Design Economic Policies that do not Create Bumpy Recovery

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This chapter studies how to make sure when the economic performance indicators approach some pre-determined target values due to the effect of adopted policies, the economy also develops reasonably well without experiencing much severe up and-down fluctuations. This problem of concern is resolved by addressing the corresponding problem of pole placement of the general control-theory model of the economic system. This chapter (1) discusses conditions under which the poles of a constant coefficient linear economic system can be arbitrarily placed, (2) provides a way to calculate the matrix of feedback gain that is useful in placing the poles by using the feedback control mechanism so that the resultant constant coefficient linear economic system possesses a good quality stability and fast response speed, and (3) investigates the problem of how to design state or output feedbacks (economic policies) so that the resultant closed-loop economic system will have the pre-determined poles. At the end, some open problems of great importance are posed for future research. This chapter relates the issue of economic policy making and the pole placement of the general control-theory model of the economic system. The results are expected to provide practically useful guidelines.

1. Introduction

Since the 1990s, with the economic integration and the financial market globalization, the frequency and contagion of financial crises have increased tremendously (Table 1). That has had profound impacts on the globalization and captured the attention of many governments and scholars from around the world. In particular, the global financial crisis, triggered by the U.S. subprime mortgage crisis in 2007, had shown a new feature that not only has a more complex transmission channel and effect but also makes it possible for a non-systemic risk of a single country or local region to become a global financial systemic risk along the path of financial globalization and economic integration. This feature created far more severe aftermath consequences than any of the previous financial crises and brings forward new challenges to the existing theories of financial crises and regulations. Therefore, it is necessary to reexamine the contagion problem of global financial crises and to consider building a supervision system for the outbreak of crises and consequent contagion. To this end, it is important for us to look for sufficient conditions of dynamic path controllability of economic systems, while this effort surely has great theoretical value for follow-up studies on the prevention of financial crises (Chen and Ying, 2012; 2014).

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Table 1. Several major financial crises since the 1990s

TIME	CRISIS NAME	COUNTRY	TYPE OF COUNTRY
1990	Bank of Scandinavia Crisis	Finland, Sweden and Norway	Developed Economy
1992	Japan's Asset Price Bubble Crisis	Japan	Developed Economy
1994	Mexico Economic Crisis	Mexico	Emerging Economy
1997	Asian Financial Crisis	Thailand	Emerging Economy
1998	Russian Financial Crisis	Russia	Emerging Economy
1999	Brazil Financial Crisis	Brazil	Emerging Economy
2001	Argentina Financial Crisis	Argentina	Emerging Economy
2001	Dotcom Bubble Crisis	America	Developed Economy
2007	Subprime Crisis	America	Developed Economy

Resources: Chen and Ying, 2012

In practical studies of the economy, we require to a certain degree not only the economic system to be controllable so that the controlled variable, such as the GDP, the inflation, etc., approaches the target value, but also the economic system performs reasonable well in its process of approaching the target value. To achieve this end, it involves addressing the problem of pole placement of general systems (Cobb, 1981; Kaczorek, 1985; Ogata, 1995; Ram, Mottershead and Tehrani, 2011; Kirk, 2012; Zubov, Mikrin and Misrikhanov, et.al, 2013).

From (Yang, Ying, and Forrest, to appear), it follows that when the real parts of the eigenvalues of matrix A are all negative, the economic system is asymptotically stable. Although the feedback control strategy that is designed by using Lyapunov method can help to materialize the goal of regulating the economy, the process for the performance of the economic system to approach the targeted values might still experience major fluctuations and might take a long time to actually reach the targets. This end has a lot to do with the eigenvalues of A which are in fact the poles of the economic system. In particular, if the poles of the economic system, or the eigenvalues of matrix A, are located on the left half plane but near the imaginary axis, then in its process of approach the targeted values, the performance of the economy will suffer from severe up and down fluctuations. Such severe up-and-down fluctuations had been experienced during the process of dealing with the 1998 Russian financial crisis.

In October 1997, a financial crisis broke out in Southeast Asia. And in the following ten months, Russia made unremitting efforts to survive. During the time period, Russia issued large amounts of national debts and sacrificed a lot of foreign exchange in the market, which drastically reduced Russian foreign exchange and gold reserves. Therefore, Russian government faced a dilemma: either continue to maintain the floating exchange rate policy of the "currency corridor" or support the bond market. Eventually Russian government decided to go with the former choice. However, the financial situation did not turn for the better; and ruble started to depreciate drastically.

Facing the continuous occurrence of new turmoil in the financial market, Russian government introduced an economic grogram to stabilize the financial situation. However, the program did not gain enough investor confidence. On August 13, 1998, George Soros, a renowned international speculator, publically suggested for Russian government to depreciate ruble in the scale of 15% - 25%. On that day, the price index of 100 industrial company stocks, as calculated by Interfax, lost over 70% of its value and fell sharply to the level of about 26% of the value at the start of the year. At the same time, the tax collected during the month of July was only about 12 billion ruble, while the operational budget for each month was no less than 20 billion ruble, showing a huge gap between the income and expense. Under the pressure of the internal and external difficulties, on August 17, the government launched three tough emergency measures:

- The first was to expand the floating range for the exchange rate of ruble while lowering the upper limit of the ruble's exchange rate against the dollar to 9.5:1. That in fact declared the depreciation of ruble against the dollar from 6.295 to 9.5, more than 50% depreciation. So, the market predicted based on this emergency measure that the exchange rate of ruble would continue to drop drastically. And indeed, in the next 10 days, ruble fell to 20 – 21:1, which busted the stable exchange rate of the past three plus years.
- The second was to delay the payments, which were due, of foreign debts, which were estimated to be around US$15 billion, for 90 days.
- The third was to lengthen the repayment periods of domestic debts, making all national debts, totaling about US$20 billion, that would mature by December 31, 2099, become mid-term debts that would mature in the next 3, 4, or 5 years. And before finishing the change of maturities, the national debt market was closed for trading.

