and Medium are themselves Type-1 fuzzy sets.

The difference between Type-1 and Type-2 sets is that Type-1 sets have an x-axis representing the domain - in this case the height of an individual. Type-2 sets employ Type-1 sets as the membership grades and these grades are in the range zero to one. Therefore, these fuzzy sets of Type-2 allow for the idea that the members of a fuzzy set do not necessarily have membership grades in [0,1] but the degree of membership for the member is itself a fuzzy set. Dubois and Prade [17] summarise Type-2 fuzzy sets in the following way: "Type-2 fuzzy sets are fuzzy sets whose grades of membership are themselves fuzzy. They are intuitively appealing because grades of membership can never be obtained precisely in practical situations."

Since for nursing assessment we wish to describe a patient as 'stable' for example, a concept which is not directly measurable it would appear that fuzzy logic and Type-2 fuzzy sets offer a more 'natural' representation than any scoring system. The next section considers the application of these ideas to the patient described earlier.

4. A Type-2 Example

The approach is to model the five domains discussed in Section 2 using Type-2 sets. For example we have a Type-2 fuzzy set 'Dependency' which instead of being scored on a scale of 1 to 5 would have possible membership grades (Type-1 fuzzy sets) 'independent', 'becoming independent', 'dependent', 'heavily dependent' and 'totally dependent'. These membership grades are held as triangular membership function as for example in Figure 2.

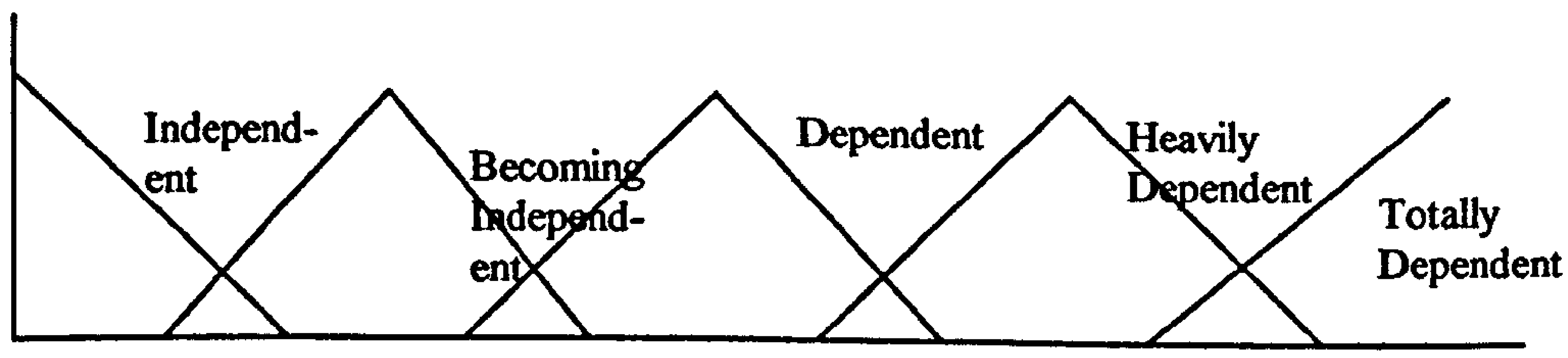

Figure 2: The Type-2 Fuzzy Membership Grades

These Type-2 fuzzy sets for the five domains can then be combined using AND to produce a Type-2 set that could be labelled, for example, 'Overall Condition'. The nursing understanding of the range of patient acuity for overall condition from 'stable' to 'critical', is also imprecise. However, nursing assessment pays attention to all aspects or domains of the patient's well being to create the summary.

For the patient in Day 1 his membership in the Type-2 sets would be as in Table 1.

Table 1 Patient Description using a Type-2 Fuzzy Representation

Day	Physical/ Medical	Complicating Factors	Clinical Intervention	Dependency	Psycho-Social
One	Potentially unstable	Potentially unstable	Complex	Heavily Dependent	Moderate Needs

Without going into the detailed mathematics (see [18] for the full detail) for each day the grades can be combined to produce an output that is itself a Type-2 representation. The degree of need within each domain is prioritised or given weighting so that the summary reflects the priority of need and yet all domains are considered in the final summing up. Using the triangular membership functions in Figure 2 the output for day one for the patient would be a triangular membership function of the same shape, and in the same place as that of 'Heavily Dependent' in Figure 2. This membership grade would not have the same label attached to it but could be described as 'unstable' or 'more acutely unwell': more unwell than 'potentially unstable', but not yet 'critical' or 'critically unwell'.

5. Conclusion

Numeric scoring of patient assessment is well established but the translation from words to integers, even when split into domains followed by a summation of integers, is crude and loses information. Nursing assessment of the patient summarises the patient's need for nursing care in simple sentences emphasising the areas of need. Each domain of nursing assessment comprises Type-2 fuzzy sets which can be combined into a summary, which in itself is a Type-2 set. The short phrases reflecting the concepts of nurse/patient interaction are imprecise and cannot be directly measured. Fuzzy logic however models imprecision, uncertainty and vagueness allowing for Computing with Words. This paper has explored the notion of using Type-2 fuzzy sets to represent nursing knowledge and experience and has demonstrated its use in an example. The example shows that words can be modelled effectively in nursing using a Type-2 fuzzy paradigm. Further work will include a detailed study on a large number of patients and an optimisation of the shape of the Type-1 membership grades.

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Type 2 Fuzzy Sets and Neuro-Fuzzy Clustering of Radiographic Tibia Images

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Abstract.

This paper is concerned with pre-processing of data for submission to neural networks. In particular the use of type 2 fuzzy sets to assist in this process is discussed and the results of using type 2 sets with FuzzyART is presented for clustering of radiographic tibia images. These results indicate that the approach outperforms a type I approach, for certain tibia problems, and that the type 2 solution assisted the expert in analysing a set of images that he was unable to classify originally.

Keywords: Type 2 Fuzzy Sets, Neural Networks, FuzzyART, Clustering, Image analysis, Tibia Stress fractures

1. Introduction

Pre-processing of the inputs for neural networks is a well known problem. This paper is concerned with an investigation into whether type 2 sets assist in presenting data in a suitable way for processing by the fuzzy neural network paradigm FuzzyART[5] for clustering of radiographic tibia images.

The paper is structured as follows: Section 2 reviews the medical background to the problem and discusses the pre-processing adopted for the tibia images, Section 3 describes how type 2 sets have been used to represent the images, Section 4 presents the results and finally in Section 5 the conclusions are presented.

2. Medical Background

There has been a marked increase in the number of people engaging in sport and exercise over the last fifteen years or so and a corresponding number of injuries presented in injury clinics. The tibia has been shown to be the most common site of stress fractures in athletes ([lo], [6], [13]). Clinicians have therefore built up significant expertise in recognising and classifying these injuries and have accumulated evidence to support them such as bone scans. This research is concerned with the analysis of bone scans from stress related injuries to the tibia. A large database of over 200 images has been established over the past fifteen years. This has not been formally analysed with respect to the type or class of injury.

The expert consultant faces a difficult task on an every day basis of classifying the images because the order and rate of presentation of injuries and associated symptoms varies and are relatively infrequent. Hence opinions as to the likely classifications may well change over time. It would not be possible with the paucity and impreciseness of the data and the state of knowledge of the expert to provide an 'automatic' classification.

