

# Geo-uninorm Consistency Control Module for Preference Similarity Network Hierarchical Clustering Based Consensus Model

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## Abstract

In order to avoid misleading decision solutions in group decision making (GDM) processes, in addition to consensus, which is obviously desirable to guarantee that the group of experts accept the final decision solution, consistency of information should also be sought after. For experts' preferences represented by reciprocal fuzzy preference relations, consistency is linked to the transitivity property. In this study, we put forward a new consensus approach to solve GDM with reciprocal preference relations that implements rationality criteria of consistency based on the transitivity property with the following twofold aim prior to finding the final decision solution: (A) to develop a *consistency control module* to provide personalized consistency feedback to inconsistent experts in the GDM problem to guarantee the consistency of preferences; and (B) to design a *consistent preference network clustering based consensus measure* based on an undirected weighted consistent preference similarity network structure with undirected complete links, which using the concept of structural equivalence will allow one to (i) cluster the experts; and (ii) measure their consensus status. Based on the uninorm characterization of consistency of reciprocal preferences relations and the geometric average, we propose the implementation of the geo-uninorm operator to derive a consistent based preference relation from a given reciprocal preference relation. This is subsequently used to measure the consistency level of a given preference relation as the cosine similarity between the respective relations' essential vectors of preference intensity. The proposed geo-uninorm consistency measure will allow the building of a consistency control module based on a personalized feedback mechanism to be implemented when the consistency level is insufficient. This consistency control module has two advantages: (1) it guarantees consistency by advising inconsistent expert(s) to modify their preferences with minimum changes; and (2) it provides *fair* recommendations individually, depending on the experts' personal level of inconsistency. Once consistency of preferences is guaranteed, a structural equivalence preference similarity network is constructed. For the purpose of representing structurally equivalent experts and measuring consensus within the group of experts, we develop an agglomerative hierarchical clustering based consensus algorithm, which can be used as a visualization tool in monitoring current state of experts' group agreement and in controlling the decision making process. The proposed model is validated with a comparative analysis with an existing literature study, from which conclusions are drawn and explained.

**Keywords:** Consistency, Consensus, Geometric Mean Uninorm, Feedback Mechanism, Social Network Analysis, Agglomerative Hierarchical Clustering.

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## 1. Introduction

Consensus group decision making theory is concerned with the description and analysis of the process by which experts' individual preferences are considered appropriately and aggregated into a decision of the group, as a whole, with sufficient level of group agreement. This process can be represented by a network of interactions between the individual experts involved in the process. An emerging research trend in this area is the development of consensus approaches within the theory of Social Network Analysis (SNA), where relationships among experts are taken into account. Several network based consensus

models have recently been developed by Shang [1], Liu et al. [2], Wu et al. [3, 4], Dong et al. [5], and Capuano et al. [6], while a brief, but informative, overview and discussion on consensus and decision making in social networks can be found in Herrera-Viedma et al. [7]. It is common to have a large number of experts in a network. However, there exists an issue of how to achieve sufficient level of consensus with these complex interactions involved. Some alternative proposals used clustering methodologies to resolve this situation, such as Kamis et al. [8], Perony et al. [9], Garcia-Lapresta and Perez-Roman [10–12], Abel et al. [13] and Li et al. [14].

In consensus group decision making models, the study and analysis of consistency is purposely conducted by its appropriate integration as a consensus criterion to avoid misleading solutions, and also to estimate information when it is unknown or missing [15–18]. When experts provide information reflecting

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their preferences over a set of feasible alternatives of a decision making problem, there are three hierarchical levels of rationality assumption: Level 1. *Indifference* – indifference when comparing an alternative  $x$  and itself  $x$ ; Level 2. *Asymmetry* – an expert cannot prefer alternative  $x$  to alternative  $y$  and alternative  $y$  to  $x$  simultaneously; Level 3. *Transitivity* – if an expert prefers alternative  $x$  to alternative  $y$ , and prefers alternative  $y$  to alternative  $z$  then this expert should prefer alternative  $x$  to alternative  $z$ . Levels 1 and 2 of rationality assumptions are verified by assuming a reciprocity property in the pairwise comparison between any two alternatives. This is seen by Saaty [19] as a ‘reasonable assumption’ when making paired comparisons. Preference relations are considered consistent when they satisfy the third level of rationality, and a property that guarantees the transitivity in the pairwise comparison among any three alternatives is called a consistency property.

A number of different ways of modelling transitivity of fuzzy preference relations have been proposed in the literature: (i) min transitivity; (ii) moderate stochastic transitivity; (iii) max transitivity; (iv) strong stochastic transitivity; (v) additive transitivity; (vi) multiplicative transitivity. Under the reciprocity property, Chiclana et al. [20] observed that max transitivity is only possible when all alternatives are equally preferred, while additive transitivity is in conflict with the unit interval scale used for measuring preference values, making them inappropriate to model consistency of reciprocal preference relations. In fact, when ‘consistency’ of preferences is considered as the ‘cardinal consistency in the strength of preferences’, which was described by Saaty in [19, page 7] as

“not merely the traditional requirement of the transitivity of preferences [...], but the actual intensity with which the preference is expressed transits through the sequence of objects in comparison,”

Chiclana et al. argued in [20] that consistency of reciprocal preference relations can be theoretically modeled via a functional equation. Under the conditions of almost associativity, continuity and monotonicity, this functional equation was proved to have the set of representable uninorms [21] as its solution. Under reciprocity property, multiplicative transitivity [22] coincides with the representable cross-ratio uninorm [23] and, therefore, it is an appropriate property to model consistency of reciprocal preference relations.

It is known that consistency needs to be integrated within consensus models in order to avoid misleading solutions in the decision making processes [24, 25]. Consensus between experts in group decision making (GDM) is obviously desirable to guarantee that the group of experts accept the final decision solution it arrives at. Consensus is often sought using the basic rationality principles that each expert presents. However, if consensus is secured and consistency criteria to fix the rationality of each expert is applied afterwards, then a divergence of the previously agreed consensus position could result, and the final solution might not be acceptable by the group of experts as a consensus solution. Thus, consistency criteria should be applied before experts’ agreement is obtained. In other words a

minimum level of rationality should be sought for each expert’s preferences before securing consensus.

The proposed group decision making methodology dealing with reciprocal preference relations consists of two consecutive main stages: (1) achieving a minimum threshold level of consistency of experts’ preferences; and (2) achieving a minimum threshold level of consensus among the group of experts. Both minimum threshold levels of consistency and consensus are to be acceptable/agreed by the group of experts, respectively, and as such are considered here to be fixed in advance. A first step towards each of these two main decision stages will be the measuring of consistency and consensus levels, so that meaningful consistency and consensus measure functions are required.

Once experts’ preferences are provided, the consistency level associated to each expert is measured. If an expert’s preferences consistency level is not sufficient, i.e., it is below the group agreed minimum threshold level of consistency, then it would be desirable to know what changes to his/her preferences to implement in order to guarantee reaching the minimum threshold level of consistency. Thus, before proceeding to the consensus stage of a GDM model, a consistency control module is activated and applied once to guarantee consistency [26, 27].

In this study, we put forward a new consensus approach to solve GDM with reciprocal preference relations that implements rationality criteria of consistency based on the transitivity property of preferences. This approach will have two key outputs: (A) to develop a *consistency control module* to provide personalized consistency feedback to inconsistent experts in order to guarantee the consistency of preferences in finding the solution to the GDM problem; and (B) to design a *consistent preference network clustering based consensus measure* by building an undirected weighted consistent preference similarity network structure with undirected complete links. This is exploited using the concept of structural equivalence to cluster the experts and to measure their consensus status.

Based on the aforementioned uninorm characterization of consistency of reciprocal preferences relations, we propose the composition of the geometric mean operator and the cross ratio uninorm operator, which we refer to as the geo-uninorm operator. This is used to derive a consistency-based preference relation from a given reciprocal preference relation, and subsequently used to measure the consistency level of a given preference relation as the cosine similarity between the respective relations’ essential vectors of preference intensities. The proposed geo-uninorm consistency measure will allow the building of a consistency control module based on a personalized feedback mechanism to be implemented when the consistency level is insufficient. This consistency control module has two advantages: (1) it guarantees consistency by advising inconsistent expert(s) to modify their preferences with minimum changes; and (2) it provides *fair* recommendations individually, depending on the experts’ personal levels of inconsistency.

Once consistency of preferences is guaranteed, the preference similarity network based on the structural equivalence concept is developed and presented comprehensively using the agglomerative hierarchical clustering algorithm. Both, internal and external cohesions, cluster consensus, clustering level of

maximum consensus and global cluster consensus degrees of the group of experts are defined in the procedure of the proposed clustering based consensus model. The agglomerative hierarchical clustering based consensus algorithm can be used as a visualization tool [28] in monitoring the current state of the experts' group agreement and in controlling the decision making process.

The rest of the paper is set out as follows: Section 2 describes the proposed consistency control module. Section 3 demonstrates the consistent preference network clustering based consensus procedure, while Section 4 presents the framework of the proposed geo-uninorm consistency control module for preference similarity network hierarchical clustering based consensus model. For validation purposes, a comparative analysis of the proposed approach with an existing literature study is carried out in Section 5 and finally, conclusions are drawn in Section 6.

