Algorithms to Detect and Rectify Multiplicative and Ordinal Inconsistencies of Fuzzy Preference Relations

Yejun Xu , Mengqi Li, Francisco Javier Cabrerizo, Francisco Chiclana, Enrique Herrera-Viedma

Abstract

Consistency, multiplicative and ordinal, of fuzzy preference relations (FPRs) is investigated. The geometric consistency index (GCI) approximated thresholds are extended to measure the degree of consistency for an FPR. For inconsistent FPRs, two algorithms are devised (1) to find the multiplicative inconsistent elements, and (2) to detect the ordinal inconsistent elements. An integrated algorithm is proposed to improve simultaneously the ordinal and multiplicative consistencies. Some examples, comparative analysis, and simulation experiments are provided to demonstrate the effectiveness of the proposed methods.

Index Terms-FPRs; Multiplicative consistency (MC); Ordinal consistency (OC); Inconsistent elements.

I. INTRODUCTION

In multiple criteria decision-making problems, pairwise comparisons on alternatives are usually used by decision makers (DMs) to express their preference information [1-3], which can be modelled mathematically with a preference relation (PR). Multiplicative PRs (MPRs) [4-9] and fuzzy PRs (FPRs) [10-13] are widely used for numerical PRs. A key drawback of pairwise comparison is that DMs fail to consider the global relationship of alternatives because they focus on two alternatives at a time. This may end in the provision of inconsistent preferences[14], especially in the presence of a large number of alternatives. Thus, in decision making scenarios with PRs, it is sensible to ensure that preferences are consistent before making the final decision.

Consistency of preferences generally has two types: ordinal consistency (OC) [15-19] and cardinal consistency (CC) [18,

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20-22]. The OC, which is based on the concept of the weak transitivity [23], is defined as follows: "if alternative x_i is preferred to alternative x_j and alternative x_j is preferred to alternative x_k, then alternative x_i should be preferred to alternative x_k". This type of consistency is considered to be the minimum requirement to be fulfilled by a 'consistent' PR [4, 21, 24]. A stronger concept than OC is CC because it requires in addition that the "intensity with which the preference is expressed transits through the sequence of objects in comparison" [4].

In the Analytic Hierarchy Process (AHP), Saaty [4] introduced the notions of perfect consistency and acceptable consistency. This was based on the consistency ratio (CR) between the consistency index (CI) of an MPR and the average CIs of a large number of random generated MPRs of the same order. An MPR with $CR < 0.1$ is considered of acceptable consistency. However, acceptable consistency does not imply OC as illustrated by Kwiesielewicz and van Uden [17] where an MPR with CR smaller than 0.1 was shown to be ordinally inconsistent, i.e., it contained inconsistent elements.

There are also studies on the consistency of FPRs that deal with different types of inconsistencies: OC [16, 25], additive consistency (AC) [16, 26-28] and multiplicative consistency (MC) [13, 29]. Ma, et al. [30] introduced a method to detect and repair the inconsistency of strict FPRs. Li, et al. [22] reviewed the AC of FPRs. Xu, et al. [15] developed an OC index (*OCI*) to measure the level of OC of FPRs and an algorithm to eliminate unreasonable 3-cycles in the relation digraph. Xia, et al. [13] adapted the geometric consistency index (*GCI*) for MPRs proposed by Aguarón and Moreno-Jiménez [31] to the case of FPRs, which was the base of a method to improve their MC. However, Xu, et al. [29] provided an example where contradictory preference values may still be present in an FPR that fulfils the MC test, which suggests that MC on its own is not sufficient to assure the consistency of FPRs. Thus, it is clear that there is a need for a study that focus on both the OC and CC at the same time.

There conflict between AC and the scale for measuring FPR values ([0,1]) led Chiclana, et al. [20] to propose a functional modelling of consistency of FPRs, that under the conditions of monotonicity and continuity was proved to be verified by a representable uninorm with strict negator operator $N(x) = 1 - x$. Because the MC is an AND-like representable uninorm with the above strict negator operator, Chiclana, et al. [20] concluded that MC is the most appropriate property for modelling CC of FPRs. Meanwhile, few researches have investigated both the OC and the MC at the same time. Because the OC can ensure the rationality and the MC can measure the level of consistency and lead to more consistent FPRs. Xu, et al. [29] developed some methods to modify the ordinal and multiplicative inconsistencies for FPRs. However, their improvement methods have two steps: ordinal inconsistency modification step, and multiplicative inconsistency modification step. The latter step aims at improving multiplicative consistency by adjusting the elements that contributes most to the multiplicative inconsistency, which it is followed by a checking of whether the adjustment destroys ordinal consistency. If the modified FPR does not have ordinal consistency, the method requires a further check and adjustment of the element second most contributor to multiplicative inconsistency. These are obviously complex and time-consuming methods. Although the OC

and MC of FPRs have been deeply investigated, there are still some open questions to be answered:

(1) In the existing work (for example, Refs. [13, 29]), the *GCI* threshold is usually set artificially and may not be reasonable. Thus, the following question needs to be addressed: is there a reasonable threshold for *GCI* to measure the level of MC?

(2) One of the two types of consistency discussed is not enough to ensure the rationality of the FPRs, while improving the two types of consistency separately will make the calculation process time-consuming and will also distort the information greatly. Thus, the following question requires also consideration: Can the OC and MC be improved simultaneously with a unique single method?

This paper aims at answering the above two research questions and fill a gap in the current research knowledge. This paper extends *GCI* to measure the MC of an FPR. A new method to detect and measure the OC is also devised. Finally, an integrated algorithm to find the inconsistent elements to improve the OC and MC simultaneously is proposed. Monte Carlo simulation are also provided to show the effectiveness and advantages of the proposed methods.

The remainder of the paper is organized as follows. Section II contains basic concepts about consistency of FPRs that will be used in subsequent sections. Section III proposes an MC-based algorithm to identify FPRs multiplicative inconsistent elements. Section IV presents a new index to measure the OC level and an algorithm to search the ordinal inconsistent elements of FPRs. Section V presents an algorithm to adjust the FPR inconsistent elements, while examples of its application and a comparison with existing methods in the literature to demonstrate the effectiveness of the proposed method are provided in Section VI. Finally, in Section VII conclusions are given.

II. Preliminaries

This section provides the basic definition related to FPRs needed in the following sections.

For simplicity, let $N = \{1, 2, \ldots, n, n > 2\}$ and $X = \{x_i, i \in N\}$ be a finite set of alternatives. In multi-attribute decision making problems, DMs aim to get ranking of alternatives based on the provided information which is assumed to be in the form of FPRs.

Definition 1 [11]: "An FPR R on a set of alternatives X is a fuzzy set on the cartesian product set $X \times X$, which is characterized by a membership function $\mu_R : X \times X \to [0,1]$, which is represented by a $n \times n$ matrix $R = (r_{ij})_{n \times n}$, with element r_{ij} interpreted as the preference degree of the alternative x_i over the alternative x_j. A value of $r_{ij} = 0.5$ indicates indifference between x_i and x_j ($x_i \sim x_j$); When $0.5 < r_{ij} \leq 1$, then x_i is strictly preferred to x_j ($x_i \succ x_j$) with $r_{ij} = 1$ indicating absolute preference of x_i over x_j; while $0 \leq r_{ij} < 0.5$ indicates that x_j is strictly preferred to x_i ($x_i \prec x_j$) with $r_{ij} = 0$, indicating absolute preference of x_j over x_i".

This interpretation underlines a reciprocity property of preference often assumed to be verified, as it is the case in the

present paper:

$$r_{ij} + r_{ji} = 1, \quad r_{ii} = 0.5, \quad r_{ij} \in [0,1], \quad \forall i, j \in N \tag{1}$$

A. Ordinal Consistency of FPRs

The OC of FPRs is described as follows:

Definition 2 [15]: "An FPR $R = (r_{ij})_{n \times n}$ is OC when for all $i, j, k \in N$, $i \neq j \neq k$, the following properties are verified:

(1) if $[\, r_{ik} > 0.5, \; r_{kj} \geq 0.5\,]$, or $[\, r_{ik} \geq 0.5, \; r_{kj} > 0.5\,]$ then $r_{ij} > 0.5$;

(2) if $[\, r_{ik} = 0.5 \;$ and $\; r_{kj} = 0.5\,]$ then $r_{ij} = 0.5$".

Xu, et al. [15] applied graph theory to investigate the OC of FPRs.

Definition 3 [15]: "The adjacency matrix of an FPR $R = (r_{ij})_{n \times n}$ is:

$$B = (b_{ij})_{n \times n}; \quad b_{ij} = \begin{cases} 1, & r_{ij} \geq 0.5, \; i \neq j \\ 0, & otherwise \end{cases} \tag{2}$$

The digraph of the FPR R is denoted by $G = (X, A)$, where the set of alternatives X is the node set and $r_{ij} > 0.5$ $A = \{(x_i, x_j) \mid i \neq j \wedge r_{ij} \geq 0.5\}$ is the directed arc set. When $r_{ij} = 0.5$, $i \neq j$, it is also $r_{ji} = 0.5$ and therefore there exist two directed arcs between x_i and x_j (one from x_i to x_j, and another from x_j to x_i)."

Remark 1: It is easy to prove that if an FPR R violates OC, then there must exist at least one directed 3-cycle in the digraph G of R, i.e. $x_i \rightarrow x_j \rightarrow x_k \rightarrow x_i$. Therefore, the OC of FPRs requires directed 3-cycles to be identified and eliminated.

B. Multiplicative consistency of FPRs

MC refers to the multiplicative transitivity property of preferences:

Definition 4 [11]: "An FPR $R = (r_{ij})_{n \times n}$ is MC if it satisfies:

$$r_{ij} r_{jk} r_{ki} = r_{ji} r_{kj} r_{ik}, \quad \forall i, j, k \in N \tag{3}$$

Which is referred to as the multiplicative transitivity property".

Remark 2: Eq.(3) requires that $r_{ij} > 0$, $\forall i, j$, and therefore it can be rewritten as:

$$\frac{r_{ij}}{r_{ji}} = \frac{r_{ik}}{r_{ki}} \cdot \frac{r_{kj}}{r_{jk}}, \quad \forall i, j, k \in N \tag{4}$$

The value r_{ij}/r_{ji} represents the ratio of the preference power for x_i to that of x_j, and it is interpreted as follows: x_i is r_{ij}/r_{ji} times preferred to x_j. This value will be referred to as the direct judgement of alternative x_i over alternative x_j, while $\frac{r_{ik}}{r_{ki}} \cdot \frac{r_{kj}}{r_{jk}}$ will be referred to as the indirect judgement of alternative x_i over alternative x_j via (though) alternative

x_k.

