Fuzzy Efficiency Measures in Data Envelopment Analysis Using Lexicographic Multiobjective Approach

Adel Hatami-Marbini*
Department of Strategic Management and Marketing
Leicester Business School
De Montfort University
Hugh Aston Building, The Gateway, Leicester LE1 9BH, UK
E-mail: adel.hatamimarbini@dmu.ac.uk

Ali Ebrahimnejad
Department of Mathematics,
Qaemshahr Branch
Islamic Azad University
Qaemshahr, Iran
E-mail: a.ebrahimnejad@srbiau.ac.ir

Sebastián Lozano
Department of Industrial Management,
University of Seville, Spain
E-mail: slozano@us.es

* *Corresponding author*

Fuzzy Efficiency Measures in Data Envelopment Analysis Using Lexicographic Multiobjective Approach

Abstract

There is an extensive literature in data envelopment analysis (DEA) aimed at evaluating the relative efficiency of a set of decision-making units (DMUs). Conventional DEA models use definite and precise data while real-life problems often consist of some ambiguous and vague information, such as linguistic terms. Fuzzy sets theory can be effectively used to handle data ambiguity and vagueness in DEA problems. This paper proposes a novel fully fuzzified DEA (FFDEA) approach where, in addition to input and output data, all the variables are considered fuzzy, including the resulting efficiency scores. A lexicographic multi-objective linear programming (MOLP) approach is suggested to solve the fuzzy models proposed in this study. The contribution of this paper is fivefold: 1) both fuzzy Constant and Variable Returns to Scale models are considered to measure fuzzy efficiencies; 2) a classification scheme for DMUs, based on their fuzzy efficiencies, is defined with three categories; 3) fuzzy input and output targets are computed for improving the inefficient DMUs; 4) a super-efficiency FFDEA model is also formulated to rank the fuzzy efficient DMUs; and 5)the proposed approach is illustrated, and compared with existing methods, using a dataset from the literature.

Keywords: Data envelopment analysis; Fuzzy mathematical programming; Lexicographic multi-objective linear programming; Fuzzy targets; Super-efficiency

1. Introduction

Data envelopment analysis (DEA), initially introduced by Charnes et al. (1978), is a widely used mathematical programming technique for estimating the frontier production for peer decision making units (DMUs) with multiple inputs and multiple outputs. Charnes et al. (1978) model, commonly referred to as CCR model, assumed constant returns to scale (CRS). Banker et al. (1984) developed the so-called BCC model for evaluating the performance of units in the case of variable returns to scale (VRS). The units are assumed to operate homogenously under similar conditions. Based on the observed data and some preliminary assumptions, DEA is able to establish an empirical efficient frontier. If a DMU lies on the frontier, it is said to be efficient, otherwise it is said to be inefficient. Computing the distance to the efficient frontier (using some metric and a certain orientation) DEA provides the relative efficiency score, as well as a target for improving for each inefficient DMU. In practice, the efficiency score might be considered as a performance indicator for continuous improvement while the target informs about the amount (percentage) by which an inefficient DMU should decrease its inputs and/or increase its outputs to become efficient. Moreover, the reference set of efficient DMUs with which the target is constructed represents best practice models that act as benchmarks to the inefficient DMU.

In conventional DEA models, such as CCR and BCC, the observed input and output data of the DMUs are often not known precisely. That may not be always the case in the real world. Imprecise evaluations may be the result of unquantifiable, incomplete and non-obtainable information. Imprecise data representation with interval, ordinal, and ratio interval data was initially proposed by Cooper et al. (1999, 2001a, 2001b), leading to so-called interval DEA (IDEA) to study the uncertainty in DEA. Numerous other researchers have also proposed and applied different DEA models with interval data (e.g. Despotis and Smirlis 2002, Entani et al. 2001, Wang et al. 2005, Shokouhi et al. 2010, 2014, Hatami-Marbini et al. 2014). However, decision makers often prefer using linguistic phrases and expressions such as ‘‘large’’ profit or ‘‘low’’ inventory in their communication, information that cannot be handled by IDEA. In general, observations are typically divided into quantitative and qualitative. Quantitative observed data are often exact, precise and specific values while qualitative data, such as “good”, “better” and “very good”, are often imprecise or vague values. The distances between qualitative data are not clear and it does not make sense to use the ordinal scaling to measure the preference linguistic terms that arise in natural language.

Fuzzy sets theory, initiated by Zadeh(1965), is a well-known tool to represent this type of data. Compared to traditional binary sets ("true" or "false", 0 or 1) fuzzy sets are based on the concept of "degree of membership", that ranges between zero and one. Natural language is not straightforwardly transformed into the absolute terms of 0 and 1. Fuzzy logic considers the membership values 0 and 1 as extreme cases but also considers possible intermediate membership values between 0 and 1. Hence, fuzzy sets have the capability of describing qualitative data as fuzzy numbers.

Numerous fuzzy sets-based methods have been proposed in DEA in the last two decades. Generally, the linear programming (LP) DEA models are converted to fuzzy LP (FLP) models when the input and/or output data are characterized by fuzzy numbers. The existing fuzzy DEA (FDEA) methods can be classified into six main categories, namely, the tolerance approach, the α-level based approach, the fuzzy ranking approach, the possibility approach, the fuzzy arithmetic, and the fuzzy random/type-2 (Hatami-Marbini et al. 2011a, Emrouznejad et al. 2014).

The tolerance approach (e.g. Sengupta1992) was the first FDEA model that used the concept of fuzziness in DEA modeling by defining tolerance levels on constraint violations. The limitation behind the tolerance approach is related to the design of a DEA model with a fuzzy objective function and fuzzy constraints which may or may not be satisfied (Triantis and Girod 1998).

The α-level approach is probably the most popular FDEA model in the literature. This approach generally tries to transform the FDEA model into a pair of parametric programs for each α-level. Kao and Liu (2000), one of the most cited studies in the α-level approach's category, used Chen and Klein (1997) method for ranking fuzzy numbers to convert the FDEA model to a pair of parametric mathematical programs for the given level of α. Saati et al. (2002) proposed a fuzzy CCR model as a possibilistic programming problem and changed it into an interval programming problem by means of the α-level based approach. Thereupon, some fuzzy DEA-based extension has been done using Saati et al. (2002) method such as a four-phase fuzzy DEA framework based on the theory of displaced ideal (Hatami-Marbini et al. 2010)or a positive-normative use of fuzzy logic in a NATO enlargement application (Hatami-Marbini et al. 2013). The α-level approach is also the one generally used in network DEA (e.g. Kao and Liu 2011, Kao and Lin 2012, Lozano 2014a, 2014b).The fuzzy ranking approach category is composed of FDEA models developed based on distinctive fuzzy ranking methods. Guo and

Tanaka (2011) was the first to develop a fuzzy CCR model based on the fuzzy ranking approach. Different fuzzy ranking methods may lead to different efficiency assessments. Hatami-Marbini et al. (2011b) proposed a fully fuzzified CCR model to get the fuzzy efficiency of the DMUs where the input-output data as well as their weights are characterized by fuzzy numbers.

The ‘‘possibility approach’’ and the ‘‘credibility approach’’ to FDEA mainly stemmed from Lertworasirikul et al. (2003), which modeled the uncertainty in fuzzy objective function and fuzzy constraints with possibility measures from both optimistic and pessimistic viewpoints.

In the fuzzy arithmetic category, Wang et al. (2009) argued that a fuzzy fractional programming in the dual FDEA model cannot simply be transformed into a LP model using conventional methods. They therefore centered on the fuzzy fractional programming form of CCR model and transformed the multiplier formulation of the fuzzy CCR model into three LP models to obtain the fuzzy efficiency of the DMUs.

In the fuzzy random/type-2 category, Qin et al. (2009) presented a DEA model with type-2 fuzzy inputs and outputs solved in two steps. First, they exploited a reduction method for type-2 fuzzy variables based on the expected value of a fuzzy variable, and then they built a FDEA model with the obtained fuzzy variables. Qin and Liu (2010) developed a fuzzy random DEA (FRDEA) model where randomness and fuzziness exist simultaneously. The authors characterized the fuzzy random data with known possibility and probability distributions. Tavana et al. (2012) also introduced three different FDEA models consisting of probability-possibility, probability-necessity and probability-credibility constraints in which input and output data entailed fuzziness and randomness at the same time.

In another category of fuzzy DEA models are those that make use of geometric properties. Thus, Biondi Neto et al. (2011) developed a method to generate fuzzy efficient frontier by the use of interval DEA frontier when a single interval input or output presents a certain degree of uncertainty. The authors used a geometrical and algebraic approach to obtain a membership degree of each DMU in lieu of its efficiency score. In the same line, several researchers have defined the fuzzy version of the production possibility set (PPS) in which all production plans have different degrees of membership (Raei Nojehdehi et al. 2011, 2012, 2013; Bagherzadeh Valami et al. 2013). To do so, the authors first use a geometrical approach to acquire the membership function of fuzzy PPS and then transform the geometrical terms into the algebraic expression using some basic relationships of DEA models.

Apart from the tolerance approach, which exploits the fuzziness concept, FDEA models are generally solved as *FLP models* with fuzzy coefficients (i.e., fuzzy input-output data) and crisp decision variables. Since FDEA models take the form of fuzzy LP (FLP) problems, the different FDEA approaches have been developed as different ways of solving the corresponding FLP models. In general, FLP problems can be classified into the following six categories to handle imprecise data:

1) FLP models when decision variables and the right-hand-side of the constraints are characterized by fuzzy numbers (e.g. Mahdavi-Amiri and Nasseri 2007);
2) FLP models when the coefficients of the decision variables in the objective function are characterized by fuzzy numbers (e.g. Wu 2008);
3) FLP models when the coefficients of the decision variables in the constraints and the right-hand-side of the constraints are characterized by fuzzy numbers(e.g. Liu 2001);
4) FLP models when the decision variables, the coefficients of the decision variables in the objective function and the right-hand-side of the constraints are characterized by fuzzy numbers (e.g. Ganesan and Veeramani 2006);
5) FLP models when the coefficients of the decision variables in the objective function, the coefficients of the decision variables in the constraints and the right-hand-side of the constraints are characterized by fuzzy numbers(e.g. Mahdavi-Amiri and Nasseri 2006);
6) FLP models when all of the parameters and variables are characterized by fuzzy numbers (e.g. HosseinzadehLotfi et al. 2009).

From the FDEA literature reviewed, with exceptions like Hatami-Marbini et al. (2011b), existing FDEA models belonging to categories (2) and (3) by considering fuzziness only in the input and output data, i.e. in the coefficients of the objective function and constraints of the DEA model. Therefore, the fuzzy aspect of performance evaluation is partially lost and the assessment process is solely limited to crisp decisions. Further, in many real life cases providing a crisp efficiency in the presence of vagueness and imprecise data such as linguistic terms is not rational for a decision-maker. If the use of precise efficiencies is not appropriate due to uncertainties in the evaluated system, it may always be desirable to provide fuzzy efficiencies by taking into consideration in such situations. Using fuzzy decision variables makes sense because, as Tanaka et al. (2000) argued, the real-world decisions are complex and there is usually a functional hierarchy that involves crisp decisions at the lower level but these are derived from fuzzy decisions at the upper decision levels. This happens also in FDEA, since DEA is basically a data-driven technique and, therefore, assessing the efficiency of a set of DMUs based on uncertain data is a complex issue that can be approached, in the first instance, as a fuzzy decision problem. Our aim in this study is more in line with Stanciulescu et al. (2003), who argued that, although it is easier and more convenient to consider crisp variables in FLP, it is preferable from a modeling point of view, to consider fuzzy decision variables. We thus propose the fuzzy efficiencies within a FDEA frame by defining fuzzy decision variables in addition to fuzzy inputs and outputs that can fully reflect, not hide, the uncertainty present in the problem, which allows the decision maker to arrive at a more meaningful (and valid) final results. Such a fuzzy efficiency will provide more detailed outcomes and flexibility with a certain degree of membership to the decision maker. As far as we know, no study in the literature has addressed this challenging research gap in the existing FDEA models.

Whereas economists are more interested in the DEA formulation in "envelopment form" because of estimating production technology using some axioms most FDEA models in the literature have been developed based on the "multiplier formulation". In this paper, both fuzzy Constant and Variable Returns to Scale models in "envelopment form" are proposed to measure fuzzy efficiencies. The resulting fuzzy LP model, with just one fuzzy variable in the objective function, is converted into a multi-objective linear programming (MOLP) problem which can be solved using a lexicographic approach. The proposed approach classifies DMUs into three distinct categories based on their fuzzy efficiencies. Fuzzy input and output targets for improving the inefficient DMUs are also computed. In addition, a super-efficiency FDEA model is formulated to rank the fuzzy efficient DMUs.

The outline of this paper is as follows. In Section 2, we provide an overview of fuzzy set theory and the original DEA models. In Section 3, a fully fuzzified DEA (FFDEA) approach based on the CRS and VRS models is proposed while in Section 4 we present the corresponding super-efficiency FFDEA model. Section 5 discusses the computational complexity of the proposed models compared with the existing methods in the literature. In Section 6, we use two dataset from the literature to illustrate the validity and applicability of the proposed approach, comparing it with several existing methods. In Section 7, conclusions are drawn and further research is envisaged.

2. Preliminaries

2.1. Fuzzy set theory

Since the seminal work of Zadeh(1967) the theory of fuzzy sets has advanced in multiple directions and in many disciplines, such as artificial intelligence, decision theory, expert systems, management science, and operations research, as an effective way of modeling the uncertainty present in the real world. In this section, we review some basic definitions of fuzzy set theory (e.g. Zimmerman 2001).