## 2. Consistency Control Module

This section focuses on the description and introduction of concepts and terminology regarding reciprocal fuzzy preference relations: the essential vector of preference intensities, the introduction of geo-uninorm consistent fuzzy preference relations, the proposal of a new consistency measure based on cosine similarity function and the design of a consistency feedback mechanism to guarantee the consistency of preferences.

### 2.1. Reciprocal Fuzzy Preference Relations and its Essential Vector of Preference Intensity

We assume that a group of experts,  $E = \{e^1, e^2, \dots, e^n\}$ , provide their opinions or preferences on a given a finite set of alternatives,  $Y = \{y_1, y_2, \dots, y_m\}$ , in the form of reciprocal fuzzy preference relations.

**Definition 1.** A reciprocal fuzzy preference relation  $P = (p_{ij})$  over a finite set of alternatives  $Y$  is a binary relation on  $Y \times Y$ , which is characterized by a membership function  $\mu_P : Y \times Y \rightarrow [0, 1]$ , where  $p_{ij}$  represents the intensity of preference of the alternative  $y_i$  over the alternative  $y_j$ . We have the following interpretation:

- $p_{ij} = 0.5$  when  $y_i$  and  $y_j$  are equally preferred (indifference);
- $p_{ij} \in (0.5, 1]$  meaning  $y_i$  is preferred to  $y_j$ ;
- $p_{ij} = 1$  when  $y_i$  is absolutely preferred to  $y_j$ ;
- $p_{ij} + p_{ji} = 1, \forall y_i, y_j \in Y$ .

Let  $P^h = (p_{ij}^h)$  be the  $m \times m$  reciprocal matrix representation of preferences over a finite set of alternatives  $Y$ , given by an

expert  $h$ :

$$P^h = \begin{bmatrix} p_{11}^h & p_{12}^h & \cdots & p_{1m}^h \\ p_{21}^h & p_{22}^h & \cdots & p_{2m}^h \\ \vdots & \vdots & \ddots & \vdots \\ p_{(m-1)1}^h & p_{(m-1)2}^h & \cdots & p_{(m-1)m}^h \\ p_{m1}^h & p_{m2}^h & \cdots & p_{mm}^h \end{bmatrix};$$

$$0 \leq p_{ij}^h \leq 1;$$

$$p_{ij}^h + p_{ji}^h = 1 \quad \forall i, j \in \{1, 2, \dots, m\}.$$

Let  $\mathbb{P}_{m \times m}$  be the set of reciprocal fuzzy preference relations matrices  $P^h$  from all experts  $E$ . A reciprocal fuzzy preference relation can be represented by its essential vector of preference intensity [29], which is mathematically defined below.

**Definition 2.** The essential vector of preference intensity of expert  $h$  associated to his/her reciprocal fuzzy preference relation  $P^h = (p_{ij}^h)_{m \times m} \in \mathbb{P}_{m \times m}$  is the vector of dimension  $\frac{m(m-1)}{2}$ ,  $VP^h \in \mathbb{R}^{m(m-1)/2}$ , form with the entries above the main diagonal of  $P^h$ :

$$VP^h = (p_{12}^h, p_{13}^h, \dots, p_{1m}^h, p_{23}^h, \dots, p_{2m}^h, \dots, p_{(m-1)m}^h) \\ = (vp_1, vp_2, \dots, vp_k, \dots, vp_{m(m-1)/2}).$$

We will denote by  $\mathbb{V}(Y)$  the set of essential vectors of preference intensity on the set of alternatives  $Y$ .

### 2.2. Geo-uninorm Consistent Fuzzy Preference Relations and Consistency Measure

The assumption that experts are able to quantify their preferences in  $[0, 1]$  instead of  $\{0, 1\}$ , carries an assumption that experts can select accurately from an infinite set of possible options. However, this has the positive outcome that the consistency of reciprocal preference relations is amenable to be modeled mathematically via a functional equation [20]. Under the conditions of associativity, almost continuity and monotonicity, via Aczél's theorem [30, page 107], Chiclana et al. [20] proved that the solution of the consistency functional equation of reciprocal fuzzy preference relations is the set of self-dual representable uninorms [21, 31]. From the properties proposed in the research literature to model transitivity of fuzzy preference relations, Tanino's multiplicative transitivity property under reciprocity becomes

$$U(x, y) = \begin{cases} 0, & \text{if } (x, y) \in \{(0, 1), (1, 0)\}, \\ \frac{xy}{xy + (1-x)(1-y)}, & \text{otherwise.} \end{cases} \quad (1)$$

This is the cross ratio uninorm, which is a conjunctive self-dual representable uninorm (with identity element 0.5), also known as the symmetric sum [23]. Uninorms are associative, which can be used to extend them to  $m$  arguments. The cross-ratio uninorm becomes the three  $\prod$  operator [21]:

$$U(x_1, x_2, \dots, x_m) = \begin{cases} 0, & \text{if } \exists i, j : (x_i, x_j) \in \{(0, 1), (1, 0)\}, \\ \frac{\prod_{i=1}^n x_i}{\prod_{i=1}^m x_i + \prod_{i=1}^m (1-x_i)}, & \text{otherwise.} \end{cases} \quad (2)$$

A key element that differentiates uninorm operators from mean operators is the property of reinforcement. An operator is a reinforcement type operator if, given a set of input values, the output is above the maximum of the input values when all input values are ‘high’ and below the minimum of the input values when all input values are ‘low’. Obviously, mean operators cannot be reinforcement type operators because they are located between the minimum and maximum of their input values; while uninorm operators are reinforcement operators when all input values are above or below their identity element. For operators different to the minimum and maximum operators, associativity and idempotency are incompatible properties [32]. The reinforcement property was not considered essential by Yager when the input values correspond to criteria measuring the same property, and added [33]

“[...] at a meta-level the use of mean type operators is appropriate in situations in which the values being aggregated are essentially multiple manifestations of the same variable. In this environment, the mean operator is acting like a *smoothing operator* to unify the different manifestations of the same concept.”

Thus, associativity is not necessary in applications where mean operators (simple or weighted averages) are required to fuse individual information into a collective one, and in particular, when the input values to aggregate measure the same property [34].

In the following, we pursue the implementation of Yager’s [33] concept of extended mean operator, and in particular of a special class of extended mean operators that include the well-known classical average operators. He terms these as *classical mean operators*, i.e., operators that verify the properties of *commutativity*, *idempotency*, *monotonicity* and *self identity*. Because all of the input values of the cross-ratio uninorm refer to the same property (preference modeling), we propose the application of a classical average operator, in this case the geometric mean operator, in conjunction with the cross-ratio uninorm operator (consistency modeling). This is possible to achieve because the cross ratio uninorm is a particular type of the more general class of operators [35, 36]

$$PI(x_1, \dots, x_m) = \begin{cases} 0, & \text{if } \exists i, j : (x_i, x_j) \in \{(0, 1), (1, 0)\}, \\ \frac{\prod_{i=1}^m M(x_i)}{\prod_{i=1}^m M(x_i) + \prod_{i=1}^m M(1 - x_i)}, & \text{otherwise,} \end{cases} \quad (3)$$

with non-negative and increasing generating function  $M$ . When the generating function  $M(z) = z^{\frac{1}{m}}$ , we obtain the following

operator:

$$G_U(x_1, x_2, \dots, x_m) = \begin{cases} 0, & \text{if } \exists i, j : (x_i, x_j) \in \{(0, 1), (1, 0)\}, \\ \frac{\prod_{i=1}^m x_i^{\frac{1}{m}}}{\prod_{i=1}^m x_i^{\frac{1}{m}} + \prod_{i=1}^m (1 - x_i)^{\frac{1}{m}}}, & \text{otherwise.} \end{cases} \quad (4)$$

For obvious reasons, this operator will be referred to in this paper as the geo-uninorm operator, which can be equivalently expressed as

$$G_U(x_1, x_2, \dots, x_m) = \begin{cases} 0, & \text{if } \exists i, j : (x_i, x_j) \in \{(0, 1), (1, 0)\}, \\ \frac{1}{1 + \prod_{i=1}^m \left( \frac{1}{x_i} - 1 \right)^{\frac{1}{m}}}, & \text{otherwise.} \end{cases} \quad (5)$$

The geo-uninorm operator is a classical mean operator as the following results prove.

**Proposition 1.** *The geo-uninorm operator verifies the following properties: (1) Commutativity (2) Idempotency (3) Monotonicity (4) Self identity.*

*Proof.* Let  $G(x_1, x_2, \dots, x_m) = \prod_{i=1}^m x_i^{\frac{1}{m}}$  be the geometric mean operator.

**Commutativity.** Because  $G$  satisfies commutativity it is clear that  $G_U$  does so also.

**Idempotency.** Because  $G$  satisfies idempotency, then for all  $x \in [0, 1]$   $G(x, \dots, x) = x$ , and  $G(1 - x, \dots, 1 - x) = 1 - x$ . Consequently  $G_U(x, \dots, x) = \frac{x}{x + (1 - x)} = x$ .

