Type-Reduced Set Structure and the Truncated Type-2 Fuzzy Set

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Abstract

In this paper, the Type-Reduced Set (TRS) of the continuous type-2 fuzzy set is considered as an object in its own right. The structures of the TRSs of both the interval and generalised forms of the type-2 fuzzy set are investigated. In each case the respective TRS structure is approached by first examining the TRS of the discretised set. The TRS of a continuous interval type-2 fuzzy set is demonstrated to be a continuous horizontal straight line, and that of a generalised type-2 fuzzy set, a continuous, convex curve. This analysis leads on to the concept of truncation, and the definition of the *truncation grade*. The *truncated type-2 fuzzy set* is then defined, whose TRS (and hence defuzzified value) is identical to that of the non-truncated type-2 fuzzy set. This result is termed the *Type-2 Truncation Theorem*, an immediate corollary of which is the *Type-2 Equivalence Theorem* which states that the defuzzified values of type-2 fuzzy sets that are equivalent under truncation are equal. Experimental corroboration of the equivalence of the non-truncated and truncated generalised type-2 fuzzy set is provided. The implications of these theorems for uncertainty quantification are explored. The theorem's repercussions for type-2 defuzzification employing the α-Planes Representation are examined; it is shown that the known inaccuracies of the α-Planes Method are deeply entrenched.

Keywords: Type-2 Fuzzy Set, Type Reduction, Defuzzification, Truncation

1. Introduction

Type-2 fuzzy sets are an extension of type-1 fuzzy sets in which the sets' membership grades are themselves type-1 fuzzy sets. The concept dates back to Zadeh's seminal paper of 1975 [39]. They take two forms, the interval, for which all secondary membership grades are 1[1], and the generalised, where the secondary membership grade may take any value between 0 and 1. For the computationally simpler interval type-2 Fuzzy Inferencing System (FIS) [31] applications in areas such as control, simulation and optimisation have been developed [1, 4–8, 25]. So far, generalised type-2 fuzzy applications are few in number [22, 26, 31]. This is attributable to the enormous computational complexity of generalised type-2 fuzzy inferencing. Strategies have been developed that reduce the computational complexity of all stages of the generalised type-2 FIS [15, 21, 27, 42]. We believe that the research presented in this paper will lead to further complexity reducing techniques, in turn leading to an increasing number of generalised type-2 FIS applications.

In a Mamdani Type-2 FIS (Figure 1), a crisp numerical input passes through three stages of processing: fuzzification, inferencing, and lastly, the crucial stage of defuzzification. Through defuzzification, the type-2 *aggregated set* produced during the inferencing stage is converted into a crisp

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[1] Interval type-2 fuzzy sets may be regarded as generalisations of interval-valued fuzzy sets [2].

output. For discretised type-1 fuzzy sets, defuzzification is a simple procedure; defuzzification of a discretised type-2 fuzzy set is more complicated, consisting of two stages [29]:

1. Type-reduction, which converts a type-2 fuzzy set to a type-1 fuzzy set known as the Type-Reduced Set (TRS), and
2. defuzzification of the type-1 TRS.

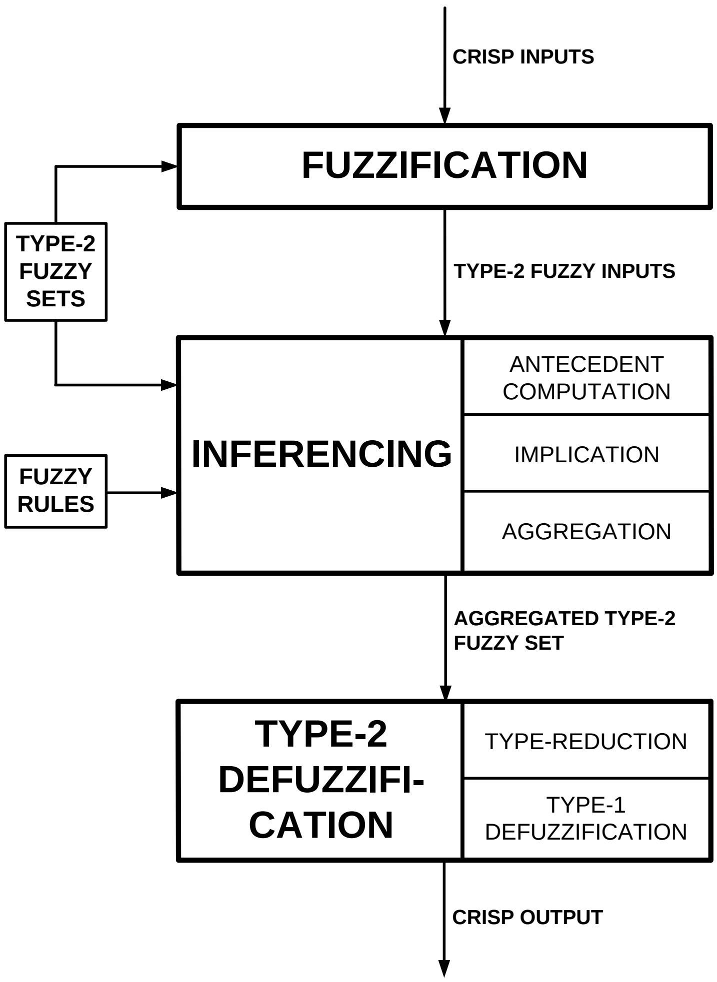


Figure 1: The Mamdani Type-2 FIS [14].