The major use of bone scanning in sports injuries is for the detection of an injury class called 'stress fractures'. Stress fractures are usually partial fractures or cracks which result from the application of habitual, non-violent, repetitive stress which exceeds the existing functional capacity of the bones. Patients who presented with exercise-induced lower leg pain were originally bone scanned in order to eliminate the possibility of stress fractures. However, it soon became apparent that the bone scan appearance was not simply a matter of normal or abnormal as several other patterns began to emerge. These have been tentatively placed into one of seven classes based on an interpretation of the image:

1. **Normal.** The image is completely uniform.
2. **Athletic Normal.** Lines of uniform increased uptake along the entire length of shin.
3. **Stress Fracture.** There is a single, dense, localised area of uptake.
4. **Medial Tibial Syndrome(MTS).** There is a line of increased uptake along some part of the lower half of the posterior cortex.
5. **Focal Multiple spots.** These could be sub-clinical multiple stress fractures.
6. **Healing Stress Fracture.** Less obvious than a stress fracture.

7. Patchy. Non-uniform uptake.

The classes other than the MTS and stress fractures are based on experiential interpretation of the images alone because there are no confirming clinical factors. Hence there can be difficulties distinguishing between them. Experience has shown that re-examination of these images may reclassify them into different categories due to perceptual variation. This indicates that these classes may really be in a dimension of overlapping sub and super-classes. Since there has not been a systematic analysis of what the database of images contains, these dimensions are unknown. This has significance since the efficacy of a particular treatment may depend on correct classification.

There is no precise way, when viewing a radiographic image, of determining the problem with any given shin. Much of the description of the images is imprecise, vague and fuzzy. For instance, the distinctiveness of a line on the image or whether a line is 'much longer' than its width are important in assisting with the diagnosis. The approach taken was to describe the images using a questionnaire developed in conjunction with the expert. In total 203 images were analysed using the questionnaire. Some of the data is essentially binary in nature whilst the rest could be described as imprecise. For input to a neural network these fuzzy categories had to be converted into a numeric format. Earlier work [8] adopted a simple approach of representing the various categories by essentially a type 1 fuzzy set. For example the question about a line on an image "Where is the line located?' was translated into a fuzzy set *Location* with values O(Lower), 0.25(Junction Lower Middle), 0.5(Middle), 0.75(Junction Middle Upper) and 1 (Upper). In other words an image which was described as 'Junction Middle Upper' had 0.75 as the degree of membership of the set *Location*. Clearly these numbers have been chosen arbitrarily to reflect the ordering of the description of where the line is located. Given that this transformation is very crude the results of this early work were encouraging indicating that, using this method, some relationship could be shown by the output from the questionnaire and the consultant's interpretation of the image.

3. Type 2 Fuzzy Pre-Processing

The initial approach to representing fuzzy categories adopted a traditional fuzzy set approach (type 1) of assigning numeric grades to represent the membership values. This approach is crude in that it reduces the consultant's imprecise terminology to an arbitrary number in [0,1]. An alternative approach is the use of type 2 fuzzy sets.

Type 2 sets[1] that allow linguistic membership grades were initially described by Zadeh [17]. A particularly clear definition of a type 2 fuzzy set is *"A fuzzy set of type 2 is defined by a fuzzy membership function, the grade (that is, fuzzy grade) of which is a fuzzy set in the unit interval [0, 1], rather than a point in [0, 1]"* [11]. A fuzzy set of type 2, A, in a set X, is the fuzzy set characterised by the fuzzy membership function μ_A as

$$\mu_A : X \rightarrow [0,1]^{[0,1]} \qquad (1)$$

where μ_A (x) is known as a *fuzzy grade*, a fuzzy set in [0,1]. Fuzzy sets of type 2, therefore, allow for the idea that the members of a fuzzy set do not necessarily have membership grades in [0, 1] but the degree of membership for the member is itself a fuzzy set. As Yager [16] points out *"The usefulness of fuzzy subsets of type II is that it enables us to extend membership grades to linguistic values"*.

The questionnaire was re-interpreted to allow the consultant's interpretation of each image to be represented using type 2 fuzzy sets for the location and length of the line in an image. These two features are very important in assisting the diagnosis. So, the following type 2 fuzzy sets were used:

- *Location* is a type 2 fuzzy set where any particular image is a member of the set with fuzzy grade
  - Low
  - Junction Low Middle
  - Middle
  - Junction Middle Upper
  - Upper
- *Ratio* (ratio of length to width) which is a type 2 fuzzy set with fuzzy grades
  - Same
  - Longer
  - Much Longer

The sets used are shown in Fig. 1.

The next section gives an overview of FuzzyART selected for the clustering process and presents some results.

4. Results

There are many neural network approaches to clustering. The algorithm chosen for this particular problem was FuzzyART as it is suitable for clustering with fuzzy data. The purpose of this paper is not to describe the paradigm. The reader is referred to Carpenter and Grossberg[4], [3], Beale[2], Bartfi[1] and Innocent[7] for a detailed description of the ART algorithm. ART networks map feature vectors

[1] Researchers sometimes describe them as type II sets. Throughout type 2 will be used and type 1 will refer to 'traditional' fuzzy sets

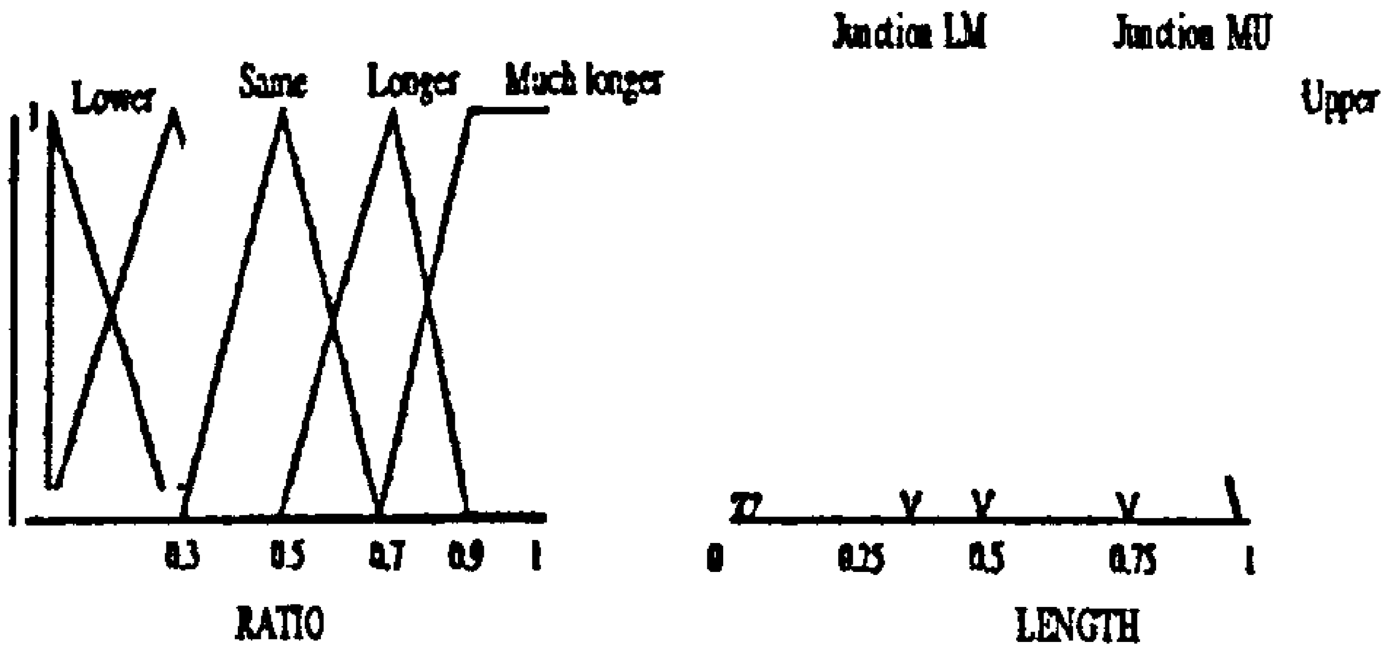

Figure 1. The type *2* fuzzy sets *'Location '* and *'Ratio'*

in the input into the input layer where, after processing, it is mapped into a single node corresponding to a cluster identifier. If a node cannot be mapped into, a new node is established. For this research the data is inherently fuzzy in nature so the modified ART algorithm FuzzyART([5]) has been adopted. The essential difference between FuzzyART and ART is that of using the fuzzy intersection operator of min where the logical AND operator is used in the original ART algorithm.