In order to measure the level of consistency, Aguarón and Moreno-Jiménez [31] proposed the geometric consistency index (GCI) for reciprocal MPRs $A = (a_{ij})_{n \times n}$ ($a_{ij} \cdot a_{ji} = 1$):

$$GCI(A) = \frac{2}{(n-1)(n-2)} \sum_{i<j} (\log a_{ij} - \log v_i + \log v_j)^2 \tag{5}$$

where $v = (v_1, v_2, \cdots, v_n)^T$ is the following priority vector derived from A:

$$v_i = \left(\prod_{j=1}^{n} a_{ij} \right)^{1/n} \bigg/ \sum_{j=1}^{n} \left(\prod_{j=1}^{n} a_{ij} \right)^{1/n}; \quad \sum_{i=1}^{n} v_i = 1 \text{ and } v_i > 0, i \in N.$$

According to the theoretical relation between CR and the GCI, Aguarón and Moreno-Jiménez [31] provided the below approximated threshold values for GCI (Table I).

<p align="center">TABLE I. THE APPROXIMATED THRESHOLDS FOR GCI</p>

CR	0.01	0.05	0.1	0.15
$\overline{GCI}(n=3)$	0.0314	0.1573	0.3147	0.4720
$\overline{GCI}(n=4)$	0.0352	0.1763	0.3526	0.5289
$\overline{GCI}(n>4)$	~0.037	~0.185	~0.370	~0.555

The following transformation provides a route to derive an MPR $A = (a_{ij})_{n \times n}$ from an FPR $R = (r_{ij})_{n \times n}$ [32]:

$$a_{ij} = \frac{r_{ij}}{r_{ji}} \tag{6}$$

Motivated by Aguarón and Moreno-Jiménez [31], Xia, et al. [13] adopted the GCI to the case of FPRs.

Definition 5 [13]: "Let $R = (r_{ij})_{n \times n}$ be an FPR, and $w = (w_1, w_2, \cdots, w_n)^T$ be the priority vector derived from R satisfying $\sum_{i=1}^{n} w_i = 1$, $w_i < 1$ and $w_i > 0$, $i \in N$, then the GCI of R is given by

$$GCI(R) = \frac{2}{(n-1)(n-2)} \sum_{i<j} (\ln r_{ij} - \ln r_{ji} - \ln w_i + \ln w_j)^2 \tag{7}$$

When $GCI(R)$ is zero, the FPR R is of perfect MC. The smaller the value of $GCI(R)$, the better the consistency of R".

In practical situations, the DMs' FPRs are not of perfect MC. In these cases, a threshold value $\overline{GCI}$ is set when $GCI(R) \leq \overline{GCI}$ the FPR R is considered of acceptable MC. Xia, et al. [13] did not provide $\overline{GCI}$ threshold values, although it is obvious from Eq.(5) and Eq.(6) that the threshold values of Table I for MPRs can be used to check acceptable MC for FPRs.

Given an FPR, checking its acceptable MC requires the computation of its priority vector in the first place. Wang and Fan [33] proposed the following logarithmic least squares model to obtain the priority vector from an FPR:

(M-1) $\min J = \sum_{i=1}^{n} \sum_{j=1}^{n} (\ln r_{ij} - \ln r_{ji} - \ln w_i + \ln w_j)^2$

s.t. $\sum_{i=1}^{n} w_i = 1, \ 0 < w_i < 1, \ i \in N$

The solution to (M-1) was also given by Wang and Fan [33]:

$$w_i = \frac{\left(\prod_{j=1}^{n} \dfrac{r_{ij}}{r_{ji}}\right)^{1/n}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} \dfrac{r_{ij}}{r_{ji}}\right)^{1/n}}, \ i \in N \tag{8}$$

It coincides, as expected, with the one given by Aguarón and Moreno-Jiménez [31] for MPRs using Eq.(6).

Xu, et al. [29] illustrated with some examples that FPR may still contain contradictory judgments in terms of OC but be regarded as MC. Thus, to study consistency of preferences properly MC is not sufficient on its own and OC should also be considered. This is the focus of Section IV where an effective method to measure OC of FPRs is put forward. Before this can be done, in next section the problem of detecting multiplicative inconsistency is first tackled.

III. MULTIPLICATIVE INCONSISTENCY DETECTION FOR FPRs

First, measuring the 'MC degree' of FPRs can help devising effective methods to detect its most inconsistent elements. Subsequently, we can improve its consistency.

A. The Multiplicative Inconsistency of FPRs

The equality $r_{ij}/r_{ji} = (r_{ik}/r_{ki}) \cdot (r_{kj}/r_{jk})$ (for all $i,j,k \in N$) holds for an FPR $R = (r_{ij})_{n \times n}$ of perfect MC. This property can be explained intuitively in Fig. 1, which combines direct and indirect judgments on a judgment scale.

If R is of perfect MC, all indirect judgments $(r_{ik}/r_{ki}) \cdot (r_{kj}/r_{jk})$ would be located in the blue dot of Fig. 1. Otherwise, there would be indirect judgments located on the right of the blue dot (as the two provided in Fig.1), which indicate preference ratios higher than the direct judgment, and/or on the left of the blue dot (as the one indirect in Fig. 1), which indicates preference ratios lower than the direct judgment.
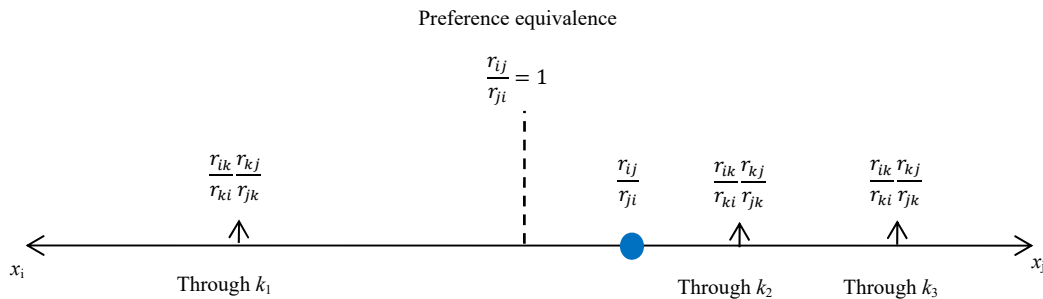

Fig. 1. The direct and indirect judgments on a judgment scale

From Fig. 1, if R is not MC, then the direct value r_{ij}/r_{ji} and the indirect preference value $(r_{ik}/r_{ki}) \cdot (r_{kj}/r_{jk})$ may be diversely scattered and their deviation is measured as follows:

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$$\delta\left(\frac{r_{ij}}{r_{ji}}, \frac{r_{ik}}{r_{ki}} \cdot \frac{r_{kj}}{r_{jk}}\right) = \left| \log\left(\frac{r_{ij}}{r_{ji}}\right) - \log\left(\frac{r_{ik}}{r_{ki}} \cdot \frac{r_{kj}}{r_{jk}}\right) \right| \tag{9}$$

or

$$\delta\left(\frac{r_{ij}}{r_{ji}}, \frac{r_{ik}}{r_{ki}} \cdot \frac{r_{kj}}{r_{jk}}\right) = \left(\log\left(\frac{r_{ij}}{r_{ji}}\right) - \log\left(\frac{r_{ik}}{r_{ki}} \cdot \frac{r_{kj}}{r_{jk}}\right) \right)^2 \tag{10}$$

Remark 3: Clearly an FPR is MC if and only if the deviation values from Eq.(9) to Eq.(10) are 0 for all $i, j, k \in N$. Meanwhile, if a particular deviation value associated with r_{ij}/r_{ji} is not 0, then we can say that r_{ij} is an inconsistent element of the FPR.

In order to measure the inconsistency degree of an FPR and detect the multiplicative inconsistent elements, we first aggregate the above deviations values:

$$\theta_{ij} = \frac{1}{n-2} \sum_{k=1, k \neq i, k \neq j}^{n} \delta\left(\frac{r_{ij}}{r_{ji}}, \frac{r_{ik}}{r_{ki}} \cdot \frac{r_{kj}}{r_{jk}}\right), \quad \forall i, j, k \in N \tag{11}$$

Alternative approaches to define θ_{ij} are possible, such as the worst deviation approach:

$$\theta_{ij} = \max_{k \neq i, k \neq j} \left(\delta\left(\frac{r_{ij}}{r_{ji}}, \frac{r_{ik}}{r_{ki}} \cdot \frac{r_{kj}}{r_{jk}}\right) \right), \quad \forall i, j, k \in N \tag{12}$$

or the geometric mean approach

$$\theta_{ij} = \left(\prod_{k=1, k \neq i, k \neq j}^{n} \delta\left(\frac{r_{ij}}{r_{ji}}, \frac{r_{ik}}{r_{ki}} \cdot \frac{r_{kj}}{r_{jk}}\right) \right)^{\frac{1}{n-2}}, \quad \forall i, j, k \in N \tag{13}$$

Different definitions of θ_{ij} and δ would result in different values of θ_{ij}, which may lead to different results and different formulas in practical problems. In any case, in the symmetrical matrix $\Theta = (\theta_{ij})_{n \times n}$ ($\theta_{ij} = \theta_{ji}$), the larger the value θ_{ij}, the more inconsistent the element r_{ij} is. Thus, the most inconsistent element of an FPR will correspond to the largest value in the matrix $\Theta = (\theta_{ij})_{n \times n}$. Thus, the following algorithm (Algorithm 1) is proposed.

Algorithm 1. Let $R = (r_{ij})_{n \times n}$ be an FPR, and $\overline{GCI(R)}$ a consistency threshold value from Table I.

Step 1. Compute $GCI(R)$ using Eq.(7). If $GCI(R) \leq \overline{GCI(R)}$, then R has acceptable MC and go to Step 5. Otherwise, continue with the next step.

Step 2. Compute the deviation δ using Eq.(9) or Eq.(10).

Step 3. Compute θ_{ij} and establish the matrix $\Theta = (\theta_{ij})_{n \times n}$ according to one of Eq.(11) to Eq.(13).

Step 4. Find the value $\theta_{i_m j_m} = \max_{i, j} \{\theta_{ij}\}$ in matrix $\Theta = (\theta_{ij})_{n \times n}$, then the corresponding $r_{i_m j_m}$ is the most multiplicative

inconsistent element in the FPR R.

Step 5. End.

B. The Accumulation of Multiplicative Inconsistent Elements

Given an FPR, the accumulation of the inconsistency values associated with its respective elements can be considered as its multiplicative inconsistency degree:

$$\theta = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}\theta_{ij} \tag{14}$$

Alternative expression for θ, such as the worst deviation is stated below

$$\theta = \max_{i,j}(\theta_{ij}) \tag{15}$$

Depending on the different mathematical formulations used for δ and θ_{ij}, θ may result in different values. In any case, when $\theta = 0$, R will be of perfect MC; while the smaller the value of θ, the more consistent it will be.