As soon as these three measures were announced, they immediately caused public outcry, the stock market crashed and closed for trading, and the exchange rate of ruble plunged. Afterward, the central bank altogether declared that it would allow ruble to float freely. That caused the public to either run for ruble in order to exchange for the U.S. dollar or buy anxiously consumer goods. Along with the fall of ruble's exchange rate, the stock market plummeted much further. As of the end of August 28, the price index of the 100 industrial company stocks, as calculated by Interfax, fell to US$15.92 billion, which represented a fall of 85% when compared to the level of US$103.356 billion reached at the start of the year. And then the market simply closed down, making the price index become worthless.

Although all the responses adopted by Russian government helped to reduce the economic loss of the nation, they also created major obstacles for the recovery of the domestic economy.

Firstly, half of the deposits of the domestic residents were lost. The prices of imported goods rose 2 to 3 times, which also made the prices of domestic goods go up drastically. In September, the consumer prices went up 40%, which was more than the 36% rise that occurred in February 1992, the highest since the start of the economic reform. People's actual wages went down 13.8%, making nearly 1/3 of all the residents live under the poverty line. The overall economy dropped 5%, industry 5.2%, agriculture 10%, and foreign trade 16.1%.

Secondly, a large number of commercial banks, especial those big banks, suffered heavy losses. The SBS agriculture bank, one of the seven financial giants, at the time held short-term national debts in the equivalent amount of US$1 billion, which became worthless instantly. It was estimated that about one half of the commercial banks were on the verge of bankruptcy.

Thirdly, this Russian financial crisis spread over its national border and affected Europe, the United States, Latin America, becoming a global effect. Foreign investors lost about US$33 billion in this huge financial storm, where American long term capital management firms (or hedge funds) lost about US$2.5 billion, George Soros' Quantum Fund lost around US$2 billion, American Bankers Trust lost somewhere near US$0.49 billion. Germany was the largest creditor of Russia, which owed Germany 75 billion marks (about US$44.4 billion), most of which were government-guaranteed bank loans. So, any trouble that appeared on the Russian financial market affected the safety of German creditors, creating shocks on the German market. Then the shock waves were spread over to the entire European financial market. For example, Frankfurt DAX index once fell over 3%, the CAC40 index of Paris stock market dropped 1.76%, Amsterdam stock market lowered 2%, Zurich stock market lost 1.6%, etc. For related details, please consult with (Dabrowski, 2012; Kenourgios and Padhi, 2012; Kenourgios, Samitas, and Paltalidis, 2011; Bisignano, Hunter, and Kaufman, 2012; Gluschenko, 2015; Razin and Rosefielde, 2011).

Therefore, in the process of actually controlling the conflict of currencies, we need the adopted feedback control strategies to show desirable good qualities in order to reduce as much as possible the severe systemic fluctuations caused by the employed control strategies, such as the excessive chaos of the market caused by the measures adopted by Russian government to control the exchange rate. Additionally, when not all of the eigenvalues of matrix A are located on the left half of the complex plane, the economic system is not asymptotically stable, and the feedback strategies designed by using Lyapunov method cannot make the performance of the economy approach the pre-determined targets. Hence, it is necessary for us to establish new methods not only to guarantee that the process for the performance of the economy (or the output of the economic system) to approach the pre-determined targets exhibits desirable qualities but also to make the performance of the economy actually approach the pre-determined targets by designing relatively good feedback control strategies, even if the economic system itself is not asymptotically stable. To this end, the design of such feedback control strategies is closely related to the poles of the economic system, which is known as a method for pole placement.

Additionally, from the discussion in (Yang, Ying, and Forrest, to appear), it follows that state feedback affects the poles of any closed loop system; and that for certain open loop instable economic systems, we can design a feedback control so that the resultant closed loop system is stable.

Results in (Yang, Ying, and Forrest, to appear) indicate that the poles of the economic system determine the stability of the system. Additionally, some other properties of the economic system are also determined by the location of the poles. For example, the dynamic characteristics of the system are greatly influenced by the location of the poles of the closed loop system. This chapter discusses the problem of how to design feedback so that the resultant closed loop economic system will have the pre-determined poles, known as the problem of pole placement. In other words, the so-called pole placement is to place the poles of the closed loop system at desirable locations by using either state feedback or the output feedback. Because the performance of the economic system is closely related to the location of the system's poles, it makes the problem of pole placement occupy an important position in the study of feedback economic policy design. Here, we need to solve two problems. First, we need to establish the conditions under which the poles can be relocated; and second, we need to determine the feedback gain matrix that plays an important role in the poles relocation.

This chapter is organized as follows: Section 2 discusses the problem of how to place the poles of the economic system by using either state feedback or output feedback, and the problem of how to determine the desirable locations for the poles. Section 3 outlines the need to study the placement of eigenvectors and so the problem of eigenstructure assignment. Then, the presentation of this work is concluded in Section 4.

2. Pole Placement

2.1. Pole Placement by Using State Feedback

First of all, let us look at the following problem: For what kind of economic system can we arbitrarily place its poles? Because the locations of the poles determine some of the very important properties of the economic system, the ability to arbitrarily relocate these points means the capability to alter at one's will some of these important characteristics of the economy through using feedback. Hence the ability to relocate the poles should be closely related to the controllability of the economic system. Based on this reasoning, we will address this problem by using the form of controllable systemic structures.

Assume that an economy can be written as a constant coefficient linear system, which takes the following structural form and is (A_1, B_1) controllable.

$$\begin{bmatrix} \frac{dx_c}{dt} \\ \frac{dx_{Nc}}{dt} \end{bmatrix} = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_c \\ x_{Nc} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u \tag{1}$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_c \\ x_{Nc} \end{bmatrix} \tag{2}$$

Evidently, the set of poles of this system is $\sigma(A_1) \cup \sigma(A_2)$. By using the following feedback

$$u = r - Kx = r + \begin{bmatrix} K_c & K_{Nc} \end{bmatrix} \begin{bmatrix} x_c \\ x_{Nc} \end{bmatrix}$$

where $-K = \begin{bmatrix} K_c & K_{Nc} \end{bmatrix}$, we obtain the following closed loop system:

$$\begin{bmatrix} \frac{dx_c}{dt} \\ \frac{dx_{Nc}}{dt} \end{bmatrix} = \begin{bmatrix} A_1 + B_1 K_c & A_{12} + B_1 K_{Nc} \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_c \\ x_{Nc} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} r$$

whose set of poles is $\sigma(A_1 + B_1 K_c) \cup \sigma(A_2)$.