**Monotonicity.** Let us assume we have  $(x_1, \dots, x_m)$  and  $(y_1, \dots, y_m)$  such that:  $0 < y_i \leq x_i \leq 1 \forall i$ . The case when some of the  $y_i$  are zero is evident because then

$$0 = G_U(y_1, \dots, y_m) \leq G_U(x_1, \dots, x_m).$$

In this case we have that

$$\begin{aligned} \forall i : 1 \leq \frac{1}{x_i} \leq \frac{1}{y_i} &\iff 0 \leq \frac{1}{x_i} - 1 \leq \frac{1}{y_i} - 1 \iff \\ 0 \leq \left( \frac{1}{x_i} - 1 \right)^{\frac{1}{m}} &\leq \left( \frac{1}{y_i} - 1 \right)^{\frac{1}{m}}. \end{aligned}$$

This implies that

$$\begin{aligned} 0 \leq \prod_{i=1}^m \left( \frac{1}{x_i} - 1 \right)^{\frac{1}{m}} &\leq \prod_{i=1}^m \left( \frac{1}{y_i} - 1 \right)^{\frac{1}{m}} \iff \\ G_U(y_1, \dots, y_m) &\leq G_U(x_1, \dots, x_m). \end{aligned}$$

**Self identity.** We prove that

$$G_U(x_1, \dots, x_m, G_U(x_1, \dots, x_m)) = G_U(x_1, \dots, x_m).$$

Again, if one of the  $x_i$  is zero then the above is evident as both left and right hand sides of the equation are zero. In all other cases, we have by definition:

$$G_U(x_1, \dots, x_m, G_U(x_1, \dots, x_m)) = \frac{1}{1 + \left( \frac{1}{G_U(x_1, \dots, x_m)} - 1 \right)^{\frac{1}{m+1}} \prod_{i=1}^m \left( \frac{1}{x_i} - 1 \right)^{\frac{1}{m+1}}}$$

However,

$$G_U(x_1, x_2, \dots, x_m) = \frac{1}{1 + \prod_{i=1}^m \left( \frac{1}{x_i} - 1 \right)^{\frac{1}{m}}} \iff \frac{1}{G_U(x_1, \dots, x_m)} - 1 = \prod_{i=1}^m \left( \frac{1}{x_i} - 1 \right)^{\frac{1}{m}}.$$

Therefore,

$$G_U(x_1, \dots, x_m, G_U(x_1, \dots, x_m)) = \frac{1}{1 + \prod_{i=1}^m \left( \frac{1}{x_i} - 1 \right)^{\frac{1}{m(m+1)}} \prod_{i=1}^m \left( \frac{1}{x_i} - 1 \right)^{\frac{1}{m+1}}} = \frac{1}{1 + \prod_{i=1}^m \left( \frac{1}{x_i} - 1 \right)^{\frac{1}{m+1} + \frac{1}{m(m+1)}}}.$$

Because  $\frac{1}{m+1} + \frac{1}{m(m+1)} = \frac{m+1}{m(m+1)} = \frac{1}{m}$ , we conclude that

$$G_U(x_1, \dots, x_m, G_U(x_1, \dots, x_m)) = \frac{1}{1 + \prod_{i=1}^m \left( \frac{1}{x_i} - 1 \right)^{\frac{1}{m}}} = G_U(x_1, x_2, \dots, x_m).$$

□

Because the geo-uniform operator is a mean operator, it is obvious that it does not satisfy the reinforcement property as discussed above. However, it satisfies weaker reinforcement properties that are desirable for transitivity of preferences. Indeed, denoting by

$$x_* = \min\{x_1, x_2, \dots, x_m\}$$

and

$$x^* = \max\{x_1, x_2, \dots, x_m\},$$

monotonicity of  $G_U$  implies that

$$G_U(x_*, x_*, \dots, x_*) \leq G_U(x_1, x_2, \dots, x_m) \leq G_U(x^*, x^*, \dots, x^*).$$

Idempotency of  $G_U$  results in

$$\min\{x_1, x_2, \dots, x_m\} \leq G_U(x_1, x_2, \dots, x_m) \leq \max\{x_1, x_2, \dots, x_m\}.$$

It is obvious then that the geo-uniform operator verifies weak stochastic transitivity, min-transitivity, and moderate stochastic transitivity:

**Proposition 2.** *The geo-uniform operator satisfies:*

- **Weak stochastic transitivity:**

$$x_i \geq 0.5 \forall i \implies G_U(x_1, x_2, \dots, x_m) \geq 0.5;$$

- **Min transitivity:**

$$G_U(x_1, x_2, \dots, x_m) \geq \min\{x_1, x_2, \dots, x_m\} \forall x_i;$$

- **Moderate stochastic transitivity:**

$$x_i \geq 0.5 \forall i \implies G_U(x_1, x_2, \dots, x_m) \geq \min_i x_i.$$

Additionally, since the function  $f(x) = \log\left(\frac{1}{x} - 1\right)$  is concave on  $[0.5, 1)$  ( $f''(x) \leq 0$ ) and convex on  $(0, 0.5]$  ( $f''(x) \geq 0$ ), the following reinforcement properties were proved in [35]:

- (a) When  $x_i \in [0.5, 1) \forall i \implies f\left(\sum_{i=1}^m \alpha_i x_i\right) \geq \sum_{i=1}^m \alpha_i f(x_i)$

subject to  $\sum_{i=1}^m \alpha_i = 1$ . Taking  $\alpha_i = \frac{1}{m}$  we have

$$\log\left(\frac{1}{\frac{1}{m} \sum_{i=1}^m x_i} - 1\right) \geq \sum_{i=1}^m \frac{1}{m} \log\left(\frac{1}{x_i} - 1\right) = \log \prod_{i=1}^m \left(\frac{1}{x_i} - 1\right)^{\frac{1}{m}}.$$

Monotonicity of  $f$  implies that

$$\frac{1}{\frac{1}{m} \sum_{i=1}^m x_i} - 1 \geq \prod_{i=1}^m \left(\frac{1}{x_i} - 1\right)^{\frac{1}{m}} \implies \frac{1}{m} \sum_{i=1}^m x_i \leq G_U(x_1, x_2, \dots, x_m).$$

- (b) The case when  $x_i \in (0, 0.5] \forall i$  is derived similarly to the above case by changing the inequality symbols.

**Proposition 3.** *The geo-uniform operator satisfies mean reinforcement properties:*

- $x_i \geq 0.5 \forall i \implies G_U(x_1, x_2, \dots, x_m) \geq \frac{1}{m} \sum_{i=1}^m x_i;$
- $x_i \leq 0.5 \forall i \implies G_U(x_1, x_2, \dots, x_m) \leq \frac{1}{m} \sum_{i=1}^m x_i.$

Thus we see that the geo-uniform operator inherits properties from both the geometric mean operator and the cross ratio uniform operator, which makes it an appropriate operator for modeling transitivity and, therefore, consistency of fuzzy preferences. In the following we introduce the geo-uniform consistency property of a reciprocal fuzzy preference relation.

**Definition 3.** *A reciprocal fuzzy preference relation,  $P = (p_{ij})$ , on a finite set of alternatives,  $Y = \{y_1, y_2, \dots, y_n\}$ , is geo-uniform consistent when*

$$p_{ij} = G_U(p_{ik}, p_{kj}) \forall i, j, k, \text{ such that } (p_{ik}, p_{kj}) \notin \{(0, 1), (1, 0)\}.$$

Following Chiclana et al.'s methodological approach to construct uninorm-based consistent reciprocal fuzzy preference relations [20], the geo-uninorm consistent fuzzy preference relation,  $C = (c_{ij})$ , based on the set of  $(m-1)$  reciprocal fuzzy preference relation values  $P = \{p_{i(i+1)}; i = 1, \dots, m-1\}$  is constructed as follows:

1. For  $(i, j)$  such that  $j > (i+1)$ :

$$c_{ij} = G_U(p_{i(i+1)}, p_{(i+1)(i+2)}, \dots, p_{(j-1)j}).$$

2. For  $(i, j)$  such that  $j < i$ :  $c_{ij} = 1 - c_{ji}$ .

This methodology can be exploited to define a measure of consistency of a given fuzzy preference relation. First, we construct its associated geo-uninorm consistent fuzzy preference relation. Second, we measure how similar these two fuzzy preference relations are. This similarity degree is defined as the level of consistency of the given fuzzy preference relation. This is summarized in the following definition, where the proposed measure of similarity is the cosine similarity between the essential vectors of preference intensity of the given fuzzy preference relation and the associated geo-uninorm consistent fuzzy preference relation.