The architecture adopted for this work consists of 2 ART networks (ARTa and ARTb) connected by a mapping control structure. During training, the input pattern is presented to the ARTa network and the desired output pattern is presented to the ARTb network. Each ART network has a controlling parameter called the 'vigilance' setting. If the vigilance is high, then this forces the ART training algorithm to make fine discriminations between patterns presented to it. If the vigilance is low, then the discriminations are coarse. The vigilance for ARTb is set high and fixed so that every class of a desired pattern is associated with a single node in the network. In ARTa there is a starting vigilance level which is quite low and then is increased as required by a 'match tracking' process.

FuzzyART is highly dependant on the order of presentation of the input data and on their respective control parameters [7]. Consequently, the order of presentation of the input vectors was randomised and 10 separate trials for each of 3 values of the control parameters were produced. Each of these 30 trials was examined with respect to the expert's classification of the images and 1 of the random clusters for each control value and each method were selected. This was done by selecting that trial that best represents the mode of all trials. Finally, for every cluster containing an expert's 'unknown' image vector, an artmap was trained and used to predict the possible class of the unknown image. The expert was then asked to comment on the neural classification.

The nature of the data and the neural network paradigms means that presenting results in a clear well defined way is difficult. A variety of factors affect the results:

- the order of presentation of vectors for the algorithm affects the clustering;
- different control parameters produce different clusters;
- for the type 1 sets, arbitrary numbers have been chosen to reflect the category ordering;
- for the type 2 sets the fuzzy membership grades chosen will affect the results.

The interest here is in assisting the expert in the diagnostic process. The nature of the classification of shin images is that the expert is constantly learning about the processes going on, often revising the nature of the groupings and exploring potential sub clusters within clusters. The clustering algorithms and the paucity of data make automatic classification impossible. However, it is hoped that the method will provide the expert some beneficial help with image classification. What is important for the shin image analysis is whether, given a small amount of data, the technique adopted can give some indication to the expert of the type of problem with the image to a reasonable degree of accuracy. In other words can this approach offer an insight to the consultant for those images which could not be classified originally? Further does the type 2 representation produce better results for FuzzyART *given random orderings and particular control parameters?*

Of the original 203 images, there were 38 images the expert could not originally classify.

The expert was asked to look at particular clusters and make a judgement in a number of categories relating to the prediction of the system. These were:

1. if the system predictions all agree on a particular prediction category: very likely, likely, possible, impossible;
2. if the system produced differing predictions: preferred prediction category.

By applying this approach to all the unknown images, we can construct a table (Table 1) showing the performance of each type of pre-processing. The results are mixed, but for MTS, Patchy and Stress Fractures type 2 appears to show an improvement on type 1 perhaps reflecting that type 2 sets capture more of the uncertainty in the expert's description of the images. However for Healing Stress Fractures the results are not consistent. This may reflect the uncertainty in the source image itself. Further work will have to be carried out to find the reasons for this.

Expert	Class	FuzzyART parameters					
		0.2		0.25		0.3	
		T1	T2	T1	T2	T1	T2
MTS	15	40	80	80	80	80	67
Patchy	2	0	0	0	50	50	0
Stress	2	0	100	50	50	0	50
Healing	15	20	33	73	40	47	53

Table 1. A comparison between the expert and the pre-processing - percent agreement

5. Conclusions

The problem of classification of images of the tibia is fraught with difficulties. There is a small amount of historical data much of which is imprecise. In particular, the images themselves are difficult to classify and, indeed, the classifications change with time. This paper presents some reults of applying neuro-fuzzy clustering techniques to the problem where some of the input data is represented by type 2 fuzzy sets. These results indicate that certain combinations of neural network paradigm and type 2 representation produce an improvement over a type 1 representation for some of those images the expert was originally unable to classify. This approach therefore offers insights to the expert to assist them in classification of shin images.

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A Type 2 Adaptive Fuzzy Inferencing System

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Abstract

For standard fuzzy sets, type 1 fuzzy sets, the membership function is represented by numbers or some function whose parameters have to be determined by knowledge acquisition or some 'learning' algorithm. Type 1 fuzzy sets make the assumption that there is full certainty in these representations. In most applications this is not the case and can be considered a serious shortcoming with the approach. As an alternative, type 2 fuzzy sets allow for linguistic grades of membership and, therefore, present a better representation of the 'fuzziness', when applied to a particular problem, than type 1 fuzzy sets.

With a type 2 representation, there is no requirement to ask an expert for numerical membership grades - they can be linguistic. However, the associated cost is that the fuzzy membership grades and rules have somehow to be determined and no recognised approach yet exists. For type 1 systems a number of approaches have been adopted. One in particular is the Adaptive Network Based Fuzzy Inferencing System (ANFIS) which has successfully been applied to a variety of applications. ANFIS takes domain data and learns the membership functions and rules for a type 1 fuzzy inferencing system. Our work aims to extend this approach for type 2 systems.

This paper presents this work. Our Type 2 Adaptive Fuzzy Inferencing System has inputs that are linguistic variables (rather than numbers) and the membership functions for these fuzzy grades are learnt from the relationship between these inputs and the given output. The paper describes the algorithm developed highlighting the theoretical and computational issues involved.

1.INTRODUCTION

This paper reports on the development of an adaptive fuzzy system that employs type 2 fuzzy sets. Neural networks are an example of *adaptive networks* which are network structures consisting of connected nodes where the links are directional. The nodes are processing units that perform a function and can be adaptive in that the result of the processing is dependent on the parameters selected for the node. Not all nodes in such a network are necessarily adaptive. The adaptive network has a 'learning' algorithm where the parameters in the nodes are learnt by training from data - known as supervised learning. It is essentially a mapping process between the inputs and outputs of the network. This approach has some advantages including:

- these networks have strong generalisation capabilities - given an input it has not seen before a trained network will usually perform well in 'guessing' the output;
- the fact that the parameters are learnt by the algorithm removes the need for the knowledge acquisition exercise from an expert(s);
- they are known to be robust.

However they have significant disadvantages. In particular neural networks learn a mapping between an input vector and an output vector and as such are 'black box' in that it is not possible to gain a good understanding of the underlying reasoning process. They do not have a knowledge base in the same way an expert system would so there is no opportunity for an expert to inspect the reasoning process. They take numerical input and can handle noisy data. Real world applications often have uncertain or fuzzy data where for example an expert would describe a property as 'low' rather than a numerical term. Lin and Lu[11] report some work on training a neural network with linguistic terms but this approach is not widely used and the resulting system is still 'black box' and difficult to interpret.