From the discussion in (Yang, Ying, and Forrest, to appear), it follows that $\sigma(A_2)$ should stay the same with the applied feedback. Therefore, it means that if the economic system in equ. (1) is not completely controllable, then we cannot arbitrarily relocate the system's poles by using feedback. Because linear transformations do not change a system's controllability and poles, this conclusion holds true generally for general constant coefficient linear systems. On the other hand, if a constant coefficient linear system is completely controllable, can the system's poles be arbitrarily placed? The answer is yes. So, we have the following important result.

Theorem 1. A sufficient and necessary condition for an economic system (A, B) to arbitrarily place its poles is that the system is completely (A, B) controllable.

Assume that A is not a stable matrix. If there is K such that the matrix $A+BK$ of the resultant closed loop system as obtained from using the feedback $u = r + Kx$ is a stable matrix, then we say that the original economy is capable of being stable.

Theorem 2. A sufficient and necessary condition for an economic system to be capable of being stable is that all elements in $\sigma(A_2)$ is of negative real parts.

This result is also a natural consequence of the previous discussions. The problem of finding K such that $A+BK$ is stable is referred to as the problem of stabilization.

In the following, let us look at the specific details of how to place the poles of a controllable economic system. First, let us look at the case of economy which involves only a single performance index and a single indicator.

Assume that the systemic representation of the economy is

$$\frac{dx}{dt} = Ax + bu \tag{3}$$

which is controllable and written in the canonical form so that we have

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \ldots & \cdots & & & & \\ 0 & & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & \cdots & -a_{n-1} \end{bmatrix}, \; b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

The characteristic polynomial of this system is

$$|sI - A| = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

Under the effect of the feedback $u = kx + r = [k_1, \cdots, k_n]x + r$, we obtain the following closed loop system

$$\frac{dx}{dt} = (A + bk)x + br$$

where

$$A+bk=\begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & & & \\ 0 & & & & 0 & 1 \\ -a_0+k_1 & -a_1+k_2 & \cdots & \cdots & \cdots & -a_{n-1}+k_n \end{bmatrix}$$

The characteristic polynomial of the resultant closed loop system is

$$|sI-(A+bk)|=s^n+(a_{n-1}-k_n)s^{n-1}+\cdots+(a_1-k_2)s+(a_0-k_1) \tag{4}$$

Assume that we like to place the poles of the system at n given locations on the complex plane: $\lambda_1,\lambda_2,\cdots,\lambda_n$. From these n points, we can calculate the characteristic polynomial

$$(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)=s^n+\alpha_{n-1}s^{n-1}+\cdots+\alpha_1 s+\alpha_0 \tag{5}$$

If we assume that when there is one complex number in the list of the pre-determined poles $\lambda_1,\lambda_2,\cdots,\lambda_n$, its conjugate needs also appear in the list, then all the coefficients α_i in the previous equation are real numbers. By comparing equs. (4) and (5), we know that in order to materialize the required poles for the closed loop, we only need to select appropriate $k_1,k_2,\cdots,k_n$ such that

$$\begin{cases} \alpha_0=a_0-k_1 \\ \alpha_1=a_1-k_2 \\ \vdots \\ \alpha_{n-1}=a_{n-1}-k_n \end{cases}$$

That is, the following vector will satisfy our need.

$$k=[a_0-\alpha_0,a_1-\alpha_1,\cdots,a_{n-1}-\alpha_{n-1}] \tag{6}$$

For the general controllable economic system in equ. (3) with a single performance indicator, we first transform it into the canonical form and then place its poles. To this end, we use the following transformation:

$$x=Tx'$$

$$T=\begin{bmatrix} q \\ qA \\ \vdots \\ qA^{n-1} \end{bmatrix}^{-1}$$

where $q=[0,0,\cdots,1]U^{-1}$ and $U=[b,Ab,\cdots,A^{n-1}b]^{-1}$. This transformation changes the original system in equ. (3) into

$$\frac{dx'}{dt}=T^{-1}ATx'+T^{-1}bu$$

Because $qU=[0,0,\cdots,1]=[qb,qAb,\cdots,qA^{n-1}b]$, we have

$$qb=0, qAb=0,\cdots,qA^{n-2}b=0, qA^{n-1}b=1$$

That implies

$$T^{-1}b=\begin{bmatrix} qb \\ qAb \\ \vdots \\ qA^{n-1}b \end{bmatrix}=\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Next we show that $T^{-1}AT$ is of our desired form. That is, we show

$$T^{-1}AT=\begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \ddots & \cdots & & & & \\ 0 & & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & \cdots & -a_{n-1} \end{bmatrix} \qquad (7)$$

Because the characteristic polynomial of A is

$$|sI-A|=s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0$$

we have

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \ddots & \cdots & & & & \\ 0 & & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & \cdots & -a_{n-1} \end{bmatrix} T^{-1}=\begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \ddots & \cdots & & & & \\ 0 & & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & \cdots & -a_{n-1} \end{bmatrix}\begin{bmatrix} q \\ qA \\ \vdots \\ qA^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} qA \\ qA^2 \\ \vdots \\ qA^{n-1} \\ q(-a_0 I - a_1 A - \cdots - a_{n-1} A^{n-1}) \end{bmatrix} = \begin{bmatrix} qA \\ qA^2 \\ \vdots \\ qA^{n-1} \\ qA^n \end{bmatrix} = T^{-1} A$$

So, equ. (7) is proven.

Therefore, we have transformed the original single input controllable system in equ. (3) into the canonical form and we can now arbitrarily place the poles according to equ. (6). In the previous equations we have used the following:

$$A^n = -a_0 I - a_1 A - \cdots - a_{n-1} A^{n-1}$$

So, based on equ. (6) we obtain the feedback $u = kx' + r$. By returning back to the original coordinate system x we produce

$$u = kT^{-1}x + r = Kx + r \tag{8}$$

That is what we wanted, where $K = kT^{-1}$.