The TRS is an intermediate by-product of type-2 defuzzification, a link between the originating type-2 fuzzy set and the type-1 TRS. In the interval case, Wu and Mendel [37] argue that the *length* of the TRS provides a measure of the uncertainty relating to the set. The most widely adopted method for type-reducing an interval type-2 fuzzy set is the *Karnik-Mendel Iterative Procedure (KMIP)* [23], which gives rise to an interval, the midpoint of which is deemed to be the defuzzified value. The endpoints of the interval are termed *uncertainty bounds* [37, page 622], reflecting the belief that the interval length quantifies the uncertainty pertaining to the type-2 fuzzy set. The iterative procedure is an efficient search method for locating these endpoints. It is not a precise technique [13], [23, Page 203], being a good approximation to the Exhaustive Method of Defuzzification (Section 2). Since the publication of the KMIP, various more efficient enhancements have been proposed [35], which differ somewhat in their search strategy whilst giving identical results.

Type-reduction does not necessarily give rise to the TRS; other type-1 fuzzy sets may result [19, 20, 33]. However this report is concerned specifically with the TRS.

The research presented in this report is essentially theoretical. Having established the structure of the TRS for a generalised type-2 fuzzy set, the innovative concept of *truncating* such a set is introduced. Two theorems relating type-2 defuzzification to truncation are presented, and their implications for type-2 fuzzy inferencing explored. The paper is structured as follows: The remainder of this section is concerned with assumptions and definitions. In Section 2 type-reduction is described. TRS structure is investigated in Section 3. In Section 4 the truncated type-2 fuzzy set is defined, leading into the *Type-2 Truncation Theorem* and the *Type-2 Equivalence Theorem*. Experiments corroborating the equivalence of the truncated type-2 fuzzy set to the non-truncated type-2 fuzzy set are described in Section 5. Implications of the Type-2 Truncation Theorem and Type-2 Equivalence Theorem for type-2 defuzzification are examined in Section 6. Lastly, Section 7 concludes the paper.

1.1. Preliminaries

1.1.1. Definitions

In type-2 fuzzy set theory, notation can be problematic [3]; we have adopted terminology that we believe to be clear and unambiguous. The following definitions are pertinent to this paper [12].

Let X be a universe of discourse, which in this paper is assumed to be the continuous closed unit interval $U = [0,1]$. A type-1 fuzzy set A on X is characterised by a membership function $\mu_A : X \to U$ and it is expressed as follows [38]:

$$A = \{(x, \mu_A(x)) |\, \mu_A(x) \in U \,\forall x \in X\}. \tag{1}$$

Note that the membership grades of A are crisp numbers. In the following we will use the notation u to refer to the membership grades in U associated to elements of set X. The *scalar cardinality* of a fuzzy set A defined on a finite universal set X is the summation of the membership grades of all the elements of X in A, i.e.

$$|A| = \sum_{x \in X} \mu_A(x).$$

Let $\tilde{P}(U)$ be the set of fuzzy sets in U. A type-2 fuzzy set $\tilde{A}$ on X is a fuzzy set whose membership grades are themselves fuzzy [39–41]. This implies that $\mu_{\tilde{A}}(x)$ is a fuzzy set in U for all x, i.e. $\mu_{\tilde{A}} : X \to \tilde{P}(U)$ and

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) |\, \mu_{\tilde{A}}(x) \in \tilde{P}(U) \,\forall x \in X\}. \tag{2}$$

In type-2 fuzzy set theory, the universal set X is referred to as the primary domain; U is the secondary domain. The support of the fuzzy set $\mu_{\tilde{A}}(x)$, $J_x = \{u \in U | \mu_{\tilde{A}}(x)(u) > 0\}$, is known as the primary membership of $x \in X$; $\mu_{\tilde{A}(x)} : J_x \to U$ defined by $\mu_{\tilde{A}(x)}(u) = \mu_{\tilde{A}}(x)(u)$ is known as the secondary membership function of x; while $\mu_{\tilde{A}}(x)(u)$ is known as a secondary membership grade of x. Thus, Applying (1), we obtain:

$$\mu_{\tilde{A}}(x) = \{(u, \mu_{\tilde{A}}(x)(u)) |\, \mu_{\tilde{A}}(x)(u) \in U \,\forall u \in J_x \subseteq U\}. \tag{3}$$

