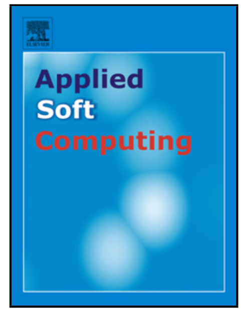
Accepted Manuscript



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PII: S1568-4946(18)30445-9
DOI: https://doi.org/10.1016/j.asoc.2018.07.057
Reference: ASOC 5025

To appear in: *Applied Soft Computing*

Received date: 19-1-2018
Revised date: 10-7-2018
Accepted date: 30-7-2018

Please cite this article as: Hatami-Marbini A, Saati S, Efficiency Evaluation in Two-stage Data Envelopment Analysis under a Fuzzy Environment: A Common-Weights Approach, *Applied Soft Computing Journal* (2018), https://doi.org/10.1016/j.asoc.2018.07.057

2nd Revision
ASOC-D-18-00163R1
Applied Soft Computing

Efficiency Evaluation in Two-stage Data Envelopment Analysis under a Fuzzy Environment: A Common-Weights Approach

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Highlights

- We look into internal structures of a production system to assess its performance.
- We present a common-weights DEA method for two-stage structures with fuzzy data.
- We assess the efficiency of the system and component processes.
- The new approach is illustrated through a numerical example.

All additions and changes to the first revision are highlighted in this revision.

Abstract

Data envelopment analysis (DEA) has been genuinely known as an impeccable technique for efficiency measurement. In practice, since many production systems such as broadcasting companies, banking and R&D activities include two processes connected in series, we have need of utilizing two-stage DEA models to identify the sources of inefficiency and explore in turn appropriate options for improving performance. The lack of the ability to generate the actual weights is not only an ongoing challenge in traditional DEA models, it can have serious repercussion for the contemporary DEA models (e.g., two-stage DEA). This paper presents a common-weights method for two-stage structures that allows us to consider equality of opportunity in a fuzzy environment when evaluating the system efficiency and the component process efficiencies. The proposed approach first seeks upper bounds on factor weights and then determines a set of common weights by a single linear programming problem. We illustrate the approach with a data set taken from the literature.

Keywords: Two-stage system; Data envelopment analysis; Fuzzy data; Common set of weights

1. Introduction

Performance evaluation has been widely used across many organizations to determine best practices in the market to improve performance and increase efficiency. Often, performance evaluation is known as a means in service operations management in the light of difficulty in defining service standards rather than manufacturing standards.

Data envelopment analysis (DEA) developed by Charnes et al. (1978) is a well-known method that has been widely and successfully used to evaluate organization's performance in the presence of multiple inputs and multiple outputs. One characteristic of the DEA models is the full flexibility in choosing weights, making the evaluated unit appears in its most favourable light. However, it is very often observed biased weight distribution across inputs or outputs in addition to weak discriminatory power which is not acceptable in the widening range of real world applications. Needless to say, controlling factor weights is of paramount importance and consequently the significant literature has been allotted to value judgement approaches. Weight restrictions which reflect the relative importance of different factors are the very popular technique in the DEA model. To assess the performance of a set of decision-making units (DMUs), Cook and Seiford (2009) classified DEA models with weight restrictions into five different groups. The first group originated by Dyson and Thanassoulis (1988) and Roll et al. (1991) is referred to as "*absolute multiplier restrictions*", in which absolute lower and upper bounds are imposed on input and output weights. The second group developed by Charnes et al. (1990) is called "*cone ratio restrictions*" and its aim is to provide more realistic weights by imposing a set of linear restrictions. The *"assurance region"* as the third group is a special case of the cone ratio concept initially presented by Thompson et al. (1995), and is intended to get rid of large differences in the values of weights. The assurance region idea has received a great deal of attention in the last decades from theoretical and practical aspects (see e.g., Khalili et al. (2010), Cook and Zhu (2011)). The fourth group firstly suggested by Bessent et al. (1988) is termed "*facet models*" that utilises the constrained facet analysis to get around the common problem in the multiplier DEA models involving the occurrence of zero weights. As DMUs are projected to the weakly efficient facets, extending facets leads to the removal of weakly efficient projections and create new unobserved DMUs. The fifth group aims at establishing different approaches for producing *new unobserved DMUs* (see e.g., Thanassoulis and Allen (1998)).

Let us now focus on *absolute multiplier restrictions* group. The use of alternative DEA-based models for finding the common set of weights (CSW) and consequently evaluating the firms' efficiencies were initially proposed by Cook et al. (1990) and Roll et al. (1991), so-called *common set of weights approach*. The common-weights idea has been leveraged into a series of approaches in the literature (see e.g., Amin and Toloo (2007), Saati et al. (2012), and Hosseinzadeh Lotfi et al. (2013); Hatami-Marbini et al. (2015)). Amin and Toloo (2007) put forward a CSW integrated DEA model to yield the most efficient DMUs. Saati et al. (2012) developed a two-phase CSW approach using an ideal virtual unit to assess Danish district heating plants. Hosseinzadeh Lotfi et al. (2013) and Hatami-Marbini et al. (2015) proposed two different allocation mechanisms using a CSW approach for allocating and reducing the fixed resources to and from the DMU.

Conventionally, DEA was developed to measure the efficiency of a system as a black box, without looking at its internal structure. However, service and manufacturing operations often consist of a combination of series and parallel processes and it is essential to take the network structure into consideration rather than a black-box system. Recently, network DEA approaches have been widely investigated in the DEA literature and the relational network DEA approach developed by Kao and Hwang (2008) is one of the most appealing and purposive approaches. Kao and Hwang (2008)'s model consisting of two processes connected in series model measures the system and processes efficiencies in tandem, so-called the two-stage DEA model, and compellingly the system efficiency is the product of those of the two processes.

While the black-box and network DEA models are attempting to estimate the best practice based upon all precise observations, the uncertainty becomes an indispensable element of real world applications. Generally speaking, there are three continuous streams in the DEA literature to deal with the uncertainty. The first category is known as *stochastic DEA approaches* which typically entails semi-parametric stochastic frontier analysis (SFA) developed by Land et al. (1993) and chance constrained DEA originated by Olesen and Petersen (1995). Lately, an exhaustive survey on *stochastic DEA approaches* has been conducted by Olesen and Petersen (2016). *Interval DEA approaches* as the second category was started off by Cooper et al. (1999) to tackle observations that are given in form of intervals. This category has received special attention due to its simplicity and ease of use from the practical and theoretical standpoints (see, e.g., Hatami-Marbini et al. (2014; 2018), and Toloo et al. (2018)). The last category, called *fuzzy DEA approaches*, is based on fuzzy logic that was initially introduced by Bellman and Zadeh (1970) offering a rich value to standard logic and opens the door to the mathematical setting for coping with vagueness and natural languages. There are many approaches for measuring efficiency in fuzzy environments, and Hatami-Marbini et al. (2011) and Emrouznejad et al. (2014) classified the fuzzy DEA models with black-box structures into six groups: the tolerance approach, the α-level based approach, the fuzzy ranking approach, the possibility and credibility approach, the fuzzy arithmetic, and the fuzzy random/type-2 fuzzy sets (see e.g., Kao and Liu (2000), Saati et al. (2002), Ignatius et al. (2016), Hatami-Marbini et al. (2017a), and Hatami-Marbini et al. (2017b)).

Also, a few studies concentrate on the network DEA problems with fuzzy data over recent years (see e.g., Kao and Liu (2011), Liu (2014), Lozano (2014a), Lozano (2014b), Shermeh et al. (2016), Hatami-Marbini (2017), and Hatami-Marbini et al. (2018)). One pioneering approach —a fuzzy relational two-stage model— was suggested in Kao and Liu (2011). This approach which is based on Kao and Hwang (2008)'s model calculates the fuzzy efficiency by the use of a pair of two-level mathematical programs developed by Kao and Liu (2000). The membership functions of fuzzy efficiencies are constructed as per the lower and upper bounds calculated from a pair of programs for various α-levels. However, Kao and Liu (2011)'s model has two flaws; (i) the model for computing the lower bounds of the system efficiencies is nonlinear that is hard to be solved by a commercial off-the-shelf DEA software package, (ii) when computing the efficiencies for two processes, the non-uniqueness problem may occur.

With this study, we make five main methodological contributions to the DEA literature. First, we reflect ambiguous and vague input and output data in DEA. Second, we partially fill the gap in the existing literature by viewing the internal structure of production systems. Third, we propose a common-weights method to compute efficiencies in two-stage structures when the data are represented by fuzzy numbers. The research idea primarily emerges from Kao and Liu (2011) with aim of overcoming the aforesaid flaws observed in Kao and Liu (2011)'s models by proposing this common-weights DEA models. Fourth, the common-weights method includes two successive steps enables us to simply contract the computational complexities. The first step specifies upper bounds on weights using a single linear programming (LP) problem and the second step is extended based on another LP problem to determine the CSW. To best of our knowledge, this is the first attempt to deal with the fuzzy two-stage DEA by pinpointing the CSW. The last contribution is to present a comparative numerical example to demonstrate the advantages and efficacy of the proposed method as well as validating our findings.

The remainder of this paper is outlined as follows: Section 2 presents the basic two-stage DEA model in both deterministic and imprecise environments. Section 3 elaborates the proposed common-weights two-stage DEA model when the data are presented by fuzzy numbers. Section 4 presents an illustrative numerical example taken from Kao and Liu (2012) and the paper is finally summarized and concluded in Section 5.

2. A Two-stage DEA model

In this section, the deterministic two-stage DEA model developed by Kao and Hwang (2008) is first reviewed and we then introduce its fuzzy version proposed by Kao and Liu (2011) along with delineating our motivations in this study.

2.1. Crisp model

Assume that the production structure includes two interconnected processes or stages where the first stage uses resources (inputs) to produce goods and services (outputs) that then become the inputs to the second stage (see Figure 1). Suppose there are *n* DMUs that the first stage of each DMU$_j$ (j=1,2,…,n) converts m inputs x_{ij} (i=1,2,…, m) into p outputs z_{dj} (d=1,2,…, p) and then these p outputs become the inputs of the second stage in order to generate s outputs y_{rj} (r=1,2,…, s). The first stage outputs and the second stage inputs which have the same measures are called *intermediate products*. The ancillary literature has been enriched with a large number of methods to tackle this certain structure. The existing two-stage models can be typically divided into (i) independent approaches, (ii) connected approaches, and (iii) relational approaches. The first category of models measures the efficiencies of the system and all processes independently (see for example Seiford and Zhu (1999) as a seminal study) and the second one introduced by Färe and Grosskopf (2000) makes an attempt to deem interactions between processes in calculating the system efficiency. The last category of models relies on existing mathematical relationships between the system efficiency and the process efficiencies which can be extracted from the mathematical programming models.