Example 1. Assume that the control-theory model of an economic system is given as follows:

$$\frac{dx}{dt} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

Find a state feedback such that the resultant closed loop system has the following poles: -1, $-2+3i$, and $-2-3i$.

Solution. Step 1: let us compute the characteristic polynomial of the given model:

$$|sI - A| = s^3 - 3s^2 - 2s + 6 .$$

Step 2: Calculate the required characteristic polynomial

$$(s+1)(s+2-3i)(s+2+3i) = s^3 + 5s^2 + 17s + 13$$

From equ. (6), it follows that

$$k = [a_0 - \alpha_0, a_1 - \alpha_1, a_2 - \alpha_2] = [-7, -19, -8] .$$

For returning to the original coordinate system, we compute

$$U=\begin{bmatrix}1&1&2\\0&1&0\\0&0&1\end{bmatrix},\ q=[0\,0\,1]U^{-1}=[0\,0\,1],\ T^{-1}=\begin{bmatrix}q\\qA\\qA^2\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&3\\1&2&9\end{bmatrix}.$$

So, we have the follow desired feedback:

$$u=kT^{-1}x+r=\begin{bmatrix}-8&-35&-136\end{bmatrix}x+r\ .$$

Because when we represent an economy by using a control-theory model, the model most likely contains many variables. That is, such a control-theory model would generally be high dimensional. So, in the following let us investigate the problem of pole placement for an economy with multiple economic policy factors as the system's input.

Assume that the multiple input control-theory model of the economy is

$$\frac{dx}{dt}=Ax+Bu \tag{9}$$

We will provide a method to first solve for a state feedback to convert a multiple input controllable system into a single input controllable system, and then to employ the developed method to place the poles of the single input system to place the poles of the original multiple input system. In particular, we have the following details.

Assume that $B=\begin{bmatrix}b_1 & b_2 & \cdots & b_m\end{bmatrix}$. Because the system in equ. (9) is (A,B) controllable, the rank of $U=\begin{bmatrix}B\vdots AB\vdots\cdots\vdots A^{n-1}B\end{bmatrix}$ is n. Next, we re-arrange column vectors of U as follows:

$$\{b_1\ Ab_1\cdots A^{n-1}b_1;b_2\ Ab_2\cdots A^{n-1}b_2;\cdots;b_m\ Ab_m\cdots A^{n-1}b_m\}$$

From these $n\times m$ columns, we select from left to right all the linearly independent vectors and obtain the following matrix:

$$Q=[b_1\ Ab_1\cdots A^{\mu_1-1}b_1;b_2\ Ab_2\cdots A^{\mu_2-1}b_2;\cdots;b_m\ Ab_m\cdots A^{\mu_m-1}b_m] \tag{10}$$

Let

$$S=[0\cdots0\ \underset{\mu_1 th\ column}{\underset{\uparrow}{e_2}}\ ;\ 0\cdots0\ \underset{(\mu_1+\mu_2)th\ column}{\underset{\uparrow}{e_3}}\ ;\ \cdots;\ 0\cdots0\ \underset{(\mu_1+\cdots+\mu_{m-1})th\ column}{\underset{\uparrow}{e_m}}\ ;\ 0\cdots0\ \underset{nth column}{\underset{\uparrow}{0}};] \tag{11}$$

where e_i is the ith column of the $m\times m$ identity matrix.

It can be shown that the matrix $\hat{K}=SQ^{-1}$ constructed by employing these previous matrices Q and S can help convert the original system into a single input controllable system.

Lemma 1. Assume that $b_1 \neq 0$. Then under the effect of the feedback $u = \hat{K}x + v = SQ^{-1}x + v$, the closed loop system

$$\frac{dx}{dt} = (A + B\hat{K})x + b_1 v_1$$

is controllable, where v_1 stands for the first component of v.

Proof. From the definition of $\hat{K}$, we have $\hat{K}Q = S$. That is,

$$\begin{aligned}
&\hat{K}[b_1\ Ab_1 \cdots A^{\mu_1 - 1}b_1;\ b_2\ Ab_2 \cdots A^{\mu_2 - 1}b_2; \cdots;\ b_m\ Ab_m \cdots A^{\mu_m - 1}b_m] \\
&\qquad = [0 \cdots 0\ e_2;\ 0 \cdots 0\ e_3; \cdots\ e_m;\ 0 \cdots 0\ 0]
\end{aligned}$$

So, we obtain the following series of equations:

$$\begin{aligned}
&\hat{K}b_1 = 0, \hat{K}Ab_1 = 0, \cdots, \hat{K}A^{\mu_1 - 2}b_1 = 0, \hat{K}A^{\mu_1 - 1}b_1 = e_2; \\
&\hat{K}b_2 = 0, \hat{K}Ab_2 = 0, \cdots, \hat{K}A^{\mu_2 - 2}b_2 = 0, \hat{K}A^{\mu_2 - 1}b_2 = e_3; \\
&\vdots \\
&\hat{K}b_{m-1} = 0, \hat{K}Ab_{m-1} = 0, \cdots, \hat{K}A^{\mu_{m-1} - 2}b_{m-1} = 0, \hat{K}A^{\mu_{m-1} - 1}b_{m-1} = e_m; \\
&\hat{K}b_m = 0, \hat{K}Ab_m = 0, \cdots, \hat{K}A^{\mu_m - 2}b_m = 0, \hat{K}A^{\mu_m - 1}b_m = 0]
\end{aligned}$$

Let $\bar{A} = A + B\hat{K}$. In the following, we show the $(\bar{A}, b_1)$ controllability. We successively compute the matrix series of the $(\bar{A}, b_1)$ controllability:

$$\begin{aligned}
&\bar{A}b_1 = (A + B\hat{K})b_1 = Ab_1 + B\hat{K}b_1 = Ab_1, \\
&\cdots, \\
&\bar{A}^{\mu_1 - 1}b_1 = \bar{A}(A^{\mu_1 - 2}b_1) = (A + B\hat{K})A^{\mu_1 - 2}b_1 = A^{\mu_1 - 1}b_1, \\
&\bar{A}^{\mu_1}b_1 = \bar{A}(A^{\mu_1 - 1}b_1) = (A + B\hat{K})A^{\mu_1 - 1}b_1 = A^{\mu_1}b_1 + B\hat{K}A^{\mu_1 - 1}b_1
\end{aligned}$$