Computer systems based on fuzzy sets [24] go under a variety of terms - fuzzy inferencing systems, fuzzy systems, fuzzy knowledge based systems - and have many of applications (e.g. [15] [2] [8]). For the purposes of this paper we use the term *fuzzy system* to describe a system that employs fuzzy sets, fuzzy if-then rules, a fuzzy

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inferencing algorithm and, usually, defuzzification. The reader is referred to Mendel's tutorial paper [12] for a detailed explanation. Fuzzy systems have a number of attractions:

- they allow for linguistic description of a problem by an expert;
- they are often more robust than traditional mathematical approaches;
- the underlying reasoning process can be examined.

They do however have a major drawback in that they do not learn and therefore require significant human intervention from an expert. In particular the membership functions of the fuzzy sets have to be determined. This problem has been tackled by a number of researchers (e.g. [22] [19] [17] [21] [13]) but these approaches are often domain dependent and still require input from human expertise. An alternative approach is an *adaptive fuzzy system* [1] which offers the ability to learn from data. This paper proposes an extension to the adaptive fuzzy system known as ANFIS [4]. ANFIS takes numeric input and output to learn a fuzzy system. The numeric inputs are fuzzified as in a fuzzy system. As discussed, in many applications it would be appropriate to allow for linguistic inputs to the network. Type 2 fuzzy sets allow for linguistic membership grades and would thus seem eminently suitable as a form of representation for the linguistic inputs to an adaptive fuzzy system. This is the approach adopted here. Type 1 fuzzy inferencing systems are well established but the theory complete type 2 fuzzy logic systems[10] is still under development. Type 2 fuzzy sets are, however, starting to be used more widely in a number of applications. For the proposed approach there are a number of theoretical results that are important but due to space limitations much of the theoretical underpinning has been left to the reader to follow up the references provided. The paper is organised as follows: Section 2 explores type 2 fuzzy sets,Section 3 describes the proposed approach combining type 2 fuzzy sets and ANFIS and Section 4 provides a conclusion.

2. TYPE 2 FUZZY SETS

Type 2 fuzzy sets[1] that allow linguistic membership grades were initially described by Zadeh [25]. A clear definition of a type 2 fuzzy set is *"A fuzzy set of type 2 is defined by a fuzzy membership function, the grade (that is, fuzzy grade) of which is a fuzzy set in the unit interval [0,1], rather than a point in [0,1]"* [14]. A fuzzy set of type 2, A, in a set X, is the fuzzy set characterised by the fuzzy membership function μ_A as

$$\mu_A : X \rightarrow [0,1]^{[0,1]}$$

where $\mu_A(x)$ is known as a *fuzzy grade*, a fuzzy set in [0,1]. These fuzzy sets of type 2, therefore, allow for the idea that the members of a fuzzy set do not necessarily have membership grades in [0,1] but the degree of membership for the member is itself a fuzzy set. As Yager [23] points out *"The usefulness of fuzzy subsets of type II is that it enables us to extend membership grades to linguistic values"*. Practical applications of type 2 sets include handling tolerances in fuzzy equations systems[20], fuzzy regression models[3], determining membership functions[19], community transport scheduling[5], the internet[18] and computer networks[16] and multiobjective decision making[23]. To enable the use of type 2 sets in a computer system that uses if-then rules a method is required for computing the intersection and union of two type 2 sets. Using the notation provided by Mizumoto and Tanaka[14] we have type 2 sets, A and B, in X and $\mu_A(x)$ and $\mu_B(x)$ are two *fuzzy grades* of A and B respectively, represented as:

$$\begin{aligned}\mu_A(x) &= f(u_1)/u_1 + f(u_2)/u_2 + ... + f(u_n)/u_n \\ &= \sum_i f(u_i)/u_i\end{aligned}$$

$$\begin{aligned}\mu_B(x) &= g(w_1)/w_1 + g(w_2)/w_2 + ... + g(w_m)/w_m \\ &= \sum_j g(w_j)/w_j\end{aligned}$$

where the functions f and g are membership functions of fuzzy grades.

The most widely used definitions for union and intersection of type 2 sets are provided by Zadeh[26]. His work relies on the use of the *extension principle*[25]. This states that if $*$ is a binary operation in X then this operation can be applied to A and B by

$$A * B = \sum_{i,j} (f(u_i) \wedge g(w_j))/(u_i * w_j)$$

This leads to the following definitions for union and intersection of type 2 sets.

Union

$$\begin{aligned}\mu_{A \cup B}(x) &= \mu_A(x) \sqcup \mu_B(x) \\ &= \sum_{i,j} (f(u_i) \wedge g(w_j))/(u_i \vee w_j)\end{aligned}$$

[1] Researchers sometimes describe them as type II sets. Throughout type 2 will be used and type 1 will refer to 'traditional' fuzzy sets

Intersection

$$\mu_{A\cap B}(x) = \mu_A(x) \sqcap \mu_B(x) = \sum_{i,j} (f(u_i) \wedge g(w_j))/(u_i \wedge w_j)$$

where ⊔ denotes *join* and ⊓ denotes *meet*[9]. There are some key points to be made when comparing type 1 sets and type 2 sets:

1. The type 1 representation requires the 'expert' to describe the membership functions using numbers, whereas the type 2 approach only requires a fuzzy grade.
2. Type 2 representations capture uncertainty or 'noise' in the membership function.
3. Type 2 membership grades still have to be determined for any real application.
4. Type 2 join and meet lead to type 2 fuzzy sets. This has the merit that the uncertainty in the the initial linguistic type 2 description leads to an imprecise, type 2, solution.

All of these points are important when considering a type 2 adaptive fuzzy system. In particular, the adaptive fuzzy system will still need to learn the membership grades. A more detailed overview of type 2 fuzzy sets is provided by previous work from the author[6][7]. The next Section describes the proposed algorithm for an adaptive fuzzy inference system using type 2 sets.

3. A TYPE 2 ANFIS

This section provides a description of the proposed method for modifying the original ANFIS using type 2 sets to represent linguistic grades.

The original ANFIS approach[4] is an adaptive network that learns the membership functions and fuzzy rules in a fuzzy system. It is a supervised network with a number of inputs and outputs and the nodes within the network perform the various functions of a fuzzy system, namely:

- fuzzification of crisp inputs;
- carrying out the fuzzy AND on the antecedents of the rules;
- composition of the rules;
- defuzzification.

It performs all the functions in a fuzzy system and, usually by some steepest descent method, learns the various parameters that have to be determined.

There are a number of issues when designing an ANFIS solution, in particular:

- which operators to use for the intersection and union (there are a number of choices);
- which format the rules should take (e.g. Mamdani, Sugeno) and
- the choice of defuzzification method.

This applies, similarly, to the type 2 ANFIS and only one approach is described here. Further research will investigate the design issues in more detail. The usual ANFIS has crisp inputs (perhaps measurements for example). In a real application it might be more appropriate to have linguistic inputs, perhaps provided by an expert, which can be modelled as type 2 sets. This is the approach adopted in this work.

For illustration purposes the method for two inputs is discussed but this is extendable. Figure 1 shows the network. The inputs are linguistic and are represented as fuzzy grades. For the example here there are two linguistic inputs x_L and y_L. These vectors are then linguistic terms describing some particular quantity that might have different representations for different type 2 fuzzy sets. For example we might have $x_L = low$ and $y_L = medium$. The grades in type 2 sets are actually *fuzzy numbers* in that rather than a membership grade being crisp in [0,1] it's a fuzzy number in [0,1] where the width of the set indicates the uncertainty attached to the number. For a two input, one output model we therefore have four fuzzy rules:

$$IF\ x_L\ is\ A_1\ and\ y_L\ is\ B_1\ THEN\ z\ is\ C_1$$
$$IF\ x_L\ is\ A_1\ and\ y_L\ is\ B_2\ THEN\ z\ is\ C_2$$
$$IF\ x_L\ is\ A_2\ and\ y_L\ is\ B_1\ THEN\ z\ is\ C_3$$
$$IF\ x_L\ is\ A_2\ and\ y_L\ is\ B_2\ THEN\ z\ is\ C_4$$

where $A_1, A_2, B_1, B_2, C_1, C_2, C_3, C_4$ are type 2 fuzzy sets.