Putting (2) and (3) together we obtain

$$\tilde{A} = \{(x, (u, \mu_{\tilde{A}}(x)(u))) |\, \mu_{\tilde{A}}(x)(u) \in U, \forall x \in X \wedge \forall u \in J_x \subseteq U\}. \tag{4}$$

An *interval type-2 fuzzy set* is a type-2 fuzzy set with secondary membership functions identically equal to 1. In the interval case, Equation 4 reduces to:

$$\tilde{A} = \{(x, (u, 1)) |\, \forall x \in X \wedge \forall u \in J_x \subseteq U\}. \tag{5}$$

Under our assumption of being $X \equiv [0,1]$, a type-2 fuzzy set is contained within a unit cube and may be viewed as a surface represented by (x, u, z) co-ordinates, where $x \in U$, $u \in J_x \subseteq U$ and $z = \mu_{\tilde{A}}(x)(u) \in U$. A *slice* of a type-2 fuzzy set $\tilde{A}$ is the intersection of a plane either vertical (through the

x-axis, parallel to the $u-z$ plane) or horizontal (through the z-axis, parallel to the $x-u$ plane) with the graph of its membership function $\mu_{\tilde{A}}$. Thus, a vertical slice of a type-2 fuzzy set $\tilde{A}$ at x will be the graph of the secondary membership function $\mu_{\tilde{A}(x)} : J_x \to U$, while a horizontal slice of a type-2 fuzzy set $\tilde{A}$ at z will be the interval type-2 fuzzy set $\tilde{A}_z$ with secondary membership functions $\mu_{\tilde{A}_z(x)} : J_x \to U$ defined as follows:

$$\mu_{\tilde{A}_z(x)}(u) = \begin{cases} 0, & \mu_{\tilde{A}}(x)(u) < z \\ z, & \mu_{\tilde{A}}(x)(u) \geq z \end{cases}.$$

The *degree of discretisation* is the separation of the slices. For a computer to process type-2 fuzzy sets, it is necessary to discretise both the primary and secondary domains, possibly with different degrees of discretisation.

1.1.2. Assumptions

1. All secondary membership functions are convex, which implies that $\forall x \in X,\ \mu_{\tilde{A}}(x) : J_x \to [0,1]$ is convex, i.e. for any $u_1, u_2 \in J_x$ and any $\lambda \in [0,1]$,

   $$\mu_{\tilde{A}}(x)[\lambda u_1 + (1-\lambda)u_2] \geq \lambda \mu_{\tilde{A}}(x)(u_1) + (1-\lambda)\mu_{\tilde{A}}(x)(u_2).$$

   Thus, $\forall x \in X$ we are assuming that $\mu_{\tilde{A}}(x)$ is continuous and J_x is a closed interval in $[0,1]$: $J_x = [\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)]$, where $\underline{\mu}_{\tilde{A}}(x) = \inf\{u|\ (x,u) \in J_x\}$; $\overline{\mu}_{\tilde{A}}(x) = \sup\{u|\ (x,u) \in J_x\}$, are known as the the lower and upper membership functions of type-2 fuzzy set $\tilde{A}$.

2. The centroid method of defuzzification for type-1 fuzzy sets is used.

3. The minimum t-norm is employed.

4. The *Grid Method of Discretisation* for generalised type-2 fuzzy sets [9, 21] is employed.

2. Type-Reduction of the Type-2 Fuzzy Set

2.1. The Wavy-Slice Representation Theorem

The concept of an *embedded type-2 fuzzy set (embedded set)* or *wavy-slice* [31] (Figure 2) is crucial to type-reduction. An embedded set is a special kind of type-2 fuzzy set, which relates to the type-2 fuzzy set in which it is embedded in this way: For every primary domain value, $x(\in X)$, there is a unique secondary domain value, $u(\in J_x \subseteq U)$, plus the associated secondary membership grade that is determined by the primary and secondary domain values, $\mu_{\tilde{A}}(x)(u)(\in [0,1])$.

Definition 1 (Embedded Set)**.** *Let $\tilde{A}$ be a type-2 fuzzy set on X. For discrete sets $X_d = \{x_1, x_2, \ldots, x_N\}$ and $U_d = \{u_1, u_2, \ldots, u_M\}$,* an embedded type-2 fuzzy set $\tilde{A}_e$ *of $\tilde{A}$ is defined as the following type-2 fuzzy set*

$$\tilde{A}_e = \{(x_i, (u_{x_i}, \mu_{\tilde{A}}(x_i)(u_{x_i})))|\ \forall i \in \{1,\ldots,N\} : x_i \in X_d \wedge\ u_{x_i} \in J_{x_i} \subseteq U_d\}. \tag{6}$$

$\tilde{A}_e$ contains exactly one element from J_{x_1}, J_{x_2}, …, J_{x_N}, namely u_{x_1}, u_{x_2}, …, u_{x_N}, each with its associated secondary grade, namely $\mu_{\tilde{A}}(x_1)(u_{x_1})$, $\mu_{\tilde{A}}(x_2)(u_{x_2})$, …, $\mu_{\tilde{A}}(x_N)(u_{x_N})$.