Based on how Q is constructed, we know that $A^{\mu_1}b_1$ is a linear combination of all the columns of Q that are located to the left b_2. Let $\tilde{b}_2 = A^{\mu_1}b_1$. Then we have

$$\begin{aligned}
&\bar{A}^{\mu_1}b_1 = \tilde{b}_2 + Be_2 = \tilde{b}_2 + b_2 \\
&\bar{A}^{\mu_1 + 1}b_1 = (A + B\hat{K})(\tilde{b}_2 + b_2) = Ab_2 + B\hat{K}\tilde{b}_2 + A\tilde{b}_2 + B\hat{K}b_2
\end{aligned}$$

where $B\hat{K}b_2 = 0$. Through detailed analysis it is not difficult to see that both $B\hat{K}\tilde{b}_2$ and $A\tilde{b}_2$ are linear combinations of the columns of Q that are located to the left of Ab_2. Let $\widetilde{Ab_2} = B\hat{K}\tilde{b}_2 + A\tilde{b}_2$. Then we have

$$\bar{A}^{\mu_1+1}b_1 = Ab_2 + \widetilde{Ab_2}$$

Similarly, we obtain

$$\bar{A}^{\mu_1+\mu_2}b_1 = b_3 + \tilde{b}_3$$
$$\vdots$$
$$\bar{A}^{n-1}b_1 = A^{\mu_m-1}b_m + \widetilde{A^{\mu_m-1}}b_m$$

Therefore,

$$\begin{aligned} rank[b_1\ \bar{A}b_1\ \cdots \bar{A}^{n-1}b_1] &= rank\big[b_1\ Ab_1\ \cdots A^{\mu_1-1}b_1;\ b_2 + \widetilde{b_2}\ Ab_2 + \widetilde{Ab_2} \cdots A^{\mu_2-1}b_2 + \widetilde{A^{\mu_2-1}}b_2;\ \cdots b_m \\ &\quad + \widetilde{b_m} \cdots A^{\mu_m-1}b_m + \widetilde{A^{\mu_m-1}}b_m\big] \\ &= rank[b_1\ Ab_1\ \cdots A^{\mu_1-1}b_1;\ b_2\ Ab_2\ \cdots A^{\mu_2-1}b_2;\ \cdots;\ b_m\ Ab_m\ \cdots A^{\mu_m-1}b_m] = n \end{aligned}$$

Therefore, we have obtained the $(\bar{A}, b_1)$ controllability. QED

By applying the result in Lemma 1, we can first compute $\hat{K}$ in order to simplify the control-theory model of the economic system into a single-input controllable system, and then place the poles by using the method of pole placement established for single-input controllable systems.

Assume that $\hat{k}$ is the calculated matrix of feedback gain and makes the resultant single-input system $(\bar{A}, b_1)$ have the desirable poles. That is, $\bar{A} + b_1\hat{k}$ has the following pre-determined poles $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then, we can show that

$$\hat{K} + \bar{K} = \hat{K} + \begin{bmatrix} \hat{k} \\ 0 \end{bmatrix}$$

will also help to place the poles of the system (A, B) onto $\lambda_1, \lambda_2, \cdots, \lambda_n$. In fact, it is ready to see that

$$A + B(\hat{K} + \bar{K}) = A + B\hat{K} + b_1\hat{k} = \bar{A} + b_1\hat{k}$$

has $\lambda_1, \lambda_2, \cdots, \lambda_n$ as poles. So, $\hat{K} + \bar{K}$ is the desired matrix of feedback gain.

Summarizing what has been discussed above, we have the following particular computational steps for placing the poles of a multiple input controllable system:

Step 1: Construct matrices Q and S according to equs. (10) and (11) and then compute

$\hat{K} = SQ^{-1}$;

Step 2: Calculate $\overline{A} = A + B\hat{K}$ and its characteristic polynomial

$$| sI - \overline{A} | = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0;$$

Step 3: For given n poles $\lambda_1, \lambda_2, \cdots, \lambda_n$, calculate the following polynomial:

$$(s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

and $k = [a_0 - \alpha_0, a_1 - \alpha_1, \cdots, a_{n-1} - \alpha_{n-1}]$;

Step 4: Compute $\tilde{k} = kT^{-1}$, where

$$T^{-1} = \begin{bmatrix} q \\ q\overline{A} \\ \vdots \\ q\overline{A}^{n-1} \end{bmatrix}$$

where q is the last row of the inverse matrix of the controllability matrix U of $(\overline{A}, b_1)$.

Step 5: Calculate

$$K = \hat{K} + \begin{bmatrix} \hat{k} \\ 0 \end{bmatrix}$$

which is the desired matrix of feedback gain.

Example 2. Assume that an given economy is written in the following analytic form:

$$\frac{dx}{dt} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u$$

Find the matrix K of feedback gain so that the resultant closed loop system has poles $\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -2, \lambda_4 = -2$

Solution. Step 1: Let us construct the following matrix

$$Q = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and the feedback matrix:

$$\hat{K} = SQ^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2: Calculate

$$\bar{A} = A + B\hat{K} = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and its characteristic polynomial

$$| sI - \bar{A} | = s^4 - 3s^3 + 2s^2 .$$

Step 3: Calculate the desired polynomial

$$(s+1)(s+1)(s+2)(s+2) = s^4 + 6s^3 + 13s^2 + 12s + 4$$

so that we have $k = [-4, -12, -11, -9]$.

Step 4: Calculate

$$q = [0 \ldots 0\,1][b_1 \;\; \bar{A}b_1 \ldots \;\; \bar{A}^{n-1}b_1]^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix},$$

$$T^{-1} = \begin{bmatrix} q \\ q\bar{A} \\ \vdots \\ q\bar{A}^{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{bmatrix},$$

and $\tilde{k} = kT^{-1} = [-16, 7, -2, -18]$.

Step 5: Calculate

$$K = \hat{K} + \begin{bmatrix} \hat{k} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -16 & 7 & -2 & -18 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -16 & 7 & -2 & -18 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

As of this point in our presentation, we have concluded our discussion on the complete controllability of economic systems that can be modeled by using constant coefficient linear control systems. We provided a method to arbitrarily place all the poles of a closed loop system by using state feedback so that the proof of Theorem 1 is finished.