As we know, *GCI* is equivalent to *CI* in that it is used to measure the degree of inconsistency of *R*, but it cannot identify inconsistent elements conveniently and intuitively. Therefore, after measuring the degree of inconsistency by *GCI*, if $GCI(R) > \overline{GCI(R)}$, $\Theta = (\theta_{ij})_{n \times n}$ can be used to further locate the FPR multiplicative inconsistent elements. The following example illustrates this process.

Example 1: Assume a DM gives the following FPR (adapted from [15, 34]):

$$R_1 = \begin{bmatrix} 0.5 & 0.7 & 0.9 & 0.5 \\ 0.3 & 0.5 & 0.6 & 0.7 \\ 0.1 & 0.4 & 0.5 & 0.8 \\ 0.5 & 0.3 & 0.2 & 0.5 \end{bmatrix}$$

Algorithm 1 is run to compute the multiplicative inconsistent degree of R_1 and to identify its most MC element. Eq.(8) results in $w = (0.4697, 0.2428, 0.1619, 0.1255)^T$. From Eq.(7), it is noticed that $GCI(R_1) = 1.4581 > 0.3526$, and consequently R_1 is not of acceptable MC. Using Eq.(9) to Eq.(13), the corresponding matrix $\Theta = (\theta_{ij})_{n \times n}$ are obtained and shown in Table II:

TABLE II. The matrix Θ obtained by different definitions of θ_{ij} and δ.

δ / Θ	δ (By Eq.(9))	δ (By Eq.(10))
Θ (By Eq.(11))	$\begin{bmatrix} 0 & 1.3195 & 2.2640 & 2.6391 \\ 1.3195 & 0 & 0.9445 & 1.3195 \\ 2.2640 & 0.9445 & 0 & 2.2640 \\ 2.6391 & 1.3195 & 2.2640 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1.8818 & 6.8668 & 7.8566 \\ 1.8818 & 0 & 0.8920 & 1.8818 \\ 6.8668 & 0.8920 & 0 & 6.8668 \\ 7.8566 & 1.8818 & 6.8668 & 0 \end{bmatrix}$
Θ (By Eq.(12))	$\begin{bmatrix} 0 & 1.6946 & 3.5835 & 3.5835 \\ 1.6946 & 0 & 0.9445 & 1.6946 \\ 3.5835 & 0.9445 & 0 & 3.5835 \\ 3.5835 & 1.6946 & 3.5835 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 2.8717 & 12.8416 & 12.8416 \\ 2.8717 & 0 & 0.8920 & 2.8717 \\ 12.8416 & 0.8920 & 0 & 12.8416 \\ 12.8416 & 2.8717 & 12.8416 & 0 \end{bmatrix}$

$$\begin{matrix} \Theta \\ (\text{By Eq.(13)}) \end{matrix} \quad \begin{bmatrix} 0 & 1.2651 & 1.8397 & 2.4643 \\ 1.2651 & 0 & 0.9445 & 1.2651 \\ 1.8397 & 0.9445 & 0 & 1.8397 \\ 2.4643 & 1.2651 & 1.8397 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1.6005 & 3.3845 & 6.0727 \\ 1.6005 & 0 & 0.8920 & 1.6005 \\ 3.3845 & 0.8920 & 0 & 3.3845 \\ 6.0727 & 1.6005 & 3.3845 & 0 \end{bmatrix}$$

Form Table II, it is obvious that θ_{14} is the largest value, thus r_{14} is treated as the most multiplicative inconsistent element in R_1.

IV. ORDINAL INCONSISTENCY DETECTION FOR FPRs

This section introduces first some basic concepts of OC of FPRs. Then, an algorithm is provided to construct a matrix to locate ordinally inconsistent elements and the ordinal inconsistent matrix of an FPR, respectively. Some properties of the ordinally inconsistent matrix are also investigated.

A. The Ordinal Inconsistency of FPRs

The OC of FPRs directly affects the alternatives ranking. As discussed in Section 2, an FPR that passes the test of consistency (GCI) does not guarantee its OC. Thus, an effective method to determine the level of OC is required.

Xu, et al. [16] proposed the definition of ordinal inconsistency of an FPR as below.

Definition 6 [16]: "FPR $R = (r_{ij})_{n \times n}$ is ordinally inconsistent if it has contradictory elements r_{ij}, r_{ik}, r_{kj} for $i, j, k \in N$, $i \neq j \neq k$ satisfying

$[\, r_{ik} > 0.5, \;\; r_{kj} \geq 0.5, \;\; r_{ij} \leq 0.5 \,]$ or

$[\, r_{ik} \geq 0.5, \;\; r_{kj} > 0.5, \;\; r_{ij} \leq 0.5 \,]$ or

$[\, r_{ik} = 0.5, \;\; r_{kj} = 0.5, \;\; r_{ij} \neq 0.5 \,]$."

Definition 6 is equivalent to Definition 7.

Definition 7: "FPR $R = (r_{ij})_{n \times n}$ is ordinally inconsistent if it has contradictory elements r_{ij}, r_{ik}, r_{kj} for $i, j, k \in N$, $i \neq j \neq k$ satisfying

$[\, r_{ik} < 0.5, \;\; r_{kj} \leq 0.5, \;\; r_{ij} \geq 0.5 \,]$ or

$[\, r_{ik} \leq 0.5, \;\; r_{kj} < 0.5, \;\; r_{ij} \geq 0.5 \,]$ or

$[\, r_{ik} = 0.5, \;\; r_{kj} = 0.5, \;\; r_{ij} \neq 0.5 \,]$."

The following result derives from Definitions 6 and 7:

Theorem 1: FPR $R = (r_{ij})_{n \times n}$ is ordinally inconsistent, if and only if one the following cases is true.

Case a1: If $r_{ik}/r_{ki} > 1$, $r_{kj}/r_{jk} > 1$, then $r_{ij}/r_{ji} < 1$;

Case a2: If $r_{ik}/r_{ki} > 1$, $r_{kj}/r_{jk} > 1$, then $r_{ij}/r_{ji} = 1$;

Case a3: If $r_{ik}/r_{ki} > 1$, $r_{kj}/r_{jk} = 1$, then $r_{ij}/r_{ji} < 1$;

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Case a4: If $r_{ik}/r_{ki} > 1$, $r_{kj}/r_{jk} = 1$, then $r_{ij}/r_{ji} = 1$;

Case a5: If $r_{ik}/r_{ki} = 1$, $r_{kj}/r_{jk} > 1$, then $r_{ij}/r_{ji} < 1$;

Case a6: If $r_{ik}/r_{ki} = 1$, $r_{kj}/r_{jk} > 1$, then $r_{ij}/r_{ji} = 1$;

Case b1: If $r_{ik}/r_{ki} = 1$, $r_{kj}/r_{jk} = 1$, then $r_{ij}/r_{ji} < 1$;

Case b2: If $r_{ik}/r_{ki} = 1$, $r_{kj}/r_{jk} = 1$, then $r_{ij}/r_{ji} > 1$;

Case c1: If $r_{ik}/r_{ki} < 1$, $r_{kj}/r_{jk} < 1$, then $r_{ij}/r_{ji} > 1$;

Case c2: If $r_{ik}/r_{ki} < 1$, $r_{kj}/r_{jk} < 1$, then $r_{ij}/r_{ji} = 1$;

Case c3: If $r_{ik}/r_{ki} < 1$, $r_{kj}/r_{jk} = 1$, then $r_{ij}/r_{ji} > 1$;

Case c4: If $r_{ik}/r_{ki} < 1$, $r_{kj}/r_{jk} = 1$, then $r_{ij}/r_{ji} = 1$;

Case c5: If $r_{ik}/r_{ki} = 1$, $r_{kj}/r_{jk} < 1$, then $r_{ij}/r_{ji} > 1$;

Case c6: If $r_{ik}/r_{ki} = 1$, $r_{kj}/r_{jk} < 1$, then $r_{ij}/r_{ji} = 1$.

Proof. On the one hand, if R is ordinally inconsistent, according to Definition 6 and Definition 7, it is easy to see that one of the above 14 possible cases is true. On the other hand, if one of the **Cases a1-b1** is true, then it is r_{ik}/r_{ki}, r_{kj}/r_{jk}, $r_{ji}/r_{ij} \geq 1$ ($i \neq j \neq k$) (with at least one of them being >1), and therefore it would be r_{ik}, r_{kj}, $r_{ji} \geq 0.5$ (with at least one of them being >0.5). i.e., there would be a directed 3-cycles $x_i \succeq x_k \succeq x_j \succeq x_i$ (with at least one $\succ$ in the cycle) and the FPR would be ordinal inconsistent. If one of the **Cases b2-c6** is true, then it is r_{ki}/r_{ik}, r_{jk}/r_{kj}, $r_{ij}/r_{ji} \geq 1$ ($i \neq j \neq k$) (with at least one of them being >1), which means that r_{ki}, r_{jk}, $r_{ij} \geq 0.5$ (with at least one of them being >0.5), i.e., there would be a direct 3-cycle $x_i \succeq x_j \succeq x_k \succeq x_i$ (with at least one $\succ$ in the cycle) and the FPR would be ordinal inconsistent. □

Theorem 1 can in practice be used to find the contradictory elements of an FPR that lead to directed 3-cycles. It can also be used to propose a measurement of ordinal inconsistency of an FPR R via the following Algorithm 2 designed to compute a symmetrical matrix $\Psi = (\psi_{ij})_{n \times n}$ by detecting all possible combinations of three judgments and analyzing the ordinally inconsistent conditions.

Algorithm 2. Let $R = (r_{ij})_{n \times n}$ be an FPR.

Set $\Psi = \text{zero}(n)$

For all i, j, k ($i \neq j \neq k \neq i$) from 1 to n

If $\log(r_{ij}/r_{ji}) \cdot \log(r_{ik}/r_{ki}) \leq 0$ and $\log(r_{ij}/r_{ji}) \cdot \log(r_{kj}/r_{jk}) < 0$, then

$\quad \psi_{ij} = \psi_{ij} + 1$

If $\log(r_{ij}/r_{ji}) \cdot \log(r_{ik}/r_{ki}) < 0$ and $\log(r_{ij}/r_{ji}) \cdot \log(r_{kj}/r_{jk}) = 0$, then

$$\psi_{ij} = \psi_{ij} + 1$$

If $\log(r_{ik}/r_{ki}) = 0$ and $\log(r_{kj}/r_{jk}) = 0$ and $\log(r_{ij}/r_{ji}) \neq 0$, then

$$\psi_{ij} = \psi_{ij} + 1$$

If $\log(r_{ik}/r_{ki}) \neq 0$ and $\log(r_{kj}/r_{jk}) = 0$ and $\log(r_{ij}/r_{ji}) = 0$, then

$$\psi_{ij} = \psi_{ij} + 1$$

If $\log(r_{ik}/r_{ki}) = 0$ and $\log(r_{kj}/r_{jk}) \neq 0$ and $\log(r_{ij}/r_{ji}) = 0$, then

$$\psi_{ij} = \psi_{ij} + 1$$

If $\log(r_{ij}/r_{ji}) = 0$ and $\log(r_{ik}/r_{ki}) \cdot \log(r_{kj}/r_{jk}) > 0$, then

$$\psi_{ij} = \psi_{ij} + 1$$

End if

End for

Output Ψ

Notice that the smaller the value of ψ_{ij}, the more ordinally consistent the element r_{ij} is. Thus, the largest value in Ψ will correspond to the most ordinally inconsistent element in R, which would need to be rectified to improve the OC of R. Notice that in a directed 3-cycle with one inconsistent arc $x_i \rightarrow x_j$, the reverse is not in G. However, in the process of consistency improvement, due to the reciprocity property we need to change both the value of r_{ij} and r_{ji}. If r_{ij} is an ordinal inconsistent element which forms a directed 3-cycle, both r_{ij} and r_{ji} are in the same 3-cycle. Hence, in the process of improvement, we can consider any one of them.