Definition 2.1: A fuzzy set $\tilde{a}$, defined on universal set of real numbers $\Re$, is said to be a fuzzy number if its membership function has the following characteristics:

i) $\tilde{a}$ is convex, i.e. $\forall x, y \in \Re, \forall \lambda \in [0,1], \mu_{\tilde{a}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{a}}(y)\}$,

ii) $\tilde{a}$ is normal, i.e., $\exists \bar{x} \in \Re;\ \mu_{\tilde{a}}(\bar{x}) = 1$,

iii) $\mu_{\tilde{a}}$ is piecewise continuous.

Definition 2.2: A fuzzy number $\tilde{a} = (a_1, a_2, a_3, a_4)$ is said to be a trapezoidal fuzzy number if its membership function is given as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & a_1 \leq x \leq a_2 \\ 1 & a_2 \leq x \leq a_3 \\ \dfrac{a_4 - x}{a_4 - a_3} & a_3 \leq x \leq a_4 \\ 0 & otherwise \end{cases}$$

The set of all these trapezoidal fuzzy numbers is denoted by $TF(\Re)$. A trapezoidal fuzzy number, $\tilde{a} = (a_1, a_2, a_3, a_4)$, reduces to a real number a if $a_1 = a_2 = a_3 = a_4 = a$. Conversely, a real number a can be considered as a trapezoidal fuzzy number $\tilde{a} = (a, a, a, a)$.

Definition 2.3: A trapezoidal fuzzy number $\tilde{a} = (a_1, a_2, a_3, a_4)$ is said to be non-negative (respectively positive) if and only if $a_1 \geq 0$ (respectively $a_1 > 0$). The sets of non-negative and positive trapezoidal fuzzy numbers are denoted by $TF(\Re)^+$ and $TF(\Re)^{++}$, respectively.

Definition 2.4: Two trapezoidal fuzzy numbers $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$ are said to be equal, $\tilde{a} = \tilde{b}$, if and only if $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$ and $a_4 = b_4$.

Definition 2.5: Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$ be two non-negative trapezoidal fuzzy numbers and $k \in \Re$. Then the arithmetic operations on $\tilde{a}$ and $\tilde{b}$ are defined as follows:

i) $k\tilde{a} = (ka_1, ka_2, ka_3, ka_4)$ if $k \geq 0$,

ii) $k\tilde{a} = (ka_4, ka_3, ka_2, ka_1)$ if $k \leq 0$,

iii) $\tilde{a} \oplus \tilde{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$,

iv) $\tilde{a} \otimes \tilde{b} = (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4)$.

2.2. Classic DEA model

The purpose of production function is to transform inputs $x = (x_1, \ldots, x_m) \in \Re_+^m$ into outputs $y = (y_1, \ldots, y_s) \in \Re_+^s$. The production possibility set (PPS) or technology *T* represents the set of feasible input-output combinations as follows:

$$T = \left\{ (x, y) \in \Re_+^{m+n} \mid x \; can \; produce \; y \right\}$$

Due to different combination of the following empirical axioms corresponding to distinctive characterizations of the PPS, different DEA models exist in DEA: (P.1) no free lunch; (P.2) boundedness; (P.3) closedness; (P.4) strong disposal of inputs and outputs; (P.5) convexity; and (P.6) *r*-returns-to-scale. The two basic DEA models are the CCR model of Charnes et al. (1978) and the BCC model of Banker et al. (1984) where CCR and BCC satisfies constant returns-to-scale (CRS) and variable returns-to-scale (VRS), respectively. In economics, returns-to-scale or economies of scale plays an imperative role for describing the production function behavior when the scale of production increases (increase in output relative to the associated increase in the inputs) in the long run. The production function has CRS if we multiply each input by a positive constant and the whole production function is multiplied with an amount that is equal to that constant. In DEA the estimation of RTS was first investigated in Banker (1984) and Banker et al. (1984) where both studies proposed adding a convexity constraint to the CCR model as well as developing a technique for estimating RTS. The empirical reference technology in terms of the minimal extrapolation principle for *n* DMUs is

$T(\gamma)=\left\{(x,y)\in\Re_{+}^{m+n} \mid x\geq\sum_{j=1}^{n}\lambda_j x_j, y\leq\sum_{j=1}^{n}\lambda_j y_j, \exists\lambda_j\in\Omega(\gamma)\right\}$. In this regard, axiom (P.6) can be used to differentiate between returns-to-scale and the shape of the frontier, that is, we can scale production with a certain factor, i.e., $t\in\Omega(\gamma)\Rightarrow t.(x,y)\in T$ where $\Omega(crs)=\{\lambda\in\Re^n \mid \lambda\geq 0\}$ and $\Omega(vrs)=\{\lambda\in\Re^n \mid \lambda\geq 0, 1\lambda=1\}$, in which VRS is the weakest assumption (no rescaling possible) and CRS is the strongest assumption.

The idea of the Farrell measures behind these radial DEA models is measuring efficiency by reducing all the inputs equi-proportionally without decreasing the outputs, $E=E\left((x,y);T(\gamma)\right)=\min\left\{\theta_p\in\Re_{+} \mid \left(\theta_p x, y\right)\in T(\gamma)\right\}$ or expanding all the outputs equi-proportionally without increasing the inputs $F=F\left((x,y);T(\gamma)\right)=\max\left\{\theta_p\in\Re_{+} \mid \left(x,\theta_p y\right)\in T(\gamma)\right\}$. The latter case is referred as the output-oriented model while the former is called the input-oriented model.

Consider a set of n DMUs, where DMU$_j$ has a production plan $T(crs)$ and consumes m inputs $x_{ij}\,(i=1,2,...,m)$ to produce r outputs $y_{rj}\,(r=1,2,...,s)$. The relative efficiency of DMU$_p$ can be obtained by using the following *input-oriented CCR* model:

$$
\begin{aligned}
\min\quad & \theta_p \\
s.t.\ & \sum_{j=1}^{n}\lambda_j x_{ij}+s_i^{-}=\theta_p x_{ip},\ i=1,2,...,m, \\
& \sum_{j=1}^{n}\lambda_j y_{rj}=s_r^{+}+y_{rp},\ r=1,2,...,s, \\
& \lambda_j\geq 0,\ j=1,2,...,n, \\
& s_i^{-}\geq 0,\ i=1,2,...,m, \\
& s_r^{+}\geq 0,\ r=1,2,...,s.
\end{aligned}
\qquad (1)
$$

In this model, an efficiency score, θ_p, is computed for DMU$_p$ by minimizing all the inputs radially (i.e. equi-proportionally) without reducing its outputs. The objective value of (1) lies within $0 < \theta_p \leq 1$. The non-negative slack variables s_i^- and s_r^+ represent the additional i^{th} input excess and the r^{th} output shortfall, respectively, as

$$s_i^- = \theta_p x_{ip} - \sum_{j=1}^{n} \lambda_j x_{ij}, \; i = 1,2,\ldots,m,$$

$$s_r^+ = \sum_{j=1}^{n} \lambda_j y_{rj} - y_{rp}, \; r = 1,2,\ldots,s,$$

To compute the efficiency score and the remaining input excesses and output shortfalls the following two-phase process is followed:

- Phase I: Calculate the optimal objective value of model (1), denoted by θ_p^*.
- Phase II: Solve the following model so as to maximize the sum of remaining input excesses and output shortfalls (after carrying out the radial input reduction θ_p^*):

$$\begin{aligned}
\max \quad & \sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_i^- \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta_p^* x_{ip}, \; i = 1,2,\ldots,m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} = s_r^+ + y_{rp}, \; r = 1,2,\ldots,s, \\
& \lambda_j \geq 0, \;\; j = 1,2,\ldots,n, \\
& s_i^- \geq 0, \; i = 1,2,\ldots,m, \\
& s_r^+ \geq 0, \;\; r = 1,2,\ldots,s.
\end{aligned} \tag{2}$$

Definition 2.6: Suppose that θ_p^* is the optimal value of model (1), and $s_r^{+*}(i = 1,2,\ldots,m,)$ and $s_i^{-*}(r = 1,2,\ldots,s)$ are the optimal values of model (2). DMU_p can be accordingly classified into one of three groups:

- DMU_p is called "*efficient*" if and only if $\theta_p^* = 1$ and $\sum_{r=1}^{s} s_r^{+*} + \sum_{i=1}^{m} s_i^{-*} = 0$.

- DMU_p is called "*weakly efficient*" if and only if $\theta_p^* = 1$ and $\sum_{r=1}^{s} s_r^{+*} + \sum_{i=1}^{m} s_i^{-*} > 0$.
- DMU_p is called "*inefficient*" if and only if $\theta_p^* < 1$.

To improve the performance of an inefficient DMU_p, we first define the reference set, RS_p, based on the solution of the Phase II (i.e. model (2)) as $RS_p=\{j|\lambda_j^* > 0\}$. The target inputs of an inefficient DMU_p can be computed in two steps: (i)its inputs are reduced proportionally by a factor θ_p^* and,(ii) remaining input excesses s_i^{-*} are removed. At the same time, the outputs of DMU_p can be increased by the output shortfalls. Thus, the *DEA projection* (a.k.a. DEA target) of an inefficient DMU_p can be computed as:

$$\hat{x}_{ip} = \theta_p^* x_{ip} - s_i^{-*},\ i = 1,2,...,m,$$

$$\hat{y}_{rp} = y_{rp} + s_r^{+*},\ r = 1,2,...,s.$$

Note that the CCR score can be called the (global) *technical efficiency* (TE) since there is no rescaling effect while BCC under the VRS assumption can be called (local) *pure technical efficiency* (PTE) in order to determine the source of inefficient DMU that may be due to the ineffective operation of the technology in transforming inputs to outputs (pure technical inefficiency)or its variance from the most productive scale size (scale inefficiency) or both (Cooper et al. 2007; Banker, 1984).In this regard, a DMU is said to be operating at *optimal returns-to-scale* or the *most productive scale size* when the BCC score is equal to the CCR score. Practically, the only difference between the BCC model and the CCR model consists in the inclusion, in both models (1) and (2), of the convexity constraint $\sum_{j=1}^{n} \lambda_j = 1$.

The above conventional DEA models, particularly BCC model, result in many efficient DMUs that can discern no difference between them. There have been developed a wide variety of methods in the literature for improving the discriminatory power of DEA, one of which is the super-efficiency method proposed by Andersen and Petersen (1993) to rank the efficient DMUs.

The super-efficiency corresponding model (1) can be defined as the optimal objective function value of the following model:

$$
\begin{aligned}
\min \quad & \theta_p \\
s.t. \quad & \sum_{\substack{j=1 \\ j \neq p}}^{n} \lambda_j x_{ij} + s_i^- = \theta_p x_{ip}, \ i = 1, 2, \dots, m, \\
& \sum_{\substack{j=1 \\ j \neq p}}^{n} \lambda_j y_{rj} = s_r^+ + y_{rp}, \ r = 1, 2, \dots, s, \\
& \lambda_j \geq 0, \ j = 1, 2, \dots, n, j \neq p \\
& s_i^- \geq 0, \ i = 1, 2, \dots, m, \\
& s_r^+ \geq 0, \ r = 1, 2, \dots, s.
\end{aligned} \tag{3}
$$

The difference between models (1) and (3) is to exclude the DMU under evaluation, DMU_p, from the PPS in model (3). In addition, efficient DMUs have efficiency scores larger than or equal to 1, and inefficient DMUs have the same efficiency scores obtained from model (1).

3. Fully Fuzzified DEA (FFDEA) Model

In this section, we propose CRS and VRS FFDEA models to measure the relative fuzzy efficiency of each DMU as well as to identify the inefficiency sources that are associated with fuzzy inputs slacks and/or fuzzy outputs shortfalls.

3.1. CRS FFDEA model

The FLP analog of *CCR* model (1) for evaluating the efficiency of DMU_p can be formulated as

$$
\begin{aligned}
\min \quad & \tilde{\theta}_p^{CRS} \\
s.t. \quad & \sum_{j=1}^{n} \tilde{\lambda}_j \otimes \tilde{x}_{ij} + \tilde{s}_i^- = \tilde{\theta}_p^{CRS} \otimes \tilde{x}_{ip}, \ i = 1, 2, \dots, m, \\
& \sum_{j=1}^{n} \tilde{\lambda}_j \otimes \tilde{y}_{rj} = \tilde{s}_r^+ + \tilde{y}_{rp}, \ r = 1, 2, \dots, s,
\end{aligned} \tag{4}
$$

$$\lambda_j \in TF(\Re)^+, \quad j = 1,2,\ldots,n,$$
$$\tilde{s}_i^- \in TF(\Re)^+, \quad i = 1,2,\ldots,m,$$
$$\tilde{s}_r^+ \in TF(\Re)^+, \quad r = 1,2,\ldots,s.$$

The structure of this FLP follows that of model (1). Thus, it uses an input orientation and computes a target fuzzy operating point is computed as a fuzzy linear combination of the observed data. The outputs of this target fuzzy operating point cannot be lower than those of the DMU_p being assessed and the corresponding inputs are expressed as a fuzzy input reduction factor $\tilde{\theta}_p^{CRS}$ applied to the observed inputs of that DMU. The differentiating feature of this model, and the reason why it is labeled a fully fuzzified approach, is the fact that the coefficients of the linear combination λ_j as well as the efficiency score $\tilde{\theta}_p^{CRS}$ and the input and output slacks $\tilde{s}_i^-$ and $\tilde{s}_r^+$ are all trapezoidal fuzzy numbers. The objective function is, as in (1), to maximize the reduction in the amount of inputs consumed by the target operating point.

There are different types of fuzzy membership functions that can be used to model fuzzy numbers (see Definition 2.1). However, membership functions form is of concern to analysts, particularly in practice the type of fuzzy membership function considered should be either chosen by the experts based on their experience (Tanaka and Guo 1999, Tanaka et al. 2000) or constructed based on available data (see Appendix A). In the case of model (3), due to their special structure with more flexibility and mathematical convenience, we assume trapezoidal fuzzy numbers to quantify the ambiguity in the decision parameters and in the input and output data. The choice of trapezoidal fuzzy numbers in this study leads to simplicity in modeling and ease of interpretation due to their *linear* membership functions (see Definition 2.2).