Let us here focus on the relational two-stage model developed by Kao and Hwang (2008). Under constant returns to scale (CRS) technology, the overall efficiency scores of the two-stage process under evaluation, denoted by subscript *o*, can be calculated by the following formulation:

$$E_o = \max_{u,v,w} \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}}, \quad (1)$$

$$\begin{aligned} s.t.\ & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \quad j = 1,2,\dots,n, \\ & \frac{\sum_{d=1}^{p} w_d z_{dj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \quad j = 1,2,\dots,n, \\ & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{d=1}^{p} w_d z_{dj}} \le 1, \quad j = 1,2,\dots,n, \\ & u_r, v_i, w_d \ge 0, \quad r = 1,2,\dots,s;\ i = 1,2,\dots,m;\ d = 1,2,\dots,p, \end{aligned}$$

where v_i, w_d and u_r are the weights (shadow prices) assigned to the i^{th} input and d^{th} intermediate product, r^{th} output. The above model aims to maximize the efficiency of the evaluated DMU as a whole subject to satisfying a set of constraints associated with the system and two processes. We point out the constraints $\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1\ (j = 1,2,\dots,n)$ are redundant and its removal cannot affect the optimal objective function value of model (1). Thus, model (1) is equivalent to the following LP problem:

$$E_o = \max_{u,v,w} \ \sum_{r=1}^{s} u_r y_{ro}, \quad (2)$$

$$\begin{aligned} s.t.\ & \sum_{i=1}^{m} v_i x_{io} = 1, \\ & \sum_{d=1}^{p} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1,2\dots,n, \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{p} w_d z_{dj} \le 0, \quad j = 1,2,\dots,n, \\ & u_r, v_i, w_d \ge 0, \quad r = 1,2,\dots,s; \quad i = 1,2,\dots,m;\ d = 1,2,\dots,p. \end{aligned}$$

On optimality (v_i^*, w_d^*, u_r^*), the system and process efficiencies of the DMU being evaluated are calculated as $E_o^* = \sum_{r=1}^{s} u_r^* y_{ro}$, $E_o^{1*} = \frac{\sum_{d=1}^{p} w_d^* z_{do}}{\sum_{i=1}^{m} v_i^* x_{io}} = 1 - s_o^{1*}$ and $E_o^{2*} = \frac{\sum_{r=1}^{s} u_r^* y_{ro}}{\sum_{d=1}^{p} w_d^* z_{do}} = 1 - s_o^{2*}$ where s_o^{1*} and s_o^{2*} are the slack variables associated with the first and second process constraints. In effect, s_j^{1*} and s_j^{2*} indicate the amount of inefficiency that are distributed between two stages. Obviously, the system efficiency is the product of process efficiencies, i.e., $E_j^* = E_j^{1*} \times E_j^{2*}$. So, the production system is called *efficient* if all its processes are efficient. Though it is likely to observe no efficient DMU, we do not find it disturbing because the main intention of efficiency measurement is to pinpoint the source of inefficiency and, then to take appropriate actions.

2.2. Imprecise model and our research motivations

All data in the conventional two-stage models are assumed to be precise. However, real-world problems inherently include imprecision and uncertainty. Kao and Liu (2011) was the first to represent the fuzzy observations in relational two-stage models and show that the system efficiency is still related to the process efficiencies under a fuzzy environment.

An LR fuzzy number $\tilde{A}$ is a fuzzy subset of real numbers in the universe of discourse X with a continuous, convex and normalized membership function $\mu_{\tilde{A}}(X)$ that is defined below:

$$\mu_{\tilde{A}}(X) = \begin{cases} L\left(\dfrac{m-x}{\alpha}\right), & x \leq m, \\ 1, & m \leq x \leq n, \\ R\left(\dfrac{x-n}{\beta}\right), & x \geq n, \end{cases} \tag{3}$$

where $[m, n]$ is the mean of $\tilde{A}$, α and β are the left and right spreads, and L and R are left- and right-shape functions, respectively. The LR fuzzy number $\tilde{A}$ can be denoted by $(m, n, \alpha, \beta)_{LR}$. It is painless to be converted a precise number into an LR fuzzy number with one value in the domain. Thus, we presume that all observations are characterized by fuzzy numbers with the aim of simplifying the notations. In this respect, one can express the inputs, outputs and intermediate measures of each DMU by the fuzzy numbers as $\tilde{x}_{ij}$,$\tilde{y}_{rj}$, and $\tilde{z}_{dj}$, respectively, and its resulting efficiency should take the form of a fuzzy number with a certain membership function $\mu_{\tilde{E}_o}$. By exploiting Zadeh's extension principle (Zimmermann, 1996), one can calculate $\mu_{\tilde{E}_o}$ of fuzzy efficiency using the following equation:

$$\mu_{\tilde{E}_o}(e) = \sup_{x,y,z} \min_{i,r,d,j} \left\{ \mu_{\tilde{x}_{ij}}(x_{ij}), \mu_{\tilde{y}_{rj}}(y_{rj}), \mu_{\tilde{z}_{dj}}(z_{dj}) | e = E_o(x,y,z) \right\} \tag{4}$$

where $E_o(x,y,z)$ is a mathematical programming model defined via model (2), and x_{ij}, y_{rj} and z_{dj} are variables to be specified. The above equation can be expressed as the following program:

$$\begin{aligned} \mu_{\tilde{E}_o}(e) = \max_{x,y,z} \ & h, \\ s.t. \ & \mu_{\tilde{x}_{ij}}(x_{ij}) \geq h, \quad \forall i,j, \\ & \mu_{\tilde{y}_{rj}}(y_{rj}) \geq h, \quad \forall r,j, \\ & \mu_{\tilde{z}_{dj}}(z_{dj}) \geq h, \quad \forall p,j, \\ & e = E_o(x,y,z). \end{aligned} \tag{5}$$

Since the existing methods are unable to solve the above model, the two-level mathematical program of Kao and Liu (2000) is employed to determine the fuzzy efficiencies. The γ-levels of $\tilde{x}_{ij}$, $\tilde{y}_{rj}$, and $\tilde{z}_{dj}$ are represented as $(\tilde{x}_{ij})_\gamma = \left[(x_{ij})_\gamma^L, (x_{ij})_\gamma^U\right]$, $(\tilde{y}_{rj})_\gamma =$

$\left[(y_{rj})_\gamma^L, (y_{rj})_\gamma^U\right]$, $(\tilde{z}_{dj})_\gamma = \left[(z_{dj})_\gamma^L, (z_{dj})_\gamma^U\right]$, respectively. Hence, it is in need of seeking the lower and upper bounds of γ-levels of $\tilde{E}_o$ to attain the membership function $\mu_{\tilde{E}_o}(e)$ in a way that $(E_o)_\gamma^U = \max\{e|\mu_{\tilde{E}_o}(e) \geq \gamma\}$ and $(E_o)_\gamma^L = \min\{e|\mu_{\tilde{E}_o}(e) \geq \gamma\}$. Although the two-level programs are developed by Kao and Liu (2011) to calculate the upper and lower bounds of the system efficiency for different γ-levels, it is essential to convert these program into one-level ones. Regarding the upper bound of the system efficiency, since both inner and outer programs are maximized, the following one-level mathematical model can be formulated:

$$(E_o)_\gamma^U = \max_{u,v,w,z} \frac{\sum_{r=1}^s u_r (y_{ro})_\gamma^U}{\sum_{i=1}^m v_i (x_{io})_\gamma^L}, \tag{6}$$

$$\begin{aligned}
s.t.\ & \frac{\sum_{d=1}^p w_d z_{do}}{\sum_{i=1}^m v_i (x_{io})_\gamma^L} \leq 1, \\
& \frac{\sum_{d=1}^p w_d z_{dj}}{\sum_{i=1}^m v_i (x_{ij})_\gamma^U} \leq 1, \qquad j = 1,2,\dots,n, j \neq o, \\
& \frac{\sum_{r=1}^s u_r (y_{rj})_\gamma^L}{\sum_{d=1}^p w_d z_{dj}} \leq 1, \qquad j = 1,2,\dots,n, j \neq o, \\
& \frac{\sum_{r=1}^s u_r (y_{ro})_\gamma^U}{\sum_{d=1}^p w_d z_{do}} \leq 1, \\
& (z_{dj})_\gamma^L \leq z_{dj} \leq (z_{dj})_\gamma^U, d = 1,2,\dots,p, j = 1,2,\dots,n, \\
& u_r, v_i, w_d \geq 0, \qquad r = 1,2,\dots,s;\ i = 1,2\dots,m;\ d = 1,2,\dots,p.
\end{aligned}$$

Though the above model is nonlinear thanks to the terms $w_d z_{dj}$, it can turn out to be linear by dint of the variable substitutions. After solving model (6) for a given γ-level, one can obtain u_r^*, v_i^*, w_d^* and z_{dj}^* which enable us to calculate the upper bound of the first and second fuzzy process efficiencies as $(E_o^1)_\gamma^U = \frac{\sum_{d=1}^p w_d^* z_{do}^*}{\sum_{i=1}^m v_i^* (x_{io})_\gamma^L}$ and $(E_o^2)_\gamma^U = \frac{\sum_{r=1}^s u_r^* (y_{ro})_\gamma^U}{\sum_{d=1}^p w_d^* z_{do}^*}$, respectively in a way that $(E_o)_\gamma^U = (E_o^1)_\gamma^U \times (E_o^2)_\gamma^U$.