We use the following example to demonstrate the usefulness of Algorithm 2.

Example 2 (Example 1 continuation):

By Algorithm 2, the matrix $\Psi = (\psi_{ij})_{n \times n}$ is established:

$$\Psi = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

By observing the matrix Ψ, it is noticed that the element r_{41} appears twice in all directed 3-cycles while the rest of values appear once. Therefore, by Theorem 4 there exist the following two directed 3-cycles:

$$L = \{x_1 \rightarrow x_2 \rightarrow x_4 \sim x_1, \ x_1 \rightarrow x_3 \rightarrow x_4 \sim x_1\}.$$

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Then, r_{41} or r_{14} are regarded as the most ordinally inconsistent elements.

B. Overall OC

The ordinal inconsistency of an FPR is measured by the number of directed 3-cycles present in its digraph. The following result provide a convenient relationship between the sum of the elements of matrix Ψ and the number directed 3-cycles present in the digraph of an FPR.

Theorem 2: For FPR R, the sum of all elements in $\Psi = (\psi_{ij})_{n \times n}$ is 6 times the number of directed 3-cycles in G.

Proof. The elements of R in a directed 3-cycle in G are contradictory and therefore by Algorithm 2 it is $\psi_{ij} = \psi_{jk} = \psi_{ki} = \psi_{ji} = \psi_{kj} = \psi_{ik} = 1$. Thus, the sum of all elements in $\Psi = (\psi_{ij})_{n \times n}$ is 6 times the number of directed 3-cycles in G. $\square$

Theorem 2 justifies the below definition of overall OC of an FPR.

Definition 8: The overall OC of an FPR $R = (r_{ij})_{n \times n}$ is:

$$\psi(R) = \frac{1}{6} \sum_{i=1}^{n} \sum_{j=1}^{n} \psi_{ij} \tag{16}$$

The following theorem is provided:

Theorem 3: An FPR $R = (r_{ij})_{n \times n}$ is OC if and only if $\psi(R) = 0$.

Proof: If R is OC, then there is no directed 3-cycles present in its digraph and consequently it will be $\psi(R) = 0$. On the contrary, if $\psi(R) = 0$ then it is $\psi_{ij} = 0$, $\forall i, j \in N$ and therefore no element of R is present in a directed 3-cycle; consequently R is OC.

The larger the value of $\psi(R)$, the less OC R is.

The FPR of Example 2 has an overall OC of $\psi(R_1) = 2$, and it is not OC.

V. AN ALGORITHM FOR IMPROVING CONSISTENCIES

When multiplicative and ordinally inconsistent judgments are detected, the next task is to remove them. In order to achieve this goal, in the following, an automatic procedure to guide the DM on how to revise inconsistent preference values are proposed.

In order to revise the inconsistent element r_{ij}/r_{ji}, all possible indirect judgments $(r_{ik}/r_{ki}) \cdot (r_{kj}/r_{jk})$ are used to derive the following revised value of r_{ij}/r_{ji}

$$\frac{r'_{ij}}{r'_{ji}} = \left(\prod_{k=1, k \neq i, k \neq j}^{n} \frac{r_{ik}}{r_{ki}} \cdot \frac{r_{kj}}{r_{jk}} \right)^{\frac{1}{n-2}} \tag{17}$$

In FPRs, DMs tend to provide their preference information using the discrete scale $S_{[0.1, 0.9]} = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6,$

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$0.7, 0.8, 0.9\}$. However, the revised value r'_{ij} using Eq.(17) is generally not in this scale. Thus, to make the revised values in this scale, values for r'_{ij}/r'_{ji} via Eq.(17) will be matched with the Table III closet to ratio value in the $S_{[0.1,0.9]}$.

TABLE III. THE RATIOS CORRESPONDING TO THE VALUE IN THE SCALE $S_{[0.1,0.9]}$

$\dfrac{r_{ij}}{r_{ji}}$	$\dfrac{0.1}{0.9}$	$\dfrac{0.2}{0.8}$	$\dfrac{0.3}{0.7}$	$\dfrac{0.4}{0.6}$	$\dfrac{0.5}{0.5}$	$\dfrac{0.6}{0.4}$	$\dfrac{0.7}{0.3}$	$\dfrac{0.8}{0.2}$	$\dfrac{0.9}{0.1}$
Ratio	0.1111	0.25	0.4286	0.6667	1	1.5	2.3333	4	9

In particular, when the obtained value r'_{ij}/r'_{ji} is greater than 9, then the following will be set $r'_{ij} = 0.9$; while for a r'_{ij}/r'_{ji} lower than 0.11111, it will be set $r'_{ij} = 0.1$. Meanwhile, r'_{ij}/r'_{ji} is close to 1 it will be $r'_{ij} = 0.6$ when $r'_{ij}/r'_{ji} > 1$, and $r'_{ij} = 0.4$ when $r'_{ij}/r'_{ji} < 1$.

Based on Algorithms 1 and 2, the automatic approach to identify and repair inconsistent judgments is detailed in Algorithm 3 and depicted in Fig. 2.

Algorithm 3. Let $R = (r_{ij})_{n \times n}$ be an FPR and let t be the number of iterations.

Step 1. Let $R^{(t)} = (r_{ij}^{(t)})_{n \times n} = (r_{ij})_{n \times n}$ and $t = 0$.

Step 2. Calculate the priority vector $w^{(t)} = (w_1^{(t)}, w_2^{(t)}, \cdots, w_n^{(t)})^T$ of $R^{(t)}$ by Eq.(8).

Step 3. Construct the matrix $\Psi^{(t)} = (\psi_{ij}^{(t)})_{n \times n}$ by Algorithm 2.

Step 4. Compute $GCI(R^{(t)})$ by Eq.(7) and $\psi(R^{(t)})$ by Eq.(16).

Step 5. $R^{(t)}$ can be classified into the following four types.

Step 5A: If $R^{(t)}$ is both multiplicative and ordinally inconsistent ($GCI(R^{(t)}) > \overline{GCI(R)}$, $\psi(R^{(t)}) \neq 0$), go to Step 6A.

Step 5B: If $R^{(t)}$ is MC but ordinally inconsistent ($GCI(R^{(t)}) \leq \overline{GCI(R)}$, $\psi(R^{(t)}) \neq 0$), go to Step 6B.

Step 5C: If $R^{(t)}$ is OC but multiplicative inconsistent ($\psi(R^{(t)}) = 0$, $GCI(R^{(t)}) > \overline{GCI(R)}$), go to Step 6C.

Step 5D: If $R^{(t)}$ is both MC and OC, that is $GCI(R^{(t)}) \leq \overline{GCI(R)}$, $\psi(R^{(t)}) = 0$, then go to Step 9.

Step 6. The treatments of inconsistencies.

Step 6A: Following Step 5A:

(1) According to Algorithm 1, establish matrix $\Theta^{(t)} = (\theta_{ij}^{(t)})_{n \times n}$.

(2) Find $\psi_{i_o j_o}^{(t)} = \max_{i,j}\{\psi_{ij}^{(t)}\}$ in matrix $\Psi^{(t)} = (\psi_{ij}^{(t)})_{n \times n}$ and $\theta_{i_m j_m}^{(t)} = \max_{i,j}\{\theta_{ij}^{(t)}\}$ in matrix $\Theta^{(t)} = (\theta_{ij}^{(t)})_{n \times n}$.

(3) If the corresponding $r_{i_m j_m}^{(t)}$ and $r_{i_o j_o}^{(t)}$ are the same element (i.e., $i_m = i_o$, $j_m = j_o$), then it is regarded as the most inconsistent element to be adjusted. Otherwise, we automatically select the corresponding $r_{i_o j_o}^{(t)}$ as the most inconsistent element to be adjusted.

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(4) Compute the value of $r_{ij}'^{(t)}/r_{ji}'^{(t)}$ using Eq.(17) and identify the closest ratio to $r_{ij}'^{(t)}/r_{ji}'^{(t)}$ in Table III. Then find the revised value $r_{ij}'^{(t)}$ and go to next step.

Step 6B: Following Step 5B:

(1) Find $\psi_{i_o j_o}^{(t)} = \max_{i,j}\{\psi_{ij}^{(t)}\}$ in matrix $\Psi^{(t)} = (\psi_{ij}^{(t)})_{n\times n}$, then the corresponding $r_{i_o j_o}^{(t)}$ in $R^{(t)}$ is regarded as the most ordinally inconsistent element. If there are two or more most ordinally inconsistent elements, choose any one to be the most ordinally inconsistent one.

(2) Compute the value of $r_{ij}'^{(t)}/r_{ji}'^{(t)}$ using Eq.(17) and identify the closest ratio to $r_{ij}'^{(t)}/r_{ji}'^{(t)}$ in Table III. Then find the revised value $r_{ij}'^{(t)}$ and go to next step.

(3) If the value of $r_{ij}'^{(t)}$ satisfies $\log(r_{ij}^{(t)}/r_{ji}^{(t)}) \cdot \log(r_{ij}'^{(t)}/r_{ji}'^{(t)}) < 0$, go to next step. Otherwise, we set $r_{ij}'^{(t)} = 1 - r_{ij}^{(t)}$, $r_{ji}'^{(t)} = r_{ij}^{(t)}$.

Step 6C: Following Step 5C:

(1) According to Algorithm 1, establish matrix $\Theta^{(t)} = (\theta_{ij}^{(t)})_{n\times n}$.

(2) Find $\theta_{i_m j_m}^{(t)} = \max_{i,j}\{\theta_{ij}^{(t)}\}$ in matrix $\Theta^{(t)} = (\theta_{ij}^{(t)})_{n\times n}$, then the corresponding $r_{i_m j_m}^{(t)}$ in $R^{(t)}$ is regarded as the most multiplicative inconsistent element.