Therefore, $\tilde{\theta}_p^{CRS}$, $\tilde{x}_{ij}$, $\tilde{y}_{rj}$, $\tilde{\lambda}_j$, $\tilde{s}_i^-$ and $\tilde{s}_r^+$ are all represented by trapezoidal fuzzy numbers $\left(\theta_{p,1}^{CRS}, \theta_{p,4}^{CRS}, \theta_{p,4}^{CRS}, \theta_{p,4}^{CRS}\right)$, $\left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\right)$, $\left(y_{rj,1}, y_{rj,2}, y_{rj,3}, y_{rj,4}\right)$ and $\left(\lambda_{j,1}, \lambda_{j,2}, \lambda_{j,3}, \lambda_{j,4}\right)$, $\left(s_{i,1}^-, s_{i,2}^-, s_{i,3}^-, s_{i,4}^-\right)$ and $\left(s_{r,1}^+, s_{r,2}^+, s_{r,3}^+, s_{r,4}^+\right)$, respectively. Note that triangular membership function is a special case of trapezoidal membership function and one can be simply applied by using the following substitutions; $\theta_{p,4}^{CRS} = \theta_{p,4}^{CRS}$, $x_{ij,2} = x_{ij,3}$, $y_{rj,2} = y_{rj,3}$ and $\lambda_{j,2} = \lambda_{j,3}$, $s_{i,2}^{-} = s_{i,3}^{-}$ and $s_{r,2}^{+} = s_{r,3}^{+}$.Note that fuzzy production plans can be constructed by a combination of both fuzzy and crisp input and output data in which crisp data has no uncertainty. In this paper, we assume that the production plans are only based on fuzzy inputs and outputs since the crisp data can be simply transformed into the fuzzy form (c.f., Definition 2.2).By the use of trapezoidal fuzzy numbers, FFDEA model (4) can be reformulated as follows:

$$\begin{aligned}
\min \quad & \left(\theta_{p,1}^{CRS}, \theta_{p,4}^{CRS}, \theta_{p,4}^{CRS}, \theta_{p,4}^{CRS}\right) \\
s.t. \; & \sum_{j=1}^{n} \left(\lambda_{j,1}, \lambda_{j,2}, \lambda_{j,3}, \lambda_{j,4}\right) \otimes \left(x_{ij,1}, x_{ij,2}, x_{ij,3}, x_{ij,4}\right) + \left(s_{i,1}^{-}, s_{i,2}^{-}, s_{i,3}^{-}, s_{i,4}^{-}\right) = \\
& \qquad \left(\theta_{p,1}^{CRS}, \theta_{p,4}^{CRS}, \theta_{p,4}^{CRS}, \theta_{p,4}^{CRS}\right) \otimes \left(x_{ip,1}, x_{ip,2}, x_{ip,3}, x_{ip,4}\right), \quad i = 1, 2, ..., m, \\
& \sum_{j=1}^{n} \left(\lambda_{j,1}, \lambda_{j,2}, \lambda_{j,3}, \lambda_{j,4}\right) \otimes \left(y_{rj,1}, y_{rj,2}, y_{rj,3}, y_{rj,4}\right) = \left(s_{r,1}^{+}, s_{r,2}^{+}, s_{r,3}^{+}, s_{r,4}^{+}\right) + \\
& \qquad \left(y_{rp,1}, y_{rp,2}, y_{rp,3}, y_{rp,4}\right), \quad r = 1, 2, ..., s, \\
& \left(\lambda_{j,1}, \lambda_{j,2}, \lambda_{j,3}, \lambda_{j,4}\right), \left(s_{i,1}^{-}, s_{i,2}^{-}, s_{i,3}^{-}, s_{i,4}^{-}\right), \left(s_{r,1}^{+}, s_{r,2}^{+}, s_{r,3}^{+}, s_{r,4}^{+}\right) \in TF(\mathbb{R})^{+}
\end{aligned} \tag{5}$$

Remark 3.1. It is assumed that each DMU has at least one positive fuzzy input and one positive fuzzy output, which can be expressed as (see Definition 2.3):

$$\begin{aligned}
& X_{j,1} \geq 0, X_{j,1} \neq 0, \quad j = 1, 2, ..., n \\
& Y_{j,1} \geq 0, Y_{j,1} \neq 0, \quad j = 1, 2, ..., n
\end{aligned} \tag{6}$$

Based on fuzzy arithmetic operations, model (5) can be rewritten as follows:

$$\begin{aligned}
\min \quad & \left(\theta_{p,1}^{CRS}, \theta_{p,2}^{CRS}, \theta_{p,3}^{CRS}, \theta_{p,4}^{CRS}\right) \\
s.t. \; & \sum_{j=1}^{n} \left(\lambda_{j,1} x_{ij,1}, \lambda_{j,2} x_{ij,2}, \lambda_{j,3} x_{ij,3}, \lambda_{j,4} x_{ij,4}\right) + \left(s_{i,1}^{-}, s_{i,2}^{-}, s_{i,3}^{-}, s_{i,4}^{-}\right) = \\
& \qquad \left(\theta_{p,1}^{CRS} x_{ip,1}, \theta_{p,2}^{CRS} x_{ip,2}, \theta_{p,3}^{CRS} x_{ip,3}, \theta_{p,4}^{CRS} x_{ip,4}\right), \quad i = 1, 2, ..., m, \quad (i)
\end{aligned}$$

$$\sum_{j=1}^{n}\left(\lambda_{j,1}y_{rj,1},\lambda_{j,2}y_{rj,2},\lambda_{j,3}y_{rj,3},\lambda_{j,4}y_{rj,4}\right)=\left(s_{r,1}^{+},s_{r,2}^{+},s_{r,3}^{+},s_{r,4}^{+}\right)+\left(y_{rp,1},y_{rp,2},y_{rp,3},y_{rp,4}\right),\quad r=1,2,\ldots,s, \quad (ii) \tag{7}$$

$$\lambda_{j,1}\geq 0;\ \lambda_{j,2}-\lambda_{j,1}\geq 0;\ \lambda_{j,3}-\lambda_{j,2}\geq 0;\ \lambda_{j,4}-\lambda_{j,3}\geq 0,\ \ j=1,2,\ldots,n, \quad (iii)$$

$$s_{i,1}^{-}\geq 0;\ s_{i,2}^{-}-s_{i,1}^{-}\geq 0;\ s_{i,3}^{-}-s_{i,2}^{-}\geq 0;\ s_{i,4}^{-}-s_{i,3}^{-}\geq 0,\ i=1,2,\ldots,m, \quad (iv)$$

$$s_{r,1}^{+}\geq 0;\ s_{r,2}^{+}-s_{r,1}^{+}\geq 0;\ s_{r,3}^{+}-s_{r,2}^{+}\geq 0;\ s_{r,4}^{+}-s_{r,3}^{+}\geq 0,\ r=1,2,\ldots,s, \quad (v)$$

$$\theta_{p,2}^{CRS}-\theta_{p,1}^{CRS}\geq 0,\ \theta_{p,3}^{CRS}-\theta_{p,2}^{CRS}\geq 0,\ \theta_{p,4}^{CRS}-\theta_{p,3}^{CRS}\geq 0. \quad (vi)$$

Constraints (*i*) and (*ii*) in model (7) are simply obtained using the multiplication of two trapezoidal fuzzy numbers. To impose the non-negative $\tilde{\lambda}_j$ preserving its form as a trapezoidal fuzzy number, we have $\lambda_{j,4}\geq\lambda_{j,3}\geq\lambda_{j,2}\geq\lambda_{j,1}\geq 0$, which corresponds to constraints*(iii)*. Analogously, the non-negativity of $\tilde{s}_i^{-}$ and $\tilde{s}_r^{+}$ are imposed through constraints (*iv*) and (*v*) of model (6). In addition, the preservation of the form of its trapezoidal fuzzy number, $\tilde{\theta}_p^{CRS}=\left(\theta_{p,1}^{CRS},\theta_{p,4}^{CRS},\theta_{p,4}^{CRS},\theta_{p,4}^{CRS}\right)$ leads to constraints (*vi*) of model (7).

Now, based on Definition 2.4, we can rewrite the constraints of model (7) as

$$\sum_{j=1}^{n}\lambda_{j,k}x_{ij,k}+s_{i,k}^{-}=\theta_{p,k}^{CRS}x_{ip,k},\ i=1,2,\ldots,m,\ k=1,2,3,4,$$

$$\sum_{j=1}^{n}\lambda_{j,k}y_{rj,k}=s_{r,k}^{+}+y_{rp,k},\ r=1,2,\ldots,s,\ \ k=1,2,3,4, \tag{8}$$

$$\lambda_{j,1}\geq 0;\ \lambda_{j,2}-\lambda_{j,1}\geq 0;\ \lambda_{j,3}-\lambda_{j,2}\geq 0;\ \lambda_{j,4}-\lambda_{j,3}\geq 0,\ \ j=1,2,\ldots,n,$$

$$s_{i,1}^{-}\geq 0;\ s_{i,2}^{-}-s_{i,1}^{-}\geq 0;\ s_{i,3}^{-}-s_{i,2}^{-}\geq 0;\ s_{i,4}^{-}-s_{i,3}^{-}\geq 0,\ i=1,2,\ldots,m,$$

$$s_{r,1}^{+}\geq 0;\ s_{r,2}^{+}-s_{r,1}^{+}\geq 0;\ s_{r,3}^{+}-s_{r,2}^{+}\geq 0;\ s_{r,4}^{+}-s_{r,3}^{+}\geq 0,\ r=1,2,\ldots,s,$$

$$\theta_{p,2}^{CRS}-\theta_{p,1}^{CRS}\geq 0,\ \theta_{p,3}^{CRS}-\theta_{p,2}^{CRS}\geq 0,\ \theta_{p,4}^{CRS}-\theta_{p,3}^{CRS}\geq 0.$$

The optimal value of the objective function of model (7) computes the relative fuzzy efficiency of DMU$_p$. Note that model (7) is still an FLP model with one fuzzy variable in the objective function. This study proposes a new method to solve model (7) using the conversion of this model into a multi-objective linear programming (MOLP) problem. Thus, minimizing the fuzzy decision variable $\tilde{\theta}_p^{CRS}$, which has four variable parameters (namely $\theta_{p,1}^{CRS}, \theta_{p,2}^{CRS}$, $\theta_{p,3}^{CRS}$ and $\theta_{p,4}^{CRS}$) leads to the following MOLP model with four objective functions for minimizing these four variables simultaneously:

$$
\begin{aligned}
\min \quad & \theta_{p,1}^{CRS} \\
\min \quad & \theta_{p,2}^{CRS} \\
\min \quad & \theta_{p,3}^{CRS} \\
\min \quad & \theta_{p,4}^{CRS} \\
s.t. \quad & \theta_{p,2}^{CRS} - \theta_{p,1}^{CRS} \geq 0, \;\; \theta_{p,3}^{CRS} - \theta_{p,2}^{CRS} \geq 0, \;\; \theta_{p,4}^{CRS} - \theta_{p,3}^{CRS} \geq 0. \\
& \textit{Remaining constraints in } (8).
\end{aligned}
\tag{9}
$$

It is worth noting that due to the last constraint of (8) the value of the four objective functions $\theta_{p,1}^{CRS}, \theta_{p,2}^{CRS}$, $\theta_{p,3}^{CRS}$ and $\theta_{p,4}^{CRS}$ always preserve the form of trapezoidal fuzzy number as $\left(\theta_{p,1}^{CRS}, \theta_{p,4}^{CRS}, \theta_{p,4}^{CRS}, \theta_{p,4}^{CRS}\right)$throughout the optimization process. Now, to find the relative fuzzy efficiency of DMU$_p$, we propose a lexicographic approach for solving MOLP model (9) so that one of the objectives (namely $\theta_{p,4}^{CRS}$) is optimized first, then, maintaining the optimal value of that variable, a second objective (namely $\theta_{p,3}^{CRS}$) is optimized, and then, following the same procedure, successively, the third ($\theta_{p,2}^{CRS}$) and fourth ($\theta_{p,1}^{CRS}$) objective functions are optimized. Therefore, the following LP model is solved first:

$$
\begin{aligned}
\min \quad & \theta_{p,4}^{CRS} \\
s.t. \quad & \theta_{p,2}^{CRS} - \theta_{p,1}^{CRS} \geq 0, \;\; \theta_{p,3}^{CRS} - \theta_{p,2}^{CRS} \geq 0, \;\; \theta_{p,4}^{CRS} - \theta_{p,3}^{CRS} \geq 0. \\
& \textit{Remaining constraints in } (7).
\end{aligned}
\tag{10}
$$

It should be noted that if $\theta_{p,1}^{CRS}$ is first minimized and takes unity, then it is possible to the optimal value of $\theta_{p,2}^{CRS}$, $\theta_{p,3}^{CRS}$ or $\theta_{p,4}^{CRS}$ is larger than unity.

Proposition 1. The optimal objective function value of model (10) lies within the interval (0,1].