**Definition 4.** The cosine-consistency degree of expert  $h$ ,  $CCD(e^h)$ , is the similarity degree between the essential vector of preference intensity,  $VP^h = (vp_k^h)$ , and the essential vector of geo-uninorm consistent preference intensity,  $VC^h = (vc_k^h)$ ,

$$CCD(e^h) = \frac{\sum_{k=1}^{m(m-1)/2} (vp_k^h \cdot vc_k^h)}{\sqrt{\sum_{k=1}^{m(m-1)/2} (vp_k^h)^2} \cdot \sqrt{\sum_{k=1}^{m(m-1)/2} (vc_k^h)^2}}. \quad (6)$$

**Example 1.** In order to demonstrate our proposed procedure in this sub-section, we use the example in Chu et al. [37], where a committee of eight (8) experts,  $E = \{e^1, e^2, \dots, e^8\}$ , give their opinions over a set of six (6) alternatives,  $Y = \{y_1, y_2, \dots, y_6\}$  in terms of reciprocal fuzzy preference relations. Below we reproduce the evaluation matrix  $P^1$  with the values of its essential vector of preference intensity  $VP^1$  boldfaced:

$$P^1 = \begin{bmatrix} 1 & \mathbf{0.4} & \mathbf{0.2} & \mathbf{0.6} & \mathbf{0.7} & \mathbf{0.8} \\ 0.6 & 1 & \mathbf{0.1} & \mathbf{0.6} & \mathbf{0.9} & \mathbf{0.7} \\ 0.8 & 0.9 & 1 & \mathbf{0.3} & \mathbf{0.1} & \mathbf{0.1} \\ 0.4 & 0.4 & 0.7 & 1 & \mathbf{0.5} & \mathbf{0.2} \\ 0.3 & 0.1 & 0.9 & 0.5 & 1 & \mathbf{0.7} \\ 0.2 & 0.3 & 0.9 & 0.8 & 0.3 & 1 \end{bmatrix}$$

$$VP^1 = (0.4, 0.2, 0.6, 0.7, 0.8, 0.1, 0.6, 0.9, 0.7, 0.3, 0.1, 0.1, 0.5, 0.2, 0.7).$$

The rest of the essential vectors of preference intensity are as

follows:

$$\begin{aligned} VP^2 &= (0.3, 0.3, 0.5, 0.6, 0.6, 0.4, 0.7, 0.2, 0.3, 0.5, 0.4, 0.2, 0.6, 0.7, 0.4); \\ VP^3 &= (0.6, 0.6, 0.6, 0.1, 0.4, 0.3, 0.6, 0.3, 0.6, 0.6, 0.1, 0.6, 0.7, 0.6, 0.2); \\ VP^4 &= (0.2, 0.1, 0.5, 0.8, 0.8, 0.2, 0.9, 0.2, 0.4, 0.8, 0.1, 0.1, 1.0, 0.8, 0.6); \\ VP^5 &= (0.6, 0.3, 0.6, 0.6, 0.7, 0.1, 0.7, 0.8, 0.4, 0.3, 0.3, 0.2, 0.5, 0.2, 0.7); \\ VP^6 &= (0.3, 0.1, 0.5, 0.7, 0.6, 0.4, 0.7, 0.2, 0.4, 0.5, 0.4, 0.2, 0.6, 0.7, 0.4); \\ VP^7 &= (0.7, 0.4, 0.6, 0.2, 0.6, 0.3, 0.7, 0.3, 0.8, 0.6, 0.1, 0.6, 0.7, 0.6, 0.2); \\ VP^8 &= (0.4, 0.3, 0.3, 0.6, 0.7, 0.2, 0.9, 0.2, 0.4, 0.8, 0.1, 0.1, 1.0, 0.8, 0.6). \end{aligned}$$

Below, we compute the geo-uninorm consistent fuzzy preference relation,  $C^1$ , associated to  $P^1$ , and the corresponding essential vector of geo-uninorm consistent preference intensity. Using the following preference values of  $P^1$ :  $\{p_{12} = 0.4, p_{23} = 0.1, p_{34} = 0.3, p_{45} = 0.5, p_{56} = 0.7\}$ , we obtain:

$$c_{15} = \frac{(p_{12} \cdot p_{23} \cdot p_{34} \cdot p_{45})^{\frac{1}{4}}}{(p_{12} \cdot p_{23} \cdot p_{34} \cdot p_{45})^{\frac{1}{4}} + [(1 - p_{12}) \cdot (1 - p_{23}) \cdot (1 - p_{34}) \cdot (1 - p_{45})]^{\frac{1}{4}}} = 0.2968.$$

Similarly, we obtain  $c_{16} = 0.3727$ ;  $c_{24} = 0.1791$ ;  $c_{36} = 0.5$ ;  $c_{46} = 0.604$ . The remaining values are obtained using the reciprocity property, resulting in:

$$C^1 = \begin{bmatrix} 1 & 0.4 & 0.2139 & 0.2405 & 0.2968 & 0.3727 \\ 0.6 & 1 & 0.1 & 0.1791 & 0.266 & 0.366 \\ 0.7861 & 0.9 & 1 & 0.3 & 0.3956 & 0.5 \\ 0.7595 & 0.8209 & 0.7 & 1 & 0.5 & 0.6044 \\ 0.7032 & 0.734 & 0.6044 & 0.5 & 1 & 0.7 \\ 0.6273 & 0.634 & 0.5 & 0.3956 & 0.3 & 1 \end{bmatrix},$$

$$VC^1 = (0.4, 0.2139, 0.2405, 0.2968, 0.3727, 0.1, 0.1791, 0.266, 0.366, 0.3, 0.3956, 0.5, 0.5, 0.6044, 0.7).$$

Similarly, we compute the rest of the experts' essential vectors of geo-uninorm consistent preference intensity:

$$\begin{aligned} VC^2 &= (0.3, 0.3483, 0.3971, 0.4472, 0.4377, 0.4, 0.4495, 0.5, 0.4747, 0.5, 0.5505, 0.5, 0.6, 0.5, 0.4); \\ VC^3 &= (0.6, 0.445, 0.497, 0.5505, 0.4713, 0.3, 0.445, 0.5337, 0.439, 0.6, 0.6517, 0.4889, 0.7, 0.433, 0.2); \\ VC^4 &= (0.2, 0.2, 0.3865, 1, 1, 0.2, 0.5, 1, 1, 0.8, 1, 1, 1, 1, 0.6); \\ VC^5 &= (0.6, 0.2899, 0.2932, 0.3408, 0.4114, 0.1, 0.1791, 0.266, 0.366, 0.3, 0.3956, 0.5, 0.5, 0.6044, 0.7); \\ VC^6 &= (0.3, 0.3483, 0.3971, 0.4472, 0.4377, 0.4, 0.4495, 0.5, 0.4747, 0.5, 0.5505, 0.5, 0.6, 0.5, 0.4); \\ VC^7 &= (0.7, 0.5, 0.5337, 0.5777, 0.4933, 0.3, 0.445, 0.5337, 0.439, 0.6, 0.6517, 0.4889, 0.7, 0.433, 0.2); \\ VC^8 &= (0.4, 0.2899, 0.4663, 1, 1, 0.2, 0.5, 1, 1, 0.8, 1, 1, 1, 1, 0.6). \end{aligned}$$

As per Definition 4, the cosine-consistency degree for each expert is measured and listed below:

$$CCD(e^1) = 0.8307; \quad CCD(e^2) = 0.9513;$$

$$CCD(e^3) = 0.9182; \quad CCD(e^4) = 0.7462;$$

$$CCD(e^5) = 0.8788; \quad CCD(e^6) = 0.9435;$$

$$CCD(e^7) = 0.9109; \quad CCD(e^8) = 0.7212.$$

### 2.3. Personalized Geo-uniform Consistency Feedback Mechanism

The consistency feedback mechanism is purposely carried out for those cases when the consistency degree of the expert is insufficient, which could impact on the quality of the decision making solution. For those ‘inconsistent expert(s)’ who have a consistency degrees lower than a consistency threshold, personalized consistency based changes to their preferences will be generated to guarantee that the consistency threshold is achieved.

The following notation is used in the description of the personalized geo-uniform consistency feedback mechanism algorithm below:

- Essential vector of preference intensity of expert  $e^h$ :

$$VP^h = \{vp_1, vp_2, \dots, vp_{(m-1)m/2}\};$$

- Essential vector of geo-uniform consistent preference intensity of expert  $e^h$ :

$$VC^h = \{vc_1, vc_2, \dots, vc_{(m-1)m/2}\};$$

- Cosine-consistency degrees of all experts:

$$\overline{CCD} = \{CCD(e^1), CCD(e^2), \dots, CCD(e^n)\};$$

- Personalized consistency parameter control:  $\gamma \in [0, 1]$ ;
- Consistency threshold:  $\eta$ .

**Example 2** (Continuation of Example 1). Let the consistency threshold  $\eta = 0.8$ ; experts  $e^4$  and  $e^8$  are classed as inconsistent. For simplicity, we use discrete values of  $\gamma$  from the set  $\{0.1, 0.2, \dots, 0.9, 1\}$ . Table 1 presents the  $CCD_\gamma$  for inconsistent experts  $e^4$  and  $e^8$ , with both requiring a  $\gamma$  value of 0.2 to become consistent. In the continuous case, a lower value of  $\gamma$  would have been required for both experts  $e_4$  and  $e_8$ , with a lower  $\gamma$  value for  $e_4$  than for  $e_8$ . A different threshold value, like  $\eta = 0.83$ , would have produced different discrete  $\gamma$  values for experts  $e^4$  and  $e^8$ , 0.2 and 0.3 respectively.

### 3. Consistent Preference Network Clustering Based Consensus Measure

The information given by the experts in Section 2 is at this stage considered ‘consistent’ for making the given decision. The consistent essential vectors of preference intensity are utilized to build an undirected weighted consistent preference similarity network structure, consisting of undirected complete links,  $L$ , connected to the vertices, representing the set of experts  $E$ , with set of weights,  $\mathbb{S}$ , attached to each of them reflecting their similarity of preference. Because the similarity functions are symmetric, which means that the preference similarity of experts  $e^c$  and  $e^d$  coincides with the preference similarity of vertices  $e^d$  and  $e^c$ , the similarity network is undirected with a unique weight attached to a pair of its nodes. This undirected weighted consistent preference similarity network,  $\mathbb{N}$ , is defined below (Fig. 1 shows an example).