(3) Compute the value of $r_{ij}'^{(t)}/r_{ji}'^{(t)}$ using Eq.(17) and identify the closest ratio value from Table III. Then find the revised value $r_{ij}'^{(t)}$.

(4) If the value of $r_{ij}'^{(t)}$ satisfies $\log(r_{ij}^{(t)}/r_{ji}^{(t)}) \cdot \log(r_{ij}'^{(t)}/r_{ji}'^{(t)}) > 0$, go to Step 7. Otherwise, go back to (2) to search the second largest value in matrix $\Theta^{(t)} = (\theta_{ij}^{(t)})_{n\times n}$ to improve.

Step 7. New improved FPR $R^{(t+1)} = (r_{ij}^{(t+1)})_{n\times n}$, where

$$(r_{ij}^{(t+1)}, r_{ji}^{(t+1)}) = \begin{cases} (r_{ij}'^{(t)}, 1 - r_{ij}'^{(t)}), & \text{if } r_{ij}^{(t)} \text{ is the inconsistent element} \\ (r_{ij}^{(t)}, r_{ji}^{(t)}), & \text{otherwise} \end{cases} \tag{18}$$

Step 8. Let $t = t + 1$, then go back to Step 2.

Step 9. Output t, $R^{(t)}$, $\psi(R^{(t)})$ and $GCI(R^{(t)})$.

Step 10. End.

Remark 4: In Step 6A, after finding the largest value $\psi_{i_o j_o}^{(t)}$ and $\theta_{i_m j_m}^{(t)}$, the most multiplicative inconsistent element $r_{i_m j_m}^{(t)}$ and the most ordinally inconsistent element $r_{i_o j_o}^{(t)}$ in $R^{(t)}$ are identified. If $r_{i_m j_m}^{(t)}$ and $r_{i_o j_o}^{(t)}$ are the same (i, j) element in R,

we naturally adopt the above improved method; Otherwise, priority is given to revise the ordinally inconsistent element because it is considered the minimum requirement for the judgement information not to be illogical. This explains why in Algorithm 3, the ordinally inconsistent element $r_{i_o j_o}^{(t)}$ is selected to be the rectified element when both inconsistencies happen in different fuzzy preference value positions.
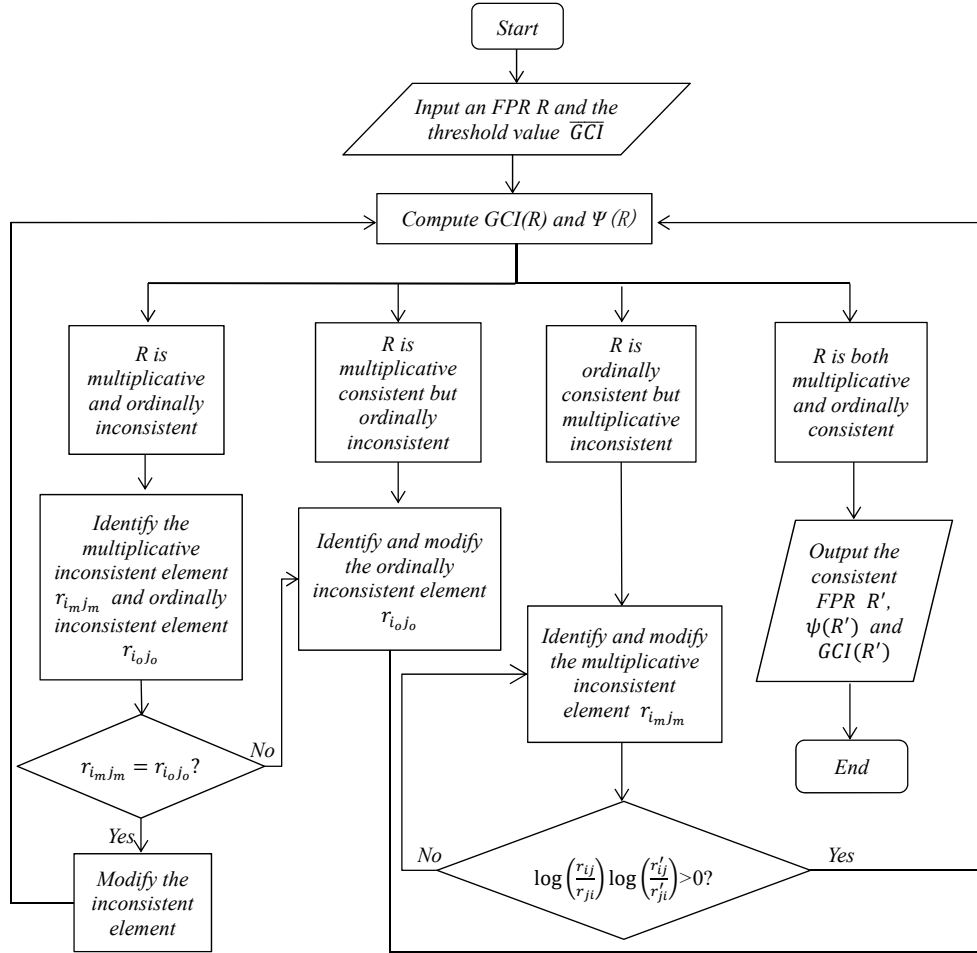

Fig. 2. The process of identifying and repairing inconsistent judgments in Algorithm 3

Remark 5: In Algorithm 3, we identify the most inconsistent element (i.e. search the maximum value in the matrix $\Psi^{(t)} = (\psi_{ij}^{(t)})_{n \times n}$ and $\Theta^{(t)} = (\theta_{ij}^{(t)})_{n \times n}$). If there exist more than one inconsistent element in $R^{(t)}$ with maximum value, then the inconsistent element with value closer to 0.5 is selected. If more than one value results again, then any of them is selected (randomly, for example) as the inconsistent element to adjust.

Remark 6: In Step 6B, when the FPR is ordinally inconsistent but MC, the revised values should reverse the direction of the ordinally inconsistent elements. If the revised values do not satisfy this requirement, i.e., $\log(r_{ij}^{(t)}/r_{ji}^{(t)}) \cdot \log(r_{ij}'^{(t)}/r_{ji}'^{(t)}) < 0$, then we just reverse its direction by changing their original values. If there are two or more

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ordinally inconsistent elements, we choose any one to reverse it, this does not affect the final result. This is also verified in our Monte Carlo simulation experiments. In Step 6C, the FPR is the OC but multiplicative inconsistent (i.e., $\psi(R^{(t)}) = 0$ and $GCI(R^{(t)}) > \overline{GCI(R)}$). Thus, the revised value $r_{ij}'^{(t)}$ should not reverse the direction of the original value $r_{ij}^{(t)}$. In order to achieve this goal, the constraint $\log\left(r_{ij}^{(t)}/r_{ji}^{(t)}\right) \cdot \log\left(r_{ij}'^{(t)}/r_{ji}'^{(t)}\right) > 0$ is imposed, i.e., if $r_{ij}^{(t)} > 0.5$ (i.e., $r_{ij}^{(t)}/r_{ji}^{(t)} > 1$), then the adjusted value $r_{ij}'^{(t)}$ also should be larger than 0.5, i.e., $r_{ij}'^{(t)}/r_{ji}'^{(t)} > 1$. Similarly, if $r_{ij}^{(t)} < 0.5$, (i.e., $r_{ij}^{(t)}/r_{ji}^{(t)} < 1$), then the adjusted value $r_{ij}'^{(t)}$ should be lower than 0.5, i.e., $r_{ij}'^{(t)}/r_{ji}'^{(t)} < 1$. Therefore, the revised element should satisfy $\log\left(r_{ij}^{(t)}/r_{ji}^{(t)}\right) \cdot \log\left(r_{ij}'^{(t)}/r_{ji}'^{(t)}\right) > 0$. Otherwise, the second largest value in matrix $\Theta^{(t)} = (\theta_{ij}^{(t)})_{n\times n}$ is chosen to be improved. This will guarantee the construction of an FPR with acceptable MC and OC.

VI. ILLUSTRATIVE EXAMPLES, COMPARATIVE ANALYSES AND DISCUSSIONS

This section includes two subsections. The first one offers some illustrative examples, and comparisons with the previous methods in literature. The second one provides some simulation experiments to additionally support the effectiveness of the proposed methods.

A. Illustrative Examples and Comparative Analyses

In this section, we will offer four examples, one for each type of inconsistency situation of Algorithm 3 (Fig. 2), to demonstrate the usefulness and feasibility of our developed methods.

- Example 3 is the continuation of Example 1, which has been studied in Xu, et al. [15], and corresponds to the case of an FPR that is both multiplicative and ordinally inconsistent, with same ordinal and multiplicative inconsistent elements.

- Example 4 was investigated by Xia, et al. [13], and corresponds to an FPR that is OC but multiplicative inconsistency.

- Example 5 was examined by Xu, et al. [29]. It corresponds to an FPR that is ordinally inconsistent but with acceptable MC.

- Example 6 is from Xu and Cai [35], and it covers the case of an FPR that is both ordinally and multiplicative inconsistent, with different ordinal and multiplicative inconsistent elements.

Comparisons with the previous approaches in the literature are also provided for each example.

Example 3 (Example 1 continuation): In this example, we will adjust the multiplicative and ordinal inconsistencies of an FPR which has the same ordinal and multiplicative inconsistent elements.

Step 1. Let $R_1^{(0)} = (r_{ij}^{(0)})_{4\times 4} = (r_{ij})_{4\times 4}$, $t = 0$, and $\overline{GCI(R_1)} = 0.3526$.

Step 2. By Eq.(8), $w^{(0)} = (0.4697, 0.2428, 0.1619, 0.1255)^T$. By Eq.(7), it is $GCI(R_1^{(0)}) = 1.4581 > \overline{GCI(R_1)}$. Thus, $R_1^{(0)}$ is multiplicative inconsistent.

Step 3. In Example 2, it was obtained the following the matrix $\Psi^{(0)} = (\psi_{ij}^{(0)})_{4\times 4}$:

$$\Psi^{(0)} = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

Step 4. By Eq.(16), $\psi(R_1^{(0)}) = 2 \neq 0$. It means that $R_1^{(0)}$ is also ordinally inconsistent.

Step 5. According to Algorithm 1, due to different definitions of δ and θ_{ij}, we may get many forms of $\Theta^{(0)} = (\theta_{ij}^{(0)})_{n\times n}$ which are shown in Table I. For the convenience of explanation, we use the following matrix established by Eq.(9) and Eq.(11) as an example.

$$\Theta^{(0)} = \begin{bmatrix} 0 & 1.3195 & 2.2640 & 2.6391 \\ 1.3195 & 0 & 0.9445 & 1.3195 \\ 2.2640 & 0.9445 & 0 & 2.2640 \\ 2.6391 & 1.3195 & 2.2640 & 0 \end{bmatrix}$$

Step 6. Observing the matrix $\Psi^{(0)}$, the largest value in $\Psi^{(0)}$ is $\psi_{14}^{(0)} = \psi_{41}^{(0)} = 2$. Meanwhile, the largest value in the matrix $\Theta^{(0)}$ is $\theta_{14}^{(0)} = \theta_{41}^{(0)} = 2.6391$. Therefore, $r_{14}^{(0)}$ is the most inconsistent element.