Mendel and John have demonstrated that a type-2 fuzzy set is definable as the union of its embedded type-2 fuzzy sets [31]. This powerful result is known as the type-2 fuzzy set *Representation Theorem* or *Wavy-Slice Representation Theorem*. The Wavy-Slice Representation Theorem is formally stated thus [31, Page 121]:

Let $\tilde{A}_e^j$ denote the jth embedded type-2 fuzzy set for type-2 fuzzy set $\tilde{A}$, i.e.,

$$\tilde{A}_e^j \equiv \left\{ \left(x_i, \left(u_i^j, \mu_{\tilde{A}}(x_i)(u_i^j) \right) \right) \middle|\ \forall i \in \{1,\ldots,N\} : u_i^j \in J_{x_i} \right\}.$$

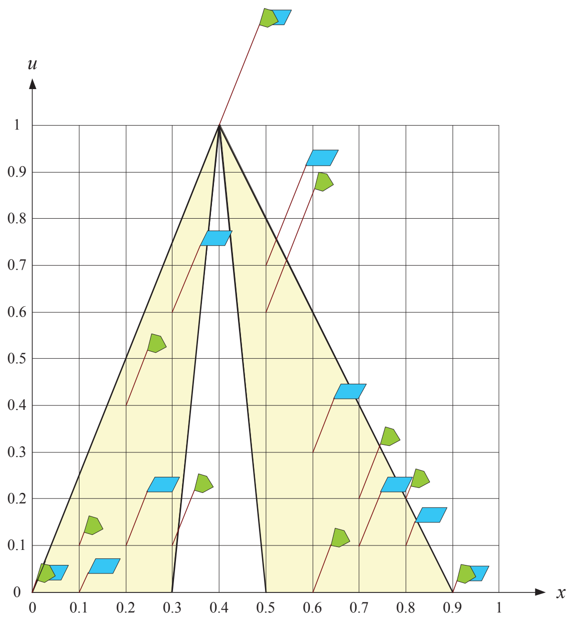


Figure 2: Two embedded type-2 fuzzy sets, indicated by different flag styles. The flag height indicates the secondary membership grade. The degree of discretisation of the primary and secondary domains is 0.1. The FOU is the shaded region.

Then $\tilde{A}$ may be represented as the union of its embedded type-2 fuzzy sets, i.e.,

$$\tilde{A} = \sum_{j=1}^{n} \tilde{A}_e^j, \text{ where } n \equiv \prod_{i=1}^{N} M_i, M_i \text{ being the cardinality of } J_{x_i}.$$

Note: An alternative way of defining an embedded type-2 fuzzy set $\tilde{A}_e$ could be via a selection function of the multimapping $x \to J_x$, say $f(x) \in J_x \ \forall x$. Then embedded type-2 fuzzy set $\tilde{A}_e$ could then be defined as the set of triplets $\{(x, f(x), \mu_{\tilde{A}}(f(x))\}$ where f is a selection of J_x. Clearly, a type-2 fuzzy set $\tilde{A}$ will be union of the embedded type-2 fuzzy sets derived from all possible different selection functions.

2.2. *Exhaustive Defuzzification*

Type-2 defuzzification strategies derive from and incorporate type-1 defuzzification techniques. The strategy known as *Exhaustive Defuzzification*, so called because every embedded set is processed in turn, is built upon the foundation of the Wavy-Slice Representation Theorem [31] and is therefore precise[2] [31]. However it is a very inefficient method owing to its high computational complexity. Its first and main stage consists of type-reduction of the type-2 fuzzy set to form the TRS [12], formally defined thus:

Definition 2 (TRS). *The TRS associated with a type-2 fuzzy set $\tilde{A}$ with primary domain X discretised into N points $X_d = \{x_1, x_2, \ldots, x_N\}$, is*

$$C_{\tilde{A}} = \left\{ \left(\frac{\sum_{i=1}^{N} x_i \cdot u_{k_i}}{\sum_{i=1}^{N} u_{k_i}}, \mu_{\tilde{A}}(x_1)(u_{k_1}) * \ldots * \mu_{\tilde{A}}(x_N)(u_{k_N}) \right) \middle| \; \forall (u_{k_1}, u_{k_2}, \ldots, u_{k_N}) \in J_{x_1} \times J_{x_2} \times \ldots \times J_{x_N} \subseteq U^N \right\}, \tag{7}$$

[2] Discretisation in itself brings an unavoidable element of approximation. However the Exhaustive Method does not introduce further inaccuracies subsequent to discretisation.

where $*$ *is a t-norm.*

Though this definition does not explicitly mention embedded sets, they appear implicitly in Equation (7). When this equation is presented in algorithmic form (Algorithm 1, adapted from Mendel [29]), explicit mention is made of embedded sets.