Proof**:** The solution $\theta_{p,1}^{CRS} = \theta_{p,4}^{CRS} = \theta_{p,4}^{CRS} = \theta_{p,4}^{CRS} = 1$, $s_{i,1}^{-} = s_{i,2}^{-} = s_{i,3}^{-} = s_{i,4}^{-} = 0, (i = 1,\ldots,m)$, $s_{r,1}^{+} = s_{r,2}^{+} = s_{r,3}^{+} = s_{r,4}^{+} = 0, (r = 1,\ldots,s)$, $\lambda_{p,1} = \lambda_{p,2} = \lambda_{p,3} = \lambda_{p,4} = 1$ and $\lambda_{j,1} = \lambda_{j,2} = \lambda_{j,3} = = \lambda_{j,4} = 0, \forall j \neq p$ is a feasible solution of model (10). Since its objective function value is unity, this implies that the optimal objective function value of model (9) $\theta_{p,4}^{CRS,*} \leq 1$. On the other hand, due to non-negativity of the fuzzy outputs, $Y_{p,4} > 0$, (see Remark 3.1), and the constraints $\sum_{j=1}^{n} \lambda_{j,4} y_{rj,4} \geq y_{rp,4}$ $(r = 1,2,\ldots,s)$ (equivalently $\sum_{j=1}^{n} \lambda_{j,4} y_{rj,4} - s_{r,4}^{+} = y_{rp,4}$ where $s_{r,4}^{+} \geq 0$) some $\lambda_{j,4}$ must take a non-zero value. Thus, for some input i $0 < \sum_{j=1}^{n} \lambda_{j,4} x_{ij,4} \leq \theta_{p,4}^{CRS,*} x_{ip,4}$, which implies $\theta_{p,4}^{CRS,*} > 0$.

□

Most likely, the optimal solution of (10) may not be unique. To handle this problem one method is to find the optimal solution, $\tilde{x}_{ij}$, $\tilde{y}_{rj}$, $\tilde{\lambda}_j, \tilde{s}_i^{-}$ and $\tilde{s}_r^{+}$ while preserving the optimal value $\theta_{p,4}^{CRS,*}$ calculated from (10). Therefore, the second step in the lexicographic optimization of model (9) consists in solving the following LP model

$$
\begin{aligned}
\min \quad & \theta_{p,3}^{CRS} \\
s.t. \quad & \theta_{p,4}^{CRS} = \theta_{p,4}^{CRS,*} \\
& \theta_{p,2}^{CRS} - \theta_{p,1}^{CRS} \geq 0, \;\; \theta_{p,3}^{CRS} - \theta_{p,2}^{CRS} \geq 0, \;\; \theta_{p,4}^{CRS} - \theta_{p,3}^{CRS} \geq 0. \\
& \textit{Remaining constraints in } (8).
\end{aligned}
\tag{11}
$$

The first constraint of (11) ensures that the optimal solution of model (11) is also a feasible solution to model (10). Put differently, excluding the first constraints of (11), models

(10) and (11) have the same constraints and the objective function of model (11) is the margin minimization of model (10).

Proposition2. The optimal value of the objective function of model (11) lies within (0, $\theta_{p,4}^{CRS,*}$].

Proof: On the one hand, from the constraints of model (11) and taking into account Proposition 1it follows that $\theta_{p,3}^{CRS} \leq \theta_{p,4}^{CRS,*} \leq 1$. On the other hand, by a similar argument as in Proposition 1, the non-negativity of the fuzzy outputs, $Y_{p,3} > 0$, and the constraints $\sum_{j=1}^{n} \lambda_{j,4} y_{rj,4} \geq y_{rp,4}$ $(r = 1,2,...,s)$ imply that some $\lambda_{j,3}$ must take a non-zero value. Therefore, for some input i $0 < \sum_{j=1}^{n} \lambda_{j,3} x_{ij,3} \leq \theta_{p,3}^{CRS,*} x_{ip,3}$, which means $\theta_{p,4}^{CRS,*} > 0$. □

Corollary 1. $\theta_{p,3}^{CRS,*} = 1$ if and only if $\theta_{p,4}^{CRS,*} = 1$.

The third and fourth steps in the lexicographic optimization of model (9) involves solving, successively, the following two LP models

$$
\begin{aligned}
\min \quad & \theta_{p,2}^{CRS} \\
s.t. \quad & \theta_{p,4}^{CRS} = \theta_{p,4}^{CRS,*} \\
& \theta_{p,3}^{CRS} = \theta_{p,3}^{CRS,*} \\
& \theta_{p,2}^{CRS} - \theta_{p,1}^{CRS} \geq 0, \quad \theta_{p,3}^{CRS} - \theta_{p,2}^{CRS} \geq 0, \quad \theta_{p,4}^{CRS} - \theta_{p,3}^{CRS} \geq 0. \\
& \textit{Remaining constraints in } (8).
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\min \quad & \theta_{p,1}^{CRS} \\
s.t. \quad & \theta_{p,4}^{CRS} = \theta_{p,4}^{CRS,*} \\
& \theta_{p,3}^{CRS} = \theta_{p,3}^{CRS,*} \\
& \theta_{p,2}^{CRS} = \theta_{p,2}^{CRS,*} \\
& \theta_{p,2}^{CRS} - \theta_{p,1}^{CRS} \geq 0, \quad \theta_{p,3}^{CRS} - \theta_{p,2}^{CRS} \geq 0, \quad \theta_{p,4}^{CRS} - \theta_{p,3}^{CRS} \geq 0. \\
& \textit{Remaining constraints in } (8).
\end{aligned}
\tag{13}
$$

It is obvious that the constraint $\theta_{p,4}^{CRS} - \theta_{p,3}^{CRS} \geq 0$ in (12) and the constraints $\theta_{p,4}^{CRS} - \theta_{p,3}^{CRS} \geq 0$ and $\theta_{p,3}^{CRS} - \theta_{p,2}^{CRS} \geq 0$ in (13) are redundant. Moreover, the following statements can be made (proof omitted).

Proposition3. The optimal value of the objective function of models (12) and (13) lie within (0, $\theta_{p,3}^{CRS,*}$] and (0, $\theta_{p,2}^{CRS,*}$], respectively.

Corollary 2. $\theta_{p,2}^{CRS,*} = 1$ if and only if $\theta_{p,3}^{CRS,*} = \theta_{p,4}^{CRS,*} = 1$.

Corollary 3. $\theta_{p,1}^{CRS,*} = 1$ if and only if $\theta_{p,2}^{CRS,*} = \theta_{p,3}^{CRS,*} = \theta_{p,4}^{CRS,*} = 1$.

Although the lexicographic optimization approach involving models (10)-(13) allows the computation of the relative fuzzy efficiency of DMU_p $\tilde{\theta}_p^{CRS,*} = \left(\theta_{p,1}^{CRS,*}, \theta_{p,2}^{CRS,*}, \theta_{p,3}^{CRS,*}, \theta_{p,4}^{CRS,*}\right)$, since this efficiency score uses a radial metric there may remain input and output slacks. Therefore, analogously to the CCR model, the following Phase II model can be used to maximize the sum of fuzzy input excesses and fuzzy output shortfalls, while preserving $\tilde{\theta}_p^{CRS,*} = \left(\theta_{p,1}^{CRS,*}, \theta_{p,2}^{CRS,*}, \theta_{p,3}^{CRS,*}, \theta_{p,4}^{CRS,*}\right)$,

$$\max \quad \sum_{i=1}^{m}\left(s_{i,1}^{-} + s_{i,2}^{-} + s_{i,3}^{-} + s_{i,4}^{-}\right) + \sum_{r=1}^{s}\left(s_{r,1}^{+} + s_{r,2}^{+} + s_{r,3}^{+} + s_{r,4}^{+}\right)$$

$$s.t. \ \sum_{j=1}^{n} \lambda_{j,k} x_{ij,k} + s_{i,k}^{-} = \theta_{p,k}^{CRS,*} x_{ip,k}, \ i = 1,2,\ldots,m, \quad k = 1,2,3,4, \tag{14}$$

$$\sum_{j=1}^{n} \lambda_{j,k} y_{rj,k} = s_{r,k}^{+} + y_{rp,k}, \ r = 1,2,\ldots,s, \quad k = 1,2,3,4$$

$$\lambda_{j,1} \geq 0; \ \lambda_{j,2} - \lambda_{j,1} \geq 0; \ \lambda_{j,3} - \lambda_{j,2} \geq 0; \ \lambda_{j,4} - \lambda_{j,3} \geq 0; \ j = 1,2,\ldots,n,$$

$$s_{i,1}^{-} \geq 0; \ s_{i,2}^{-} - s_{i,1}^{-} \geq 0; \ s_{i,3}^{-} - s_{i,2}^{-} \geq 0; \ s_{i,4}^{-} - s_{i,3}^{-} \geq 0, \ \ i = 1,2,\ldots,m,$$

$$s_{r,1}^{+} \geq 0; \ s_{r,2}^{+} - s_{r,1}^{+} \geq 0; \ s_{r,3}^{+} - s_{r,2}^{+} \geq 0; \ s_{r,4}^{+} - s_{r,3}^{+} \geq 0, \ \ r = 1,2,\ldots,s.$$

Definition 3.1: Let $\tilde{\theta}_p^{CRS,*} = \left(\theta_{p,1}^{CRS,*}, \theta_{p,2}^{CRS,*}, \theta_{p,3}^{CRS,*}, \theta_{p,4}^{CRS,*}\right)$, the relative fuzzy efficiency of DMU_p computed from Phase I and $\tilde{s}_i^{-*} = \left(s_{i,1}^{-*}, s_{i,2}^{-*}, s_{i,3}^{-*}, s_{i,4}^{-*}\right)$ and $\tilde{s}_r^{+*} = \left(s_{r,1}^{+*}, s_{r,2}^{+*}, s_{r,3}^{+*}, s_{r,4}^{+*}\right)$ the optimal solution of Phase II calculated from model (13), then DMU_p can be classified into one of these three groups:

- DMU_p is "*FFDEA efficient"* if and only if $\tilde{\theta}_p^{CRS,*} = (1,1,1,1)$, $\tilde{s}_i^{-*} = (0,0,0,0)\ \forall i$ and $\tilde{s}_r^{+*} = (0,0,0,0)\ \forall r.$
- DMU_p is called "*FFDEA weakly efficient"* if and only if $\tilde{\theta}_p^{CRS,*} = (1,1,1,1)$ and the optimal value of the objective function of (14) is positive (i.e. $\sum_{i=1}^{m} \tilde{s}_i^{-*} + \sum_{r=1}^{s} \tilde{s}_r^{+*} \neq (0,0,0,0)$)
- DMU_p is called "*FFDEA inefficient"* if and only if $\tilde{\theta}_p^{CRS,*} \neq (1,1,1,1)$.

To provide a complete ranking of *FFDEA efficient* DMUs, we will develop a new super-efficiency FFDEA method in Section 4. Nonetheless, we use the following definition to rank *FFDEA inefficient* DMUs:

Definition 3.2: Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers. According to the lexicography method, $\tilde{a} > \tilde{b}$ if and only if one of the following cases apply

i) $a_4 > b_4,$

ii) $a_4 = b_4, a_3 > b_3,$

iii) $a_4 = b_4, a_3 = b_3, a_2 > b_2,$

iv) $a_4 = b_4, a_3 = b_3, a_2 = b_2, a_1 > b_1.$

Definition 3.3: Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers. According to the lexicography method, $\tilde{a} = \tilde{b}$ if and only if $a_4 = b_4, a_3 = b_3, a_2 = b_2, a_1 = b_1$

Definition 3.4: Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers. According to the lexicography method, $\tilde{a} \geq \tilde{b}$ if and only if $\tilde{a} > \tilde{b}$ or $\tilde{a} = \tilde{b}$

In addition to computing relative fuzzy efficiencies and identifying FFDEA efficient, weakly efficient and inefficient DMUs, the proposed approach allows the identification of benchmarks and the computation of fuzzy input and output targets for each inefficient DMU_p. This can be done based on the optimal solution of model (14). Thus, the benchmarks of an inefficient DMU_p correspond to the DMUs in its corresponding reference set

$$R_p = \left\{ j \middle| \tilde{\lambda}_j^* = (\lambda_{1,j}^*, \lambda_{2,j}^*, \lambda_{3,j}^*, \lambda_{4,j}^*) \neq (0,0,0,0) \right\}.$$

Analogously, the fuzzy input and output targets of an inefficient DMU_p can be computed as:

$$\hat{x}_{ip,k} = \theta_{p,k}^{CRS,*} x_{ip,k} - s_{i,k}^{-*} = \sum_{j \in R_p} \lambda_{j,k} x_{ij,k}, \quad k = 1,2,3,4 \tag{15}$$

$$\hat{y}_{rp,k} = y_{rp,k} + s_{r,k}^{+*} = \sum_{j \in R_p} \lambda_{j,k} y_{rj,k}, \quad k = 1,2,3,4, \tag{16}$$

3.2. VRS FFDEA model

In this sub-section, the VRS case will be discussed. In principle, the only difference with respect to the CRS case presented in Sub-section3.1 is the inclusion of the convexity constraint $\sum_{j=1}^{n} \tilde{\lambda}_j = (1,1,1,1)$ in models (4), (5) and (7) as well as labeling the corresponding relative fuzzy efficiency as $\tilde{\theta}_p^{VRS} = \left(\theta_{p,1}^{VRS}, \theta_{p,4}^{VRS}, \theta_{p,4}^{VRS}, \theta_{p,4}^{VRS} \right)$.Moreover, according to definition 2.4, including this convexity constraint is equivalent to adding the following constraints to model (9)

$$\sum_{j=1}^{n} \lambda_{j,1} = 1, \; \sum_{j=1}^{n} \lambda_{j,2} = 1, \; \sum_{j=1}^{n} \lambda_{j,3} = 1, \; \sum_{j=1}^{n} \lambda_{j,4} = 1,$$

However, since $\lambda_{j,1} \geq 0; \; \lambda_{j,2} - \lambda_{j,1} \geq 0; \; \lambda_{j,3} - \lambda_{j,2} \geq 0; \; \lambda_{j,4} - \lambda_{j,3} \geq 0, \;\; j = 1,2,\ldots,n,$ it follows that, necessarily, $\lambda_{j,1} = \lambda_{j,2} = \lambda_{j,3} = \lambda_{j,4}, \;\; j = 1,2,\ldots,n$. This reduces the numbers of constraints and variables of model (9). Thus, relabeling