Dissimilarly, the lower bound of the system efficiency cannot be transformed into the LP program because the inner and outer programs have opposite directions for optimization. To deal with the problem, Kao and Liu (2011) use the duality theorem and the inner program is substituted with its dual with the aim of making the minimization programs for both the inner and outer programs. As a result of such concerted effort, Kao and Liu (2011) arrive at the following nonlinear program (NLP):

$$(E_o)_\gamma^L = \min_{\varphi,\beta,\gamma,z} \theta, \tag{7}$$

$$s.t.\ \ \theta(x_{io})^U_\gamma - \left[\varphi_o(x_{io})^U_\gamma + \sum_{\substack{j=1\\ j\neq k}}^{n}\varphi_j\left(x_{ij}\right)^L_\gamma\right] - \left[\beta_o(x_{io})^U_\gamma + \sum_{\substack{j=1\\ j\neq k}}^{n}\beta_j\left(x_{ij}\right)^L_\gamma\right] \geq 0, i = 1, \dots, m,$$

$$\sum_{j=1}^{n}\beta_j z_{dj} - \sum_{j=1}^{n}\gamma_j z_{dj} \geq 0, \quad d = 1,2,\dots,p,$$

$$\left[\varphi_O(y_{rO})^L_\gamma + \sum_{\substack{j=1\\ j\neq k}}^{n}\varphi_j\left(y_{rj}\right)^U_\gamma\right] - \left[\gamma_o(y_{ro})^L_\gamma + \sum_{\substack{j=1\\ j\neq k}}^{n}\gamma_j\left(y_{rj}\right)^U_\gamma\right] \geq (y_{ro})^L_\gamma, \quad r = 1,2,\dots,s,$$

$$\left(z_{dj}\right)^L_\gamma \leq z_{dj} \leq \left(z_{dj}\right)^U_\gamma, \quad d = 1,2,\dots,p; \ \ j = 1,2,\dots,n,$$

$$\varphi_j, \beta_j, \gamma_j \geq 0, \quad j = 1,2,\dots,n.$$

Furthermore, Kao and Liu (2011) take the shadow prices of model (7) into consideration at optimality to get the values weights u_r^*, v_i^* and w_d^* for calculating the lower bound of the two fuzzy process efficiencies as $(E_o^1)^L_\gamma = \frac{\sum_{d=1}^{p} w_d^* z_{do}^*}{\sum_{i=1}^{m} v_i^*(x_{io})^U_\gamma}$ and $(E_o^2)^U_\gamma = \frac{\sum_{r=1}^{s} u_r^*(y_{ro})^L_\gamma}{\sum_{d=1}^{p} w_d^* z_{dj}^*}$, respectively, where $(E_o)^U_\gamma = (E_o^1)^U_\gamma \times (E_o^2)^U_\gamma$.

At present, let us draw attention to a number of drawbacks of Kao and Liu (2011)'s approach. First, it might be hard and complicated from a computational perspective to gain the lower bounds of the system efficiencies, especially for real-world problems with excessive observations, because model (7) is nonlinear. Second, when multiple solutions occur in models (6) and (7), the interval efficiencies of two processes may not be unique and this issue would lead to confusion about actual measures in performance evaluation. The last flaw occurs when the NLP solvers are not able to yield the shadow prices for model (7). Our motivation for this study is derived from the interest in tackling these drawbacks by developing an enhanced assessment method, which is capable of seeking the common weights for inputs, outputs and intermediate products.

3. Common-weights two-stage DEA model with fuzzy data

As mentioned in the earlier section, the aim of the fuzzy two-stage DEA model is to allow each DMU to adopt the full weight flexibility that sets itself in the most favourable light against the other units. However, solving a pair of mathematical models for each DMU is not only beneficial from computation perspective, but also leads to the different sets of weights that normally have high variance. Inspired by Saati and Memariani (2005), we present a common-weights method for two-stage structures that allow us to equitably evaluate the system efficiency and the component process efficiencies on the same scale. As a matter of fact, the common-weights method is a special case of weight restrictions when inter-unit weight flexibility is completely eliminated. The existing common-weights methods are generally compliant with two principles; it is first essential to ensure that the CSW is set out to be within the weight intervals, and the second one states that the CSW is able to precisely reflect the components of each DMU to a large extent, that is, at least one DMU turns out to be 100% efficient using the CSW.

Assume that the inputs, outputs and intermediate measures of the j^{th} DMU are characterized by the fuzzy numbers as $\tilde{x}_{ij}, i = 1,2,\dots,m, j = 1,2,\dots,n$, $\tilde{y}_{rj}, r = 1,2,\dots,s, j = 1,2,\dots,n$ and $\tilde{z}_{dj}, d = 1,2,\dots,p, j = 1,2,\dots,n$, respectively. Imposing absolute lower and upper bounds on input and output weights, the fuzzy two-stage DEA model is then given by:

$$\max_{u,v,w} \sum_{r=1}^{s} u_r \tilde{y}_{ro}, \tag{8}$$

$$\begin{aligned} s.t \quad & \sum_{i=1}^{m} v_i \tilde{x}_{io} = 1, \\ & \sum_{d=1}^{p} w_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \le 0, \quad j = 1,2,\dots,n, \\ & \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{p} w_d \tilde{z}_{dj} \le 0, \quad j = 1,2,\dots,n, \\ & u_r^l \le u_r \le u_r^u, \quad r = 1,2,\dots,s, \\ & v_i^l \le v_i \le v_i^u, \quad i = 1,2,\dots,m, \\ & w_d^l \le w_d \le w_d^u, \quad d = 1,2,\dots,p. \end{aligned}$$

Our common-weights method includes two steps; the first step aims at determining an upper bound on each weight using a LP problem, and the second step intends to determine the CSW using a LP problem where additional constraints are permitted on each factor weight.

3.1. First step

The following mathematical model is formulated to obtain the upper bounds on all the output weights:

$$\max_{u,v,w} u_r, \tag{9}$$

$$\begin{aligned} s.t \quad & \sum_{i=1}^{m} v_i \tilde{x}_{ij} \le 1, \quad j = 1,2,\dots,n, \\ & \sum_{d=1}^{p} w_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \le 0, \quad j = 1,2,\dots,n, \\ & \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{p} w_d \tilde{z}_{dj} \le 0, \quad j = 1,2,\dots,n, \\ & u_r, v_i, w_d \ge 0, \quad r = 1,2,\dots,s\,; \ i = 1,2,\dots,m\,; \ d = 1,2,\dots,p, \end{aligned}$$

where $u_r, r = 1,2,\dots,s,$ presents the upper bound associated with the r^{th} output weights. The objective of the above model is to maximize each output weight under a given production possibility set. Without loss of generality, let $\tilde{x}_{ij} = \left(x_{ij}^m, x_{ij}^\alpha, x_{ij}^\beta\right), i = 1,2,\dots,m, j = 1,2,\dots,n$, $\tilde{y}_{rj} = \left(y_{rj}^m, y_{rj}^\alpha, y_{rj}^\beta\right)$, $r = 1,2,\dots,s$, $j = 1,2,\dots,n$ and $\tilde{z}_{dj} = \left(z_{dj}^m, z_{dj}^\alpha, z_{dj}^\beta\right), d = 1,2,\dots,p$, $j = 1,2,\dots,n$ are the triangular fuzzy numbers that have been widely used in practice and theoretical studies. To calculate the upper bounds on weights associated with the i^{th} input and the d^{th} intermediate measure, one can only replace the objective function of model (9) with "$\max_{u,v,w} \ v_i$" and "$\max_{u,v,w} \ w_d$", respectively. To this end, it is required to solve $s+i+p$ fuzzy linear programming (FLP) problems. However, solving the FLP problems is a long-standing challenge in the literature and many methods have been developed by researchers (see Luhandjula (1989) for an overview).

As earlier mentioned, the fuzzy DEA models by way of FLP problems can be classified into six groups: the tolerance approach, the γ-level based approach, the fuzzy ranking approach, the possibility and credibility approach, the fuzzy arithmetic, and the fuzzy random/type-2 fuzzy sets (Hatami-Marbini et al., 2011; Emrouznejad et al., 2014). Amongst them, we rely on the γ-level based approach developed by Saati et al. (2002) to be adopted for computing upper bounds of weights through the FLP problems since this approach makes concerted effort to preserve the fuzzy information without detriment of the decision-makers' intuition and subjective judgements in the performance assessment. In what follows, the inputs, outputs and intermediate measures are represented by different levels of confidence intervals and the fuzzy model is ultimately transformed as follows:

$$
\begin{aligned}
&\max_{u,v,w} \quad u_r && .(10)\\
&s.t \quad \sum_{i=1}^{m} v_i \hat{x}_{ij} \leq 1, \quad j = 1,2,\dots,n,\\
&\quad \sum_{d=1}^{p} w_d \hat{z}_{dj} - \sum_{i=1}^{m} v_i \hat{x}_{ij} \leq 0, \quad j = 1,2,\dots,n,\\
&\quad \sum_{r=1}^{s} u_r \hat{y}_{rj} - \sum_{d=1}^{p} w_d \hat{z}_{dj} \leq 0, \quad j = 1,2,\dots,n,\\
&\quad y_{rj}^{m} - (1-\gamma) y_{rj}^{\alpha} \leq \hat{y}_{rj} \leq y_{rj}^{m} + (1-\gamma) y_{rj}^{\beta}, \quad r = 1,2,\dots,s; j = 1,2,\dots,n,\\
&\quad x_{ij}^{m} - (1-\gamma) x_{ij}^{\alpha} \leq \hat{x}_{ij} \leq x_{ij}^{m} + (1-\gamma) x_{ij}^{\beta}, \quad i = 1,2,\dots,m; j = 1,2,\dots,n,\\
&\quad z_{dj}^{m} - (1-\gamma) z_{dj}^{\alpha} \leq \hat{z}_{dj} \leq z_{dj}^{m} + (1-\gamma) z_{dj}^{\beta}, \quad d = 1,2,\dots,p; j = 1,2,\dots,n,\\
&\quad u_r, v_i, w_d \geq 0, \quad r = 1,2,\dots,s; \ i = 1,2,\dots,m; \ d = 1,2,\dots,p,
\end{aligned}
$$

where $\hat{x}_{ij} \epsilon \left[x_{ij}^{m} - (1-\gamma) x_{ij}^{\alpha}, x_{ij}^{m} + (1-\gamma) x_{ij}^{\beta}\right]$, $\hat{y}_{rj} \epsilon \left[y_{rj}^{m} - (1-\gamma) y_{rj}^{\alpha}, y_{rj}^{m} + (1-\gamma) y_{rj}^{\beta}\right]$, $\hat{z}_{dj} \epsilon \left[z_{dj}^{m} - (1-\gamma) z_{dj}^{\alpha}, z_{dj}^{m} + (1-\gamma) z_{dj}^{\beta}\right]$ are interval alteration variables. Since terms $v_i \hat{x}_{ij}$, $w_d \hat{z}_{dj}$ and $u_r \hat{y}_{rj}$ render the above model nonlinear, they can be substituted by $\bar{x}_{ij}$, $\bar{y}_{rj}$ and $\bar{z}_{dj}$, respectively, in order to arrive at the following parametric LP model:

$$
\begin{aligned}
&\max \quad u_r && (11)\\
&s.t \quad \sum_{i=1}^{m} \bar{x}_{ij} \leq 1, \quad j = 1,2,\dots,n,
\end{aligned}
$$

$\sum_{d=1}^{p} \bar{z}_{dj} - \sum_{i=1}^{m} \bar{x}_{ij} \leq 0, \qquad j = 1,2,\dots,n,$

$\sum_{r=1}^{s} \bar{y}_{rj} - \sum_{d=1}^{p} \bar{z}_{dj} \leq 0, \qquad j = 1,2,\dots,n,$

$u_r\left(y_{rj}^m - (1-\gamma)y_{rj}^\alpha\right) \leq \bar{y}_{rj} \leq u_r\left(y_{rj}^m + (1-\gamma)y_{rj}^\beta\right), \quad r = 1,2,\dots,s; j = 1,2,\dots,n,$

$v_i\left(x_{ij}^m - (1-\gamma)x_{ij}^\alpha\right) \leq \bar{x}_{ij} \leq v_i\left(x_{ij}^m + (1-\gamma)x_{ij}^\beta\right), \quad i = 1,2,\dots,m; j = 1,2,\dots,n,$

$w_d\left(z_{dj}^m - (1-\gamma)z_{dj}^\alpha\right) \leq \bar{z}_{dj} \leq w_d\left(z_{dj}^m + (1-\gamma)z_{dj}^\beta\right), \quad d = 1,2,\dots,p; j = 1,2,\dots,n,$

$\bar{y}_{rj}, \bar{x}_{ij}, \bar{z}_{dj}, u_r, v_i, w_d \geq 0, \quad r = 1,2,\dots,s; i = 1,2,\dots,m; d = 1,2,\dots,p; j = 1,2,\dots,n,$

where $\gamma \in [0,1]$ is a parameter. We point out that the analogous set of constraints in model (11) is used to formulate the models for calculating the upper bound on input weights and intermediate measure weights in which their objective functions are "$\max_{u,v,w} v_i$" and "$\max_{u,v,w} w_d$", respectively.