**Definition 5.** Let  $E$  be a set of experts and  $\mathbb{C} = \{VC^1, VC^2, \dots, VC^m\}$  the corresponding set of consistent essential vector of preference intensity on a set of alternatives  $Y$ . Let  $S$  be an essential vectors of preference intensity similarity function, i.e., a reflexive and symmetric function  $S : \mathbb{V}(Y) \times \mathbb{V}(Y) \rightarrow [0, 1]$ . Then, the set of experts,  $E$ , can be connected by a set of links,  $L = \{l_{12}, \dots, l_{1m}, l_{23}, \dots, l_{2m}, \dots, l_{(m-1)m}\}$ , with the following set of consistent preference similarity weights attached,  $\mathbb{S} = S(\mathbb{C} \times \mathbb{C}) = \{S_1, S_2, \dots, S_{m(m-1)/2}\}$ . The resulting undirected weighted consistent preference similarity network will be denoted by  $\mathbb{N} = \langle E, L, \mathbb{S} \rangle$ .

From the perspective of Social Network Analysis (SNA), for the purpose of deriving the above preference similarity network, the concept of structural equivalence [38] is applied. We rely on the definition of structural equivalence: two experts are structurally equivalent if they are connected to the same experts (have the same neighbors), which is seen as evidence of them have similar characteristics in their own social environments [39]. In constructing our structural equivalence preference similarity network, we compute the cosine similarity degree,  $CS^{cd}$  between the pair of consistent essential vectors of preference intensity from expert  $e^c$ ,  $VC^c = \{vc_1^c, \dots, vc_k^c, \dots, vc_{\frac{m(m-1)}{2}}^c\}$ , and  $e^d$ ,  $VC^d = \{vc_1^d, \dots, vc_k^d, \dots, vc_{\frac{m(m-1)}{2}}^d\}$ ,

$$CS^{cd} = CS(VC^c, VC^d) = \frac{\sum_{k=1}^{m(m-1)/2} (vc_k^c \cdot vc_k^d)}{\sqrt{\sum_{k=1}^{m(m-1)/2} (vc_k^c)^2} \cdot \sqrt{\sum_{k=1}^{m(m-1)/2} (vc_k^d)^2}}.$$

The cosine similarity was chosen because it is one of the well-known functions applied in representing structural equivalence but also because, in contrast to Euclidean-based similarity, its stability in measuring consensus regardless of the number of experts involved has been proven [40].

In order to present the structural equivalence preference similarity network comprehensively, an agglomerative hierarchical clustering method is implemented. This approach has the ability to partition structural equivalent experts discretely into clusters with an explicit algorithm and interpretation [41]. In our case, a cluster will be interpreted as a collection of experts which have similar preferences among them and have dissimilar preferences to the experts from different clusters. The resulting dendrogram is a convenient graphical visualization of the hierarchical sequence of clustering solution (Fig. 2 shows an example). The dendrogram is horizontally cut at a certain  $\alpha$ -level (dendrogram’s height) according to the chosen number of clusters at the level of the consistent preference similarity matrix,  $CS$ . After the clustering solution is obtained, we construct a procedure of measuring consensus based on the concept of clusters’ homogeneity. It seems reasonable to reach cohesiveness of preferences (consensus) because experts are clustered according to their structural equivalence relation, where they are expected to have strong connections within their cluster’s members rather than with the outsider experts [8].

STEP 1: Identify inconsistent expert(s),  $e_{low}$ :

$$e_{low} = \{e_{low}^h \mid CCD(e^h) < \eta\}. \quad (7)$$

STEP 2: Recommend that each inconsistent expert,  $e_{low}^h$ , change his/her essential vector of preference intensity,  $VP^h$ , closer to the associated essential vector of geo-uniform consistent preference intensity,  $VC^h$ , according to the following linear combination with personalized consistency parameter control,  $\gamma$ :

$$VP_{\gamma}^h = (1 - \gamma) \cdot VP^h + \gamma \cdot VC^h. \quad (8)$$

STEP 3: Compute the new cosine-consistency degree,  $CCD_{\gamma}(e^h)$ , between  $VP_{\gamma}^h = \{vp_{\gamma_1}^h, \dots, vp_{\gamma_k}^h, \dots, vp_{\gamma_{\frac{m(m-1)}{2}}}^h\}$  and  $VC_{\gamma}^h = \{vc_1^h, \dots, vc_k^h, \dots, vc_{\frac{m(m-1)}{2}}^h\}$ :

$$CCD_{\gamma}(e_h) = \frac{\sum_{k=1}^{m(m-1)/2} (vp_{\gamma_k}^h \cdot vc_k^h)}{\sqrt{\sum_{k=1}^{m(m-1)/2} (vp_{\gamma_k}^h)^2} \cdot \sqrt{\sum_{k=1}^{m(m-1)/2} (vc_k^h)^2}}. \quad (9)$$

Note that when  $\gamma = 0$ , we have  $VP_0^h = VP^h$  and  $CCD_0(e_h) = CCD(e_h)$ , while when  $\gamma = 1$ ,  $VP_1^h = VC^h$  and  $CCD_1(e_h) = 1$ . The larger the value of  $\gamma$ , the closer  $VP_{\gamma}^h$  will be to  $VC^h$ , and therefore the higher  $CCD_{\gamma}(e_h)$  will be.

STEP 4: Choose the optimal control parameter,  $\hat{\gamma}$ :

$$CCD_{\hat{\gamma}}(e_h) = \eta.$$

The optimal control parameter corresponds to the gamma value that will optimize the change cost (difference between the original preference values and new personalized preference values) for an inconsistent expert to be classed as consistent. Different inconsistent experts will have different optimal control parameter values, making the feedback process common but at the same time personalized.

---

Table 1: Cosine-consistency degrees,  $CCD_{\gamma}$ , for inconsistent experts  $e^4$  and  $e^8$ .

$\gamma$	0	0.1	<b>0.2</b>	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$CCD_{\gamma}(e_4)$	0.7462	0.7927	<b>0.8363</b>	0.8757	0.9103	0.9393	0.9625	0.9798	0.9914	0.9980	1
$CCD_{\gamma}(e_8)$	0.7212	0.7726	<b>0.8207</b>	0.8643	0.9025	0.9344	0.9596	0.9784	0.9909	0.9979	1



The agglomerative hierarchical clustering method has no predetermined number of clusters. Instead it has a set of distinct  $\alpha$ -levels in the agglomerative hierarchical clustering:  $\mathbb{L} = \{\alpha_l : l = 2, \dots, m-1\}$ . Level  $\alpha_1$  represents the extreme case of having a single cluster containing all experts, while level  $\alpha_m$  is the initial partition of the agglomerative hierarchical clustering where each member belongs to its own cluster. In practice, for these two extreme levels, no clustering technique effectively applies.

Let  $K_l = \{K_{lr} : r = 1, \dots, l\}$  be the set of clusters at level  $\alpha_l$ . In order to measure consensus with capability of structural equivalent relations, presented by the agglomerative hierarchical clustering, we define experts' cluster homogeneity based on their internal and external cohesions, and combine both elements to obtained the collective group cluster consensus measure. Let  $\#K_{lr}$  denotes the cardinality of  $K_{lr}$ . The cluster internal cohesion degree,  $(\delta_{int})$ , the cluster external cohesion degree,  $(\delta_{ext})$ , and cluster-consensus degree,  $(\delta_{CC})$ , are defined next.

**Definition 6.** The  $\alpha_l$ -level cluster internal cohesion degree of cluster  $K_{lr}$  is

$$\delta_{int}(K_{lr}) = \frac{\sum_{i \in K_{lr}} \sum_{j \in K_{lr}} S^{ij}}{(\#K_{lr})^2},$$

where  $S^{ij}$  is cosine similarity degree between expert  $i$  and  $j$  in the cluster  $K_{lr}$ .

**Definition 7.** The  $\alpha_l$ -level cluster external cohesion degree of cluster  $K_{lr}$  is

$$\delta_{ext}(K_{lr}) = \frac{\sum_{i \in K_{lr}} \sum_{j \notin K_{lr}} S^{ij}}{\#K_{lr}(n - \#K_{lr})},$$

where  $n = \#E$  is the total number of experts and  $S^{ij}$  is cosine similarity degree between expert  $i$  in the cluster  $K_{lr}$  and the expert  $j$  outside the cluster  $K_{lr}$ .

**Definition 8.** The  $\alpha_l$ -level cluster consensus degree of cluster  $K_{lr}$ ,

$$\delta_{CC}(K_{lr}) = \frac{\#K_{lr}(\delta_{int}(K_{lr}) - \delta_{ext}(K_{lr}))}{n} + \delta_{ext}(K_{lr}).$$

Groups of experts are clustered based on their preference similarities. Thus it is expected that  $\delta_{int}(K_{lr}) > \delta_{ext}(K_{lr})$  will be satisfied in the proposed consensus framework. In this case, the similarities between experts in a group are greater internally, meaning that they are more closely attached within their group members than with the outsider experts, and that they are a very homogeneous group.

The group of experts' consensus degree at each  $\alpha$ -level cluster,  $\delta_{LC}(l)$ , is then determined to represent the preference homogeneity between experts at that cluster level.

**Definition 9.** The  $\alpha_l$ -level cluster consensus degree of the group of experts  $E$  is

$$\delta_{LC}(l) = \frac{\sum_{r=1}^l \delta_{CC}(K_{lr})}{l}.$$

We are aiming at achieving consensus. Thus the maximum of all the  $\alpha_l$ -level cluster consensus degrees of the group of experts  $E$  is chosen as the optimal level. The definition is given below.