$\lambda_{j,1}=\lambda_{j,2}=\lambda_{j,3}=\lambda_{j,4}=\lambda_j,\quad j=1,2,...,n$ the VRS equivalent of model (9) can be formulated as

$$
\begin{aligned}
\min\quad & \theta_{p,1}^{VRS}\\
\min\quad & \theta_{p,2}^{VRS}\\
\min\quad & \theta_{p,3}^{VRS}\\
\min\quad & \theta_{p,4}^{VRS}\\
s.t.\ & \sum_{j=1}^{n}\lambda_j x_{ij,k}+s_{i,k}^{-}=\theta_{p,k}^{VRS}x_{ip,k},\ i=1,2,...,m,\quad k=1,2,3,4,\\
& \sum_{j=1}^{n}\lambda_j y_{rj,k}=s_{r,k}^{+}+y_{rp,k},\ r=1,2,...,s,\quad k=1,2,3,4 \qquad (17)\\
& \sum_{j=1}^{n}\lambda_j=1\\
& \lambda_j\geq 0,\quad j=1,2,...,n,\\
& s_{i,1}^{-}\geq 0;\ s_{i,2}^{-}-s_{i,1}^{-}\geq 0;\ s_{i,3}^{-}-s_{i,2}^{-}\geq 0;\ s_{i,4}^{-}-s_{i,3}^{-}\geq 0,\ i=1,2,...,m,\\
& s_{r,1}^{+}\geq 0;\ s_{r,2}^{+}-s_{r,1}^{+}\geq 0;\ s_{r,3}^{+}-s_{r,2}^{+}\geq 0;\ s_{r,4}^{+}-s_{r,3}^{+}\geq 0,\ r=1,2,...,s,\\
& \theta_{p,2}^{VRS}-\theta_{p,1}^{VRS}\geq 0,\ \theta_{p,3}^{VRS}-\theta_{p,2}^{VRS}\geq 0,\ \theta_{p,4}^{VRS}-\theta_{p,3}^{VRS}\geq 0.
\end{aligned}
$$

This FLP can be transformed into a MOLP problem and solved using a lexicographic optimization approach just as in the CRS case. Based on the solutions of the corresponding models, i.e. relative fuzzy efficiency $\tilde{\theta}_p^{VRS,*}=\left(\theta_{p,1}^{VRS,*},\theta_{p,2}^{VRS,*},\theta_{p,3}^{VRS,*},\theta_{p,4}^{VRS,*}\right)$, fuzzy input excesses $\tilde{s}_i^{-*}=\left(s_{i,1}^{-*},s_{i,2}^{-*},s_{i,3}^{-*},s_{i,4}^{-*}\right)$ and fuzzy output shortfalls $\tilde{s}_r^{+*}=\left(s_{r,1}^{+*},s_{r,2}^{+*},s_{r,3}^{+*},s_{r,4}^{+*}\right)$ the classification of DMUs and the computation of fuzzy input and output targets can be similarly done. Thus,

Definition 3.5: A DMU_p can be classified into one of these three groups:

- DMU_p is "*VRS FFDEA efficient*" if and only if $\tilde{\theta}_p^{VRS,*} = (1,1,1,1)$, $\tilde{s}_i^{-*} = (0,0,0,0)$ $\forall i$ and $\tilde{s}_r^{+*} = (0,0,0,0)$ $\forall r$.
- DMU_p is called "*VRS FFDEA weakly efficient*" if and only if $\tilde{\theta}_p^{VRS,*} = (1,1,1,1)$ and the optimal value of the objective function of (14) is positive (i.e. $\sum_{i=1}^{m} \tilde{s}_i^{-*} + \sum_{r=1}^{s} \tilde{s}_r^{+*} \neq (0,0,0,0)$)
- DMU_p is called "*VRS FFDEA inefficient*" if and only if $\tilde{\theta}_p^{VRS,*} \neq (1,1,1,1)$.

Proposition4. $\tilde{\theta}_{p,4}^{VRS,*} \geq \tilde{\theta}_{p,4}^{CRS,*}$

Proof: Model (17) is equivalent to model (9) with the additional constraints $\sum_{j=1}^{n} \lambda_{j,1} = 1$, $\sum_{j=1}^{n} \lambda_{j,2} = 1$, $\sum_{j=1}^{n} \lambda_{j,3} = 1$, $\sum_{j=1}^{n} \lambda_{j,4} = 1$, and $\lambda_{j,1} = \lambda_{j,2} = \lambda_{j,3} = \lambda_{j,4} \quad \forall j$. Therefore, because of this reduced feasibility region, in the first step of the lexicographic optimization approach, when computing $\theta_{p,4}^{VRS,*}$, a larger optimal value would necessarily ensue, i.e. $\tilde{\theta}_{p,4}^{VRS,*} \geq \tilde{\theta}_{p,4}^{CRS,*}$. □

4. Super-efficiency FFDEA method

In the proposed approach, especially in the VRS case, several DMUs may be assessed as FFDEA efficient. This is a common feature in DEA and precludes a complete ranking of the DMUs. There are many DEA ranking methods (see, for example, Adler et al. 2002 for a review) among which a simple and often used approach is super-efficiency (e.g. Andersen and Petersen 1993). The basic idea in the super-efficiency approach is that the DMU under evaluation should be eliminated from DMUs that determine the reference set. In this section, a radial, input-oriented super-efficiency FFDEA model is formulated. From model (7) we get the corresponding super-efficiency formulation

$$\min \quad \left(\theta_{p,1}^{SE-CRS}, \theta_{p,4}^{SE-CRS}, \theta_{p,4}^{SE-CRS}, \theta_{p,4}^{SE-CRS}\right)$$

$$\begin{aligned}
s.t. \quad & \sum_{\substack{j=1 \\ j\neq p}}^{n} \left(\lambda_{j,1}x_{ij,1}, \lambda_{j,2}x_{ij,2}, \lambda_{j,3}x_{ij,3}, \lambda_{j,4}x_{ij,4}\right) + \left(s_{i,1}^{-}, s_{i,2}^{-}, s_{i,3}^{-}, s_{i,4}^{-}\right) = \\
& \qquad \left(\theta_{p,1}^{SE-CRS}x_{ip,1}, \theta_{p,2}^{SE-CRS}x_{ip,2}, \theta_{p,3}^{SE-CRS}x_{ip,3}, \theta_{p,4}^{SE-CRS}x_{ip,4}\right), \quad i=1,2,...,m, \\
& \sum_{\substack{j=1 \\ j\neq p}}^{n} \left(\lambda_{j,1}y_{rj,1}, \lambda_{j,2}y_{rj,2}, \lambda_{j,3}y_{rj,3}, \lambda_{j,4}y_{rj,4}\right) = \left(s_{r,1}^{+}, s_{r,2}^{+}, s_{r,3}^{+}, s_{r,4}^{+}\right) + \\
& \qquad \left(y_{rp,1}, y_{rp,2}, y_{rp,3}, y_{rp,4}\right), \quad r=1,2,...,s, \\
& \lambda_{j,1} \geq 0;\ \lambda_{j,2}-\lambda_{j,1} \geq 0;\ \lambda_{j,3}-\lambda_{j,2} \geq 0;\ \lambda_{j,4}-\lambda_{j,3} \geq 0,\ j=1,2,...,n,\ j \neq p \\
& s_{i,1}^{-} \geq 0;\ s_{i,2}^{-}-s_{i,1}^{-} \geq 0;\ s_{i,3}^{-}-s_{i,2}^{-} \geq 0;\ s_{i,4}^{-}-s_{i,3}^{-} \geq 0,\ i=1,2,...,m, \\
& s_{r,1}^{+} \geq 0;\ s_{r,2}^{+}-s_{r,1}^{+} \geq 0;\ s_{r,3}^{+}-s_{r,2}^{+} \geq 0;\ s_{r,4}^{+}-s_{r,3}^{+} \geq 0,\ r=1,2,...,s,
\end{aligned} \tag{18}$$

Although the modification is simple (note just the $j \neq p$ in the first three constraints) it has some important implications. Thus, the equivalent to Proposition no longer holds, i.e. $\theta_{p,4}^{SE-CRS,*}$ can be greater than or equal to unity (see Proposition 5). Two additional remarks should be taken into account. One is that the super-efficiency model is solved only for the FFDEA efficient DMUs. For FFDEA weakly efficient and inefficient DMUs there is not any change because of their exclusion from the reference set. This is because only efficient DMUs define the efficient frontier and can thus be part of the reference set anyway. The second remark is that since the super-efficiency model does not project on the true efficient frontier (but on that which results when the DMU being assessed is removed) the super-efficiency approach is not aimed at computing input and output targets. This means that there is no need to carry out a Phase two. We apply the proposed lexicographic method to solve MOLP model (18) and obtain the optimal fuzzy super-efficiency score of FFDEA efficient DMUs. In addition, we can employ definitions 3.2-3.4 to determine a complete ranking of these DMUs. Moreover, the following statement can be made (proof omitted).

Proposition 5. If DMU_p is FFDEA efficient, then $\theta_{p,4}^{SE-CRS,*} \geq 1$.

Model (18) corresponds to the CRS case. The corresponding model for the VRS case just requires adding the convexity constraint $\sum_{j=1,\, j\neq p}^{n} \lambda_j = 1$. It is well known, however, that this type of radial VRS super-efficiency models can have infeasibilities, thus preventing ranking the total set of efficient DMUs (e.g. Seiford and Zhu 1999). Moreover, infeasibilities can also occur, even in the CRS case, if an efficient DMU has zero input value (Zhu 1996). Another recent discussion finds the super-efficiency model unsuccessful in ranking efficient units by conducting simulation experiments (Banker and Chang, 2006). How to overcome all these problems has been the subject of much research in DEA and many alternative super-efficiency DEA models have been proposed to overcome them (e.g. Seiford and Zhu, 1999; Chen 2004; Cook et al. 2009; Lee and Zhu 2012; Chen et al. 2013; Lin and Chen 2015; Pourmahmoud et al. 2016; Aldamak et al. 2016). As indicated above, in this paper the basic Andersen and Petersen (1993) super-efficiency approach is used to be consistent with the radial, input-oriented character of the FFDEA model (7). Although desirable, exploring other super-efficiency variants or other DEA ranking methods is out of the scope of this paper and is a topic for further research.

5. Computational discussion

In this section, we show that our approach is computationally economical compared to the existing FDEA methods involving Kao and Liu (2000), Guo and Tanaka (2001), Saati et al. (2002), León et al. (2003) and Lertworasirikul et al. (2003).

Every DEA model can be represented in the envelopment form (primal model) or multiplier form (dual model) form by dint of a primal-dual transformation. The primal VRS model involves $m+s+1$ constraints and $n+1$ variables where m, s and n are the numbers of inputs, outputs and observations, respectively, while the dual VRS model involves $n+1$ constraints and $m+n$ variables. Generally, in DEA the number of DMUs is considerably larger than the number of inputs and outputs thanks to the rule of thumb (Golany and Roll, 1989). The computational burden of the LP problem directly depends on the number of constraints and variables but since the memory size required for preserving the basis is the square of the number of constraints, the LP problem with less number of constraints is more appropriate in terms of memory saving purposes. Therefore, the primal DEA model is computationally economical contrary to the dual DEA model. Since the solving of FDEA models is basically based on transforming the problem into the crisp LP problems, the computational burden that mostly hinges on the number of constraints can be studied.

Kao and Liu (2000) developed a fuzzy (dual) VRS model to measure the efficiencies of DMUs with fuzzy inputs and outputs. They used α-level's idea and Zadeh's extension principle to convert the fuzzy DEA model into a pair of crisp parametric programs with different α-level sets to estimate the membership function of the efficiency measure. Kao and Liu (2000)'s method requires solving $2nq$LP models[1] where each model involves n constraints, $m+s$ nonnegative constraints and $m+s+1$variables that is computationally expensive. Although Kao and Liu considered a simple example to show the strength of their method in constructing the membership function of the efficiency using different α-level sets there is no guarantee that the monotonicity property will be generally satisfied.

Guo and Tanaka (2001) proposed a fuzzy (dual) CRS model for estimating fuzzy efficiency in which fuzzy constraints (including fuzzy equalities and fuzzy inequalities) were converted into crisp constraints by using the comparison rule for fuzzy numbers for a given possibility α-level. Guo and Tanaka (2001) used the possibilistic programming to formulate a LP problem involving a primary objective function and a secondary objective function. This can be expressed with a primary objective function by introducing an added constraint based on the optimal objective function value of the secondary model that is independently calculated. Therefore, Guo and Tanaka (2001)'s method requires solving $2nq$LP models where each secondary model involves 2constraints, m nonnegative constraints and m variables and each primary model involves $3+2n$ constraints, $m+s$ nonnegative constraints and $m+s$ variables. Furthermore, the authors generalized their method with considering the relationship between DEA and regression analysis (RA), which requires solving the primary model with $4n$ more variables. Despite heavy computational burden of Guo and Tanaka (2001)'s method, it is solely limited to the case of symmetrical triangular fuzzy variables and cannot be applied to non-symmetrical triangular fuzzy.

Saati et al. (2002) extended a fuzzy (dual) CRS model based on the α-level set approach in which they first defined a possibilistic-programming problem and then transformed it into a crisp linear programming model. Saati et al. (2002)'s method requires solving nq LP models where each model includes $n(1+2m+2s)+1$ constraints, $m+s$ nonnegative constraints and $(m+s)(n+1)$ variables, leading to a very high computational burden.

---

[1] n and q represent the number of observations and α levels, respectively.

León et al. (2003) developed a fuzzy version of the primal VRS model in the frame of possibilistic-programming optimization by applying two ranking methods in terms of α-level sets. León et al. (2003) 's method requires solving nq LP models where each model includes $4(m+s)+1$ constraints, n nonnegative constraints and $n+1$ variables over and above we remark that the zero value of the right-hand-side of $4m$ constraints leads to increase in degeneracy and the computational complexity. This method yields a crisp efficiency score for a given DMU for each α-level.