The network in Figure 1 describes the type 2 fuzzy system encoded in the rules above. A square node indicates an adaptive node with modifiable parameters whereas a circle indicates a fixed node that simply performs a function. As with the original ANFIS the approach consists of a forward pass and a backward pass with a learning algorithm which uses a combination of least square error and gradient descent to determine the parameters that are in the adaptive nodes. Due to space limitations we only describe the forward pass in detail and then describe briefly the learning algorithm.

LAYER 1

Every node in this layer is an adaptive node. Each node has attached to it three linguistic grades of membership of the type 2 fuzzy set which will match, from a linguistic point of view, the inputs supplied for training. There are clearly a number of possibilities for the 'shape' of the

LAYER 4
Layer 4 is a single fixed node that composes all the outputs of the rules. This is simply the union (or *join* as it is known in the case of type 2 sets) of membership grades of the C_i. So,

$$O_4 = \sqcup_i O_{3,i}$$

where for two type 2 fuzzy sets $\mu_A(x) = \sum_i f_x(u_i)/u_i$ and $\mu_B(x) = \sum_j g_x(v_j)/v_j$ the join is defined by

$$\begin{aligned}\mu_{A \cup B}(x) &= \mu_A(x) \sqcup \mu_B(x) \\ &= \sum_{i,j} (f_x(u_i) \vee g_x(w_j))/(u_i \vee w_j)\end{aligned}$$

The output of this layer is thus also a type 2 set.

LAYER 5
This layer is fixed and performs the defuzzification. Here the *type reduction process* described by Karnik and Mendel[10] is adopted. The idea behind this approach is to reduce the type 2 set to a type 1 set and then defuzzify that set in the normal way to produce a crisp value that can then be compared with the target output to calculate an error for 'back propagating' through the network for parameter modification.

4. THE LEARNING ALGORITHM

The network described has a number of parameters to be learnt. Firstly there are the parameters for the membership grades of the antecedent type 2 fuzzy sets in layer 1. Layer 3 has the conequent type 2 fuzzy sets and there are parameters to be learnt there as well. Since the output of the network is numeric this can be compared with the expected output from a teacher and backpropagation used to feed the error back to adjust the nodes in the parameters.

5. CONCLUSION

This paper has reported an approach that uses an adaptive network to learn a type 2 fuzzy system based on linguistic inputs and numeric output. This overcomes the shortfalls of the current approaches in that the linguistic inputs take the form of type 2 sets and rather than determining the membership grades by knowledge acquisition they can be learnt by using a supervised training algorithm. Future work will apply this method to real data sets to investigate its usefulness in applications.

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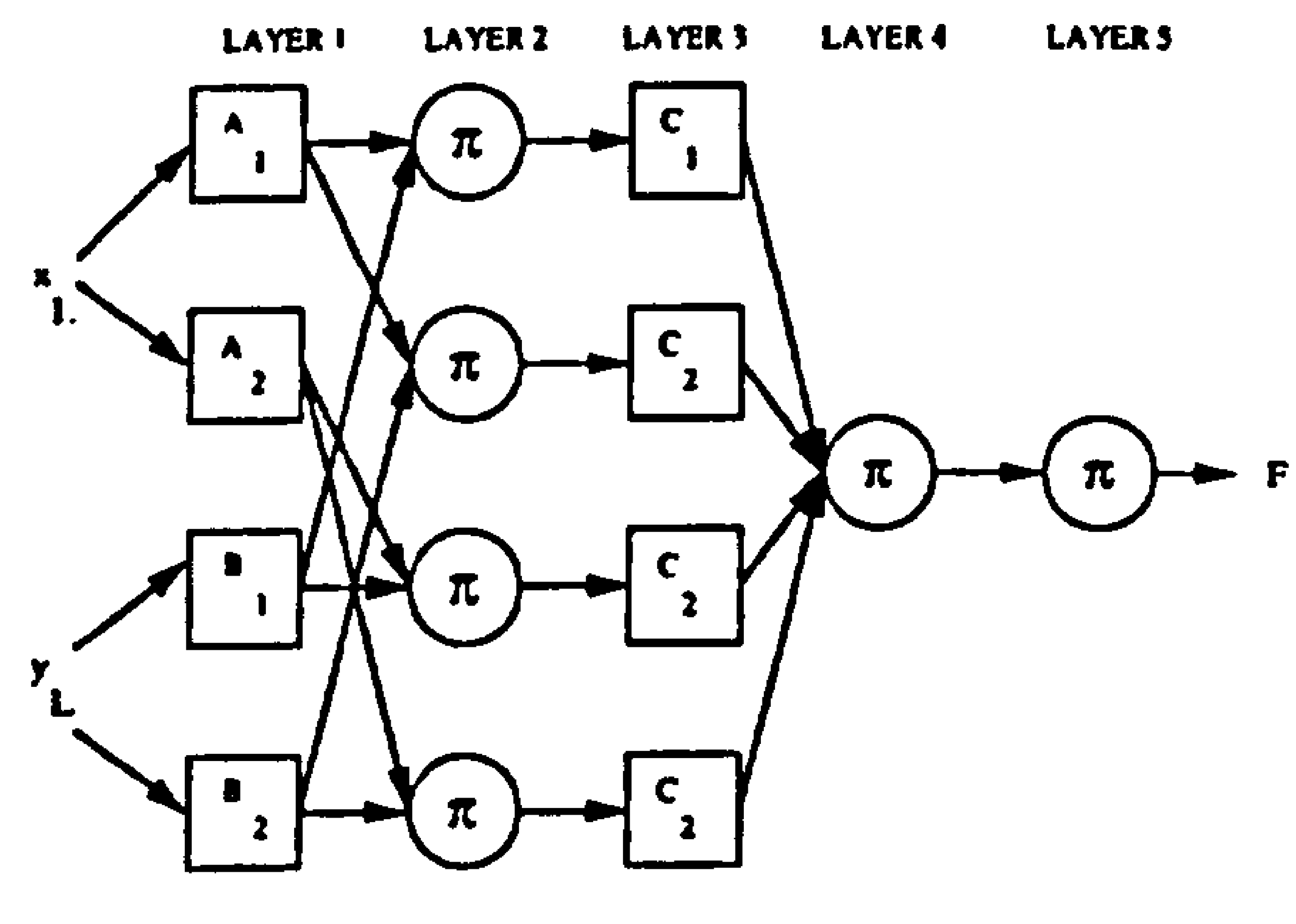

Figure 1. Type 2 ANFIS

fuzzy set. This approach adopts three linguistic grades (labelled low, medium and high). The membership functions are represented by sigmoidal and bell shaped functions:

$$O_{i,l}(x) = \frac{1}{1+exp(-a_{i,l}(x-c_{i,l}))} \quad for\ a_{i,l} < 0,\ 0 \le x \le 1$$

$$O_{i,h}(x) = \frac{1}{1+exp(-a_{i,h}(x-c_{i,h}))} \quad for\ a_{i,h} > 0,\ 0 \le x \le 1$$

$$O_{i,m} = \frac{1}{1+\left|\frac{x-c_{i,m}}{a_{i,m}}\right|^{2b_{i,m}}} \quad 0 \le x \le 1$$

So, there are a total of seven parameters to be learnt for each node ($a_{i,l}, c_{i,l}, a_{i,m}, b_{i,m}, c_{i,m}, a_{i,h}, c_{i,h}$). For the sigmoidal function the parameter a controls the slope at the crossover point c. For the bell shaped function the parameter c determines the centre, a the width and b is used to control the slope at the crossover points. Figure 2 shows some example membership functions.