In the discussions above, among the conditions of controllability we mainly employed that the rank of the matrix $U = [B\ AB \ldots\ A^{n-1}B]$ is n. As for the following constant coefficient linear economic system of discrete time

$$x(k+1) = Ax(k) + Bu(k)$$

this particular condition is equivalent to that of complete reachability. So, for discrete economic systems, we have the following result:

Theorem 3. A sufficient and necessary condition for a constant coefficient linear economic system (A, B) of discrete time to arbitrarily place its poles through using state feedback is that is completely reachable.

For economic systems of discrete time, Theorem 2 also holds true. And, the afore-described computational scheme for placing poles is also applicable to constant coefficient linear economic systems of discrete time. All the relevant details are omitted here. In the following, let us look at a much simplified method for placing poles, which can greatly reduce the amount of computations.

2.2. An Improvement on How to Place Poles

In step 4 of placing the poles, we needed to calculate the last row q of the inverse of the controllability matrix of $(\bar{A}, b_1)$. In Example 2 we found that q is equal to the last row of Q^{-1}.

This discovery is not a coincidence. In fact, it is a common phenomenon. To this end, we have the following result.

Lemma 2. The last row of the inverse of the controllability matrix U of the system $(\bar{A}, b_1)$ is equal to the last row of the inverse of the matrix Q as constructed according to equ. (10).

To show this result, we only need to prove that the determinants of U and Q are the same and the corresponding cofactors of the *i*th row and *n*th column are also the same. All the details are omitted here.

By applying this lemma and the derivations in (Wang, 1985), we have the following simplified scheme for placing poles.

Step 1: Construct matrices Q and S according to equs. (10) and (11) and then calculate $\hat{K} = SQ^{-1}$ and denote the last row of Q^{-1} as q.

Step 2: Assume that the desired poles are $\lambda_1, \lambda_2, \ldots, \lambda_h$, $a_1 \pm b_1 i, a_2 \pm b_2 i, \ldots, a_l \pm b_l i$, $h + 2l = n$. Calculate

$$\hat{k} = -q(\bar{A} - \lambda_1 I)\cdots(\bar{A} - \lambda_h I)[\bar{A}^2 - 2a_1\bar{A} + (a_1^2 + b_1^2)]\cdots[\bar{A}^2 - 2a_l\bar{A} + (a_l^2 + b_l^2)]$$

Step 3: The following is what is desired:

$$K = \hat{K} + \begin{bmatrix} \hat{k} \\ 0 \end{bmatrix}.$$

This simplified scheme omits many of the operational steps, such as computing the inverse of the controllability matrix, the characteristic polynomial of $\bar{A}$, and the desired characteristic polynomial, when compared to the previous scheme. When the magnitude of n is large, the amount of reduced computation becomes noticeable.

Example 3. Let us still look at the same problem of pole placement as discussed in Example 2.

Solution. Step 1, which is the same as before, produces

$$\hat{K} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$q = \begin{bmatrix} 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

Step 2:

$$\hat{k} = -\begin{bmatrix} 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} (A+I)(A+I)(A+2I)(A+2I) = \begin{bmatrix} -16 & 7 & -2 & -18 \end{bmatrix}.$$

Step 3:

$$K = \hat{K} + \begin{bmatrix} \hat{k} \\ 0 \end{bmatrix} = \begin{bmatrix} -16 & 7 & -2 & -18 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.3. Pole Placement through Output Feedback

When some of the states of the economic system cannot be used as feedback, such as the performance of the underground economy, we can consider using output feedback to place the poles. From comparing equs. (4) and (5) in (Yang, Ying, and Forrest, to appear), it follows that this is equivalent to replacing K_0 by KC within the closed loop system. Let K_s stand for the gain matrix of state feedback, and K_0 the gain matrix of output feedback. When $K_s = K_0 C$, that is, when we solve for K_0 from $K_s = K_0 C$, we can use the output feedback to arbitrarily place the poles of the controllable economic system. From the knowledge of algebra, it follows that that end is only possible when K_S^T belongs to $\text{span}(C^T)$. So, generally speaking, it means that not all poles can be arbitrarily placed; and how many poles of a closed loop system can be arbitrarily placed through output feedback becomes an important question. (Zheng, 1990) has surveyed the results obtained along this research line and provided two computational schemes on how to place $m+r-1$ poles through using output feedback, where m is the dimension of the control vector, and r the dimension of the output vector. So, it can be seen that when $m+r-1 \geq n$, the economic system of concern can arbitrarily place all of its poles through using output feedback.

In the following, let us look at a computational scheme on how to arbitrarily place $m+r-1$ poles of a controllable economic system (C, A, B) by applying output feedback. For all the technical details of the reasoning behind this scheme, please consult with (Zheng, 1990).

Step 1: The particular way to place $m-1$ poles $\lambda_1, \ldots, \lambda_{m-1}$ by solving for the matrix $K_1 = q_1 k_1 = [q_1^1 \ldots q_m^1]^T [k_1^1 \ldots k_r^1]$ is given below.

Determine k_1 such that $(A, k_1 C)$ is observable. And arbitrarily select one element of q_1 and the other $m-1$ elements of q_1 are determined based on $\lambda_1, \ldots, \lambda_{m-1}$ as follows:

$$p_1(s) = p_0(s) - k_1 W_0(s) q_1$$

where $p_1(s)$ stands for the characteristic polynomial of the closed loop economic system that is obtained from the output feedback with K_1 as the matrix of feedback gain. That is, we have

$$p_1(s) = \det(sI - A - BK_1C) = \det(sI - A - Bq_1k_1C)$$
$$p_0(s) = \det(sI - A)$$

$$W_0(s) = C\text{Adj}(sI - A)B$$

In particular, by solving the system of equations

$$p_1(\lambda_i) = p_0(\lambda_i) - k_1 W_0(\lambda_i) q_1, i = 1, \cdots, m-1$$

we obtain the other $m-1$ elements of q_1.