As an alternative approach, Lertworasirikul et al. (2003) proposed a fuzzy (dual) CCR model using possibility theory and the concept of chance-constrained programming in which constraints are assumed to be fuzzy events at predefined acceptable levels of possibility for constraints. Though each constraint can have a predefined acceptable level of possibility Lertworasirikul et al. (2003) presumed the same possibility level for all fuzzy constraints and under this assumption their method requires solving nq LP models where each model includes $3+n$ constraints, $m+s$ nonnegative constraints and $m+s+1$ variables.

Let us now investigate the computational complexity of the proposed approach in this study involving models (9), (14) and (17). Regarding the fuzzy (primal) CRS model (9), we require solving $4n$ LP models since this model for each DMU is transformed into four LP models to calculate the fuzzy efficiency, in which model (10) contains $7(s+m)+3(n+1)$ constraints, $n+m+s$ nonnegative variable constraints and $4(m+s+n+1)$ variables. Compared to model (10), models (11), (12) and (13) entail one, two and three less decision variables, respectively, and models (12) and (13) entail one and two less constraints, respectively. Next, model (14) can be run n times to distinguish weakly efficient units where each model involves $7(s+m)+3(n+1)$ constraints, $n+m+s$ nonnegative variable constraints and $4(m+s+n)$ variables. Similarly, we require solving $4n$ LP models for the fuzzy (primal) VRS model (9), where the primary LP model entails $7(m+s)+n+3$ constraints, $n+m+s$ nonnegative variable constraints and $4(m+s+1)+n$ variables. The computational complexity of the existing methods and our method is summarized in Table1. Obviously, in the proposed approach in this study we need much less computational effort in terms of the number LP models that can be solved for each unit. In addition, although the number of constraints and variables are generally increased in our proposed models we think of more fuzziness in the modelling to fully reflect the uncertainty.

----Insert Table 1 Here----

6. Numerical examples

In this section, we present two numerical examples to illustrate the applicability and efficacy of the proposed method. We first consider a hypothetical example proposed by Guo and Tanaka (2001) followed by a second example from Azadi et al. (2015) for evaluating a real case study in sustainable supply chain management (SSCM).

6.1. Example 1

Let us first consider the dataset used by Guo and Tanaka (2001) in this subsection, in which five DMUs consume two fuzzy inputs to produce two fuzzy outputs under CRS (see Table 2). Note that, in this example, all the inputs and outputs are triangular fuzzy numbers, which are a subset of the trapezoidal fuzzy numbers considered in this paper. Since this dataset was also used by Saati et al. (2002) and Lertworasirikul et al. (2003) we will be able to compare our results with those of these three FDEA methods.

----Insert Table 2 Here----

Table 3 shows the results of the proposed approach. Note that DMUs B, D and E are found FFDEA efficient because their fuzzy efficiencies are (1, 1, 1, 1) and zero optimal value obtained for Phase II while DMUs A and C are assessed as FFDEA inefficient as presented in the last column of Table 2.

----Insert Table 3 Here----

It may be interesting to report also the fuzzy input and output targets of the two FFDEA inefficient DMUs. These are shown in Table 4.

----Insert Table 4 Here----

Finally, before commenting on the results of existing FDEA methods applied to this example, let us see the results of the super-efficiency FFDEA method proposed in Section 4. These are shown in Table 5, together with the corresponding ranking using definition 3.2.

----Insert Table 5 Here----

Guo and Tanaka (2001) proposed a fuzzy CCR model with fuzzy constraints, including fuzzy equalities and fuzzy inequalities. The fuzzy constraints were converted into crisp constraints by predefining a possibility level and using the ranking system for fuzzy numbers. They defined a DMU as α-possibilistic efficient if the maximum value of the fuzzy efficiency at that α-level is greater than or equal to 1.The set of all possibilistic efficient DMUs was called the α-possibilistic nondominated set, denoted by S_α. The fuzzy efficiencies of Guo and Tanaka (2001) for $\alpha =$ {0,0.5,0.75,1} are reported in Table 5. As it can be seen in Table 6, $S_0 = \{B, C, D, E\}$, $S_{0.5} = \{B, D\}, S_{0.75} = \{B, D\}$ and $S_1 = \{B, D, E\}$ are the nondominated sets for different α values. Note that DMU_B and DMU_D are in the possibilistic nondominated set for all α-levels. As for DMU E, there might me a typo in the reported results because, it is to be expected (and it occurs in all cases except for DMU E and α=0.5 and α=0.75 and for DMU B and α=0.5), that the upper limit of the α-cuts are decreasing with α.

----Insert Table 6 Here----

Lertworasirikul et al. (2003) developed a CCR DEA model using the possibility approach and they defined a DMU as α-possibilistic efficient if the objective value of the model is greater than or equal to 1 at the specific α level. Hence, as shown in Table 6, DMU_B and DMU_D and DMU_E are possibilistic efficient at all possibility levels while DMU_A and DMU_C are possibilistic efficient only at some αlevels.

Finally, Saati et al. (2002) used their fuzzy CCR model with anα-level based approach to compute the efficiency of the five DMUs as presented in Table 6. Note that they found DMUs B, D and E efficient for all α values while DMUs A and C were efficient just for some α values. Although not shown in Table 6, the method of Saati et al. (2002) also allowed ranking the efficient DMUs. Their ranking is dependent on the α level considered but, in this example, their method ranked DMU D first, DMU E second and DMU B third for all α values. This is only partially consistent with the rank derived by super-efficiency FFDEA as shown in Table 5.

Let us now compare the existing methods with the proposed models in terms of computational complexity. Table 7 gives the number of LP models that are solved for this example when α=0,0.1,0.2,...,1 as well as the number of constraints, nonnegative variable constraints and variables. We require solving 20 LP CRS models corresponding model (9) while it is necessary to solve 110, 110, 55, 55 and 55 LP models for KL., GT., S., LE. and L. methods, respectively. Therefore, the proposed method in this study has the capability to obtain the fuzzy efficiency with less computational effort compared to the existing methods. However, we need to define more constraints and decision variables for each LP model so as to fully reflect the uncertainty.

----Insert Table 7 Here----

6.2. Example2 (Case study)

In this sub-section, we also illustrate the approach proposed in this paper with a real case study in sustainable supply chain management (SSCM) taken from Azadi et al. (2015). Evaluating the performance of suppliers in terms of the sustainability aspect plays an important role to establish a sustainable (i.e. green) SCM. Azadi et al. (2015) have studied the ARCIC Company with the aim of selecting the most sustainable suppliers among 26 potential suppliers for raw materials. The inputs consist of *total cost of shipments* (TC), and *number of shipments per month* (NS), *eco-design cost* (ED) and *cost of work safety and labor health* (CS. Note that TC and NS are the economic criteria while ED and CS are the environmental and social criteria, respectively. The outputs consist of *number of shipments to arrive on time* (NOT) and *number of bills received from the supplier without errors* (NB). While all inputs are precisely measured, the outputs are characterized by triangular fuzzy numbers (L, M, U) to handle the uncertainty as shown in (Azadi et al. 2015, Table 4, P. 279).

The fuzzy efficiency scores of our proposed approach for all the suppliers are reported in the 2^{nd} column of Table 8. The classification of suppliers in the 3^{rd} column of Table 8 shows that suppliers {6, 7, 8, 9, 16, 17, 19, 22} are FFDEA efficient and serve as the "benchmark" units that can be used to improve the inefficient units. In addition, the fuzzy input and output targets of the inefficient supplier are shown in Table 8.

----Insert Table 8 Here----

7. Conclusions

This paper proposes a novel FDEA framework that handles fuzzy input and output data and computes fuzzy efficiency scores as well as fuzzy input and output targets. CRS and VRS cases are considered. DMUs are classified into three groups (namely FFDEA inefficient, FFDEA weakly efficient and FFDEA efficient) depending on whether all inputs, only some inputs or outputs or none of the inputs or outputs can be improved. A super-efficiency model to rank efficient DMUs is also presented. The proposed approach has been illustrated using a dataset from the literature comparing the proposed approach with some existing FDEA methods. The results show a notable consistency with those other approaches, having as an advantage over them that both the efficiency scores and the computed targets are fuzzy. The proposed approach has also been applied to a case study, thus showing its usefulness in practical settings.

As possible topics for further research we envisage extending the proposed approach to other DEA formulations, different from the radial CCR and BCC models. Also, especially in the VRS case, other approaches, different from super-efficiency, should be developed for ranking the FFDEA efficient DMUs.

References

Adler, N., Friedman, L. & Sinuany-Stern, Z. (2002). Review of ranking methods in the data envelopment analysis context, *European Journal of Operational Research*, 140, 249-265.

Aldamak, A., Hatami-Marbini, A., & Zolfaghari, S. (2016). Dual frontiers without convexity. *Computers & Industrial Engineering*, *101*, 466-478.

Andersen, P. & Petersen, N.C. (1993). A procedure for ranking efficient units in data envelopment analysis,*Management Science*, 39, 1261-1264.

Azadi, M., Jafarian, M., Saen, R. F. & Mirhedayatian, S. M. (2015). A new fuzzy DEA model for evaluation of efficiency and effectiveness of suppliers in sustainable supply chain management context. *Computers & Operations Research*, 54, 274-285.

Bagherzadeh Valami, H., Nojehdehi, R. R., Abianeh, P. M. M. & Zaeri, H. (2013). Production possibility of production plans in DEA with imprecise input and output. *Research Journal of Applied Sciences, Engineering and Technology*, 5(17), 4264-4267.

Banker, R. D. (1984). Estimating most productive scale size using data envelopment analysis. *European Journal of Operational Research*, 17(1), 35-44.

Banker, R. D., & Chang, H. (2006). The super-efficiency procedure for outlier identification, not for ranking efficient units. *European Journal of Operational Research*, *175*(2), 1311-1320.

Banker, R. D., Charnes, A. & Cooper, W. W. (1984). Some models for estimating technical and scale inefficiency in data envelopment analysis. *Management Science*, 30, 1078–1092.

Biondi Neto, L., Meza, L. A., Gomes, E. G., & Bergiante, N. C. R. (2011). A geometrical approach for fuzzy DEA frontiers using different T norms. *WSEAS Transactions on Systems*, 10(5), 127-136.

Chameau, J. L. & Santamarina, J. C. (1987a). Membership functions I: Comparing methods of measurement. *International Journal of Approximate Reasoning*, 1(3), 287-301.

Chameau, J. L. & Santamarina, J. C. (1987b). Membership functions II: Trends in Fuzziness and Implications. *International Journal of Approximate Reasoning*, 1(3), 303-317.

Charnes, A., Cooper, W. W. & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research*, 2, 429-444.

Chen, C.B., & Klein, C.M. (1997). A simple approach to ranking a group of aggregated fuzzy utilities. *IEEE Transactions on Systems, Man, and Cybernetics–Part B: Cybernetics*, 27, 26-35.

Chen, Y. (2004). Ranking efficient units in DEA. *Omega*, 32, 213-219.

Chen, Y., Du, J., Huo, J. (2013). Super-efficiency based on a modified directional distance function. *Omega*, 41(3), 621-625.

Cook, W.D., Liang, L., Zha, Y. & Zhu, J. (2009). A Modified Super-efficiency DEA Model for Infeasibility. *Journal of Operational Research Society*, 60, 276-281.

Cooper, W.W., Park, K.S. & Yu, G. (1999). IDEA and AR-IDEA: models for dealing with imprecise data in DEA. *Management Science,* 45, 597-607.

Cooper, W.W., Park, K.S. &Yu, G. (2001a). An illustrative application of IDEA (imprecise data envelopment analysis) to a Korean mobile telecommunication company. *Operations Research*, 49, 807-820.

Cooper, W.W., Park, K.S. &Yu, G. (2001b) IDEA (imprecise data envelopment analysis) with CMDs (column maximum decision making units). *Journal of the Operational Research Society*, 52, 176-181.

Cooper, W.W., Seiford, L.M. & Tone, K. (2007). *Data envelopment analysis: a comprehensive text with models, applications, references and DEA-solver software*. Springer Science & Business Media.

Despotis, D.K. & Smirlis, Y.G. (2002). Data envelopment analysis with imprecise data. *European Journal of Operational Research*, 140, 24-36.

Dubois, D. & Prade, H. (1993). Fuzzy sets and probability: misunderstandings, bridges and gaps. In: *Proceedings of the Second IEEE International Conference on Fuzzy Systems*, IEEE, pp. 1059-1068.

Dubois, D. & Prade, H. (Eds.). (2012). *Fundamentals of fuzzy sets* (Vol. 7). Springer Science & Business Media.

Emami, M. R., Türksen, I. B. & Goldenberg, A. A. (1999). A unified parameterized formulation of reasoning in fuzzy modeling and control. *Fuzzy Sets and Systems*, 108(1), 59-81.

Emrouznejad, A., Tavana, M. & Hatami-Marbini, A. (2014). The state of the art in fuzzy data envelopment analysis. In: *Performance measurement with fuzzy data envelopment analysis, Studies in Fuzziness and Soft Computing*, vol. 309, Springer-Verlag, Berlin, pp. 1-45.

Entani, T., Maeda, Y. & Tanaka, H. (2002). Dual models of interval DEA and its extension to interval data. *European Journal of Operational Research*, 136, 32-45.

Ganesan, K. & Veeramani, P. (2006). Fuzzy linear programs with trapezoidal fuzzy numbers. *Annals of Operations Research*,143(1), 305-315.

Golany, B. & Roll, Y. (1989). An Application Procedure for DEA. *Omega*, 17, 237-250.

Guo, P., & Tanaka, H. (2001). Fuzzy DEA: a perceptual evaluation method. *Fuzzy Sets and Systems*, 119 (1), 149-160.

Hatami-Marbini, A., Agrell, P.J. & Emrouznejad, A. (2014). Interval data without sign restrictions. *Applied Mathematical Modelling*, 38(7-8), 2028-2036.