Input: a discretised generalised type-2 fuzzy set
Output: a discrete type-1 fuzzy set (the TRS)

forall *embedded sets* **do**
    find the minimum secondary membership grade (z) ;
    calculate the primary domain value (x) of the type-1 centroid of the embedded type-2 fuzzy set ;
    pair the secondary grade (z) with the primary domain value (x) to give set of ordered pairs (x,z) {some values of x may correspond to more than one value of z} ;
end
forall *primary domain (x) values* **do**
    select the maximum secondary grade {make each x correspond to a unique value} ;
end

Algorithm 1: Exhaustive type-reduction of a discretised type-2 fuzzy set to a type-1 fuzzy set, using the minimum t-norm, (adapted from [29]).

For the TRS of an interval type-2 fuzzy set, Definition 2 reduces to:

Definition 3 (TRS of an Interval Type-2 Fuzzy Set)**.** *The TRS associated with an interval type-2 fuzzy set* $\tilde{A}$ *with primary domain* X *discretised into* N *points* $X_d = \{x_1, x_2, \ldots, x_N\}$*, is*

$$C_{\tilde{A}} = \left\{ \left(\frac{\sum_{i=1}^{N} x_i \cdot u_{k_i}}{\sum_{i=1}^{N} u_{k_i}}, 1 \right) \middle| \; \forall (u_{k_1}, u_{k_2}, \ldots, u_{k_N}) \in J_{x_1} \times J_{x_2} \times \ldots \times J_{x_N} \subseteq U^N \right\}. \quad (8)$$

3. Characterisation of the TRS Structure

3.1. Structure of the TRS of a Discretised Interval Type-2 Fuzzy Set

Starting from the previous result in Section 2.2, which states that a type-2 fuzzy set is the union of its embedded type-2 fuzzy sets, a type-2 fuzzy set is reduced to a type-1 fuzzy set by computing the centroid of all its type-2 fuzzy embedded sets. Thus, if $\tilde{A}_e = \{(x_i, (u_{x_i}, \mu_{\tilde{A}}(x_i)(u_{x_i})))| \; \forall i \in \{1, \ldots, N\} : x_i \in X_d \wedge \; u_{x_i} \in J_{x_i} \subseteq U_d\}$ is an embedded type-2 fuzzy set of $\tilde{A}$, we notice that the first component of the elements of set (7), $\frac{\sum_{i=1}^{N} x_i \cdot u_{k_i}}{\sum_{i=1}^{N} u_{k_i}}$, which is the centroid of the set of values $\{(x_i, u_{x_i} | x_i \in X; u_{x_i} \in J_{x_i}\}$, is computed over a cartesian product of closed intervals. Thus, in the general continuous case, the output values of such expression will form a closed interval. Each element in such closed interval output will have associated a membership degree, as per the second component of the elements of set (7), after the t-norm minimum is applied to the secondary membership values of the embedded set $\tilde{A}_e$. As we are assuming that $\mu_{\tilde{A}}(x)$ is continuous and all primary memberships, J_x, are closed interval in $[0,1]$, the range of $\mu_{\tilde{A}}(x)$ will be a closed interval, and consequently the set of secondary membership values of type-2 fuzzy set $\tilde{A}$ will be a closed interval subset of the unit interval U. This means that the set of values $\mu_{\tilde{A}}(x_1)(u_{k_1}) * \ldots * \mu_{\tilde{A}}(x_N)(u_{k_N})$ when $N \to \infty$ will be bounded above and below, and therefore it will have a supremum and infimum. Consequently, expression (7) in the continuous case is a type-1 fuzzy set. When $\tilde{A}$ is an interval type-2 fuzzy set, its TRS will be a crisp closed interval subset of the unit interval U, with membership function identically equal to 1. In this section we investigate how the TRS tuples are positioned along the interval of the TRS.

Figure 3 shows an interval type-2 fuzzy set. The primary domain X is discretised into N vertical slices with degree of discretisation d_x such that $N = \frac{1}{d_x} + 1$. The codomain (U) degree of discretisation is d_u. E is an embedded set whose codomain value at the I^{th} vertical slice $x = x_I$ is u_{I_E}. E_I^1 is another embedded set, identical to E apart from the codomain value on vertical slice x_I, which is $u_{I_E} + d_u$. For embedded set E_I^2 the codomain value at x_I is $u_{I_E} + 2 \cdot d_u$, and for embedded set E_I^n the codomain value at x_I is $u_{I_E} + n \cdot d_u$. Let X_E be the (centroid) defuzzified value of embedded set E. $X_{E_I^1}$, $X_{E_I^2}$, $\ldots X_{E_I^n}$ are similarly defined.