Step 7. By Eq.(17), the new value of $r_{14}'^{(0)} / r_{41}'^{(0)} = 14 > 9$ is derived, and therefore it is set $r_{14}'^{(0)} = 0.9$.

Step 8. According to Eq.(18), the following improved FPR $R_1^{(1)}$ is derived:

$$R_1^{(1)} = \begin{bmatrix} 0.5 & 0.7 & 0.9 & 0.9 \\ 0.3 & 0.5 & 0.6 & 0.7 \\ 0.1 & 0.4 & 0.5 & 0.8 \\ 0.1 & 0.3 & 0.2 & 0.5 \end{bmatrix}$$

Step 9. By Eq.(8), $w^{(1)} = (0.6303, 0.1881, 0.1254, 0.0562)^T$, and the ranking of alternatives is $x_1 \succ x_2 \succ x_3 \succ x_4$.

Step 10. By Algorithm 2, the matrix $\Psi^{(1)} = (\psi_{ij}^{(1)})_{4\times 4}$ is:

$$\Psi^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 11. By Eq.(7), $GCI(R_1^{(1)}) = 0.3299 < \overline{GCI(R_1)}$. By Eq.(16), $\psi(R_1^{(1)}) = 0$. Thus, $R_1^{(1)}$ is OC and of acceptable MC.

Through the proposed consistency improvement process, an FPR $R_1^{(1)}$ with OC and acceptable MC is eventually obtained. Furthermore, the ranking of the alternatives that is derived from $R_1^{(1)}$ coincides with the initial ranking result.

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In this example, because there exist equal preferences (i.e. $r_{14} = r_{41} = 0.5$), Ma, et al. [30]'s method fails to improve consistency. Xu, et al. [15]'s method can only remove the ordinal inconsistency because MC is not considered. Furthermore, Xu, et al. [15] derived the following adjusted OC consistent FPR:

$$R_1 = \begin{bmatrix} 0.5 & 0.7 & 0.9 & 0.6 \\ 0.3 & 0.5 & 0.6 & 0.7 \\ 0.1 & 0.4 & 0.5 & 0.8 \\ 0.4 & 0.3 & 0.2 & 0.5 \end{bmatrix}$$

When Eq.(7) is applied, it is observed that $GCI(R_1) = 1.1288 > \overline{GCI(R_1)}$, i.e., R_1 is not of acceptable MC. It is evident that the proposed method in this paper is more efficient than existing ones, because it can improve the OC and MC simultaneously in a single model.

Example 4: Suppose the following an FPR on four alternatives x_1, x_2, x_3 and x_4 (adapted from Xia, et al. [13]):

$$R_2 = \begin{bmatrix} 0.5 & 0.4 & 0.7 & 0.3 \\ 0.6 & 0.5 & 0.6 & 0.8 \\ 0.3 & 0.4 & 0.5 & 0.3 \\ 0.7 & 0.2 & 0.7 & 0.5 \end{bmatrix}$$

In this example, we will deal with an ordinal consistent but multiplicative inconsistent FPR.

Step 1. Let $R_2^{(0)} = (r_{ij}^{(0)})_{4\times4} = (r_{ij})_{4\times4}$, $t = 0$, and $\overline{GCI(R_2)} = 0.3526$.

Step 2. By Eq.(8), the priority vector of $R_2^{(0)}$ is $w^{(0)} = (0.2098, 0.4021, 0.1373, 0.2508)^T$.

Step 3. By Algorithm 2, the following matrix $\Psi = (\psi_{ij})_{n\times n}$ was obtained:

$$\Psi^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 4. By Eq.(7), $GCI(R_2^{(0)}) = 0.6767 > \overline{GCI(R_2)}$. By Eq.(16), the value of $\psi(R_2^{(0)})$ is equal to 0. Thus, $R_2^{(0)}$ is OC but multiplicative inconsistent. Thus, only MC of $R_2^{(0)}$ is to be improved.

Step 5. According to Eq.(9) and Eq.(11), the matrix $\Theta^{(0)} = (\theta_{ij}^{(0)})_{n\times n}$:

$$\Theta^{(0)} = \begin{bmatrix} 0 & 1.3377 & 0.8473 & 1.3377 \\ 1.3377 & 0 & 1.3377 & 1.8281 \\ 0.8473 & 1.3377 & 0 & 1.3377 \\ 1.3377 & 1.8281 & 1.3377 & 0 \end{bmatrix}$$

Step 6. The largest value in $\Theta^{(0)}$ is $\theta_{24}^{(0)} = \theta_{42}^{(0)} = 1.8281$. Thus, $r_{24}^{(0)}$ in $R_2^{(0)}$ is the most multiplicative inconsistent element that requires to be improved.

Step 7. By Eq.(17), $r_{24}'^{(0)} / r_{42}'^{(0)} = 0.6429$ and it is closet to $0.4/0.6$ in Table III. Then $r_{24}'^{(0)} = 0.4$.

Step 8. However, it is $\log(r_{24}^{(0)}/r_{42}^{(0)}) \cdot \log\left(r_{24}'^{(0)}/r_{42}'^{(0)}\right) < 0$, which means that the new value $r_{24}'^{(0)}$ changes the original order.

Thus, the second largest value in matrix $\Theta^{(0)}$ is to be found and improved.

Step 9. The second largest value in matrix $\Theta^{(0)}$ is $\theta_{12}^{(0)} = \theta_{21}^{(0)} = \theta_{14}^{(0)} = \theta_{41}^{(0)} = \theta_{23}^{(0)} = \theta_{32}^{(0)} = \theta_{34}^{(0)} = \theta_{43}^{(0)} = 1.3377$. According to

Remark 5, any of the values $r_{12}^{(0)}$, $r_{14}^{(0)}$, $r_{23}^{(0)}$, $r_{34}^{(0)}$ can be selected as the most multiplicative inconsistent element. Here,

$r_{23}^{(0)}$ is selected.

Step 10. By Eq.(17), we get the value of $r_{23}'^{(0)}/r_{32}'^{(0)} = 5.7155$, which is closest to $0.8/0.2$ in Table III. This means that

$r_{23}'^{(0)} = 0.8$.

Step 11. Because the values of $r_{23}^{(0)}$ and $r_{23}'^{(0)}$ satisfy $\log\left(r_{23}^{(0)}/r_{32}^{(0)}\right) \cdot \log\left(r_{23}'^{(0)}/r_{32}'^{(0)}\right) > 0$, the value of $r_{23}'^{(0)}$ does not

modify the OC.

Step 12. By Eq.(18), the following improved FPR $R_2^{(1)}$ is obtained:

$$R_2^{(1)} = \begin{bmatrix} 0.5 & 0.4 & 0.7 & 0.3 \\ 0.6 & 0.5 & 0.8 & 0.8 \\ 0.3 & 0.2 & 0.5 & 0.3 \\ 0.7 & 0.2 & 0.7 & 0.5 \end{bmatrix}$$

Step 13. By Eq.(8), it is $w^{(1)} = (0.1939, 0.4750, 0.0993, 0.2318)^T$.

Step 14. By Eq.(7), because $GCI(R_2^{(1)}) = 0.3996 > \overline{GCI(R_2)}$, then $R_2^{(1)}$ is still of unacceptable MC. Therefore, the MC

needs further improvement.

Step 15. According to Eq.(9) and Eq.(11), the new matrix $\Theta^{(1)} = (\theta_{ij}^{(1)})_{n \times n}$ is:

$$\Theta^{(1)} = \begin{bmatrix} 0 & 0.9808 & 0.4904 & 1.3377 \\ 0.9808 & 0 & 0.4904 & 1.3377 \\ 0.4904 & 0.4904 & 0 & 0.8473 \\ 1.3377 & 1.3377 & 0.8473 & 0 \end{bmatrix}$$

Step 16. The maximum value in the matrix $\Theta^{(1)}$ is $\theta_{14}^{(1)} = \theta_{41}^{(1)} = \theta_{24}^{(1)} = \theta_{42}^{(1)} = 1.3377$. As per Remark 5, $r_{24}^{(1)}$ in $R_2^{(1)}$ is

selected as the multiplicative inconsistent element.

Step 17. By Eq.(17), it is $r_{24}'^{(1)}/r_{42}'^{(1)} = 1.0498 > 1$, which leads to $r_{24}'^{(1)} = 0.6$.

Step 18. Because the values of $r_{24}^{(1)}$ and $r_{24}'^{(1)}$ satisfy $\log\left(r_{24}^{(1)}/r_{42}^{(1)}\right) \cdot \log\left(r_{24}'^{(1)}/r_{42}'^{(1)}\right) > 0$, $r_{24}'^{(1)}$ does not change the OC.

Step 19. By Eq.(18), the new improved FPR $R_2^{(2)}$ is:

$$R_2^{(2)} = \begin{bmatrix} 0.5 & 0.4 & 0.7 & 0.3 \\ 0.6 & 0.5 & 0.8 & 0.6 \\ 0.3 & 0.2 & 0.5 & 0.3 \\ 0.7 & 0.4 & 0.7 & 0.5 \end{bmatrix}$$

Step 20. By Eq.(8), $w^{(2)} = (0.2017, 0.3867, 0.1034, 0.3082)^T$, i.e., $x_2 \succ x_4 \succ x_1 \succ x_3$.

Step 21. By Eq.(7), $GCI(R_2^{(2)}) = 0.1226 < \overline{GCI(R_2)}$, which means that $R_2^{(2)}$ is of acceptable MC.

Finally, $R_2^{(2)}$ is both MC and OC. Furthermore, the derived ranking of alternatives $x_2 \succ x_4 \succ x_1 \succ x_3$ coincides with the initial ranking.

Xia, et al. [13]'s method requires 4 iterations to derive the following relation of acceptable MC [for $\overline{GCI(R_2)} = 0.3526$, $\theta = 0.1$ (θ is a parameter in Xia, et al. [13]'s method)]:

$$R_2^{(4)} = \begin{bmatrix} 0.5000 & 0.3799 & 0.6685 & 0.3503 \\ 0.6201 & 0.5000 & 0.6536 & 0.7450 \\ 0.3315 & 0.3464 & 0.5000 & 0.3181 \\ 0.6497 & 0.2550 & 0.6819 & 0.5000 \end{bmatrix}, \quad GCI(R_2^{(4)}) = 0.2915 < \overline{GCI(R_2)}.$$

This shows that Xia, et al. [13]'s approach is computationally more expensive than the present method, in addition to modifying all the original preference values except those on the main diagonal. However, the proposed method requires the modification of only 4 original preference values; the improved FPR has a lower $GCI(R_2^{(2)})$ of 0.1226, and therefore it is more MC than the improved FPR derived by Xia, et al. [13].