**Definition 10.** The optimal agglomerative hierarchical clustering level,  $\alpha_{\hat{l}}$ -level, is the solution to the following optimization problem

$$\max_{\alpha_l \in \mathbb{L}} \delta_{LC}(l).$$

As the numbers of levels is finite, the above optimization problem is solvable. However, the solution might not be unique, with more than one  $\alpha_l$ -level with same maximum cluster consensus degree. In this case, we use the  $\alpha_l$ -level cluster consensus coefficient of variation,  $CCV_{LC}(l)$ , to discriminate between  $\alpha_l$ -levels, with the one with lowest  $CCV_{LC}(l)$  being the optimal  $\alpha_{\hat{l}}$ -level amongst all the  $\alpha_l$ -levels with maximum cluster consensus degree.

**Definition 11.** The  $\alpha_l$ -level cluster consensus coefficient of variation is

$$CCV_{LC}(l) = \frac{CS D_{LC}(l)}{\delta_{LC}(l)}$$

where

$$CS D_{LC}(l) = \sqrt{\frac{\sum_{r=1}^l [\delta_{CC}(K_{lr}) - \delta_{LC}(l)]^2}{l}}.$$

In the case of having two or more  $\alpha_l$ -levels with same maximum cluster level consensus degree and cluster consensus coefficient of variation, then we select the lowest  $\alpha_l$ -level value because it will involve the lowest number of clusters, which will subsequently require a lower number of rounds in the feedback mechanism for the minimum threshold value of consensus to be achieved.

Finally, the global cluster consensus degree of a group of experts  $E$ ,  $\delta_{LC}(\hat{l})$ , is defined next.

**Definition 12.** The global cluster consensus degree of a group of experts  $E$  is  $\delta_{LC}(\hat{l})$ , with  $\alpha_{\hat{l}}$ -level being the optimal clustering level.

The clustering based consistent preference similarity network consensus algorithm is presented in Algorithm 1.

**Example 3.** In order to measure structural equivalence experts in the network, the consistent cosine similarity degrees between all pairs of expert preferences are computed, resulting in the following symmetric consistent preference similarity matrix:

$$CS = \begin{bmatrix} 1 & 0.833 & 0.762 & 0.863 & 0.973 & 0.844 & 0.810 & 0.855 \\ 0.833 & 1 & 0.873 & 0.963 & 0.871 & 0.991 & 0.886 & 0.965 \\ 0.762 & 0.873 & 1 & 0.837 & 0.805 & 0.844 & 0.984 & 0.880 \\ 0.863 & 0.963 & 0.837 & 1 & 0.864 & 0.971 & 0.868 & 0.989 \\ 0.973 & 0.871 & 0.805 & 0.864 & 1 & 0.866 & 0.835 & 0.873 \\ 0.844 & 0.991 & 0.844 & 0.971 & 0.866 & 1 & 0.876 & 0.962 \\ 0.810 & 0.886 & 0.984 & 0.868 & 0.835 & 0.876 & 1 & 0.901 \\ 0.855 & 0.965 & 0.880 & 0.989 & 0.873 & 0.962 & 0.901 & 1 \end{bmatrix}.$$

The undirected weighted consistent preference similarity network,  $\mathbb{N}$ , is computed and demonstrated as in Figure 1. For

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**Algorithm 1:** The clustering based consistent preference similarity network consensus procedure

---

**Data:** A set of consistent essential vector of preference intensity,  $\mathbb{C} = \{VC^1, VC^2, \dots, VC^m\}$  of a set of experts,  $E = \{e^1, e^2, \dots, e^m\}$ , on a set of alternatives,  $Y = \{y_1, y_2, \dots, y_n\}$ .

**Result:** A hierarchical sequence of clustering solution:  $Z^m, Z^{m-1}, \dots, Z^1$ .

**begin**

```

1  | Begin the clustering with partition  $Z^m = \{K_1, K_2, \dots, K_m\}$  where each cluster  $K_c$  has exactly one element  $e^c$ :
   |    $Z^m = \{\{e^1\}, \{e^2\}, \dots, \{e^m\}\} = \{\{K_1\}, \{K_2\}, \dots, \{K_m\}\}$ ;
   |    $i \leftarrow m$ ;
   |   while  $i > 1$  do
2  |       Determine clusters  $K_c$  and  $K_d$  in  $Z^i = \{K_1, K_2, \dots, K_i\}$  with maximal distance  $(D^{cd})$ ;
3  |       Merge clusters  $K_c$  and  $K_d$  into cluster  $K_r$ ;
4  |       Construct new partition  $Z^{i-1}$  by removing  $K_c$  and  $K_d$  and adding cluster  $K_r$ ;
   |        $i \leftarrow i - 1$ ;
   |   end while

```

**end**

**Input:** *The consistent preference similarity matrix, CS and a dendrogram with:*

A set of experts:  $E = \{e^1, e^2, \dots, e^n\}$ ;  
A set of all distinct  $\alpha$ -levels:  $\mathbb{L} = \{\alpha_l : l = 2, 3, \dots, n-1\}$ ;  
A set of clusters at each  $\alpha_l$ -level:  $K_l = \{K_{lr} : r = 1, 2, \dots, l\}$ ;  
Consensus threshold value:  $= \gamma$ .

**Output:** A global cluster consensus degree of a group of experts:  $\delta_{LC}(\hat{l})$ .

**begin**

```

5  | Determine experts in each cluster of  $\alpha_l$ -level;
6  | Calculate  $\delta_{int}(K_{lr})$  and  $\delta_{ext}(K_{lr})$  for each cluster in  $K_l$ ;
7  | Determine  $\delta_{CC}(K_{lr})$  by combining  $\delta_{int}(K_{lr})$  and  $\delta_{ext}(K_{lr})$  for each cluster in  $K_l$ ;
8  | Compute  $\delta_{LC}(l)$  for all  $\alpha_l$ -level in  $\mathbb{L}$ ;
9  | Identify optimal agglomerative hierarchical clustering level:  $\alpha_{\hat{l}}$ -level;
10 | Identify the global cluster consensus degree of a group of experts:  $\delta_{LC}(\hat{l})$ ;

```

**end**

---

simplicity only a few link weights are presented. As shown in matrix  $CS$ , the similarity degree between pair of experts  $e^1$  and  $e^5$ ,  $e^2$  and  $e^6$ ,  $e^3$  and  $e^7$ , and  $e^4$  and  $e^8$  are among the highest, thus they are closely attached with each other in the similarity network  $\mathbb{N}$ . Figure 2 shows the dendrogram resulting from the agglomerative hierarchical clustering procedure. It is useful to visualize the clustering solution as it is utilized as one of the input elements in measuring global cluster consensus degree.

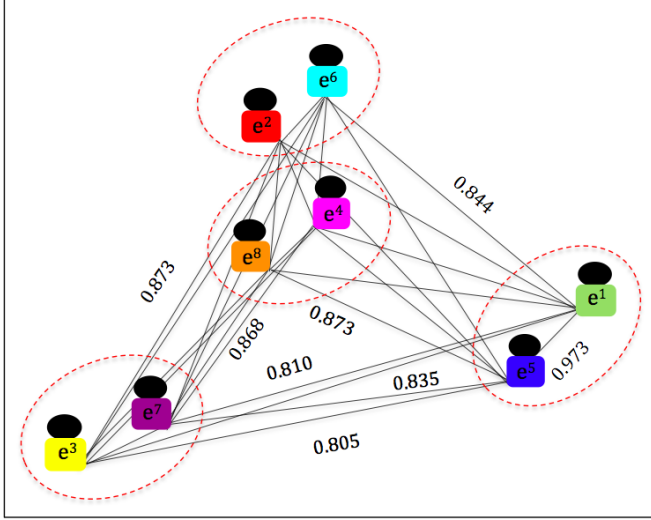


Figure 1: The undirected weighted consistent preference similarity network,  $\mathbb{N}$

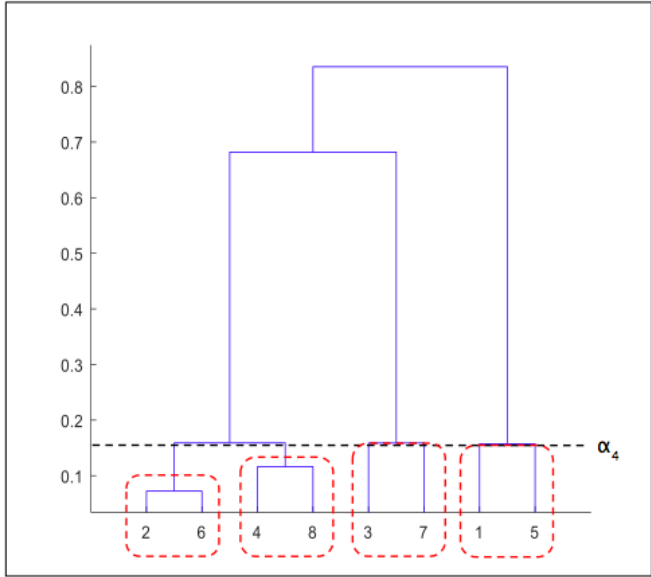


Figure 2: Dendrogram generated from the agglomerative hierarchical clustering procedure

Table 2 provides the internal and external cohesions for each cluster, cluster consensus and global cluster consensus degree for all  $\alpha$ -levels. The maximum consensus degree, obtained at the  $\alpha_4$ -level (optimal clustering level), is 0.901 and shows that the global cluster consensus degree of a group of experts is in line with network pattern in Figure 1.