LAYER 2

The nodes in this layer are fixed in that there are no parameters to be learnt. The purpose of this layer is to do the AND in the rules. The inputs to each node are two linguistic grades which are the outputs of the nodes in layer 1. Let's denote the grade from A_i to be f_i and from B_i to be g_i. Then using Zadeh's definition[27] we have

$$O_{2,i} = \sum_{j,k}(f_i(u_j) \wedge g_i(w_k))/(u_j \wedge w_k)\ i = 1,2$$

The output of this layer is a type 1 fuzzy set which is a membership grade in a type 2 fuzzy set.

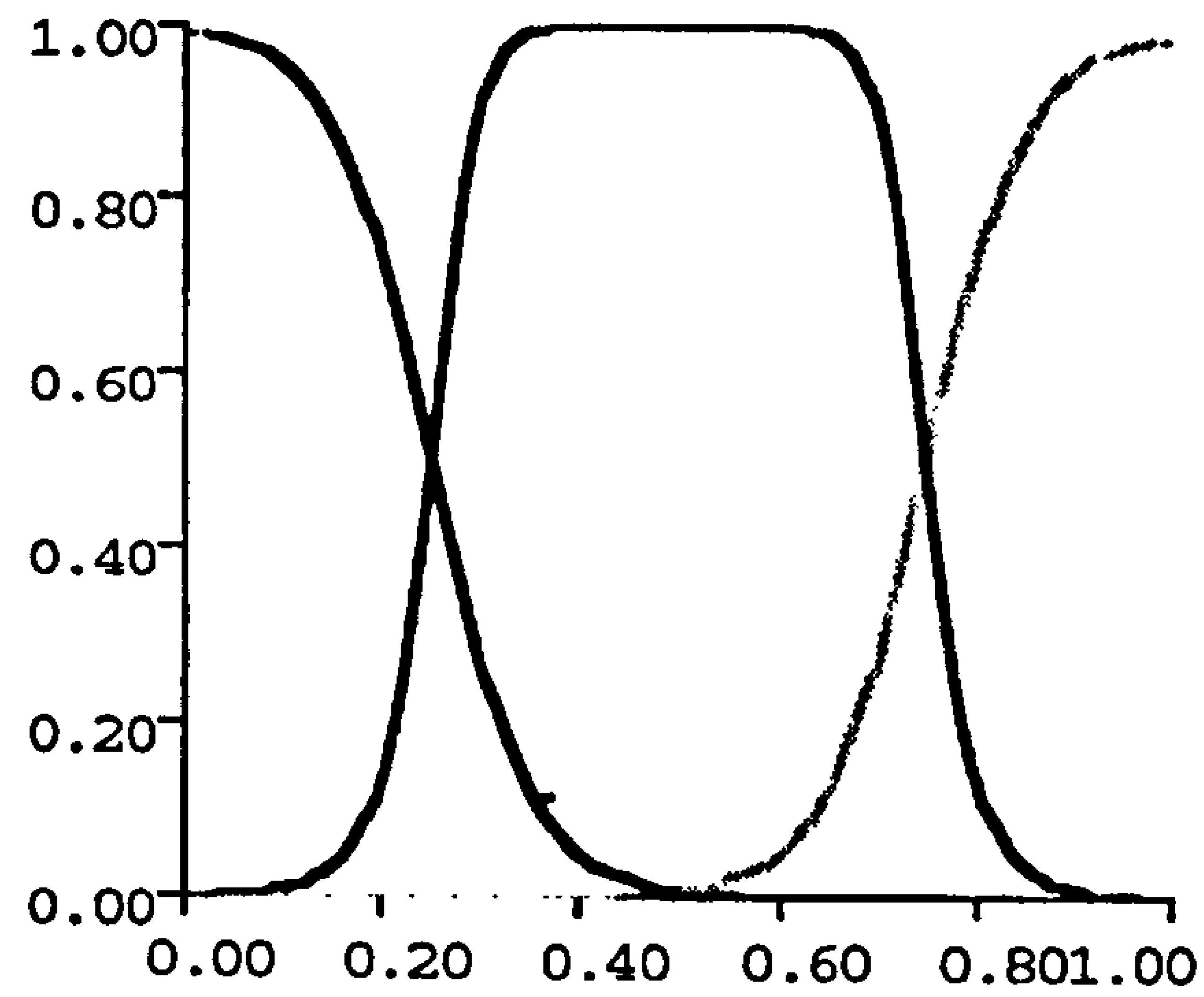

Figure 2. Membership functions for $a_l = -20; c_l = 0.25; a_h = 20; c_h = 0.75; a_m = 0.25; b_m = 5; c_m = 0.5$

LAYER 3

Layer 3 are adaptive nodes for learning the type 2 fuzzy set C_i. Each node has three linguistic grades, assumed to be of the same form as for those in layer 1, represented by:

$$O_{C_i,l}(x) = \frac{1}{1+exp(-a_{i,l}(x-c_{i,l}))} \quad for\ a_{i,l} < 0,\ 0 \le x \le 1$$

$$O_{C_i,h}(x) = \frac{1}{1+exp(-a_{i,h}(x-c_{i,h}))} \quad for\ a_{i,h} > 0,\ 0 \le x \le 1$$

$$O_{C_i,m} = \frac{1}{1+\left|\frac{x-c_{i,m}}{a_{i,m}}\right|^{2b_{i,m}}} \quad 0 \le x \le 1$$

Using Zadeh's extension principle we have for $O_{2,i} \Rightarrow C_i$ that the membership grade $O_{3,i}$ is given by:

$$O_{3,i} = \sum_{i,j}(\alpha_i \wedge \beta_j)/(1-v_i) \vee (v_i \wedge w_j)$$

where $O_{2,i}$ and $O_{C_i,z}(z = l, m \text{ or } h)$ have been dicretised to

$$O_{2,i} = \sum_i \alpha_i/v_i$$

$$O_{C_i,z} = \sum_j \beta_j/w_j$$

The output of this layer is thus a type 2 fuzzy set.

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Type 2 Fuzzy Sets for Knowledge Representation and Inferencing

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Abstract

Type 2 fuzzy sets allow for linguistic grades of membership thus assisting in knowledge representation. They also offer improvement on inferencing with type 1 sets. The various approaches to knowledge representation and inferencing are discussed, with worked examples, and some of the applications of type 2 sets are reported.
Keywords: Type 2 Fuzzy Sets, Fuzzy Normal Forms, Knowledge Representation, Inferencing

1. Introduction

Any fuzzy system employing fuzzy sets represents the fuzziness of the particular problem using a 'non-fuzzy' representation. As Klir and Folger[9] point out, " *..it may seem problematical, if not paradoxical, that a representation of fuzziness is made using membership grades that are themselves precise real numbers.* " As well as this paradox there are two particular problems that are faced by fuzzy systems developers.

Membership functions have to be determined. Attaching numbers to features of a problem is notoriously difficult and experts in a domain are unlikely to arrive at a consensus. The traditional statistical approaches[28] use averages or percentages based on informed guesses made by experts. 'Machine learning' approaches to determining membership functions, such as neural networks[16][27] and genetic algorithms[10][8] provide an alternative. These methods offer help in determining membership functions and have been applied successfully to certain control problems but they do not utilise fully the knowledge available from the expert.