Theorem 1. The program (11) has a feasible solution.

Proof. It can be straightforwardly proved due to the fact that $(\bar{x}, \bar{y}, \bar{z}, u, v, w) = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$ is a feasible solution to program (11). ■

Theorem 2. The optimal objective solution of program (11) is bounded and positive.

Proof. Let us look into the dual of model (11) as:

$$\min \sum_{j=1}^{n} \beta_j, \tag{12}$$

$$\begin{aligned}
s.t \quad & \beta_j - \omega_j - \delta_{ij} + \mu_{ij} \geq 0, && i = 1,2,\dots,m; j = 1,2,\dots,n,\\
& \lambda_j - \pi_{rj} + \rho_{rj} \geq 0, && r = 1,2,\dots,s; j = 1,2,\dots,n,\\
& \omega_j - \lambda_j - \sigma_{pj} + \varphi_{pj} \geq 0, && d = 1,2,\dots,p; j = 1,2,\dots,n,\\
& \textstyle\sum_{j=1}^{n}\left(x_{ij}^m - (1-\gamma)x_{ij}^\alpha\right)\delta_{ij} - \sum_{j=1}^{n}\left(x_{ij}^m + (1-\gamma)x_{ij}^\beta\right)\mu_{ij} \geq 0, && i = 1,2,\dots,m,\\
& \textstyle\sum_{j=1}^{n}\left(z_{dj}^m - (1-\gamma)z_{dj}^\alpha\right)\sigma_{dj} - \sum_{j=1}^{n}\left(z_{dj}^m + (1-\gamma)z_{dj}^\beta\right)\varphi_{dj} \geq 0, && d = 1,2,\dots,p,\\
& \textstyle\sum_{j=1}^{n}\left(y_{rj}^m - (1-\gamma)y_{rj}^\alpha\right)\pi_{rj} - \sum_{j=1}^{n}\left(y_{rj}^m + (1-\gamma)y_{rj}^\beta\right)\rho_{rj} \geq 0, && r = 1,2,\dots,s; r \neq t,\\
& \textstyle\sum_{j=1}^{n}\left(y_{tj}^m - (1-\gamma)y_{tj}^\alpha\right)\pi_{tj} - \sum_{j=1}^{n}\left(y_{tj}^m + (1-\gamma)y_{tj}^\beta\right)\rho_{tj} \geq 1, &&\\
& \beta_j, \omega_j, \lambda_j, \delta_{ij}, \mu_{ij}, \pi_{rj}, \rho_{rj}, \sigma_{pj}, \varphi_{pj} \geq 0, && r = 1,2,\dots,s; i = 1,2,\dots,m\ d = 1,2,\dots,p.
\end{aligned}$$

Given that each DMU has at least one positive output, i.e. $y_{rj}^m > y_{rj}^\alpha$, we can show that there is a h, $1 \leq h \leq n$ such that $H = y_{th}^m - (1-\gamma)y_{th}^\alpha \neq 0$. A feasible solution of the dual problem is $\beta_j = 0\ (\forall j \neq h), \beta_h = \frac{1}{H}, \omega_j = 0\ (\forall j \neq h), \omega_h = \frac{1}{H}, \delta_{ij} = \mu_{ij} = 0\ (\forall i,j), \sigma_{dj} = \varphi_{dj} = 0 (\forall d,j),\ \rho_{rj} = 0 (\forall r,j), \pi_{rj} = 0 (\forall r \neq t, j \neq h), \pi_{th} = \frac{1}{H}$. Therefore, the optimal objective solution of (11) has an upper bound according to the duality theorem.

Let us now focus on program (10) to prove the positivity of optimal solutions of (11). It is not hard to show that the point $\hat{x}_{ij} = x_{ij}^m + (1-\gamma)x_{ij}^\beta (\forall i, j)$, $\hat{z}_{dj} = z_{dj}^m + (1-\gamma)z_{dj}^\beta (\forall d, j), \hat{y}_{rj} = y_{rj}^m + (1-\gamma)y_{rj}^\beta (\forall r, j), v_i = \frac{1}{mb_i}(\forall i)$, $u_r = \frac{1}{sz_r}(\forall r)$, $w_d = \frac{1}{pl_d}(\forall d)$ is a feasible solution for model (10) where $z_r = \max_j\left\{y_{rj}^m + (1-\gamma)y_{rj}^\beta\right\} \neq 0$, $b_i = \max_j\left\{x_{ij}^m + (1-\gamma)x_{ij}^\beta\right\} \neq 0$, $l_d = \left\{\max_j z_{dj}^m + (1-\gamma)z_{dj}^\beta\right\} \neq 0$, and m, s and p represent the number of inputs, outputs and intermediate measures. In view of $u_r = \frac{1}{sz_r} > 0$, the optimal objective value of (11) is always positive. ■

Similarly, theorems (1) and (2) can be employed to the corresponding models for computing the upper bounds on input weights and intermediate measure weights.

3.2. Second step

Let us consider the algorithm in turn for determining a CSW. The existing common-weights methods are developed based on the two principles; (i) it needs to make sure that the CSW lies within the weight intervals, and (ii) the CSW is capable of accurately reveal the components of each DMU in a way that at least one DMU is technically efficient. In some real-life applications, prior preference and knowledge of decision-makers on any factors are not accessible. In such case, the central values across all the weights are an easy-to-use and yet powerful approach to generate a CSW and rigidly control the flexibility in the selection of weights even in the case of the fuzzy two-stage DEA models. This idea was initially formulated by Roll and Golany (1993) in conventional DEA. Considering the fuzzy two-stage DEA model (8) with bounded intervals for all the weights as $u_r \in [u_r^l, u_r^u]$, $v_i \in [v_i^l, v_i^u]$ and $w_d \in [w_d^l, w_d^u]$, the CSW can be determined by defining the identical deviation, denoted by Q, from the upper and lower bounds of weights. Employing the identical deviations among all DMUs, the following FLP program is proposed to obtain a CSW:

$$\max_{u,v,w,Q} \quad Q \tag{13}$$

$$\begin{aligned}
s.t \quad & \textstyle\sum_{d=1}^{p} w_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0, && j = 1,2,\ldots,n,, \\
& \textstyle\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{p} w_d \tilde{z}_{dj} \leq 0, && j = 1,2,\ldots,n, \\
& u_r^l + Q(u_r^u - u_r^l) \leq u_r \leq u_r^u - Q(u_r^u - u_r^l), && r = 1,2,\ldots,s, \\
& v_i^l + Q\left(v_i^u - v_i^l\right) \leq v_i \leq v_i^u - Q\left(v_i^u - v_i^l\right), && i = 1,2,\ldots,m, \\
& w_d^l + Q\left(w_d^u - w_d^l\right) \leq w_d \leq w_d^u - Q\left(w_d^u - w_d^l\right), && d = 1,2,\ldots,p,
\end{aligned}$$

where Q is a virtual variable that lies within [0, 0.5]. In the case of Q=0.5, all weights ideally take the centre of their bounded intervals. In other words, the increase in Q from 0 to 0.5 tightens the range of each weight. It is worth noting that the optimal objective value of model (13) cannot take a value greater than 0.5 since this leads to a contradiction to our earlier assumption on bounded intervals for weights. Because of unknown bounds on weights, model (11) is a NLP problem. To linearise model (11), we hence suppose the lower bounds of factor weights take a zero value, and the upper bounds of weights are calculated in the first step that is delineated in the previous subsection. The resulting fuzzy LP program is simplified as follows:

$$\max_{u,v,w,Q} \quad Q \tag{14}$$

$$\begin{aligned}
s.t \quad & \textstyle\sum_{d=1}^{p} w_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0, && j = 1,2,\dots,n, \\
& \textstyle\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{p} w_d \tilde{z}_{dj} \leq 0, && j = 1,2,\dots,n, \\
& Q u_r^u \leq u_r \leq (1-Q) u_r^u, && r = 1,2,\dots,s, \\
& Q v_i^u \leq v_i \leq (1-Q) v_i^u, && i = 1,2,\dots,m, \\
& Q w_d^u \leq w_d \leq (1-Q) w_d^u\,, && d = 1,2,\dots,p,
\end{aligned}$$

when $\tilde{x}_{ij}$, $\tilde{z}_{dj}$ and $\tilde{y}_{rj}$ are represented by fuzzy numbers, the aforesaid γ-level based method can be accommodated to reach the following crisp programming model:

$$\max \; Q \tag{15}$$

$$\begin{aligned}
s.t \quad & \textstyle\sum_{d=1}^{p} \bar{z}_{dj} - \sum_{i=1}^{m} \bar{x}_{ij} \leq 0, \quad j = 1,2,\dots,n, \\
& \textstyle\sum_{r=1}^{s} \bar{y}_{rj} - \sum_{d=1}^{p} \bar{z}_{dj} \leq 0, \quad j = 1,2,\dots,n, \\
& u_r\left(y_{rj}^m - (1-\gamma) y_{rj}^\alpha\right) \leq \bar{y}_{rj} \leq u_r\left(y_{rj}^m + (1-\gamma) y_{rj}^\beta\right), \quad r = 1,2,\dots,s; j = 1,2,\dots,n, \\
& v_i\left(x_{ij}^m - (1-\gamma) x_{ij}^\alpha\right) \leq \bar{x}_{ij} \leq v_i\left(x_{ij}^m + (1-\gamma) x_{ij}^\beta\right), \quad i = 1,2,\dots,m; j = 1,2,\dots,n, \\
& w_p\left(z_{dj}^m - (1-\gamma) z_{dj}^\alpha\right) \leq \bar{z}_{dj} \leq w_p\left(z_{dj}^m + (1-\gamma) z_{dj}^\beta\right), \quad d = 1,2,\dots,p; j = 1,2,\dots,n, \\
& Q u_r^u \leq u_r \leq (1-Q) u_r^u, \quad r = 1,2,\dots,s, \\
& Q v_i^u \leq v_i \leq (1-Q) v_i^u, \quad i = 1,2,\dots,m, \\
& Q w_d^u \leq w_d \leq (1-Q) w_d^u\,, \quad d = 1,2,\dots,p, \\
& \bar{y}_{rj}, \bar{x}_{ij}, \bar{z}_{dj} \geq 0, \quad r = 1,2,\dots,s; i = 1,2,\dots,m; d = 1,2,\dots,p; j = 1,2,\dots,n.
\end{aligned}$$