Step 2: Solve for matrix $K_2 = q_2 k^2 = [q_1^2 \cdots q_m^2]^T [k_1^2 \cdots k_r^2]$ so that under the effect of the output feedback with K_2 as the matrix of feedback gain the already placed $m-1$ poles will be maintained. And then we place the additional r poles $\lambda_m, \lambda_{m+1}, \cdots, \lambda_{m+r-1}$. The particular details are given below:

Maintaining the already placed $m-1$ poles is materialized by appropriately selecting q_2, while placing the additional r poles is done through selecting k_2. The specific method is to apply the following equation

$$p_2(s) = p_1(s) - k_2 W_1(s) q_2,$$

where $p_2(s) = \det(sI - A - BK_1C - BK_2C)$ and $W_1(s) = C\text{Adj}(sI - A - BK_1C)B$.

In order to maintain $\lambda_1, \cdots, \lambda_{m-1}$, we need $p_2(\lambda_i) = 0 (i = 1, \cdots, m-1)$, which can be satisfied by $w_i q_2 = 0 (i = 1, \cdots, m-1)$, where w_i is the only linearly independent row of $W_1(s)$. After having determined an arbitrary element of q_2, all other elements of q_2 are solved out of the equations $w_i q_2 = 0 (i = 1, \cdots, m-1)$. After q_2 is determined, we obtain k_2 from solving the following system of equations:

$$p_2(\lambda_i) = p_1(\lambda_i) - k_2 W_1(\lambda_i) q_2 = 0, i = m, m+1, \cdots, m+r-1$$

Step 3: Let $K = K_1 + K_2$, which is exactly the matrix of output feedback gain useful for placing the $m+r-1$ additional poles $\lambda_1, \cdots, \lambda_{m+r-1}$.

As a matter of fact, other than its advantages, using the method of placing poles to design macro-economic policies also suffers from some weaknesses. One advantage of the method is that the feedback mechanism is of good qualities, such as involving less fluctuations, etc.. However, the method suffers from the following major weaknesses:

Weakness #1: When it is desirable for the performance of the economy to approach its target rapidly, where the fast speed could be reached in theory when all the eigenvalues are zero after placing the poles, the needed strength of policy interference might be very high. For example, during the time when the Russian government needed to react to the crisis in order to avoid its currency to depreciate quickly, they needed the support of a huge amount of U.S. dollars within a short moment of time. However, at that particular moment, the government did not have a means to satisfy that demand.

Weakness #2: Each theoretical model for the economic system is only an approximate description of the reality. So, as long as the estimate of the parameters of the model is not accurate, there is always an expected deviation between the model and the reality. When the parameters are not accurate or when the parameters change with time, the method of pole placement will not be able to guarantee that the regulated economic variables can still accurately track the pre-determined targets.

To overcome Weakness #1, we can design feedback control strategies by using different methods of pole placement after we first select the poles of desired qualities or other appropriate methods. To overcome Weakness #2, we can employ the design method of robust regulators.

2.4. Determination of Pole Locations

In the previous sections, we have studied the computational schemes on how to place poles. In the following, let us look at the problem of how to determine the expected poles of the closed loop. In other words, we like to know how to convert the expected performance indicators of the closed loop economic system into desired locations of the poles.

One method is established on the premises that the pair of dominant poles determines the properties of the economic system, while other poles bear little influence on the system. To this end, assume that the transfer function of the second order economic system with dominant poles λ_1, λ_2 is given as follows:

$$G(s) = \frac{w_n^2}{s^2 + 2\xi w_n s + w_n^2}$$

where w_n stands for the natural frequency of economic cycles without any artificial interference, and ξ the artificial interference ratio. Then, λ_1, λ_2, ξ and w_n satisfy the following relationship:

$$|\lambda_1| = |\lambda_2| = w_n, \quad \theta = \cos^{-1}\xi .$$

To continue our discussion, let us look at the concept of overshoot or the maximum overshoot of systems. The so-called overshoot is defined as the difference between the maximum response of the system when it receives a step signal input and the system's steady-state value divided by the steady-state value. The magnitude of the overshoot of a system represents the system's ability to adjust itself and to react to occasional and sudden changes in the input. It indicates how much the system's operation could withstand extraordinary shocks.

Most systems should have its overshoot capability. However, if a system does not have its appropriate overshoot, it is equivalent to say that the system does not have any ability to bear workload beyond its load. It is relatively easy for such a system to crash. Therefore, allowing the existence of a certain level overshoot in reality corresponds to some capability for the system to adjust itself along with the up-and-down waves and cycles of oscillation. Other than including the concept of overloading, the concept of overshoot also contains the requirement of returning to the steady state: Within the allowed ratio of the overshoot, even if an overload appears, it will not lead to unexpected crashes or interruption of the system's operation.

The global financial crisis, triggered by the U.S. subprime mortgage crisis in 2008, is a very good example on how the overshoot of the then-current financial system could not handle the severe volatility of the market. Because of the levering manipulation and excessive trading of the highly efficient and greatly leveraged financial derivatives with significant price effects of the U.S. credit market, the consequent risks were transferred and spread over into the rest of the world by the innovative credit risk tools. So, the initial risk of the U.S. capital market became a risk of the entire financial market of the world. At the same time, with the ever expansion of influence over the global market, the volatility of the risk got bigger and eventually made it difficult for the overshoot of the financial system to absorb and to meet the abnormal large-scale fluctuating demand. That eventually made the initial U.S. subprime mortgage crisis a global financial crisis.

As a matter of fact, within the socio-economic development, the evolution of most systems has their naturally attached allowance of overshoot. Because temporary deviations from the normality exist commonly in socio-economic systems, appropriately setting overshoots is practically meaningful. A basic requirement for socio-economic systems is to set their individual overshoots at the right levels. Hence, for the general economic operation, if one wishes to set the overshoot to a small value, it means that he/she expects relatively small fluctuations and relative low levels of risk for the process of development. In realistic economic operations, the economic system with slow response speed generally has small overshoot necessary for dealing with minor fluctuations; while for economic systems of high financial efficiency, the encountered overshoots are generally very large.