Example 5: Assume that a DM provides the following preferences over a set of six alternatives (adapted from [29]):

$$R_3 = \begin{bmatrix} 0.5 & 0.7 & 0.3 & 0.7 & 0.3 & 0.7 \\ 0.3 & 0.5 & 0.6 & 0.5 & 0.2 & 0.5 \\ 0.7 & 0.4 & 0.5 & 0.8 & 0.5 & 0.8 \\ 0.3 & 0.5 & 0.2 & 0.5 & 0.2 & 0.5 \\ 0.7 & 0.8 & 0.5 & 0.8 & 0.5 & 0.8 \\ 0.3 & 0.5 & 0.2 & 0.5 & 0.2 & 0.5 \end{bmatrix}$$

In this example, an FPR that is ordinally inconsistent but with acceptable MC needs to be repaired.

Step 1. Let $R_3^{(0)} = (r_{ij}^{(0)})_{6 \times 6} = (r_{ij})_{6 \times 6}$, $t = 0$, and the Table I threshold value $\overline{GCI(R_3)} = 0.37$.

Step 2. By Eq.(8), the priority vector of $R_3^{(0)}$ is $w^{(0)} = (0.1646, 0.1054, 0.2443, 0.0782, 0.3293, 0.0782)^T$. By Eq.(7), $GCI(R_3^{(0)}) = 0.2419 < \overline{GCI(R_3)}$. Thus, $R_3^{(0)}$ is acceptable MC.

Step 3. According to Algorithm 2, the matrix $\Psi^{(0)} = (\psi_{ij}^{(0)})_{6 \times 6}$:

$$\Psi^{(0)} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 4 & 1 & 1 & 1 \\ 1 & 4 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Step 4. By Eq.(16), $\psi(R_3^{(0)}) = 4$. It means that $R_3^{(0)}$ is ordinally inconsistent.

Step 5. In $\Psi^{(0)}$, the largest value is $\psi_{23}^{(0)} = \psi_{32}^{(0)} = 4$, then $r_{23}^{(0)}$ is the most ordinally inconsistent element.

Step 6. By Eq.(17), we have: $r_{23}'^{(0)}/r_{32}'^{(0)} = 0.2315$ and it is closest to $0.2/0.8$ in Table III. Then $r_{23}'^{(0)} = 0.2$.

Step 7. According to Eq.(18), the following improved FPR $R_3^{(1)}$ is derived:

$$R_3^{(1)} = \begin{bmatrix} 0.5 & 0.7 & 0.3 & 0.7 & 0.3 & 0.7 \\ 0.3 & 0.5 & \textcolor{red}{0.2} & 0.5 & 0.2 & 0.5 \\ 0.7 & \textcolor{red}{0.8} & 0.5 & 0.8 & 0.5 & 0.8 \\ 0.3 & 0.5 & 0.2 & 0.5 & 0.2 & 0.5 \\ 0.7 & 0.8 & 0.5 & 0.8 & 0.5 & 0.8 \\ 0.3 & 0.5 & 0.2 & 0.5 & 0.2 & 0.5 \end{bmatrix}$$

Step 8. By Eq.(8), it is $w^{(1)} = (0.1556, 0.0739, 0.3113, 0.0739, 0.3113, 0.0739)^T$.

Step 9. By Algorithm 2, the matrix $\Psi^{(1)} = (\psi_{ij}^{(1)})_{6\times6}$ is:

$$\Psi^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 10. By Eq.(7), $GCI(R_3^{(1)}) = 0.0095 < \overline{GCI(R_3)}$. By Eq.(16), $\psi(R_3^{(1)}) = 0$. Thus, $R_3^{(1)}$ is now both MC and OC.

The ranking of alternatives will be: $x_3 \sim x_5 \succ x_1 \succ x_2 \sim x_4 \sim x_6$.

This example was investigated by Xu, et al. [29], where they derived the following improved FPR:

$$R_3' = \begin{bmatrix} 0.5 & 0.7 & 0.3 & 0.7 & 0.3 & 0.7 \\ 0.3 & 0.5 & \textcolor{red}{0.4} & 0.5 & 0.2 & 0.5 \\ 0.7 & \textcolor{red}{0.6} & 0.5 & 0.8 & 0.5 & 0.8 \\ 0.3 & 0.5 & 0.2 & 0.5 & 0.2 & 0.5 \\ 0.7 & 0.8 & 0.5 & 0.8 & 0.5 & 0.8 \\ 0.3 & 0.5 & 0.2 & 0.5 & 0.2 & 0.5 \end{bmatrix}$$

By Eq. (7), the weighting vector of R_3' is $w = (0.1611, 0.0901, 0.2736, 0.0765, 0.3222, 0.0765)^T$ and it has $GCI(R_3') = 0.0837$. Then the ranking of alternatives for R_3' would be: $x_5 \succ x_3 \succ x_1 \succ x_2 \succ x_4 \sim x_6$. As we can see in R_3, the original

indifference between pairs of alternatives (for example between x_5 and x_3: $x_5 \sim x_3$, $r_{35} = 0.5$) is kept by the proposed method but not by Xu, et al. [29]'s method. Again, this example shows that the proposed method is able to improve the OC and the MC simultaneously. After adjusting the inconsistent elements, the inconsistent elements are eliminated, and the degree of *GCI* is greatly improved from 0.2419 to 0.0095.

Example 6: Suppose a DM gives an FPR on the set of 6 alternative as follows (adapted from [35]):

$$R_4 = \begin{bmatrix} 0.5 & 0.3 & 0.3 & 0.7 & 0.8 & 0.5 \\ 0.7 & 0.5 & 0.2 & 0.7 & 0.8 & 0.6 \\ 0.7 & 0.8 & 0.5 & 0.7 & 0.7 & 0.8 \\ 0.3 & 0.3 & 0.3 & 0.5 & 0.9 & 0.7 \\ 0.2 & 0.2 & 0.3 & 0.1 & 0.5 & 0.4 \\ 0.5 & 0.4 & 0.2 & 0.3 & 0.6 & 0.5 \end{bmatrix}$$

In this example, we will rectify the multiplicative and ordinal inconsistencies of an FPR that has different ordinal and multiplicative inconsistent elements.

Step 1. Let $R_4^{(0)} = (r_{ij}^{(0)})_{6 \times 6} = (r_{ij})_{6 \times 6}$, $t = 0$, and Table I value $\overline{GCI(R_4)} = 0.37$.

Step 2. By Eq.(8), the priority vector of $R_4^{(0)}$ is $w^{(0)} = (0.1548, 0.2007, 0.3430, 0.1538, 0.0501, 0.0975)^T$. By Eq.(7), $GCI(R_4^{(0)}) = 0.5051 > \overline{GCI(R_4)}$, which means that is not of acceptable MC.

Step 3. According to Algorithm 2, the matrix $\Psi^{(0)} = (\psi_{ij}^{(0)})_{6 \times 6}$:

$$\Psi^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 4. By Eq.(16), $\psi(R_4^{(0)}) = 1$. It means that $R_4^{(0)}$ is also ordinally inconsistent.

Step 5. By Eq.(9) and Eq.(11), the matrix $\Theta^{(0)} = (\theta_{ij}^{(0)})_{n \times n}$ is derived:

$$\Theta^{(0)} = \begin{bmatrix} 0 & 0.8807 & 1.0397 & 1.2619 & 1.2182 & 0.9141 \\ 0.8807 & 0 & 1.2758 & 1.2952 & 1.2515 & 0.6779 \\ 1.0397 & 1.2758 & 0 & 1.1848 & 1.6133 & 0.5493 \\ 1.2619 & 1.2952 & 1.1848 & 0 & 1.6145 & 1.0591 \\ 1.2182 & 1.2515 & 1.6133 & 1.6145 & 0 & 0.8613 \\ 0.9141 & 0.6779 & 0.5493 & 1.0591 & 0.8613 & 0 \end{bmatrix}$$

Step 6. The largest value in $\Psi^{(0)}$ is $\psi_{14}^{(0)} = \psi_{41}^{(0)} = \psi_{16}^{(0)} = \psi_{61}^{(0)} = \psi_{46}^{(0)} = \psi_{64}^{(0)} = 1$. Meanwhile, the largest value in the matrix $\Theta^{(0)}$ is $\theta_{45}^{(0)} = \theta_{54}^{(0)} = 1.6145$. Because $r_{16}^{(0)}$ and $r_{45}^{(0)}$ are different elements, $r_{16}^{(0)}$ is regarded as the most inconsistent element.

Step 7. By Eq.(17), $r_{16}'^{(0)}/r_{61}'^{(0)} = 2$ and it is closest to $0.7/0.3$ in Table III. Then $r_{16}'^{(0)} = 0.7$.

Step 8. According to Eq.(18), the improved FPR $R_4^{(1)}$ is obtained:

$$R_4^{(1)} = \begin{bmatrix} 0.5 & 0.3 & 0.3 & 0.7 & 0.8 & {\color{red}0.7} \\ 0.7 & 0.5 & 0.2 & 0.7 & 0.8 & 0.6 \\ 0.7 & 0.8 & 0.5 & 0.7 & 0.7 & 0.8 \\ 0.3 & 0.3 & 0.3 & 0.5 & 0.9 & 0.7 \\ 0.2 & 0.2 & 0.3 & 0.1 & 0.5 & 0.4 \\ {\color{red}0.3} & 0.4 & 0.2 & 0.3 & 0.6 & 0.5 \end{bmatrix}$$

Step 9. By Eq.(8), it is $w^{(1)} = (0.1248, 0.1994, 0.3408, 0.1883, 0.0498, 0.0969)^T$. By Eq.(7), it is $GCI(R_4^{(1)}) = 0.4747$

$> \overline{GCI(R_4)}$ and it is still of not acceptable MC.

Step 10. By Algorithm 2, the matrix $\Psi^{(1)} = (\psi_{ij}^{(1)})_{6\times 6}$ is:

$$\Psi^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 11. By Eq.(16), $\psi(R_4^{(1)}) = 0$. Thus, $R_4^{(1)}$ is OC.