Note that, for a certain γ-level, u_r^u, v_i^u and w_d^u are the optimal objective values of those models developed in Step 1. The optimal solution of the above model, i.e., u_r^*, v_i^* and w_d^*, is the CSW for a given γ-level, which is used to find the fuzzy system efficiency and fuzzy process efficiencies. In this respect, the upper bound of system and process efficiencies of DMU$_o$ is defined as $(E_o)_\gamma^U = \sum_{r=1}^{s} u_r^* (y_{ro})_\gamma^U / \sum_{i=1}^{m} v_i^* (x_{io})_\gamma^L$, $(E_o^1)_\gamma^U =$

$\sum_{d=1}^{p} w_d^* \bar{z}_{do}^* / \sum_{i=1}^{m} v_i^* (x_{io})_\gamma^L$ and $(E_o^2)_\gamma^U = \sum_{r=1}^{s} u_r^* (y_{ro})_\gamma^U / \sum_{d=1}^{p} w_d^* \bar{z}_{do}^*$ where $(E_o)_\gamma^U = (E_o^1)_\gamma^U \times (E_o^2)_\gamma^U$ and its respective lower bound is in turn defined as $(E_o)_\gamma^L = \sum_{r=1}^{s} u_r^* (y_{ro})_\gamma^L / \sum_{i=1}^{m} v_i^* (x_{io})_\gamma^U$, $(E_o^1)_\gamma^L = \sum_{d=1}^{p} w_d^* \bar{z}_{do}^* / \sum_{i=1}^{m} v_i^* (x_{io})_\gamma^U$ and $(E_o^2)_\gamma^L = \sum_{r=1}^{s} u_r^* (y_{ro})_\gamma^L / \sum_{d=1}^{p} w_d^* \bar{z}_{do}^*$ where $(E_o)_\gamma^L = (E_o^1)_\gamma^L \times (E_o^2)_\gamma^L$. Contrary to Kao and Liu (2011)'s method, we think of the similar factor weights to calculate the upper and lower bounds of efficiencies, which would be more rational from the managerial perspective, rather than utilizing different weights. Put differently, unconstrained factor weights in Kao and Liu (2011)'s method might lead to biased weight distribution across factors which are usually unaccepted from decision-makers, especially in centralised organisations. Not surprisingly, we still observe the relational mathematical relationships between the system efficiency and the process efficiencies. Figure 2 sums up the proposed procedure in this study using three structured successive phases.

4. A comparative numerical example

In this section, we analyse the performance of 24 non-life insurance companies in Taiwan where the operation of each company includes two distinct processes; (i) premium acquisition and (ii) profit generation. The inputs of the first process are *operating expenses (x_1)* and *insurance expenses (x_2)* to produce the two intermediate measures; *direct written premiums (z_1)* and *reinsurance premiums (z_2)*. All these intermediate measures are then consumed by the second process to produce the two final outputs; *underwriting profit (y_1)* and *investment profit (y_2)*. The fuzzy data has been created based on the data of 2001 and 2002 to deal with imprecision to some appropriate extent. The data taken from Kao and Liu (2011) is shown in Table 1. Note that a fuzzy number in Table 1 that is by way of (a, b, c) can be written as (b, α, β) where $\alpha = b - a$ and $\beta = b - c$. Let us define 11 γ-levels {0, 0.1, 0.2,…,1} from the outset with the aim of evaluating 24 insurance companies. Solving the proposed model in the first step, we obtain the optimal input weights v_1^* and v_2^*, intermediate measures weights w_1^* and w_2^*, and output weights u_1^* and u_2^* as reported in Table 2. As can been spotted in Table 2, the ranges that all possible weight values can lie within are $v_1 \in$ [7.5E-5, 7.9E-5],$v_2 \in$ [1.49E-4, 1.58E-4],$u_1 \in$ [1.1E-5, 1.4E-5],$u_2 \in$ [9.2E-5, 1.22E-4],$w_1 \in$ [1.2E-5, 1.4E-5] and $w_2 \in$ [5.4E-5, 6.4E-5] where the increase in the γ-level always leads to the drop in the weights. By employing these weights, model (15) enables to calculate the upper and lower bounds of fuzzy system efficiency $[(E_j)_\gamma^L, (E_j)_\gamma^U]$ and two fuzzy process efficiencies, i.e., $[(E_o^1)_\gamma^L, (E_o^1)_\gamma^U]$ and $[(E_o^2)_\gamma^L, (E_o^2)_\gamma^U]$ as presented in Table 3. Note that S, $P1$, and $P2$ are indicated system, Process 1 and Process 2, respectively. Contrary to (Kao and Liu, 2011, Table 2, p. 31]'s results, in many cases our calculated efficiencies become smaller which bespeaks higher discrimination among insurance companies at large. In fact, the weak discriminatory power of Kao and Liu (2011) with eight efficient companies in Process 1 (DMU 19) and three efficient companies in Process 2 is significantly improved to one efficient company in Process 1 and one efficient company in Process 2 (DMU 22). The performance evaluation literally lies within two extremes of the continuum with respect to the value of γ-levels. When $\gamma = 1$, the bounded data turns out to be one value and consequently the deterministic case occurs. That is to say that at $\gamma = 1$ the upper and lower bounds of efficiency scores are identical (see the last column of Table 3). Oppositely, at $\gamma = 0$ it is supposed to appear efficiency scores within the widest range for each unit. For example, the bounded system efficiency of DMU6 is (0.214,0.273) and (0.209,0.209) at $\gamma = 0$ and $\gamma = 1$, respectively, while they are (0.279,0.514) and (0.390,0.390) as per Kao and Liu (2011)'s method.

The membership functions of the fuzzy system and process efficiencies can be determined by using the lower and upper bounds at different γ-levels. Figure 3 shows the membership functions associated with the fuzzy system and process efficiencies for DMU6. To increase the precision of the membership functions, one unavoidably needs to define more γ-levels, leading to a higher calculation burden.

The relationship between the upper (lower) bound of the system efficiency and the upper (lower) of the two process efficiencies is observed at all γ-levels. For instance, let us consider DMU 6 at $\gamma = 0.3$. The product of the lower bound efficiencies of Process 1 (0.575) and Process 2 (0.365) is equal to the system efficiency (0.21), and the analogous relationship can be seen in its upper bounds. i.e. 0.626×0.387=0.242.

Let us observe the results in the case of $\gamma = 1$ as reported in the last column of Table 2. As per the system and process efficiencies, the insurance companies can be ranked and indicated in bold. DMU24 is known as the worst system efficiency among the companies. The prime question arises from the management team: what are the sources of inefficiency in order to make appropriate actions? To answer this vital question, it is need to look into the performance of two decompositions; (1) premium acquisition and (2) profit generation. According to our findings, its inefficiency primarily stems from Process 2, showing that this company has the weakest performance in generating profit in comparison with other companies. However, DMU24's performance is very effective in Process 1 by taking the second largest efficiency among all 24 companies. Therefore, this method enables the decision makers to be aware of the core of inefficiency sources in order to take appropriate actions.

5. Conclusions

Having minutely gone through production systems in the real-world applications, we observe many situations where a system consists of two entwined processed in series. Contrary to tradition black-box structures, we need to attend to intermediate products when evaluating the performance of entities. Although the inherent uncertainty can be characterised by fuzzy numbers to reflect the decision-makers' subjective judgements, the complexity of the problem increases and it is awkward to fairly measure the relative efficiencies.

This paper makes an attempt to extend the fuzzy two-stage DEA approach proposed by Kao and Liu (2011) since their method suffers from two major flaws; firstly, the model for calculating the lower bound of the system efficiency is nonlinear and secondly, the interval efficiencies of two processes may not be unique in the light of multiple solutions. Dealing with these flaws incites us to develop a common-weights method for two-stage structures in a fuzzy environment. The proposed approach is twofold; the first step calculates the upper bounds on weights associated with inputs, intermediate measures and outputs and the second step determines a set of common weights by a LP problem. We illustrate the approach with a dataset taken from the literature.

The framework proposed in this paper could lend itself to real-world problems where the primary and/or secondary data collected contain vague and natural languages. For future study, it would be also compelling to develop other structures of production systems in the presence of fuzzy data.

ACKNOWLEDGEMENT

The author would like to thank the four anonymous reviewers for their insightful and constructive comments.