Now, to continue our previous discussion, the overshoot of the system is defined as follows:

$$\sigma = e^{-\xi z} / \sqrt{1-\xi^2}$$

and the system's adjustment time, which is defined to be the time needed for the absolute difference between the output and stable value to be smaller than and equal to $\Delta\%$ of the stable value:

$$t_s = 4/\xi w_n (\Delta = 2),$$
$$t_s = 3/\xi w_n (\Delta = 5).$$

When $0 < \xi < 0.9$, we can determine λ_1, λ_2 based on σ, t_s by using these equations.

Of course, practical problems of economic systems are far more complicated than what is just described. In such a case, after successfully determined the two dominant poles, other poles and zeros can also influence the properties of the economic system. So, the locations of complex poles need to be adjusted appropriately. Sometimes, repeated adjustments are needed before ideal locations can be determined.

Another method is to select poles by optimization with particular significance. For example, we can select poles through optimization by using the criterion of minimizing the integral of time and absolute error (ITAE):

$$J = \int_0^\infty t\,|\,e\,|\,dt \tag{12}$$

By employing this criterion, the characteristic polynomials of order 1 through 6 for systems of type 1 are given as follows:

$$s + w_0$$
$$s^2 + 1.4 w_0 s + w_0^2$$
$$s^3 + 1.75 w_0 s^2 + 2.15 w_0 s + w_0^3$$
$$s^4 + 2.1 w_0 s^3 + 3.5 w_0^2 s^2 + 2.7 w_0^3 s + w_0^4$$
$$s^5 + 2.8 w_0 s^4 + 5.0 w_0^2 s^3 + 5.5 w_0^3 s^2 + 3.4 w_0^4 s + w_0^5$$
$$s^6 + 3.25 w_0 s^5 + 6.6 w_0^2 s^4 + 8.6 w_0^3 s^3 + 7.45 w_0^4 s^2 + 3.95 w_0^5 s + w_0^6$$

Next, we use these polynomials as the characteristic polynomials of the desired closed loop systems in our design of the matrices of feedback gains. The value w_0 of these polynomials can be selected appropriately in order for the resultant closed loop system to satisfy some other necessary properties, such as certain requirement on the circulation of money throughout the economic system.

(Shanhian and Hassul, 1992) provides a detailed procedure for calculating the characteristic polynomial by using this ITAE criterion.

3. The Problem of Eigenstructure Assignment

Other than the influence of the poles, the eigenvectors of the matrix A of the control-theory model of the economy also bear great effects on the time response of the performance of the economic system. Assume that A has n different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and their corresponding eigenvectors are respectively $v_1, v_2, \ldots, v_n$. Then, by using the following transformation

$$x = Tx' = [v_1\ v_2\ \ldots\ v_n]x'$$

we can simplify our economic system of concern

$$\frac{dx}{dt} = Ax + Bu, y = Cx$$

into

$$\frac{dx'}{dt} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} x' + T^{-1}Bu, y = CTx'$$

If we denote

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \Lambda$$

then we have

$$e^{At} = I + (T^{-1}\Lambda T)t + \frac{1}{2!}(T^{-1}\Lambda T)^2 t^2 + \cdots = Te^{\Lambda t}T^{-1}.$$

Letting

$$T^{-1} = \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}$$

produces

$$e^{At} = [v_1\ v_2 \cdots v_n]e^{\Lambda t}\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}$$

So, the system's output is given as follows:

$$y(t) = Cx = \sum_{i=1}^{n} Cv_i e^{\lambda_i t} w_i^T x(0) + \sum_{j=1}^{m}\sum_{i=1}^{n} Cv_i w_j^T \int_0^t e^{\lambda_j t} uj(t-\tau)d\tau$$

That is, the performance output of the economic system is affected not only by the eigenvalues λ_i of the system matrix but also by the eigenvectors v_i. Therefore, a natural question of how to place the eigenvectors becomes clear.

The problem of how to place the poles of the closed loop economic system and the problem of how to place the corresponding eigenvectors together are known as the problem of eigenstructure assignment. As for how to specifically place eigenvectors, please consult with (Wang, 1985; D'Azzo and Houpia, 1981).

In this portion of the presentation we introduced a computational scheme on how to place poles and how to design observers. For other different schemes, please consult with (Zheng, 1990; Jamshidi, Tarokh, and Shafai, 1992). What needs to be noticed is that for economic systems with multiple policy inputs and multiple performance indicators different schemes generally lead to different pole placements and different designs of observers. This end actually provides a theoretical explanation for why when facing an economic or financial emergency, there tend to be different policy tools available, although they might lead to totally different outcomes.

4. Conclusion

The dynamic characteristics of the economic system depend on where the poles of the closed loop are located. When the poles of a closed loop economic system are placed at some pre-determined locations on the left half plane, we can make the economic system have not only certain desired dynamic characteristics but also an important degree of stability so that the system can withstand the shock of interference information of the environment on the performance of the economic system. Because of this reason, this chapter discusses the conditions under which the poles of a constant coefficient linear economic system can be arbitrarily placed, and provides a way to calculate the matrix of feedback gain that is useful in placing the poles by using the feedback control mechanism so that the resultant constant coefficient linear economic system possesses a good quality stability and fast response speed.

As a matter of fact, the method, discussed in this chapter, mainly resolves the problem of regulating an economy through using feedback of the matrix's elements of the closed loop system based on the eigenvalues of the closed loop. However, when a control-theory model is employed to model the actual economy, the parameters of the model are often affected by the outside environment so that it makes it difficult for the seemingly accurate placement of the poles to meet the requirements of the expected economic performance indices. Hence, considering the existence of many uncertain factors that influence the economy, a very important question for future research regarding the pole placement of an economic system is: How can we place the poles of the system within a pre-determined region such that it provides the needed possibility for designing practically useful economic policies, while it also makes the choice of poles have much greater freedom.

Additionally, at the same time when considering how to place the poles, we should also think about the potential economic losses that could be associated with the adoption of relevant policies. In other words, a placement of poles is considered good, if it does not bear a lot of potential economic losses while it makes the stability of the economy more able to withstand adverse effects of the outside world. Therefore, we should consider combining the methods of optimal control and pole placement in order to make the matrix of feedback gain able to not only place the poles of the closed loop system at expected locations but also lead to the design of policies that could potentially result in optimal regulation consequences.

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