Step 12. By Eqs.(9) and (11), the matrix $\Theta^{(1)} = (\theta_{ij}^{(1)})_{n\times n}$ is constructed:

$$\Theta^{(1)} = \begin{bmatrix} 0 & 1.0925 & 0.9820 & 1.0500 & 1.0063 & 0.6446 \\ 1.0925 & 0 & 1.2758 & 1.2952 & 1.2515 & 0.8898 \\ 0.9820 & 1.2758 & 0 & 1.1848 & 1.6133 & 0.4916 \\ 1.0500 & 1.2952 & 1.1848 & 0 & 1.6145 & 0.8473 \\ 1.0063 & 1.2515 & 1.6133 & 1.6145 & 0 & 0.6495 \\ 0.6446 & 0.8898 & 0.4916 & 0.8473 & 0.6495 & 0 \end{bmatrix}$$

Step 13. The largest value in the matrix $\Theta^{(1)}$ is $\theta_{45}^{(1)} = 1.6145$. Thus, the corresponding $r_{45}^{(1)}$ in $R_4^{(1)}$ is the multiplicative

inconsistent element.

Step 14. By Eq.(17), $r_{45}'^{(1)}/r_{54}'^{(1)} = 1.7908$, which is closest to $0.6/0.4$ in Table III. Then $r_{45}'^{(1)} = 0.6$.

Step 15. $r_{45}^{(1)}$ and $r_{45}'^{(1)}$ satisfy $\log(r_{45}^{(1)}/r_{54}^{(1)}) \cdot \log(r_{45}'^{(1)}/r_{54}'^{(1)}) > 0$, hence the value of $r_{45}'^{(1)}$ does not modify the OC.

Step 16. By Eq.(18), the new improved FPR $R_4^{(2)}$ is:

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$$R_4^{(2)} = \begin{bmatrix} 0.5 & 0.3 & 0.3 & 0.7 & 0.8 & \textcolor{red}{0.7} \\ 0.7 & 0.5 & 0.2 & 0.7 & 0.8 & 0.6 \\ 0.7 & 0.8 & 0.5 & 0.7 & 0.7 & 0.8 \\ 0.3 & 0.3 & 0.3 & 0.5 & \textcolor{red}{0.6} & 0.7 \\ 0.2 & 0.2 & 0.3 & \textcolor{red}{0.4} & 0.5 & 0.4 \\ \textcolor{red}{0.3} & 0.4 & 0.2 & 0.3 & 0.6 & 0.5 \end{bmatrix}$$

Step 17. By Eq.(8), it is $w^{(2)} = (0.1803, 0.2031, 0.3471, 0.1155, 0.0684, 0.0856)^T$. By Eq.(7), $GCI(R_4^{(2)}) = 0.3030$

$< \overline{GCI(R_4)}$, and $R_4^{(2)}$ is of acceptable MC. Thus, $R_4^{(2)}$ is MC and OC.

The following ranking of alternatives is obtained: $x_3 \succ x_2 \succ x_1 \succ x_4 \succ x_6 \succ x_5$, which coincides with the improved FPR

$R_4^{(2)}$.

According to Xu, et al. [29] 's method, the following induced matrix C' and improved matrix R_4' are obtained:

$$C' = \begin{bmatrix} 1 & 2.1667 & 1.5764 & 0.5539 & 1.4152 & 1.4965 \\ 0.5893 & 1 & 3.0595 & 0.8416 & 1.7153 & 1.7346 \\ 1.3036 & 0.5521 & 1 & 1.5299 & 4.0714 & 1.0208 \\ 2.5910 & 2.5514 & 2.1733 & 1 & 0.4802 & 0.9762 \\ 1.7169 & 1.8757 & 0.4826 & 4.0119 & 1 & 1.0142 \\ 1.0077 & 0.7944 & 1.1859 & 1.8349 & 1.4193 & 1 \end{bmatrix}; \ R_4' = \begin{bmatrix} 0.5 & 0.3 & 0.3 & 0.7 & 0.8 & 0.6 \\ 0.7 & 0.5 & 0.2 & 0.7 & 0.8 & 0.6 \\ 0.7 & 0.8 & 0.5 & 0.7 & 0.9 & 0.8 \\ 0.3 & 0.3 & 0.3 & 0.5 & 0.9 & 0.7 \\ 0.2 & 0.2 & 0.1 & 0.1 & 0.5 & 0.4 \\ 0.4 & 0.4 & 0.2 & 0.3 & 0.6 & 0.5 \end{bmatrix}.$$

The priority vector of R_4' is $w = (0.1532, 0.1857, 0.3974, 0.1423, 0.0370, 0.0843)^T$ and $GCI(R_4') = 0.3097 < \overline{GCI(R_4)}$. In

each iteration, a pair of elements is adjusted. Observing the induced matrix C', since $c_{35} \neq c_{53}$, r_{35} is regarded as the most

multiplicative inconsistent element while r_{53} is not the most multiplicative inconsistent element. Thus, it is unreasonable to

adjust r_{35} and r_{53}. However, in Algorithm 2, Θ is a symmetrical matrix, which can easily identify a pair of multiplicative

inconsistent elements. Moreover, the proposed method derives a modified FPR with $GCI(R_4^{(2)}) = 0.3030$, and therefore

with better MC. It means that adjusting r_{45} is more effective than adjusting r_{35}.

In comparison with Xia, et al. [13]'s method, for values $\overline{GCI(R_4)} = 0.37$, $\theta = 0.1$ (θ is a parameter in Xia, et al. [13]'s

method), the following matrix is derived:

$$R_4^{(2)} = \begin{bmatrix} 0.5000 & 0.3240 & 0.3021 & 0.6654 & 0.7920 & 0.5220 \\ 0.6760 & 0.5000 & 0.2272 & 0.6764 & 0.8000 & 0.6144 \\ 0.6979 & 0.7728 & 0.5000 & 0.6981 & 0.7411 & 0.7960 \\ 0.3346 & 0.3236 & 0.3019 & 0.5000 & 0.8800 & 0.6841 \\ 0.2080 & 0.2000 & 0.2589 & 0.1200 & 0.5000 & 0.3882 \\ 0.4780 & 0.3856 & 0.2040 & 0.3159 & 0.6118 & 0.5000 \end{bmatrix}$$

This matrix has a larger value $GCI(R_4^{(2)}) = 0.3312 < \overline{GCI(R_4)}$ than the corresponding one for the matrix derived from

the proposed method. In addition, the proposed method retains most of the original judgments. It only needs to adjust 4

elements. Therefore, the proposed method is not only simpler and effective but also has better consistency and retains more

original preference information.

B. Monte Carlo Simulation and Further Discussion

This subsection further discusses the effectiveness of the proposed methods. Monte Carlo simulations depends on the

random sampling to generate the results, and is an important method to model uncertainty phenomena. In order to do this,

1,000 FPRs for different dimension ranging from 3 to 9 are randomly generated and their consistencies (ordinal and/or

multiplicative) are improved using the proposed method. When randomly generating FPRs, the continuous domain (0,1) is

used and therefore the values of entries will not be in the discrete scale $S_{[0.1,0.9]}$. Thus, in the adjustment process, the

adjusted values are in the domain (0,1). The average numbers of iterations of Algorithm 3 with different thresholds for

different dimension values *n* (of Table I) are provided in Table IV and plotted in Fig. 13. For $n=3$, the average number of

iterations is lower than 1, denoting that some of the randomly generated FPRs are of OC and MC (these FPRs are not needed

to be revised); otherwise, only one iteration is required to achieve the OC and MC. From Table IV and Fig. 3, it is clear that

the average number of iterations increases as *n* increases. The percentages of the entries changed in the matrices are listed in

Table V against their dimension value *n*, with an increasing tendency observed. In any case, the majority of the original

values are not modified (largest percentage change value is 36.56%) with the proposed method. This also indicates the

effectiveness of the proposed method.

TABLE IV. THE AVERAGE ITERATIONS FOR DIFFERENT *n*

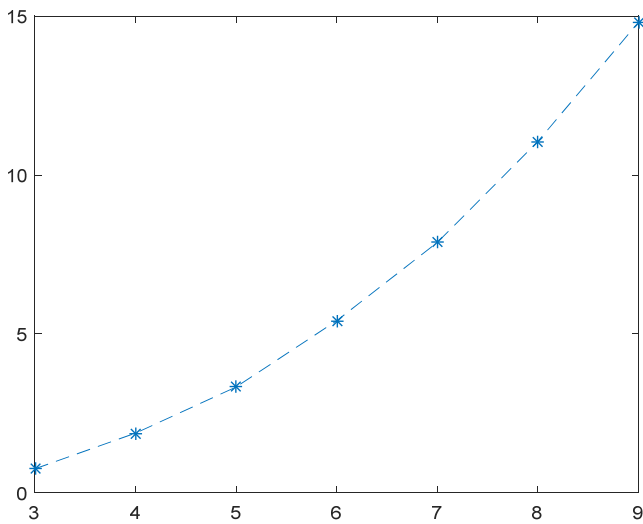| *n*	3	4	5	6	7	8	9
iterations	0.758	1.879	3.332	5.417	7.88	11.057	14.807

Fig. 3. The average number of iterations of the proposed method

TABLE V. THE PERCENTAGE OF THE VALUES CHANGED FOR DIFFERENT *n*

n	3	4	5	6	7	8	9
percentage	16.84%	23.49%	26.66%	30.09%	32.16%	34.55%	36.56%

VII. CONCLUSIONS

In this paper, algorithms to rectify the multiplicative and ordinal inconsistencies for FPRs have been proposed. Firstly, we have analyzed the MC property of FPRs. Geometric consistency has been applied to measure the MC degree of an FPR. The approximated thresholds of *GCI* for MPRs were extended to the case of FPRs to test whether an FPR is of acceptable MC. To improve the level of MC:

- First, the degree of inconsistency is measured with *GCI*, and then a matrix $\Theta = (\theta_{ij})_{n \times n}$ is established to identify the multiplicative inconsistent elements.

- Secondly, we presented the ordinal inconsistent conditions of FPRs. An effective algorithm to find the ordinally inconsistent elements was developed. Meanwhile, we proposed an approach based on the matrix $\psi(R)$ to measure the level of OC for FPRs.

- Thirdly, we designed an integrated method to improve simultaneously the OC and MC of an FPR.

- Finally, we demonstrated the validity of our proposed integrated consistency improvement method with its superior performance against existing methods using both examples that represent the four different scenarios of ordinal and multiplicative inconsistencies and simulation experiments. The following advantages noticed:

(1) Only the MC is considered in [13], and only the OC is considered in [15]. Our proposed method can improve multiplicative and OC simultaneously.

(2) Our method is simple and straightforward. In the process of improving of MC and OC, most of the original judgements are kept unchanged. This is possible because, at each iteration of the proposed method, inconsistent elements are accurately located and only one pair of preference values are modified. In contrast, other existing methods [13, 30, 32] change all the original judgments given by DM.

(3) The developed method can be applied not only to strict FPRs but also to non-strict FPRs. However, Ma, et al. [30] can only be applied to strict FPRs.

In the future, we will study the MC and OC for incomplete FPRs [25, 36-41] and will consider the application of the proposed method in group consensus models [42-50].

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