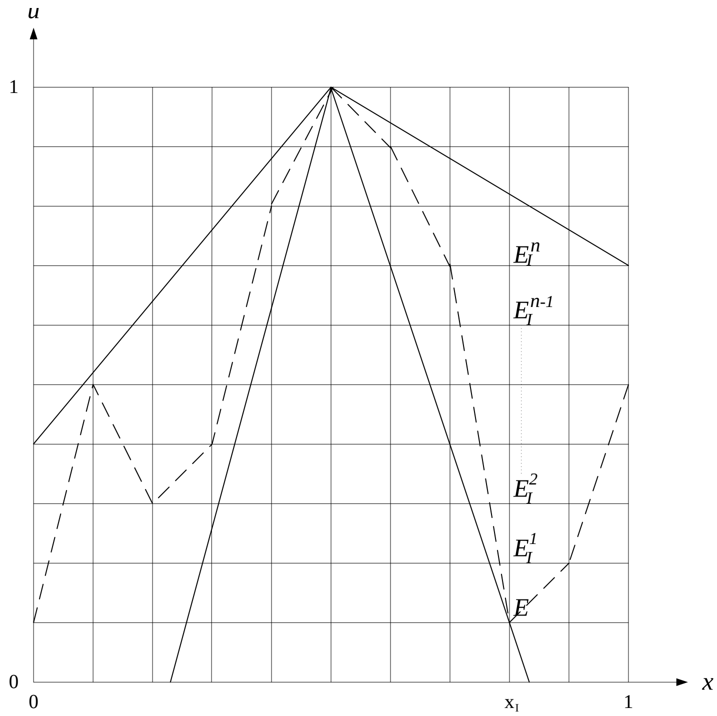


Figure 3: Embedded set E (dashed line), with related embedded sets $E_I^1, E_I^2, \ldots, E_I^{n-1}, E_I^n$.

The formula for the difference between two consecutive defuzzified values is $(\forall j \in \{1, \ldots, n\})$:

$$X_{E_I^j} - X_{E_I^{j-1}} = \frac{|E| \cdot d_u \cdot (x_I - X_E)}{(|E| + j \cdot d_u) \cdot (|E| + (j-1) \cdot d_u)}. \tag{9}$$

This formula is derived as follows:
Define $P = \sum_{i=1}^{N} x_i \cdot u_{i_E} + j \cdot d_u \cdot x_I$ and $Q = |E| + j \cdot d_u$.

$$X_{E_I^j} = \frac{\sum_{i=1}^{I-1} x_i \cdot u_{i_E} + \sum_{i=I+1}^{N} x_i \cdot u_{i_E} + x_I \cdot (u_{I_E} + j \cdot d_u)}{\sum_{i=1}^{N} u_{i_E} + j \cdot d_u} = \frac{\sum_{i=1}^{N} x_i \cdot u_{i_E} + j \cdot d_u \cdot x_I}{|E| + j \cdot d_u} = \frac{P}{Q}.$$

Similarly, we have $X_{E_I^{j-1}} = \frac{P - d_u \cdot x_I}{Q - d_u}$. Because $\sum_{i=1}^{N} x_i \cdot u_{i_E} = X_E \cdot |E|$, it follows that

$$X_{E_I^j} - X_{E_I^{j-1}} = \frac{|E| \cdot d_u (\cdot x_I - X_E)}{(|E| + j \cdot d_u) \cdot (|E| + (j-1) \cdot d_u)}.$$

3.2. Structure of the TRS of a Continuous Interval Type-2 Fuzzy Set

Because $(\|E\| + j \cdot d_u)(|E| + (j-1) \cdot d_u) > |E|^2$, it can be concluded that

$$d(X_{E_I^j}, X_{E_I^{j-1}}) = |X_{E_I^j} - X_{E_I^{j-1}}| < \frac{d_u \cdot |x_I - X_E|}{|E|}. \quad (10)$$

In Inequation (10), as n increases, so does the number of points on the vertical slice x_I. Consequently the distance between these points, d_u, decreases, and $\frac{d_u \cdot |x_I - X_E|}{|E|}$ decreases with it. Thus, $\forall \varepsilon > 0, \exists n \in \mathbb{N}$ such that $\frac{d_u \cdot |x_I - X_E|}{|E|} < \varepsilon$. It follows that $\forall j \in \{1, 2, \ldots, n\}$, $d(X_{E_I^j}, X_{E_I^{j-1}}) < \varepsilon$, which shows that the TRS of an interval type-2 fuzzy set with continuous domain and codomain is a closed interval of $[0,1]$. Indeed, each vertical slice will produce a set of weighted averages (one for each embedded set for the vertical slice) $\{X_{E_I^j} | j = 1, \ldots, n\}$. When n tends to infinity, $d(X_{E_I^j}, X_{E_I^{j-1}})$ tends to zero, and therefore the set $\{X_{E_I^j} | j = 1, \ldots, n\}$ tends towards a closed interval of $[0,1]$. We conclude that the TRS of a continuous interval type-2 fuzzy set is an interval of the line $u = 1$ with least domain value ≥ 0 and greatest domain value ≤ 1, i.e. the TRS of a continuous interval type-2 fuzzy set is a closed interval subset of the unit interval U with membership function identically equal to 1.