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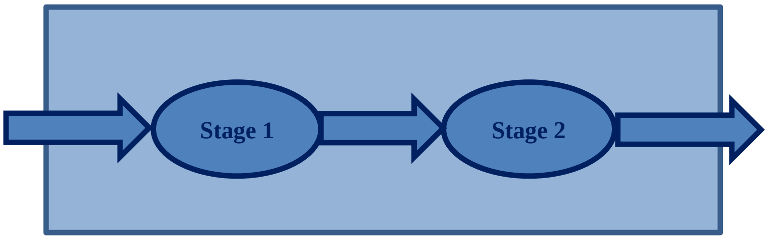

Figure 1. A two-stage structure

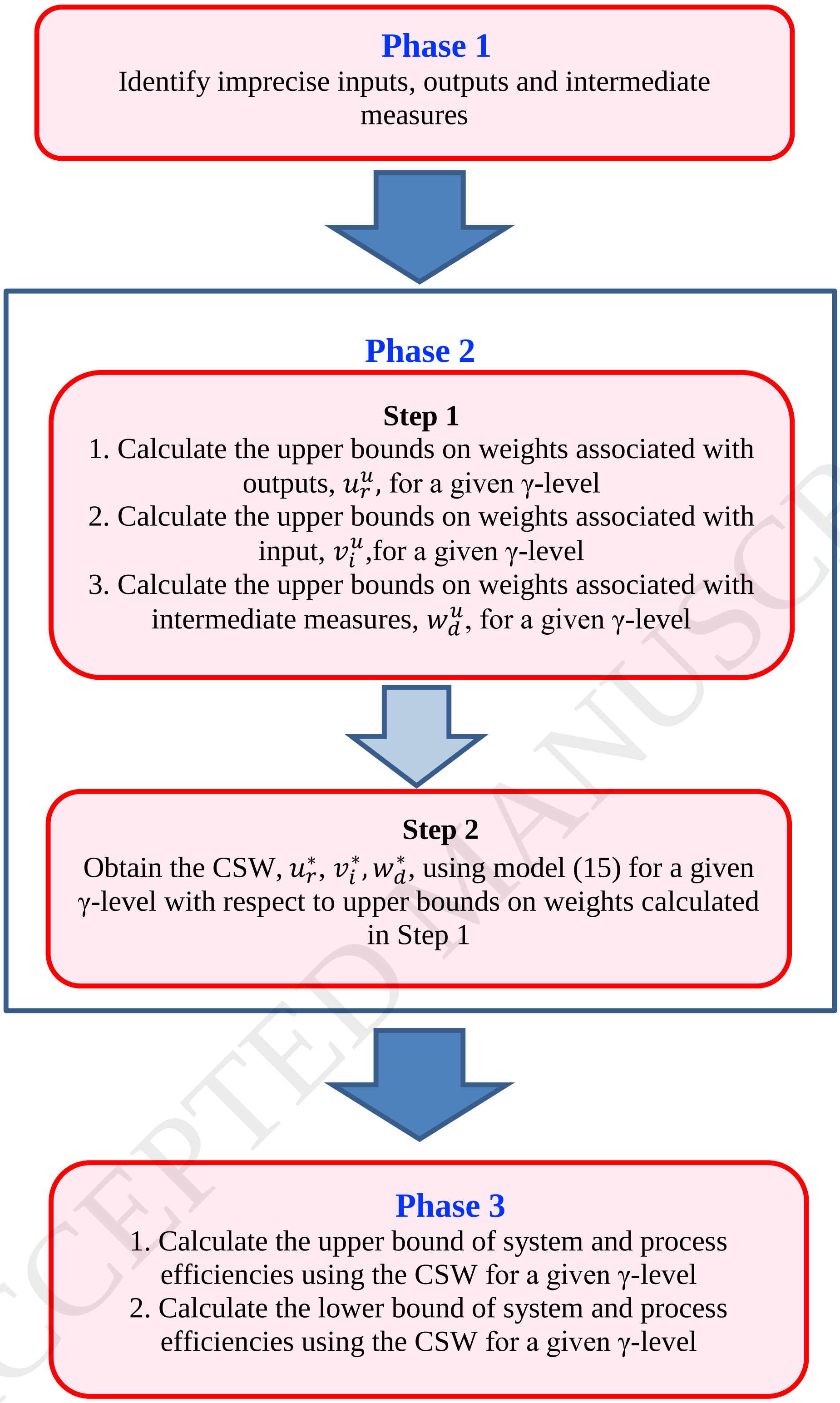

Figure 2. The proposed method.

Figure 3. The membership functions of system and two processes efficiencies for DMU6

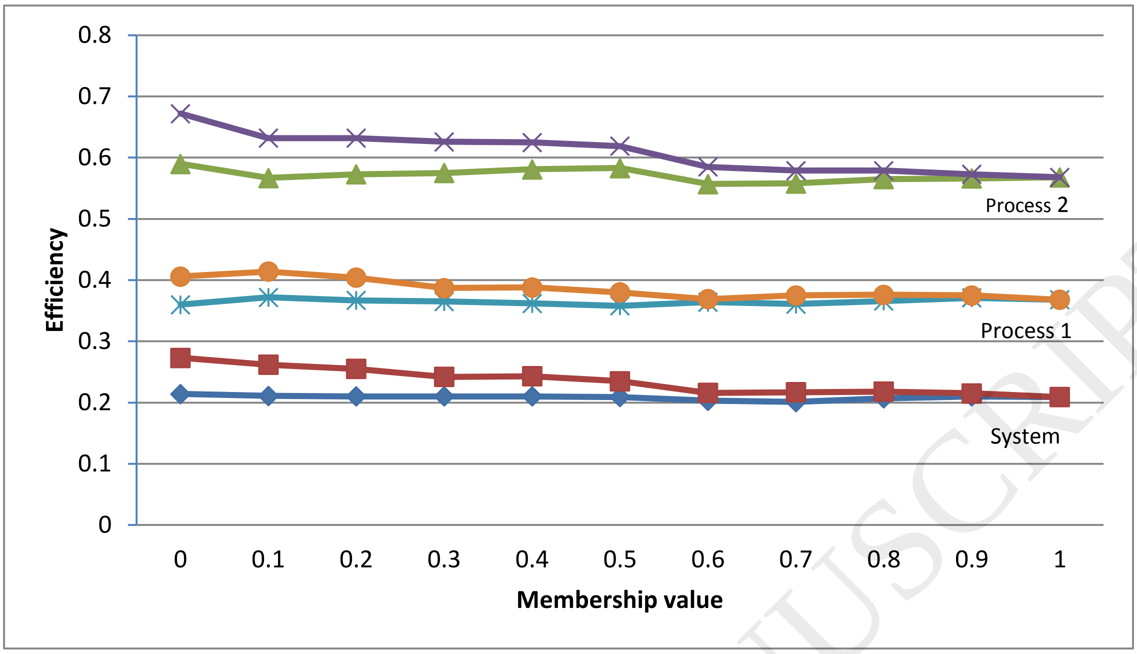

Table 1. Triangular fuzzy numbers of 24 insurance companies in Taiwan.

Co.	x_1	x_2	z_1	z_2	y_1	y_2
1	(1113,1178, 1256)	(636,673, 717)	(7041,7451, 7943)	(809,856, 912)	(930,984, 1049)	(644,681, 726)
2	(1305,1381, 1472)	(1278,1352, 1441)	(9469,10020, 10681)	(1712,1812, 1932)	(1160,1228, 1309)	(788,834, 889)
3	(1112,1117, 1255)	(559,592, 631)	(4513,4776, 5091)	(529,560, 597)	(277,293, 312)	(622,658, 701)
4	(568,601, 641)	(561,594, 633)	(2999,3174, 3383)	(351,371, 395)	(234,248, 264)	(167,177, 189)
5	(6331,6699, 7141)	(3167,3351, 3572)	(35335,37362, 39680)	(1657,1753, 1869)	(7419,7851, 8369)	(3709,3925, 4184)
6	(2483,2627, 2800)	(631,668, 712)	(9211,9747, 10390)	(900,952, 1015)	(1619,1713, 1826)	(392,415, 442)
7	(1853,1942, 2047)	(1377,1443, 1521)	(10193,10685, 11262)	(613,643, 678)	(2136,2239, 2360)	(419,439, 463)
8	(3615,3789, 3994)	(1787,1873, 1974)	(16473,17267, 18199)	(1082,1134, 1195)	(3720,3899, 4110)	(593,622, 656)
9	(1495,1567, 1652)	(906,950, 1001)	(10945,11473, 12093)	(521,546, 575)	(995,1043, 1099)	(252,264, 278)
10	(1243,1303, 1373)	(1238,1298, 1368)	(7832,8210, 8653)	(481,504, 531)	(1619,1697, 1789)	(529,554, 584)
11	(1872,1962, 2068)	(641,672, 708)	(6890,7222, 7612)	(613,643, 678)	(1418,1486, 1566)	(17,18, 19)
12	(2473,2592, 2732)	(620,650, 685)	(9000,9434, 9943)	(1067,1118, 1178)	(1502,1574, 1652)	(867,909, 958)
13	(2481,2609, 2739)	(1301,1368, 1436)	(13239,13921, 14617)	(771,811, 852)	(3432,3609, 3789)	(212,223, 234)
14	(1328,1369, 1466)	(940,988, 1037)	(7034,7396, 7766)	(442,465, 488)	(1332,1401, 1471)	(316,332, 349)
15	(2077,2184, 2293)	(619,651, 684)	(9911,10422, 10943)	(712,749, 786)	(3191,3355, 3523)	(528,555, 583)
16	(1152,1211, 1272)	(395,415, 436)	(5331,5606, 5886)	(382,402, 422)	(812,854, 897)	(187,197, 207)
17	(1382,1453, 1526)	(1032,1085, 1139)	(7318,7695, 8080)	(325,342, 359)	(2990,3144, 3301)	(353,371, 390)
18	(720,757, 795)	(520,547, 574)	(3453,3631, 3813)	(947,995, 1045)	(658,692, 727)	(155,163, 171)
19	(151,159, 167)	(173,182, 191)	(1083,1141, 1196)	(458,483, 506)	(493,519, 544)	(44,46, 48)
20	(138,145, 152)	(50,53, 56)	(300,316, 331)	(124,131, 137)	(337,355, 372)	(25,26, 27)
21	(80,84, 88)	(25,26, 27)	(214,225, 236)	(38,40, 42)	(48,51, 53)	(6,6,6)
22	(14,15, 16)	(9,10, 10)	(49,52, 54)	(13,14, 15)	(78,82, 86)	(4,4,4)
23	(51,54, 57)	(27,28, 29)	(233,245, 257)	(47,49, 51)	(1,1,1)	(17,18, 19)
24	(155,163, 171)	(223,235, 246)	(452,476, 499)	(611,644, 675)	(135,142, 149)	(15,16, 17)

Table 2.The CSW for different γ-levels.

Weight	γ										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
v_1	7.9E-5	7.9E-5	7.8E-5	7.8E-5	7.7E-5	7.7E-5	7.6E-5	7.6E-5	7.5E-5	7.5E-5	7.5E-5
v_2	1.58E-4	1.57E-4	1.56E-4	1.55E-4	1.54E-4	1.53E-4	1.53E-4	1.52E-4	1.51E-4	1.50E-4	1.49E-4
u_1	1.4E-5	1.4E-5	1.3E-5	1.3E-5	1.3E-5	1.2E-5	1.2E-5	1.2E-5	1.1E-5	1.1E-5	1.1E-5
u_2	1.22E-4	1.19E-4	1.15E-4	1.12E-4	1.09E-4	1.06E-4	1.03E-4	1.00E-4	9.8E-5	9.5E-5	9.2E-5
w_1	1.4E-5	1.3E-5	1.3E-5	1.3E-5	1.3E-5	1.3E-5	1.2E-5	1.2E-5	1.2E-5	1.2E-5	1.2E-5
w_2	6.4E-5	6.3E-5	6.2E-5	6.1E-5	6.0E-5	5.9E-5	5.8E-5	5.7E-5	5.6E-5	5.5E-5	5.4E-5

Table 3. The upper and lower bounds of system and two processes efficiencies for different γ-levels.