Hatami-Marbini, A., Emrouznejad, A. & Tavana, M. (2011a). A taxonomy and review of the fuzzy data envelopment analysis literature: two decades in the making. *European Journal of Operational Research*, 214(3), 457-472.

Hatami-Marbini, A., Saati, S. & Tavana, M. (2010). An ideal-seeking fuzzy data envelopment analysis framework. *Applied Soft Computing*, 10(4), 1062-1070.

Hatami-Marbini, A., Tavana, M. & Ebrahimi, A. (2011b). A fully fuzzified data envelopment analysis model. *International Journal of Information and Decision Sciences*, 3(3), 252-264.

Hatami-Marbini, A., Tavana, M., Agrell, P.J. & Saati, S. (2013). Positive and normative use of fuzzy DEA-BCC models: a critical view on NATO enlargement. *International Transactions in Operational Research*, 20, 411-433

Hersh, H. M. & Caramazza, A. (1976). A fuzzy set approach to modifiers and vagueness in natural language. *Journal of Experimental Psychology: General*, 105 (3), 254-276.

Hosseinzadeh Lotfi, F., Allahviranloo, T., Jondabeh, M. A. & Alizadeh, L. (2009). Solving a full fuzzy linear programming using lexicography method and fuzzy approximate solution. *Applied Mathematical Modelling*, 33(7), 3151-3156.

Kao, C. & Lin, P.H. (2012). Efficiency of parallel production systems with fuzzy data. *Fuzzy Sets and Systems*, 198, 83-98.

Kao, C. & Liu, S.T. (2000). Fuzzy efficiency measures in data envelopment analysis. *Fuzzy Sets and Systems*, 113 (3), 427–437.

Kao, C. & Liu, S.T. (2011). Efficiencies of two-stage systems with fuzzy data. *Fuzzy Sets and Systems*, 176, 20-35.

Kochen, M. & Badre, A. N. (1974). On the precision of adjectives which denote fuzzy sets. *Journal of Cybernetics*, 4(1), 49-59.

Kruse, R. & Meyer, K. D. (1987). *Statistics with vague data*. Springer Science & Business Media, Series: Theory and Decision Library B, Vol. 6.

Lee, H.S. & Zhu, J. (2012). Super-efficiency infeasibility and zero data in DEA. *European Journal of Operational Research*, 216(2) 429-433.

León, T., Liern, V., Ruiz, J. L. & Sirvent, I. (2003). A fuzzy mathematical programming approach to the assessment of efficiency with DEA models. *Fuzzy Sets and Systems*, 139(2), 407-419.

Lertworasirikul, S., Fang, S.C., Joines, J.A.& Nuttle, H.L.W. (2003). Fuzzy data envelopment analysis (DEA): a possibility approach. *Fuzzy Sets and Systems*, 139 (2), 379–394.

Lin, R. & Chen, Z. (2015). Super-efficiency measurement under variable return to scale: an approach based on a new directional distance function. *Journal of the Operational Research Society*, 66(9), 1506-1510.

Liu, X. (2001). Measuring the satisfaction of constraints in fuzzy linear programming. *Fuzzy Sets and Systems*,122(2), 263-275.

Lozano, S. (2014a). Process efficiency of two-stage systems with fuzzy data. *Fuzzy Sets and Systems*, 234, 36-49.

Lozano, S. (2014b). Computing fuzzy process efficiency in parallel systems. *Fuzzy Optimization and Decision Making*, 13 (1), 73-89.

Mahdavi-Amiri, N. & Nasseri, S.H. (2006). Duality in fuzzy number linear programming by use of a certain linear ranking function. *Applied Mathematics and Computation*, 180(1), 206-216.

Mahdavi-Amiri, N.& Nasseri, S.H. (2007). Duality results and a dual simplex method for linear programming problems with trapezoidal fuzzy variables. *Fuzzy Sets and Systems*, 158(17), 1961-1978.

Nakanishi, H., Turksen, I.B.& Sugeno, M. (1993). A review and comparison of six reasoning methods. *Fuzzy Sets and Systems*, 57(3), 257-294.

Norwich, A.M.& Turksen, I.B. (1982). The fundamental measurement of fuzziness. In: *Fuzzy Sets and Possibility Theory: Recent Developments*, R.R. Yager (ed.), Pergamon Press, New York, pp. 49-60.

Norwich, A.M.& Turksen, I.B. (1984). A model for the measurement of membership and the consequences of its empirical implementation. *Fuzzy Sets and Systems*, 12(1), 1-25.

Oden, G.C. (1979). Fuzzy propositional approach to psycholinguistic problems: An application of fuzzy set theory in cognitive science. In: *Advances in fuzzy set theory and applications*, North-Holland, Amsterdam, pp. 409-419.

Pourmahmoud, J., Hatami-Marbini, A., & Babazadeh, E. (2016). A comment on a new super-efficiency model in the presence of negative data. *Journal of the Operational Research Society*, *67*, 530-534.

Qin, R., & Liu, Y.K. (2010). A new data envelopment analysis model with fuzzy random inputs and outputs. *Journal of Applied Mathematics and Computing*, 33(1–2), 327-356.

Qin, R., Liu, Y., Liu, Z. & Wang, G. (2009). Modeling fuzzy DEA with type-2 fuzzy variable coefficients. In: *Lecture Notes in Computer Science*, vol. 5552, Springer, Berlin, pp. 25-34.

Raei Nojehdehi, R., Maleki Moghadam Abianeh, P. & Bagherzadeh Valami, H. (2012). A geometrical approach for fuzzy production possibility set in data envelopment analysis (DEA) with fuzzy input-output levels. *African Journal of Business Management*, 6(7), 2738-2745.

Raei Nojehdehi, R., Maleki Moghadam, P. & Bagherzadeh Valami, H. (2011). Fuzzy efficient frontier in data envelopment analysis with interval data. *American Journal of Scientific Research*, 40,126-133.

Raei Nojehdehi, R., Valami, H. & Najafi, S. E. (2011). Evaluating the membership degree of the DMU's related to the fuzzy frontier in DEA with fuzzy input-output levels: A geometrical approach. *Trends in Advanced Science and Engineering*, 4, 7-15.

Saati, S., Memariani, A. & Jahanshahloo, G.R. (2002). Efficiency analysis and ranking of DMUs with fuzzy data. *Fuzzy Optimization and Decision Making*, 1, 255-267.

Seiford, L. M., & Zhu, J. (1999). Infeasibility of super-efficiency data envelopment analysis models. *INFOR: Information Systems and Operational Research*, *37*(2), 174-187.

Seiford, L.R., & Zhu, J. (1999). Infeasibility of super-efficiency data envelopment analysis models, *INFOR*, 37, 174-187.

Sengupta, J.K. (1992). A fuzzy systems approach in data envelopment analysis. *Computers & Mathematics with Applications*, 24 (8-9), 259-266.

Shokouhi, A.H., Hatami-Marbini, A., Tavana, M. & Saati, S. (2010). A robust optimization approach for imprecise data envelopment analysis. *Computers & Industrial Engineering*, 59, 387-397.

Shokouhi, A.H., Shahriari, H., Agrell, P.J. & Hatami-Marbini, A. (2014). Consistent and robust ranking in imprecise data envelopment analysis under perturbations of random subsets of data.*OR Spectrum*, 36 (1), 133-160.

Stanciulescu, C., Fortemps, Ph., Installé, M. & Wertz, V. (2003). Multiobjective fuzzy linear programming problems with fuzzy decision variables. *European Journal of Operational Research*, 149, 654-675.

Sugeno, M. & Yasukawa, T. (1993). A fuzzy-logic-based approach to qualitative modeling. *IEEE Transactions on Fuzzy Systems*, *1*(1), 7-31.

Tanaka, H. & Guo, P. (1999). *Possibilistic Data Analysis for Operations Research*. Physica-Verlag, Heidelberg.

Tanaka, H., Guo, P. & Zimmermann, H.J. (2000). Possibility distributions of fuzzy decision variables obtained from possibilistic linear programming problems. *Fuzzy Sets and Systems*, 113, 323-332.

Tavana, M., Khanjani Shiraz, R., Hatami-Marbini, A., Agrell, P.J.& Paryab, K. (2012). Fuzzy stochastic data envelopment analysis with application to base realignment and closure (BRAC). *Expert System with Applications*, 39, 12247-12259.

Tone, K. (2002). A slacks-based measure of super-efficiency in data envelopment analysis. *European Journal of Operational Research*, 143, 32-41.

Triantis, K.P. & Girod, O. (1998). A mathematical programming approach for measuring technical efficiency in a fuzzy environment. *Journal of Productivity Analysis*, 10(1), 85–102.

Turksen, I. B. (1988). Stochastic fuzzy sets: A survey. In: *Combining Fuzzy Imprecision with Probabilistic Uncertainty in Decision Making*, Springer, Berlin, pp. 168-183.

Turksen, I. B. (1991). Measurement of membership functions and their acquisition. *Fuzzy Sets and Systems*, 40(1), 5-38.

Wang, Y.M., Greatbanks, R. & Yang, J.B. (2005). Interval efficiency assessment using data envelopment analysis. *Fuzzy Sets and Systems*, 153 (3), 347-370.

Wang, Y.M., Luo, Y. & Liang, L. (2009). Fuzzy data envelopment analysis based upon fuzzy arithmetic with an application to performance assessment of manufacturing enterprises. *Expert Systems with Applications*, 36, 5205-5211.

Wu, H. C. (2008). Optimality conditions for linear programming problems with fuzzy coefficients. *Computers & Mathematics with Applications*, 55(12), 2807-2822.

Zadeh, L.A. (1965). Fuzzy sets. *Information and Control*, 8, 338–353.

Zhu, J. (1996). Robustness of the efficient DMUs in data envelopment analysis. *European Journal of Operational Research*, *90*(3), 451-460.

Zimmermann, H.J. (2001). *Fuzzy Sets Theory and Its Applications*. Kluwer Academic Publishers, Boston.

Zysno, P. (1981). Modelling membership functions. *Empirical Semantics*, 1, 350-375.

Appendix A

Membership functions form in practice

In practice, constructing membership functions form is an important concern to most practitioners who may want to incorporate fuzzy set theory into decision support tools. In the experimental research, there are two main streams to construct membership functions; (i) one stream makes an attempt to experimentally validate the assumptions of fuzzy set theory, and (ii) a given interpretation for the "degree if membership" is adopted and then tries to provide its elicitation. Due to the necessity of eliciting both approaches, Dubois and Prade (2012) categorized the existing methods into six groups that can be used to subjectively interpret membership function in experiments (Norwich and Türkşen 1982; Chameau and Satamarina 1978a, 1978b, Türkşen 1991) as follows:

1) *Polling* studies whether fuzziness arise from interpersonal disagreements by means of asking the questions to various persons engaged in the process, for instance, "do you agree that production quality is good?". The responses are then polled and an average is

taken into account to construct membership functions (e.g., Herch and Caramazza, 1976; Norwich and Türkşen, 1984).

2) *Direct rating* as the simplest method to constructing membership function describes fuzziness from individual subjective vagueness. In this regard, one must meticulously design the experiment with the aim of being harder to retain past responses. According to this approach, Herch and Caramazza (1976) implemented a repetition of their experiment for a single subject over and over in time. The answers of one certain subject lead to differentiation between linguistic and logistic interpretation of membership functions. Alternatively, direct rating can be applied to make a comparison between a subject and a predefined membership function (Türkşen 1988, 1991).
3) *Reverse rating* uses for individuals or a group of individuals by repeating a certain question for a certain membership function so that the subject is given a membership degree and then looked for the object. When the answer of the subjects is determined the conditional distributions can be normally distributed and the unknown mean and variance can be estimated as usual. Chameau and Satamarina (1978a) viewed reverse rating as powerful tool to verify the membership function estimated by another approach.
4) *Interval estimation* articulates the random set-view of the membership function against its vagueness view. This approach is more suitable to circumstances where there is a linear ordering for measuring the fuzzy concept exists such as heat and time (book). In addition, this approach is a straightforward way to construct the membership function that is *less fuzzy* in comparison with polling and direct rating. Chameau and Satamarina (1978a) considers interval estimation more effective compared with polling and direct rating where the response essentially precise (Yes/No). Interval estimation bridges the gap between probability theory and fuzzy set theory using random sets and possibility measures (Dubois and Prade 1993). Also, Kruse and Meyer (1987) discussed the set-values statistics methods as a likelihood interpretation of the membership function.
5) *Membership exemplification* has no repetition to drop the effects of noise (Hersh and Caramazza, 1976). Using exemplification in Kochen and Badre (1974) resulted in a more precise membership function. Zysno (1981) applied this approach by inquiring 64 subjects from 21 to 25 years old to rate 53 various statements of age by means of one of the following four sets: very young man, young man, old man, and vey old man, by using

a scale from 0 to 100. Next, he tested the earlier hypothesis on the nature of the membership function.

6) *Pairwise comparison* is an experimental-based approach for the "precision" of membership function. Kochen and Badre (1974) reported experimental results for expressing the precision of greater, much greater and very much greater. Oden (1979) argued the advantage of fuzzy set theory using comparisons: "which is a better example of a bird: an eagle or a pelican?", and after saying "an eagle", "how much more of a bird is an eagle than a pelican?". Note that in pairwise comparison it is a need to provide many comparison experiments even in a quite simple situation.