Inferencing with type 1 sets presents another choice for the fuzzy system developer. The way of combining type 1 sets using for example 'AND', 'OR' or for implication in an if-then rule traditionally takes one particular form. However there are two forms, that are equivalent for boolean inferencing, that should be adopted. If these two forms are used an upper and lower bound are determined for the result of inferencing with fuzzy sets.

Zadeh[31] introduced the idea of type 2 fuzzy sets that help with both these problems (researchers differ in their terminology with some calling them Type II). The rest of this paper is structured as follows: Section 2 defines type 2 sets; Section 3 considers their use in knowledge representation; Section 4 describes inferencing with type 2 sets and Section 5 reports on some of the applications of type 2 sets.

2. Definition Of Type 2 Fuzzy Sets

Type 2 sets that allow linguistic membership grades were initially defined by Zadeh[31]. However a particularly clear definition of a type 2 fuzzy set is provided by Mizumoto and Tanaka[11] - *"A fuzzy set of type 2 is defined by a fuzzy membership function, the grade (that is, fuzzy grade) of which is a fuzzy set in the unit interval [0, 1], rather than a point in [0,1]"*. A fuzzy set of type 2, A, in a set X, is the fuzzy set characterised by the fuzzy membership function μ_A as

$$\mu_A : X \rightarrow [0,1]^{[0,1]}$$

where $\mu_A(x)$ is known as a *fuzzy grade*, a fuzzy set in [0,1].

Type 1 sets have a base axis representing the *domain* whereas type 2 sets employ type 1 sets as the membership grades and these grades are on the domain [0,1]. Therefore, these fuzzy sets of type 2 allow for the idea that the members of a fuzzy set do not necessarily have membership grades in [0,1] but the degree of membership for the member is itself a fuzzy set. As Yager[30] points out *"The usefulness of fuzzy subsets of type II is that it enables us to extend membership grades to linguistic values"*. A form of type 2 sets, *interval-valued fuzzy sets,* also relaxes the requirement for precise membership functions. In this case for each x, μ(x) is an interval in [0, 1]. Nguyen *et al*[13] argue that the use of intervals is necessary to to describe an expert's degree of belief. Gehrke *et al*[2] state *"Many people believe that assigning an exact number to an expert's opinion is too restrictive, and that the assignment of an interval of values is more realistic "*. As well as playing an important role in representing expert's knowledge, interval valued fuzzy sets also play an important role in inferencing with type 1 fuzzy sets as will be seen in Section 4. The theoretical properties of type 2 sets are described extensively by Mizumoto and Tanaka[11][12].

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3 Knowledge Representation with Type 2 Fuzzy Sets

To enable the use of type 2 sets in a computer system that uses if-then rules a method is required for computing the intersection and union of two type 2 sets. There are two different approaches reported in the literature - that of Hisdal[4] and Zadeh. Hoever the usual method is that proposed by Zadeh[32]. Using the notation provided by Mizumoto and Tanaka[11] we have type 2 sets, A and B, in X and $\mu_A(x)$ and $\mu_B(x)$ are two *fuzzy grades* of A and B respectively, represented as:

$$\mu_A(x) = f(u_1)/u_1 + f(u_2)/u_2 + \ldots + f(u_n)/u_n = \sum f(u_i)/u_i$$

$$\mu_B(x) = g(w_1)/w_1 + g(w_2)/w_2 + \ldots + g(w_m)/w_m = \sum_j g(w_j)/w_j$$

where the functions f and g are membership functions of fuzzy grades.

Zadeh's work relies on the use of the *extension principle*[31]. This states that if $*$ is a binary operation in X then this operation can be applied to A and B by

$$A * B = \sum_{i,j} (f(u_i) \wedge g(w_j))/(u_i * w_j)$$

This leads to the following definitions for union and intersection of type 2 sets.

Union

$$A \cap B \Leftrightarrow \mu_{A \cup B}(x) = \mu_A(x) \sqcup \mu_B(x) = \sum_{i,j} (f(u_i) \wedge g(w_j))/(u_i \vee w_j)$$

Intersection

$$A \cap B \Leftrightarrow \mu_{A \cap B}(x) = \mu_A(x) \sqcap \mu_B(x) = \sum_{i,j} (f(u_i) \wedge g(w_j))/(u_i \wedge w_j)$$

where $\sqcup$ denotes *join* and $\sqcap$ denotes *meet*[7]. Of particular interest are the implications these definitions have for handling fuzzy sets. This is best illustrated by using a simple example.

Consider the type 1 situation first. Suppose there are three people a, b and c where a is tall to degree 0.2 and heavy to degree 0.4, b is tall (0.7) and heavy (0.5) and c is tall (0.95) and heavy (0.8). The type 1 sets representing tall and heavy are

$$tall = 0.2/a + 0.7/b + 0.95/c$$

$$heavy = 0.4/a + 0.5/b + 0.8/c.$$

So, suppose we describe somebody who is tall and heavy as being big then we would arrive at

$$big = tall \cap heavy.$$

Using min for intersection

$$big = 0.2/a + 0.5/b + 0.8/c.$$

The outcome is that, for example, a is big to degree 0.2. Now consider tall and heavy as type 2 sets where

$$tall = low/a + medium/b + high/c$$

$$heavy = medium/a + medium/b + high/c$$

where for example

$$\text{low} = 1.0/0.0 + 0.75/0.1 + 0.5/0.2 + 0.25/0.3$$

$$medium = 0.33/0.3 + 0.67/0.4 + 1/0.5 + 0.67/0.6 + 0.33/0.7$$

$$high = 0.25/0.7 + 0.5/0.8 + 0.75/0.9 + 1.0/1.0$$

As with the type 1 example, big is interpreted as tall and heavy *(big = tall* $\sqcap$ *heavy)* then for person a

$$big_a = tall_a \sqcap heavy,$$

which is the meet of the two fuzzy grades low and medium. Using the definition for meet described above we have

$$big_a = 1.0/0.0 + 0.75/0.1 + 0.5/0.2 + 0.25/0.3 = \text{low}$$

There are various points to be made about this comparison:

1. The type 1 representation requires the 'expert' to attach a number to each person to describe their 'tallness' or 'heaviness' whereas the type 2 approach only requires a fuzzy grade.
2. The fuzzy grades low, medium and high are the same for both tall and heavy for simplicity purposes. They need not be.
3. The fuzzy grades low, medium and high are membership functions that still have to be determined.
4. The set 'big' for the type 1 approach is a type 1 set whereas the type 2 approach yields a type 2 solution.

An alternative definition for intersection is provided by Zadeh[32][33]. This definition relies on the level-set form of the extension principle[32]). If A is a subset of U then an α level set is a non-fuzzy set, A, which comprises all elements of U whose grade of membership in A is greater than or equal to a . A fuzzy set A can be decomposed into its constituent level sets

$$A = \sum_{\alpha} \alpha A_{\alpha}$$

and states that, suppose that f is a mapping from U to V and A is a subset of U, then

$$f(A) = \sum_{\alpha} \alpha f(A_{\alpha}).$$

This leads to the following definition for the intersection of type 2 sets

$$\mu_{A \cap B} = \sum_{\alpha} \alpha(\mu_A^{\alpha} \wedge \mu_B^{\alpha})$$

μ_A^{α} means the membership function for the α level set of A.