If the consensus threshold was set at 0.9, then no consensus

feedback process would be required because the global cluster consensus degree is above this number. However, if the global cluster consensus degree was lower than the consensus threshold, a feedback mechanism procedure has to be carried out with the purpose of obtaining a high level of group agreement. This is not part of the discussion in this paper.

#### 4. Framework Overview of the Proposed Consensus Model

This section provides the framework of the proposed geo-uniform consistency control module for a preference similarity network clustering-based consensus model. As shown in Figure 3, the model has two main phases: (A) a consistency control module, which was discussed in Section 2; and (B) a consistent preference similarity network clustering-based consensus measure, presented in Section 3. Brief explanations on both phases are presented below.

- *Phase A:* A group of experts give their evaluation concerning a set of alternatives, by means of reciprocal fuzzy preference relations, from which their essential vectors of preference intensity are extracted. The geo-uniform operator is applied to check for consistency of preferences by means of the computation of consistent-based fuzzy preference relations, that satisfy uniform properties and transitivity properties as well. Experts' consistency is checked by measuring the similarity between the constructed essential vectors of consistent preference intensities and the provided ones. The consistency degree based on the cosine similarity function is proposed. This is compared to a threshold value, which is set in advance. This comparison may or may not activate a novel personalized geo-uniform consistency feedback system. This consistency module control starts by identifying those expert(s) with consistency degree lower than the threshold. Personalized changes are then provided to the identified inconsistent experts. They then adjust their original preferences to move closer to their constructed geo-uniform consistent preferences. The amount of change cost as measured by the difference between the recommended preference value and the original ones is optimized subject to achieving a consistency threshold value, which is controlled by a personalized  $\gamma$  parameter. Different experts will receive different values of  $\gamma$ , and therefore will be treated differently. The consistency feedback process thus guarantees that consistency will reach the threshold value if the recommendations of change are effectively implemented by the inconsistent experts for use in the next consensus measuring phase.
- *Phase B:* Once consistency of preferences has been achieved, consensus is measured. To do this, the symmetric consistent preference similarity matrix,  $CS$ , is constructed by computing the cosine similarity degree between the reciprocal fuzzy preference relations of all pairs of experts. This is subsequently exploited via the SNA structural equivalence concept. Thus, an undirected weighted

Table 2: The cluster internal and external cohesions, cluster consensus and global cluster consensus degree at all  $\alpha$ -levels

$\alpha$	$K$	$E$	$\delta_{int}$	$\delta_{ext}$	$\delta_{CC}$	$\delta_{LC}$
2	1	$e^2, e^3, e^4, e^6, e^7, e^8$	0.933	0.840	0.910	0.893
	2	$e^1, e^5$	0.986	0.840	0.877	
3	1	$e^2, e^4, e^6, e^8$	0.980	0.865	0.922	0.894
	2	$e^3, e^7$	0.992	0.848	0.884	
	3	$e^1, e^5$	0.986	0.840	0.877	
4	1	$e^2, e^6$	0.996	0.896	0.921	<b>0.901</b>
	2	$e^4, e^8$	0.995	0.9	0.924	
	3	$e^3, e^7$	0.992	0.848	0.884	
	4	$e^1, e^5$	0.986	0.840	0.877	
5	1	$e^2, e^6$	0.996	0.896	0.921	0.898
	2	$e^4, e^8$	0.995	0.9	0.924	
	3	$e^3$	1	0.855	0.873	
	4	$e^7$	1	0.880	0.895	
	5	$e^1, e^5$	0.986	0.840	0.877	
6	1	$e^2, e^6$	0.996	0.896	0.921	0.894
	2	$e^4, e^8$	0.995	0.9	0.924	
	3	$e^3$	1	0.855	0.873	
	4	$e^7$	1	0.880	0.895	
	5	$e^1$	1	0.848	0.867	
	6	$e^5$	1	0.869	0.886	
7	1	$e^2, e^6$	0.996	0.896	0.921	0.899
	2	$e^4$	1	0.908	0.919	
	3	$e^8$	1	0.918	0.928	
	4	$e^3$	1	0.855	0.873	
	5	$e^7$	1	0.880	0.895	
	6	$e^1$	1	0.848	0.867	
	7	$e^5$	1	0.869	0.886	

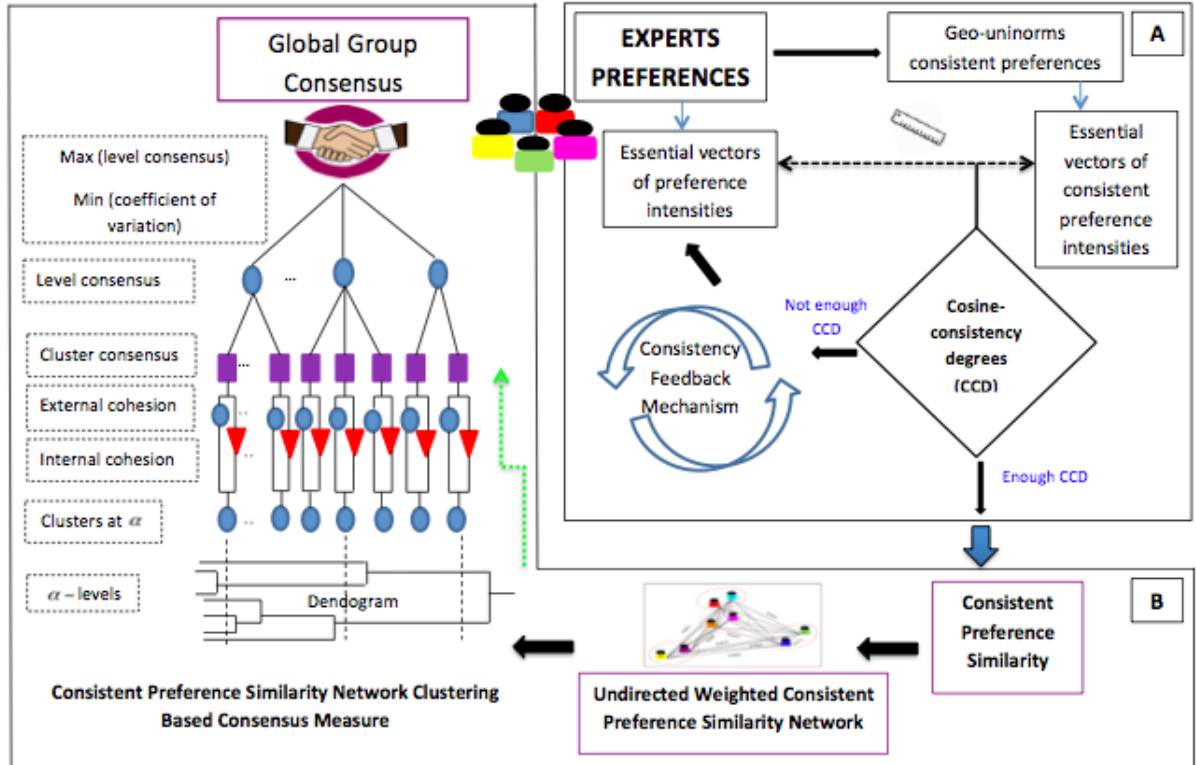


Figure 3: A framework of geo-uniform consistency control system for preference similarity network clustering based consensus model

consistent preference similarity network connecting all experts using their similarity of preference degrees is constructed. Experts are classed into subsets of structurally equivalent experts by using an agglomerative hierarchical clustering algorithm. The dendrogram of this clustering is used as a visualization tool to find the optimal consensus  $\alpha$ -level clustering solution. This is achieved by computing for each cluster, at each  $\alpha$ -level, their internal and external cluster cohesions, which are combined to determine the clusters' consensus degrees and  $\alpha$ -level consensus degree. The  $\alpha$ -level with maximum consensus is identified as the optimal one reflecting the global consensus of the group of experts at this stage.

## 5. Analysis of the Consistency Control Module and its Impact on Consensus

In this validation section, we revisit the work of Chu et al. [37], and compare our results related to the proposed consistency control module. We remark on five comparison elements of both models as depicted in Table 3 below:

Chu et al. in [37] studied reciprocity, additive consistency and acceptable consistency properties for collective preference relations. We have proposed the construction of consistent fuzzy preference relations based on the geo-uniform operator, which is related to multiplicative consistency rather than additive consistency. In computing consistency degree, Chu et al. compared individual expert's preferences with collective preferences, while in the proposed approach in this paper, an expert's preferences are compared only with his own geo-uniform consistent preferences. These differences explain the slight differences in the consistency degrees obtained by the two models. For a consistency threshold of 0.8, both approaches identify the same number and the same inconsistent experts ( $e^4$  and  $e^8$ ). Notice that if the threshold value had been set as 0.83, for example, both methods would have produced quite different sets of inconsistent experts in number. While our proposed method will only identify the previous two identified experts as inconsistent, Chu et al. would have added expert  $e_5$ . More notable differences between the approaches are apparent when the threshold value is increased.