The techniques explained above strive to provide the subjective interpretation of the membership function. However, the way of constructing the membership functions form from a particular data set is in question, that is, a question may come to mind over and over when using the fuzzy sets: how to come up with the appropriate membership function form in the presence of the proper amount of data? Hence, one can proceed with the following steps to form the membership functions:

1) Data collection includes the data sampled from the system that provides valuable information but three cautions be necessarily taken in the collecting data procedure:
   i. Checking dependency or contingency between the input and output variables to make sure the validity of the model. The trend analysis or cross-correlation analysis can be applied to identify the type of the relationship; non-linear or linear relationship.
   ii. Checking interrelationship between the input variables to make sure that those variables are independent in the data set.
   iii. Removing the highly effective noise from the model such as the major information preserves intact since reducing data often bring a negative effect on the efficiency of the results.
2) Once the adequate partition of the output space is provided the membership functions form will be approximated for the entire output space. The fuzzy clustering algorithm by means of appropriate trapezoids is a prevalent method to approximate the membership grades (Sugeno and Yasukawa, 1993). After making

the partition the entire space, one can approximate the classified data by trapezoidal functions in which convex points are used for each fuzzy cluster to fit a trapezoid (Nakanishi et al., 1993).

3) To opt for the significant input variables, for each input candidate p_j the membership functions K_{ij}(i=1,2,…,n) are constructed so that an index can be defined as $h_j = \prod_{i=1}^{n} \eta_{ij} / \eta_j ,\ j = 1,...,l$ where η_{ij} is the range when the membership functions K_{ij} equals to one, η_j stands for the range ofp_j, n is the number of rules and l is the number of input candidates. h_j with the small value indicates more dominant variable p_j.
4) The convex membership functions K_{ij} for significant inputs p_j are constructed by means of η_{ij} and performing "fuzzy line clustering" (Emami, 1999).
5) It is a need to consider a testing set on those data, which is beyond the range of all membership functions.

List of tables

Table 1. Computational comparison for different existing FDEA models

Table 2. Five DMUs with two fuzzy inputs and two fuzzy outputs

Table 3. FFDEA relative fuzzy efficiencies and classification

Table 4. Fuzzy input and output targets

Table 5. CRS FFDEA super-efficiency for Guo and Tanaka (2001) dataset

Table 6. Fuzzy efficiencies computed by other existing FDEA methods

Table 7. Computational comparison for Example 1 (alpha=0,0.1,0.2,...,1)

Table 8. Results for the case study (Example 2)

Table 1. Computational comparison for different existing FDEA models

Method		Membershipf unc.	No. of LP models	No. of constraints	No. of nonnegativec onstraints	No. of variables
Kao and Liu		Trapezoidal	$2nq$	n	$m+s$	$m+s+1$
Guo and Tanaka		Symmetric triangular	nq (Primary model) nq (Secondary model)	$3+2n$ (Primary model) 2 (Secondary model)	$m+s$ (Primary model) m (Secondary model)	$m+s$ (Primary model) m (Secondary model)
Saati et al.		Triangular	nq	$n(1+2m+2s)+1$	$m+s$	$(m+s)(n+1)$
León et al.		Trapezoidal	nq	$4(m+s)+1$	n	$n+1$
Lertworasirikul et al.		Trapezoidal	nq	$3+n$	$m+s$	$m+s+1$
Model (9)	Model (10)	Trapezoidal	n	$7(s+m)+3(n+1)$	$n+m+s$	$4(m+s+n+1)$
	Model (11)		n	$7(s+m)+3(n+1)$	$n+m+s$	$4(m+s+n)+3$
	Model (12)		n	$7(s+m)+3n+2$	$n+m+s$	$4(m+s+n)+2$
	Model (13)		n	$7(s+m)+3n+1$	$n+m+s$	$4(m+s+n)+1$
Model (14)			n	$7(s+m)+3(n+1)$	$n+m+s$	$4(m+s+n)$
Model (17)	1st model		n	$7(m+s)+n+3$	$n+m+s$	$4(m+s+1)+n$
	2nd model		n	$7(m+s)+n+3$	$n+m+s$	$4(m+s)+n+3$
	3rd model		n	$7(m+s)+n+2$	$n+m+s$	$4(m+s)+n+2$
	4th model		n	$7(m+s)+n+1$	$n+m+s$	$4(m+s)+n+1$

Table 2. Five DMUs with two fuzzy inputs and two fuzzy outputs

DMU	A	B	C	D	E
Input 1	(3.5,4.0,4.5)	(2.9,2.9,2.9)	(4.4,4.9,5.4)	(3.4,4.1,4.8)	(5.9,6.5,7.1)
Input 2	(1.9,2.1,2.3)	(1.4,1.5,1.6)	(2.2,2.6,3.0)	(2.2,2.3,2.4)	(3.6,4.1,4.6)
Output 1	(2.4,2.6,2.8)	(2.2,2.2,2.2)	(2.7,3.2,3.7)	(2.5,2.9,3.3)	(4.4,5.1,5.8)
Output 2	(3.8,4.1,4.4)	(3.3,3.5,3.7)	(4.3,5.1,5.9)	(5.5,5.7,5.9)	(6.5,7.4,8.3)

Table 3. FFDEA relative fuzzy efficiencies and classification

DMU	$\tilde{\theta}_p^{CRS,*}$	Optimal obj. func. value Phase II	RS_p	CRS Classif. FFDEA
A	(0.911, 0.911, 0.911)	2.058	B, D	inefficient
B	(1, 1, 1)	0.000	-	efficient
C	(0.821, 0.873, 0.911)	0.554	B, D, E	inefficient
D	(1, 1, 1)	0.000	-	efficient
E	(1, 1, 1)	0.000	-	efficient

Table 4. Fuzzy input and output targets

DMU	Input 1	Input 2	Output 1	Output 2
A	(3.188, 3.497, 3.818)	(1.672, 1.854, 2.036)	(2.40, 2.60, 2.80)	(4.016, 4.410, 4.808)
C	(3.574, 4.243, 4.882)	(1.805, 2.270, 2.733)	(2.70, 3.20, 3.70)	(4.300, 5.197, 6.114)

Table 5. CRS FFDEA super-efficiency for Guo and Tanaka (2001) dataset

DMU	$\theta_{p,4}^{SE-CRS,*}$	Rank
B	1.503	2
D	1.615	1
E	1.081	3

Table 6. Fuzzy efficiencies computed by other existing FDEA methods

Ref.	α	DMU				
		A	**B**	**C**	**D**	**E**
GT	0.0	(0.66, 0.81, 0.99)	(0.88, 0.98, 1.09)	(0.60, 0.82, 1.12)	(0.71, 0.93, 1.25)	(0.61, 0.79, 1.02)
	0.5	(0.75, 0.83, 0.92)	(0.94, 0.97, 1.00)	(0.71, 0.83, 0.97)	(0.85, 0.97, 1.12)	(0.72, 0.82, 0.93)
	0.75	(0.80, 0.84, 0.88)	(0.96, 0.99, 1.02)	(0.77, 0.83, 0.90)	(0.92, 0.98, 1.05)	(0.78, 0.83, 0.89)
	1.0	(0.85, 0.85, 0.85)	(1.00, 1.00, 1.00)	(0.86, 0.86, 0.86)	(1.00, 1.00, 1.00)	(1.00, 1.00, 1.00)
L.	0.0	1.107	1.238	1.276	1.520	1.296
	0.25	1.032	1.173	1.149	1.386	1.226
	0.5	0.963	1.112	1.035	1.258	1.159
	0.75	0.904	1.055	0.932	1.131	1.095
	1.0	0.855	1.000	0.861	1.000	1.000
S.	0.0	1.000	1.000	1.000	1.000	1.000
	0.5	0.954	1.000	1.000	1.000	1.000
	0.75	0.901	1.000	0.929	1.000	1.000
	1.0	0.855	1.000	0.862	1.000	1.000

Note: GT = Guo and Tanaka (2001), L = Lertworasirikul et al. (2003), S = Saati et al. (2002)

Table 7. Computational comparison for Example 1 (alpha=0,0.1,0.2,...,1)

Method		No. of LP models	No. of constraints	No. of nonnegativeconstraints	No. of variables
KL		110	5	4	5
GT.		55 (Primarymodel) 55 (Secondarymodel)	13 (Primarymodel) 2 (Secondarymodel)	4 (Primarymodel) 2 (Secondarymodel	4 (Primarymodel) 2 (Secondarymodel)
S.		55	46	4	24
LE.		55	17	5	6
L.		55	8	4	5
Model (9)	Model (10)	5	36	9	30
	Model (11)	5	36	9	29
	Model (12)	5	35	9	28
	Model (13)	5	34	9	27
Model (14)		5	36	9	30
Model (17)	1st model	5	30	9	20
	2nd model	5	30	9	19
	3rd model	5	29	9	18
	4th model	5	28	9	17

Note: GT = Guo and Tanaka (2001), L = Lertworasirikul et al. (2003), S = Saati et al. (2002), KL=Kao and Liu (2000), LE= León et al. (2003)

Table 8. Results for the case study (Example 2)

Suppliers	Efficiency ($\tilde{\theta}_p^{CRS,*}$)	CRS Classif. FFDEA	Input 1(TC)	Input 2(NS)	Input 1(ED)	Input 2(CS)	Output 1(NOT)	Output 2(NB)
1	(1,1,1)	efficient.	-–	–	–	–	–	–
2	(0.920, 0.943, 0.963)	efficient	(258.410, 265.053, 270.568)	(150.816, 154.693, 157.911)	(38.929, 39.930, 40.761)	(19.312, 19.808, 20.220)	(153, 173, 193)	(55.929, 62.990, 70.041)
3	(0.936, 0.936, 0.936)	inefficient	(289.140, 289.140, 289.140)	(178.526, 178.526, 178.526)	(32.668, 32.668, 32.668)	(37.429, 37.429, 37.429)	(203, 224.828, 246.656)	(78, 85.640, 93.279)
4	(0.836, 0.842, 0.847)	inefficient	(243.323, 245.110, 246.582)	(149.833, 150.916, 151.808)	(29.137, 29.326, 29.482)	(32.538, 32.776, 32.972)	(167, 187, 207)	(85, 92, 99)
5	(0.970, 0.970, 0.970)	inefficient	(414.410, 414.410, 414.410)	(172.602, 172.602, 172.602)	(50.423, 50.423, 50.423)	(28.120, 28.120, 28.120)	(197, 220.774, 244.548)	(163, 171.321, 179.642)
6	(1,1,1)	efficient	-–	–	–	–	–	–
7	(1,1,1)	efficient	-–	–	–	–	–	–
8	(1,1,1)	efficient	-–	–	–	–	–	–
9	(1,1,1)	efficient	-–	–	–	–	–	–
10	(0.788, 0.829, 0.863)	inefficient	(208.602, 218.219, 226.217)	(134.906, 142.016, 147.929)	(32.317, 34.002, 35.404)	(12.612, 13.269, 13.816)	(113, 133, 153)	(88, 95.061, 102.111)
11	(0.938, 0.952, 0.963)	inefficient	(288.063, 292.032, 295.304)	(113.493, 115.151, 116.518)	(48.598, 49.379, 50.023)	(26.504, 26.947, 27.311)	(125, 145, 165)	(153, 160, 167)
12	(0.855, 0.855, 0.855)	inefficient	(281.237, 281.237, 281.237	(172.313, 172.313, 172.313)	(32.483, 32.483, 32.483)	(37.545, 37.545, 37.545)	(195, 216.459, 237.919)	(90, 97.511, 105.021)
13	(0.782, 0.782, 0.782)	inefficient	(338.945, 338.945, 338.945)	(154.78, 154.78, 154.78)	(25.04, 25.04, 25.040)	(32.866, 32.866, 32.866)	(156, 177.549, 199.097)	(139, 146.542, 154.084)
14	(0.765, 0.788, 0.807)	inefficient	(198.086, 204.133, 209.115)	(124.665, 128.33, 131.35)	(27.614, 28.253, 28.779)	(26.667, 27.472, 28.135)	(129, 149, 169)	(97, 104, 111)
15	(0.486, 0.531, 0.568)	inefficient	(133.165, 145.552, 155.758)	(83.224, 90.732, 96.917)	(18.468, 19.777, 20.856)	(17.820, 19.469, 20.827)	(85, 105, 125)	(68, 75, 82)
16	(1,1,1)	efficient	-–	–	–	–	–	–
17	(1,1,1)	efficient	-–	–	–	–	–	–
18	(0.549, 0.567, 0.582)	inefficient	(235.506, 243.314, 249.760	(113.636, 117.403, 120.514)	(31.291, 32.328, 33.185)	(24.091, 24.893, 25.555)	(142, 162, 182)	(46, 53, 60)
19	(1,1,1)	efficient	-–	–	–	–	–	–
20	(0.566, 0.592, 0.614)	inefficient	(218.040, 228.095, 236.453)	(130.470, 136.686, 141.852)	(32.737, 34.469, 35.909)	(12.459, 13.034, 13.512)	(106, 126, 146)	(119, 126.349, 133.639)
21	(0.895, 0.910, 0.923)	inefficient	(222.966, 226.672, 229.725)	(138.469, 140.715, 142.565)	(28.396, 28.787, 29.110)	(29.893, 30.387, 30.793)	(150, 170, 190)	(90, 97, 104)
22	(1,1,1)	efficient	-–	–	–	–	–	–
23	(0.782, 0.782, 0.782)	inefficient	(285.335, 285.335, 285.335	(179.892, 179.892, 179.892)	(40.235, 40.235, 40.235)	(38.434, 38.434, 38.434)	(185, 206.962, 228.924)	(143, 150.687, 158.373)
24	(0.875, 0.894, 0.909)	inefficient	(258.991, 264.443, 268.975)	(136.255, 139.748, 142.651)	(35.265, 36.190, 36.959)	(15.749, 16.089, 16.372)	(112, 132, 152)	(177, 184.133, 191.244)
25	(0.583, 0.629, 0.667)	inefficient	(174.048, 185.804, 195.551)	(101.546, 109.852, 116.737)	(22.743, 24.533, 26.017)	(12.829, 13.839, 14.676)	(94, 114, 134)	(78, 85, 92)
26	(0.796, 0.796, 0.796)	inefficient	(260.226, 260.271, 260.307)	(162.315, 162.342, 162.364)	(34.219, 34.224, 34.228)	(34.931, 34.937, 34.941)	(173, 193, 213)	(113, 120, 127)