4. Inferencing and Type 2 Fuzzy Sets

This section considers the approaches that have been adopted for inferencing using type 2 sets. The standard method for inferencing with type 1 sets is based on the idea of *generalised modus ponens (gmp)* and is well established[33]. Hisdal[4] argues that this standard method for implication does, in many situations, not reflect the inherent uncertainty in any particular si tuation and that type 2 sets allow for inferencing with type 1 sets that adequately deals with this uncertainty. In particular the work shows that the type 1 relation for if-then often produces solutions that are incorrect. Another hypothesis, put forward by Turksen, is that gmp is inadequate since type 1 sets only express first order semantics and that the introduction of type 2 sets provides increased expressive power[24]. Gorzalcazany (e.g.[3]) also describes an interval valued fuzzy inference method. However, the rest of this section dicusses how intervalued fuzzy sets assist in the inferencing with type 1 sets.

4.1 Numerical Representation Inferencing

Hisdal[4] presents a long and detailed case for a method of inferencing on type 1 sets using type 2 relations. Hisdal's interpretation of inferencing with if-then rules relies on two ideas:

1. Mathematical logic allows for the implication to be true or false and the truth value of the consequent is specified, not inferred.

Table 1. The truth table for a non fuzzy if-then statement as used in natural language. The if-then statement is assumed to be true.

$A \rightarrow C$	A	C
T	T	T
T	F	BLANK

2. In fuzzy set theory the if-then statement is assumed to be true and infers a value of y from x. This is a better representation of the if-then statement as used in natural language.

These two points lead to table 1 where the if-then statement is assumed to be true and that, therefore, when the antecedent is false then the consequent is a 'don't know' state (BLANK using Hisdal's notation).

The if-then statement 'If x = A Then y = C' is written as $C|A$ where A and C are fuzzy subsets of the two universes U and V respectively. The entries of the C $|A$ type 2 relation are given by ([4] Equ. 6.1.2):

$$\Pi_{C|A}(\Pi(v_i) \mid u_i),$$

$$i=1,\ldots,I0 \qquad j=1,\ldots,J0 \qquad \Pi(v_j) \in M^{t1}.$$

Assuming we have the C $|A$ relation then we choose a particulent $P \subset U$ and deduce a set $D \subset V$. The inference operation is carried out in two steps:

1. The C $|A$ relation is particulated by P and this is given by

$$\Pi_{(A,\,C)_{,}}(u_i, \Pi(v_j)) = \Pi_P(u_i) \wedge \Pi_{(C|A)}(\Pi(v_j) \mid u_i).$$

2. Every row of the A, C relation is a fuzzy subset of V. The union of all the fuzzy subsets gives the deduced set D which is

$$\Pi_D(\Pi(v_j)) = \bigvee_{u_i} \Pi_{A,C}(u_i, \Pi(v_j)).$$

The C $|A$ and the A, C relations are, in general interval valued type 2 sets. Before we infer on an if-then statement we are in complete ignorance. Given this set the if-then statement adds new knowledge and the solution of the C $|A$ relation is a restriction of this set. Hisdal applies this idea to some situations without tackling a real problem. Clearly this approach produces a *different* solution to the traditional gmp however without the application to a real problem the efficacy of this approach is unproven.

4.2 Inferencing with Interval Valued Type 2 Fuzzy Sets

Turksen has produced a body of work relating to the use of type 2 sets when inferencing with fuzzy systems [24][19][23][25][22][18][21]. He argues that type 1 fuzzy sets and logics present concerns that can be tackled using type 2 sets. In particular he states[24]" *uncertainty models represented by the interval-valued Type II fuzzy sets and logics have a more expressive power*". He proposes that the combined linguistic expression of the linguistic values with the linguistic connectives should not be set arbitrarily to just one of the two fuzzy normal forms but to the interval generated by both.

The gmp for type 1 can be expressed as:

$$B' = A' o (A \rightarrow B)$$

where A, A', B and B' are fuzzy sets and o is the compostional rule of inference[33]. In other words we need a solution to

$$A \text{ AND } (A \rightarrow B) = B.$$

The fuzzy normal forms that mirror the Zadeh max-min approach for this particular situation are:

$$FDNF(A \rightarrow B) = (A \cap B) \cup (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B})$$
$$FCNF(A \rightarrow B) = \overline{A} \cup B$$

The result of this inference is an interval valued fuzzy set [FDNF,FCNF] . Turksen [19] uses this approach to describe four methods for approximate reasoning using interval-valued sets. The four methods are:

1. A and A' are point valued and o is crisp

2. A and A' are point valued and o is linguistic

3. A and A' are interval valued and o is crisp

4. A and A' is interval valued and o is linguistic

where 'point valued' means a type 1 set and o is the compositional rule of inference. He presents the results of using FDNF and FCNF to represent the interval valued fuzzy sets that arise from these four approaches. He then uses the simplest approach (as in 1 above) to tackle the problem of production planning in a paint factory and concludes that the results are 'good' and 'robust' and better represent the way a manager would describe the problem (linguistically). What does need investigating is whether the other approaches offer better solutions.

4.3 Type 2 Inferencing: A Summary

The Numerical Representation approach of Hisdal works from the premise that so as not to introduce false information into an if-then statement type 2 sets allow for a suitable representation of the state of ignorance before applying the statement. By using type 2 sets to represent this ignorance Hisdal provides an alternative approach to the conventional gmp and compositional rule of inference that reflects the uncertainty. Hisdal does not inference with type 2 sets as the antecedents but uses type 2 sets to enhance the inferencing with type 1.

In contrast the Turksen approach highlights a deficiency in the traditional fuzzy inferencing approaches in that they only adopt one of the normal forms. For two valued logic this is not a problem as the normal forms are equivalent but for fuzzy logic they are not. His approach of using the normal forms to generate interval valued fuzzy sets appears to capture second order imprecision.

5 Applications of Type 2 Fuzzy Sets

The number of applications of type 2 sets reported in the literature is growing. Practical applications of type 2 sets include handling tolerances in fuzzy equations systems[26], fuzzy regression models[1], multi-objective decision making[30], determining membership functions[20], community transport scheduling[5], the internet[17] and computer networks[15]. Rocha[14] extends interval valued fuzzy sets to simulate human cognitive categorisation and concept combination with the notion of evidence sets. As an example of using interval valued fuzzy sets in control Wu[29] reports on their use for the control of mobile robots. He implements a new fuzzy control methodology that he calls fuzzy interval control (FIC). He implemented the FIC for successfully navigating a miniature robot in an unknown maze without touching the walls. Type 2 sets have been used to assist in the pre-processing of data for use with neural networks[6]. This research reports the results of the analysis of bone scans from stress related injuries to the tibia of athletes. Neural network based clustering techniques are used to assist the consultant in classifying the images. The work was motivated by the situation where there is a relatively small amount of relevant data and difficulties are faced by consultants in classifying the various types of injuries. For this particular problem the consultant's interpretation of the image lends itself to representation using type 2 fuzzy sets. This research addresses whether, with fuzzy neuro-clustering techniques, some insights may be provided to the consultant that they can use along with their experience and knowledge. The results of this approach indicate that the use of neural clustering using a type 2 representation can improve the classification of shin images.

6. Conclusion

This paper has described type 2 sets and discussed various issues surrounding their use in real systems for knowledge representation and inferencing. Type 2 sets offer advantages over type 1 sets. Firstly, in knowledge representation an expert is likely to be more at home using linguistic grades or by representing their belief using intervals. Secondly, the uncertainty inherent in the fuzzy inferencing process can be captured in a number of ways by using interval valued fuzzy sets. The applications of type 2 sets in both control and decision making is on the increase - especially using interval valued fuzzy sets - and we can expect type 2 sets to play a more prominent role in applications generally.

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