Co.		γ										
		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
1	S	(0.437,0.559)	(0.428,0.532)	(0.431,0.522)	(0.429,0.507)	(0.427,0.493)	(0.727,0.483)	(0.411,0.453)	(0.409,0.439)	(0.423,0.444)	(0.427,0.437)	(0.423,0.423)**4**
	P1	(0.753,0.849)	(0.720,0.802)	(0.727,0.800)	(0.730,0.793)	(0.746,0.802)	(0.738,0.785)	(0.705,0.740)	(0.708,0.734)	(0.714,0.732)	(0.717,0.725)	(0.719,0.719)**3**
	P2	(0.581,0.658)	(0.595,0.663)	(0.593,0.652)	(0.588,0.639)	(0.572,0.615)	(0.579,0.615)	(0.583,0.612)	(0.577,0.598)	(0.592,0.607)	(0.596,0.603)	(0.589,0.589)**8**
2	S	(0.331,0.422)	(0.325,0.404)	(0.326,0.395)	(0.325,0.384)	(0.323,0.373)	(0.325,0.366)	(0.312,0.343)	(0.309,0.333)	(0.325,0.337)	(0.324,0.331)	(0.322,0.322)**8**
	P1	(0.749,0.845)	(0.722,0.804)	(0.726,0.800)	(0.728,0.792)	(0.733,0.788)	(0.735,0.781)	(0.706,0.740)	(0.707,0.733)	(0.712,0.729)	(0.713,0.722)	(0.715,0.715)**4**
	P2	(0.442,0.499)	(0.450,0.502)	(0.449,0.494)	(0.446,0.485)	(0.441,0.473)	(0.442,0.469)	(0.442,0.463)	(0.437,0.454)	(0.456,0.462)	(0.454,0.459)	(0.450,0.450)**10**
3	S	(0.406,0.501)	(0.398,0.495)	(0.401,0.486)	(0.398,0.472)	(0.395,0.456)	(0.399,0.449)	(0.382,0.421)	(0.378,0.407)	(0.394,0.415)	(0.398,0.407)	(0.392,0.392)**5**
	P1	(0.519,0.568)	(0.497,0.554)	(0.501,0.552)	(0.503,0.548)	(0.508,0.546)	(0.510,0.541)	(0.487,0.511)	(0.488,0.507)	(0.493,0.505)	(0.495,0.501)	(0.496,0.496)**16**
	P2	(0.783,0.883)	(0.801,0.893)	(0.800,0.880)	(0.792,0.861)	(0.778,0.836)	(0.782,0.830)	(0.785,0.824)	(0.775,0.803)	(0.800,0.821)	(0.804,0.813)	(0.791,0.791)**3**
4	S	(0.160,0.203)	(0.156,0.195)	(0.157,0.191)	(0.156,0.185)	(0.155,0.180)	(0.156,0.177)	(0.150,0.163)	(0.149,0.160)	(0.155,0.162)	(0.156,0.160)	(0.154,0.154)**20**
	P1	(0.456,0.514)	(0.436,0.486)	(0.441,0.484)	(0.441,0.480)	(0.445,0.479)	(0.447,0.475)	(0.426,0.447)	(0.428,0.444)	(0.432,0.442)	(0.433,0.439)	(0.435,0.435)**22**
	P2	(0.350,0.396)	(0.359,0.401)	(0.357,0.394)	(0.354,0.386)	(0.349,0.376)	(0.350,0.372)	(0.352,0.370)	(0.348,0.361)	(0.358,0.366)	(0.360,0.364)	(0.355,0.355)**15**
5	S	(0.499,0.637)	(0.491,0.609)	(0.492,0.597)	(0.489,0.579)	(0.488,0.565)	(0.490,0.552)	(0.471,0.518)	(0.468,0.503)	(0.484,0.507)	(0.489,0.501)	(0.486,0.486)**3**
	P1	(0.565,0.638)	(0.563,0.628)	(0.542,0.597)	(0.545,0.593)	(0.552,0.594)	(0.556,0.590)	(0.525,0.551)	(0.529,0.548)	(0.535,0.548)	(0.539,0.545)	(0.542,0.542)**13**
	P2	(0.884,0.998)	(0.871,0.970)	(0.908,1.00)	(0.898,0.977)	(0.884,0.951)	(0.881,0.935)	(0.897,0.941)	(0.885,0.918)	(0.904,0.926)	(0.908,0.919)	(0.896,0.896)**2**
6	S	(0.214,0.273)	(0.211,0.262)	(0.210,0.255)	(0.210,0.242)	(0.210,0.243)	(0.209,0.235)	(0.203,0.216)	(0.201,0.217)	(0.207,0.218)	(0.210,0.215)	(0.209,0.209)**13**
	P1	(0.595,0.672)	(0.567,0.632)	(0.573,0.631)	(0.575,0.626)	(0.581,0.625)	(0.583,0.619)	(0.557,0.585)	(0.558,0.579)	(0.565,0.579)	(0.566,0.573)	(0.568,0.568)**12**
	P2	(0.360,0.406)	(0.372,0.414)	(0.367,0.404)	(0.365,0.387)	(0.362,0.388)	(0.358,0.380)	(0.364,0.369)	(0.361,0.375)	(0.366,0.376)	(0.371,0.375)	(0.368,0.368)**14**
7	S	(0.217,0.250)	(0.210,0.241)	(0.200,0.234)	(0.200,0.230)	(0.199,0.225)	(0.197,0.218)	(0.190,0.206)	(0.190,0.202)	(0.194,0.202)	(0.197,0.201)	(0.196,0.196)**15**
	P1	(0.510,0.531)	(0.479,0.524)	(0.456,0.494)	(0.459,0.492)	(0.464,0.492)	(0.466,0.490)	(0.440,0.458)	(0.443,0.456)	(0.447,0.456)	(0.450,0.454)	(0.452,0.452)**20**
	P2	(0.425,0.470)	(0.420,0.460)	(0.438,0.474)	(0.435,0.467)	(0.430,0.457)	(0.424,0.446)	(0.433,0.450)	(0.429,0.443)	(0.433,0.442)	(0.437,0.442)	(0.434,0.434)**12**
8	S	(0.201,0.246)	(0.198,0.273)	(0.196,0.230)	(0.196,0.226)	(0.197,0.221)	(0.194,0.215)	(0.188,0.203)	(0.188,0.199)	(0.191,0.198)	(0.193,0.197)	(0.193,0.193)**17**
	P1	(0.503,0.556)	(0.477,0.522)	(0.482,0.522)	(0.484,0.519)	(0.490,0.520)	(0.492,0.517)	(0.466,0.485)	(0.468,0.482)	(0.473,0.482)	(0.475,0.479)	(0.477,0.477)**19**
	P2	(0.400,0.442)	(0.415,0.454)	(0.407,0.441)	(0.405,0.435)	(0.402,0.426)	(0.395,0.416)	(0.403,0.419)	(0.401,0.413)	(0.403,0.411)	(0.407,0.412)	(0.405,0.405)**13**
9	S	(0.157,0.191)	(0.154,0.184)	(0.154,0.180)	(0.153,0.176)	(0.153,0.172)	(0.152,0.168)	(0.146,0.158)	(0.145,0.154)	(0.149,0.155)	(0.151,0.154)	(0.150,0.150)**21**
	P1	(0.680,0.752)	(0.644,0.704)	(0.651,0.705)	(0.654,0.702)	(0.662,0.702)	(0.665,0.699)	(0.628,0.653)	(0.631,0.650)	(0.638,0.651)	(0.642,0.648)	(0.645,0.645)**7**
	P2	(0.231,0.254)	(0.240,0.262)	(0.236,0.255)	(0.234,0.251)	(0.231,0.245)	(0.228,0.240)	(0.233,0.242)	(0.230,0.237)	(0.233,0.238)	(0.235,0.237)	(0.233,0.233)**22**
10	S	(0.272,0.332)	(0.267,0.320)	(0.266,0.314)	(0.266,0.305)	(0.264,0.298)	(0.263,0.291)	(0.253,0.274)	(0.252,0.267)	(0.258,0.269)	(0.262,0.267)	(0.260,0.260)**11**
	P1	(0.455,0.503)	(0.432,0.473)	(0.436,0.473)	(0.439,0.470)	(0.443,0.470)	(0.445,0.468)	(0.421,0.438)	(0.423,0.436)	(0.427,0.436)	(0.430,0.434)	(0.432,0.432)**23**
	P2	(0.598,0.661)	(0.619,0.677)	(0.611,0.664)	(0.606,0.649)	(0.597,0.634)	(0.592,0.622)	(0.602,0.626)	(0.596,0.613)	(0.605,0.617)	(0.609,0.615)	(0.602,0.602)**7**