The consistency feedback mechanism applied by Chu et al. was developed in Wu and Xu [24], with both experts receiving the same parameter value ( $\gamma = 0.6$ ) controlling the preference change recommendations to reach the consistency threshold level. In the case of the proposed model in this paper, each expert will receive a different parameter value of  $\gamma$ . For illustrative purposes, in the example used in this paper, we restrict ourselves to the set of discrete values of  $\gamma$ :  $\{0.1, 0.2, \dots, 0.9, 1\}$ . In this case, the proposed model returns both experts with the same value of  $\gamma = 0.2$ . This is significantly lower than the 0.6 from the comparison model. The impact of these differences on the recommended changes experts received via the proposed model are twofold: (1) it provides *fair* recommendations individually, depending on the experts' personal level of inconsistency; (2) it guarantees consistency by advising inconsistent expert(s) to modify their preferences with minimum changes.

Thus, the proposed approach is promising because every inconsistent expert is reasonably treated, and will be more willing to accept the recommendations because only minimal changes are needed.

The new preference relations obtained, after the feedback on preference changes are implemented, are different in the two methods. The proposed method in this paper is based on a uniform-based construction of consistent preference relations that makes use of  $(m - 1)$  original preference relation values. These remain unchanged in the next stage of the proposed process. This is not the case in the method implemented by Chu et al., since inconsistent experts are advised to change all of their preferences. Thus, the proposed model in this paper is less expensive, not only in terms of the magnitude of the change recommended, but also computationally because a lower number of changes is required for an inconsistent expert to achieve the consistency threshold.

Finally, for the purpose of analyzing the impact of the proposed consistency control module in achieving consensus, Table 4 shows the results on consensus when the preference similarity network structural equivalence clustering based consensus model is implemented with and without the consistency control module.

The outcome clearly shows that consistency control module positively contributes to achieving sufficient consensus level. We can see that in the consensus process first round, the consensus degree is above the threshold value when consistency is checked and improved, meaning that experts' consistency also contributed to expert's agreement. The consensus second round, which involves the activation of a consensus feedback process, was not needed when the consistency control module was activated. On the other hand, without the consistency control module, the second round of consensus was needed. Thus, consensus models with the consistency control module are expected to be computationally less expensive for achieving consensus than those without.

## 6. Conclusion

In decision making, *correctness* of information, i.e information that is consistent, is a necessary element to be taken into account. The transitivity property has been suggested in order to model consistency because of its hierarchy status with other basic properties of pairwise comparisons, asymmetry and indifference. In this paper a geo-uniform operator has been proposed and proved to satisfy desirable properties for modelling consistency of preferences. This operator is a hybrid operator that is obtained by combining the best of the geometric average, a mean operator that assures that moderate stochastic transitivity is satisfied, and of the cross-ratio uniform, which allows for the mean reinforcement property. Approaches for deriving consistent-based fuzzy preference relations from a given fuzzy preference relation, and for measuring its consistency level are developed. These are subsequently exploited to design a personalized geo-uniform consistency feedback mechanism, with personalized recommendations depending on the

Table 3: Comparative results of consistency control module

Elements	Chu et. al[37]	Proposed model
Consistency degrees	$CCD(e^1) = 0.9400, CCD(e^2) = 0.8533,$ $CCD(e^3) = 0.8311, CCD(e^4) = \mathbf{0.7089},$ $CCD(e^5) = 0.8289, CCD(e^6) = 0.8400,$ $CCD(e^7) = 0.8422, CCD(e^8) = \mathbf{0.7511}.$	$CCD(e^1) = 0.8307, CCD(e^2) = 0.9513,$ $CCD(e^3) = 0.9182, CCD(e^4) = \mathbf{0.7462},$ $CCD(e^5) = 0.8788, CCD(e^6) = 0.9435,$ $CCD(e^7) = 0.9109, CCD(e^8) = \mathbf{0.7212}.$
Inconsistent expert(s) (Threshold 0.8)	$e^4, e^8$	$e^4, e^8$
Controlled parameter	$\gamma = 0.6$ for both $e_4, e_8$	Personalized: $\gamma(e^4) = 0.2, \gamma(e^8) = 0.2$
Feedback preferences	$CP^4 = \begin{bmatrix} 1 & \mathbf{0.3133} & 0.24 & 0.4877 & 0.6733 & 0.6867 \\ 0.6867 & 1 & \mathbf{0.3067} & 0.7333 & 0.32 & 0.4533 \\ 0.76 & 0.6933 & 1 & \mathbf{0.6867} & 0.27 & 0.2867 \\ 0.5133 & 0.2667 & 0.3133 & 1 & \mathbf{0.8067} & 0.7 \\ 0.3267 & 0.68 & 0.7267 & 0.1933 & 1 & \mathbf{0.5733} \\ 0.3133 & 0.5467 & 0.7133 & 0.3 & 0.4267 & 1 \end{bmatrix}$ $CP^8 = \begin{bmatrix} 1 & \mathbf{0.44} & 0.3667 & 0.3467 & 0.5333 & 0.6133 \\ 0.56 & 1 & \mathbf{0.3067} & 0.7067 & 0.2933 & 0.4333 \\ 0.6333 & 0.6933 & 1 & \mathbf{0.66} & 0.2467 & 0.2667 \\ 0.6533 & 0.2933 & 0.34 & 1 & \mathbf{0.8067} & 0.7067 \\ 0.4667 & 0.7067 & 0.7533 & 0.1933 & 1 & \mathbf{0.58} \\ 0.3867 & 0.5667 & 0.7333 & 0.2933 & 0.42 & 1 \end{bmatrix}$	$CP^4 = \begin{bmatrix} 1 & \mathbf{0.2} & 0.12 & 0.4773 & 0.84 & 0.84 \\ 0.8 & 1 & \mathbf{0.2} & 0.82 & 0.36 & 0.52 \\ 0.88 & 0.8 & 1 & \mathbf{0.8} & 0.28 & 0.28 \\ 0.5227 & 0.18 & 0.2 & 1 & \mathbf{1} & 0.84 \\ 0.16 & 0.64 & 0.72 & 0 & 1 & \mathbf{0.6} \\ 0.16 & 0.48 & 0.72 & 0.16 & 0.4 & 1 \end{bmatrix}$ $CP^8 = \begin{bmatrix} 1 & \mathbf{0.4} & 0.298 & 0.3333 & 0.68 & 0.76 \\ 0.6 & 1 & \mathbf{0.2} & 0.82 & 0.36 & 0.52 \\ 0.702 & 0.8 & 1 & \mathbf{0.8} & 0.28 & 0.28 \\ 0.6667 & 0.18 & 0.2 & 1 & \mathbf{1} & 0.84 \\ 0.32 & 0.64 & 0.72 & 0 & 1 & \mathbf{0.6} \\ 0.24 & 0.48 & 0.72 & 0.16 & 0.4 & 1 \end{bmatrix}$
Feedbacked consistency indexes	$newCCD(e^4) = 0.8253, newCCD(e^8) = 0.8507$	$newCCD(e^4) = 0.8363, newCCD(e^8) = 0.8207$

Table 4: Comparison of results in analyzing the impact of consistency control module towards consensus

Elements	With consistency module	Without consistency module [8]		
Consensus - 1st round (Threshold = 0.9)	0.901	0.893		
Feedback consensus (2nd round)	Not required	$\beta$	Consensus Index	Cluster Level
		0	0.893	Level 4
		<b>0.1</b>	<b>0.907</b>	<b>Level 4</b>
		0.2	0.921	Level 4
		0.3	0.934	Level 4
		0.4	0.946	Level 4
		0.5	0.956	Level 6
		0.6	0.965	Level 4
		0.7	0.972	Level 4
		0.8	0.976	Level 7
		0.9	0.978	Level 7
		1	0.973	Level 2

experts' current individual level of inconsistency, that guarantees the achievement of a desired consistency threshold value with minimum cost.

Once consistency is considered acceptable by a group of experts, a consistent-based preference similarity network clustering based consensus model is developed. This is based on the building of an undirected weighted consistent preference similarity network structure between the experts, which is exploited using the concept of *structural equivalence*. The experts are partitioned into subsets of structurally equivalent experts by using an agglomerative hierarchical clustering algorithm, whose dendrogram is used as a visualization tool to find the optimal consensus clustering solution. This is achieved by computing for each cluster, at each hierarchical clustering level, their internal and external cluster cohesions, which are combined to determine the clusters' consensus degrees, the optimal cluster level, and ultimately the consensus of the group of experts.

The proposed model is validated via comparison with the existing study in [37]. This clearly showed that the proposed consistency module is less expensive, not only on the magnitude of the change recommended, but also computationally because a smaller number of changes is required for an inconsistent expert to achieve effectively the consistency threshold. Finally, the effect on measuring consensus has also been shown as positive, as the preference similarity network structural equivalence clustering-based consensus model, with consistency control, achieves consensus more rapidly than when consistency control is not implemented.

In this work we utilized a symmetric similarity function to measure experts' preference similarities. Because of this, the construction of the preference similarity network produced undirected connections. In the future, asymmetric similarity functions, such as the Tversky index can be used to handle directed cases. Trust and influence networks, which are not symmetric, can also be considered. Furthermore, *dynamic* consensus decision making is a relevant topic to be explored due to the demand of current web technologies that require real-time communications or time-varying individual relationships.

## Acknowledgements

The authors would like to thanks the anonymous referees for providing constructive comments and suggestions. The authors also acknowledge valuable support from the University of Leicester (UK), De Montfort University (UK), Ministry of Higher Education Malaysia and Universiti Teknologi MARA (Malaysia).

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