In Subsection 3.3.1 below, Figures 4 to 6 show how the TRS from a generalised type-2 fuzzy set is built up point by point. The number of embedded sets of the originating type-2 fuzzy set runs into many millions; the centroids of each embedded set, paired with the associated minimum secondary membership grade, have been plotted as dots for 50 embedded sets (Figure 4), 500 embedded sets (Figure 5), and 5000 embedded sets (Figure 6). Each horizontal line is analogous to the TRS of an interval type-2 fuzzy set. It can clearly be seen that the points merge into a continuous line as the number of TRS tuples increases[3].

3.2.1. Ratio of Defuzzified Value Increments

Definition 4 (j^{th} Defuzzified Value Increment on Vertical Slice $x = x_I$ (DVI_I^j)). *The* j^{th} Defuzzified Value Increment (DVI_I^j) *on vertical slice* $x = x_I$ *is defined as* $X_{E_I^j} - X_{E_I^{j-1}}$.

From Equation (9),

$$DVI_I^j = X_{E_I^j} - X_{E_I^{j-1}} = \frac{|E| \cdot d_u \cdot (x_I - X_E)}{(|E| + j \cdot d_u)(|E| + (j-1) \cdot d_u)}.$$

It follows that

$$DVI_I^{j+1} = X_{E_I^{j+1}} - X_{E_I^j} = \frac{|E| \cdot d_u \cdot (x_I - X_E)}{(|E| + (j+1) \cdot d_u)(|E| + j \cdot d_u)}. \quad (11)$$

From Equations (9) and (11), we may calculate the ratio of DVI_I^{j+1} to DVI_I^j for domain value x_I:

$$\frac{DVI_I^{j+1}}{DVI_I^j} = \frac{X_{E_I^{j+1}} - X_{E_I^j}}{X_{E_I^j} - X_{E_I^{j-1}}} = \frac{|E| + (j-1) \cdot d_u}{|E| + (j+1) \cdot d_u}.$$

For any domain value x_I, this ratio depends only on the scalar cardinality of the embedded set E and the co-domain degree of discretisation d_u. Moving up the x_I^{th} vertical slice, as j increases, the ratio of a given DVI to its preceding DVI increases, tending to 1, i.e. the defuzzified values tend to become evenly spaced out along the interval. Were d_u to be decreased, this would have the effect of increasing the value of n. The continuous case is approached in which the defuzzified values merge together along the interval of the line $u = 1$.

[3]In these figures, the lines appear continuous because the relatively large size of the dots eliminates the spaces between them. In the analysis presented above, the spaces between the tuples are eliminated as the codomain degree of discretisation tends to 0.

3.3. *Structure of the TRS of a Discretised Generalised Type-2 Fuzzy Set*

In 2008 Greenfield and John reported on the stratified structure exhibited by the TRS of a generalised type-2 fuzzy set [16]. This structure was observed during investigations into the sampling method of type-2 defuzzification [18, 21]. The membership function of the TRS of a discretised type-2 fuzzy set may be regarded as a set of tuples (Section 3.1).

3.3.1. *Stratification in the Discretised TRS*

The following shows how structure is revealed to exist in the TRS. Figures 4 to 6 show typical TRSs derived from randomly generated samples of embedded sets of size 50, 500, and 5000, originating from the same discretised type-2 fuzzy set. Each tuple is shown as a dot; the dots clearly align themselves into strata. The reason for the strata's appearance is that since during type-reduction the minimum secondary grade of each embedded set is selected, the same minimum values appear repeatedly, but in association with different domain values.

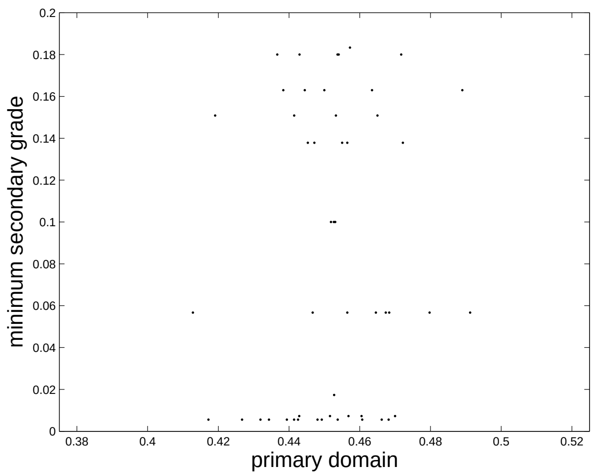


Figure 4: The TRS strata. A sample of 50 TRS tuples is shown.