Co.		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
11	S	(0.081,0.099)	(0.080,0.096)	(0.078,0.092)	(0.080,0.091)	(0.081,0.090)	(0.070,0.085)	(0.076,0.082)	(0.077,0.082)	(0.076,0.079)	(0.078,0.080)	(0.079,0.079)**23**
	P1	(0.519,0.573)	(0.494,0.540)	(0.499,0.540)	(0.500,0.536)	(0.505,0.536)	(0.507,0.532)	(0.482,0.502)	(0.484,0.498)	(0.489,0.498)	(0.490,0.495)	(0.491,0.491)**17**
	P2	(0.156,0.172)	(0.163,0.178)	(0.157,0.170)	(0.159,0.170)	(0.160,0.169)	(0.138,0.160)	(0.158,0.164)	(0.159,0.164)	(0.155,0.158)	(0.159,0.161)	(0.161,0.161)**23**
12	S	(0.392,0.484)	(0.388,0.464)	(0.389,0.457)	(0.386,0.444)	(0.384,0.434)	(0.384,0.424)	(0.369,0.400)	(0.365,0.401)	(0.378,0.394)	(0.381,0.388)	(0.554,0.554)**1**
	P1	(0.631,0.697)	(0.602,0.659)	(0.608,0.659)	(0.609,0.653)	(0.615,0.653)	(0.615,0.647)	(0.589,0.613)	(0.589,0.607)	(0.595,0.607)	(0.596,0.601)	(0.596,0.596)**9**
	P2	(0.621,0.695)	(0.644,0.705)	(0.640,0.693)	(0.634,0.680)	(0.625,0.664)	(0.624,0.655)	(0.627,0.653)	(0.620,0.661)	(0.636,0.649)	(0.639,0.646)	(0.633,0.633)**5**
13	S	(0.169,0.206)	(0.167,0.200)	(0.164,0.193)	(0.166,0.190)	(0.167,0.187)	(0.162,0.180)	(0.158,0.171)	(0.159,0.169)	(0.160,0.166)	(0.163,0.166)	(0.164,0.164)**19**
	P1	(0.557,0.615)	(0.528,0.577)	(0.534,0.578)	(0.536,0.574)	(0.542,0.575)	(0.544,0.572)	(0.515,0.536)	(0.517,0.533)	(0.523,0.533)	(0.525,0.531)	(0.528,0.528)**14**
	P2	(0.303,0.335)	(0.317,0.347)	(0.308,0.334)	(0.309,0.331)	(0.308,0.326)	(0.298,0.314)	(0.307,0.320)	(0.308,0.317)	(0.305,0.311)	(0.310,0.313)	(0.311,0.311)**18**
14	S	(0.207,0.253)	(0.204,0.244)	(0.203,0.238)	(0.202,0.233)	(0.202,0.227)	(0.200,0.221)	(0.193,0.209)	(0.193,0.204)	(0.197,0.205)	(0.199,0.203)	(0.198,0.198)**14**
	P1	(0.477,0.526)	(0.453,0.494)	(0.457,0.495)	(0.459,0.492)	(0.464,0.492)	(0.466,0.490)	(0.441,0.459)	(0.443,0.456)	(0.448,0.457)	(0.450,0.454)	(0.452,0.452)**20**
	P2	(0.435,0.481)	(0.451,0.493)	(0.444,0.480)	(0.441,0.473)	(0.436,0.462)	(0.430,0.451)	(0.438,0.456)	(0.435,0.447)	(0.439,0.448)	(0.443,0.447)	(0.439,0.439)**11**
15	S	(0.383,0.466)	(0.376,0.450)	(0.373,0.438)	(0.373,0.428)	(0.374,0.421)	(0.369,0.407)	(0.357,0.377)	(0.356,0.378)	(0.362,0.377)	(0.368,0.375)	(0.366,0.366)**6**
	P1	(0.670,0.740)	(0.636,0.696)	(0.643,0.697)	(0.645,0.692)	(0.653,0.693)	(0.655,0.688)	(0.622,0.647)	(0.624,0.643)	(0.631,0.644)	(0.633,0.639)	(0.635,0.635)**8**
	P2	(0.571,0.630)	(0.591,0.646)	(0.580,0.628)	(0.578,0.619)	(0.573,0.607)	(0.563,0.592)	(0.574,0.583)	(0.571,0.588)	(0.574,0.586)	(0.581,0.587)	(0.577,0.577)**9**
16	S	(0.205,0.250)	(0.201,0.241)	(0.200,0.289)	(0.199,0.229)	(0.200,0.224)	(0.198,0.218)	(0.191,0.207)	(0.190,0.202)	(0.194,0.203)	(0.197,0.201)	(0.195,0.195)**16**
	P1	(0.615,0.679)	(0.584,0.639)	(0.591,0.639)	(0.593,0.635)	(0.626,0.664)	(0.601,0.632)	(0.571,0.594)	(0.573,0.590)	(0.579,0.591)	(0.581,0.587)	(0.583,0.583)**10**
	P2	(0.333,0.368)	(0.344,0.377)	(0.339,0.453)	(0.336,0.361)	(0.319,0.338)	(0.329,0.345)	(0.334,0.349)	(0.332,0.342)	(0.335,0.343)	(0.339,0.342)	(0.334,0.334)**16**
17	S	(0.287,0.349)	(0.283,0.338)	(0.280,0.328)	(0.280,0.322)	(0.281,0.316)	(0.276,0.305)	(0.268,0.290)	(0.268,0.284)	(0.273,0.282)	(0.276,0.281)	(0.276,0.276)**9**
	P1	(0.432,0.476)	(0.411,0.450)	(0.413,0.447)	(0.415,0.445)	(0.420,0.445)	(0.422,0.444)	(0.398,0.414)	(0.400,0.412)	(0.405,0.413)	(0.407,0.411)	(0.409,0.409)**24**
	P2	(0.664,0.733)	(0.688,0.752)	(0.677,0.734)	(0.675,0.724)	(0.670,0.711)	(0.654,0.688)	(0.673,0.700)	(0.669,0.689)	(0.675,0.683)	(0.677,0.684)	(0.674,0.674)**4**
18	S	(0.189,0.227)	(0.183,0.218)	(0.182,0.213)	(0.181,0.208)	(0.181,0.204)	(0.180,0.198)	(0.173,0.187)	(0.173,0.183)	(0.175,0.184)	(0.178,0.182)	(0.177,0.177)**18**
	P1	(0.746,0.824)	(0.722,0.790)	(0.726,0.785)	(0.725,0.777)	(0.729,0.773)	(0.728,0.765)	(0.703,0.732)	(0.703,0.724)	(0.705,0.720)	(0.705,0.712)	(0.704,0.704)**5**
	P2	(0.253,0.275)	(0.253,0.276)	(0.251,0.271)	(0.250,0.268)	(0.248,0.264)	(0.247,0.259)	(0.246,0.256)	(0.246,0.253)	(0.249,0.255)	(0.253,0.256)	(0.252,0.252)**20**
19	S	(0.280,0.307)	(0.284,0.338)	(0.256,0.256)	(0.256,0.256)	(0.256,0.256)	(0.268,0.268)	(0.253,0.253)	(0.254,0.254)	(0.265,0.265)	(0.271,0.271)	(0.272,0.272)**10**
	P1	(1.000,1.000)	(1.000,1.000)	(1.000,1.000)	(1.000,1.000)	(1.000,1.000)	(1.000,1.000)	(1.000,1.000)	(1.000,1.000)	(1.000,1.000)	(1.000,1.000)	(1.000,1.000)**1**
	P2	(0.280,0.307)	(0.284,0.338)	(0.256,0.256)	(0.256,0.256)	(0.256,0.256)	(0.268,0.268)	(0.253,0.253)	(0.254,0.254)	(0.265,0.265)	(0.271,0.271)	(0.272,0.272)**19**
20	S	(0.378,0.458)	(0.373,0.445)	(0.368,0.429)	(0.369,0.423)	(0.372,0.418)	(0.363,0.400)	(0.353,0.381)	(0.355,0.376)	(0.356,0.370)	(0.363,0.370)	(0.365,0.365)**7**
	P1	(0.612,0.679)	(0.586,0.643)	(0.588,0.639)	(0.586,0.631)	(0.589,0.627)	(0.587,0.618)	(0.573,0.597)	(0.571,0.589)	(0.573,0.585)	(0.571,0.577)	(0.579,0.579)**11**
	P2	(0.617,0.675)	(0.638,0.692)	(0.625,0.672)	(0.630,0.671)	(0.631,0.666)	(0.618,0.647)	(0.617,0.639)	(0.621,0.639)	(0.621,0.633)	(0.636,0.642)	(0.630,0.630)**6**
Co.		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1

21	S	(0.127,0.146)	(0.125,0.141)	(0.123,0.137)	(0.123,0.135)	(0.123,0.134)	(0.121,0.127)	(0.117,0.123)	(0.117,0.122)	(0.117,0.120)	(0.119,0.121)	(0.119,0.119) **22**
	P1	(0.509,0.556)	(0.489,0.529)	(0.492,0.528)	(0.492,0.523)	(0.495,0.522)	(0.495,0.517)	(0.476,0.493)	(0.476,0.488)	(0.479,0.487)	(0.478,0.483)	(0.478,0.478) **18**
	P2	(0.249,0.262)	(0.255,0.267)	(0.251,0.260)	(0.250,0.258)	(0.248,0.256)	(0.244,0.245)	(0.245,0.250)	(0.245,0.249)	(0.245,0.247)	(0.249,0.250)	(0.249,0.249) **21**
22	S	(0.563,0.679)	(0.557,0.629)	(0.544,0.631)	(0.546,0.622)	(0.548,0.613)	(0.531,0.583)	(0.517,0.556)	(0.518,0.548)	(0.515,0.543)	(0.525,0.535)	(0.528,0.528) **2**
	P1	(0.603,0.679)	(0.580,0.629)	(0.578,0.635)	(0.573,0.622)	(0.572,0.613)	(0.550,0.583)	(0.531,0.556)	(0.529,0.548)	(0.531,0.543)	(0.529,0.535)	(0.528,0.528) **14**
	P2	(0.934,1.000)	(0.961,1.000)	(0.941,0.994)	(0.953,1.000)	(0.959,1.000)	(0.966,1.000)	(0.973,1.000)	(0.979,1.000)	(0.970,1.000)	(0.993,1.000)	(1.000,1.000) **1**
23	S	(0.234,0.285)	(0.227,0.272)	(0.230,0.262)	(0.227,0.261)	(0.225,0.253)	(0.226,0.250)	(0.217,0.234)	(0.214,0.227)	(0.223,0.232)	(0.224,0.228)	(0.220,0.220) **12**
	P1	(0.723,0.792)	(0.696,0.755)	(0.700,0.731)	(0.700,0.746)	(0.704,0.743)	(0.704,0.736)	(0.677,0.702)	(0.683,0.695)	(0.680,0.693)	(0.680,0.686)	(0.679,0.679) **6**
	P2	(0.323,0.360)	(0.327,0.361)	(0.328,0.358)	(0.324,0.350)	(0.319,0.340)	(0.321,0.340)	(0.320,0.334)	(0.314,0.326)	(0.328,0.335)	(0.329,0.333)	(0.324,0.324) **17**
24	S	(0.072,0.089)	(0.071,0.086)	(0.070,0.084)	(0.071,0.081)	(0.070,0.080)	(0.070,0.077)	(0.067,0.074)	(0.068,0.072)	(0.068,0.072)	(0.070,0.071)	(0.075,0.075) **24**
	P1	(0.867,0.957)	(0.900,0.983)	(0.899,0.972)	(0.895,0.958)	(0.893,0.940)	(0.889,0.934)	(0.874,0.909)	(0.869,0.895)	(0.867,0.884)	(0.862,0.871)	(0.931,0.931) **2**
	P2	(0.083,0.093)	(0.079,0.088)	(0.078,0.086)	(0.079,0.085)	(0.079,0.085)	(0.079,0.083)	(0.077,0.081)	(0.078,0.080)	(0.079,0.081)	(0.081,0.081)	(0.081,0.081) **24**