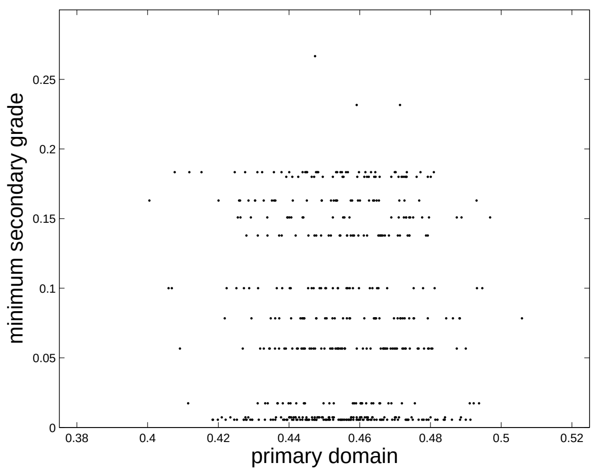


Figure 5: The TRS strata. A sample of 500 TRS tuples is shown.

Definition 5 (Stratum [16]). *Let T be the TRS of a discretised generalised type-2 fuzzy set. A* stratum*,*

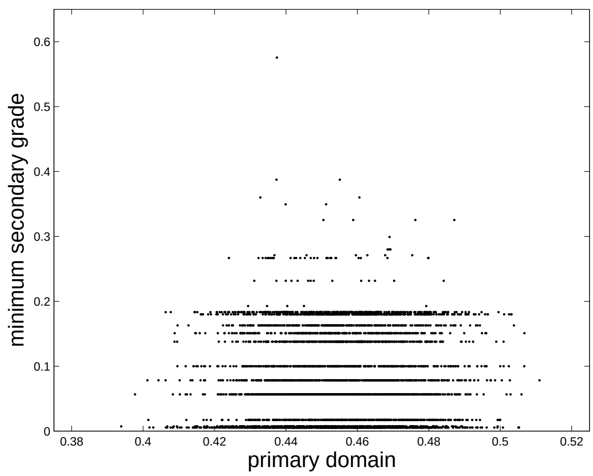


Figure 6: The TRS strata. A sample of 5000 TRS tuples is shown.

S_ω, is a subset[4] of T for which every element has the same membership grade.

$$S_\omega = \{(x, \mu_T(x)) \in T \mid \mu_T(x) = \omega\} \text{ for some } \omega \in [0,1].$$

3.3.2. Mendel's Probability Analysis of the Stratified Structure

Mendel's observations on the stratified structure [30] are interesting and relevant. In relation to the Sampling Method of Defuzzification [21], he has deduced that as the domain discretisation becomes finer, the probability of a randomly selected embedded set containing at least one of the minimum secondary grades (from one of the secondary membership functions) approaches 1. Consequently the lower strata contain more tuples than the higher ones. Mendel's argument is reproduced here:

> "Assumptions:
>
> 1. Primary variable x is discretized into N values $x_1, x_2, \ldots, x_N$.
> 2. We are free to choose N, e.g. make it as small as we choose.
> 3. All primary memberships are discretized into the same number of levels, M. (Even if you do not do this, the analysis below is interesting, and can be modified to the case of non-equal discretization.)
> 4. The smallest secondary grade for each of the N secondary MFs occurs exactly one time in each of the secondary MFs (this is controversial, but it could be changed with a more complicated analysis).
>
> My first goal is to compute the probability of choosing an embedded T2 FS that contains at least one of the minimum secondary grades.
>
> 1. The total number of embedded T2 FSs is M^N.
> 2. The total number of embedded T2 FSs that do not contain at least one of the minimum secondary values is $(M-1)^N$.
> 3. The total number of embedded T2 FSs that contain at least one of the minimum secondary values is $M^N - (M-1)^N$.
> 4. The probability of choosing an embedded T2 FS that contains at least one of the minimum secondary values is $p(1 \text{ or more}|M,N)$, where

[4]In the commonly accepted crisp sense of the word. Klir and Folger [24, Page 19] give a different definition of subset in the type-1 fuzzy context.

$$p(1 \text{ or more}|M,N) = \frac{M^N - (M-1)^N}{M^N}. \tag{12}$$

Next, I want to study $p(1 \text{ or more}|M,N)$, especially as N increases. Note from (12) that

$$p(1 \text{ or more}|M,N) = 1 - \left(\frac{M-1}{M}\right)^N. \tag{13}$$

Because $(M-1)/M < 1$, it is true that:

Fact: As N increases

$$p(1 \text{ or more}|M,N) \to 1'' \text{ [for a fixed natural number } M > 0.] \blacksquare \tag{14}$$

Conversely, if N is fixed and M increases, the opposite result holds, i.e. $p(1 \text{ or more }|N,M) \to 0$ for $M \to \infty$.

The Sampling Method was devised in order to circumvent the defuzzification bottleneck engendered by the proliferation of embedded sets. Type-reduction (Algorithm 1) requires that every embedded set be processed. The number of embedded sets within a type-2 fuzzy set is $\prod_{i=1}^{N} M_i$, where N is the number of vertical slices into which the primary domain has been discretised, and M_i is the number of elements on the i^{th} slice. The finer the discretisation, the better the representation of a given fuzzy set, but the greater the number of embedded sets generated. A reasonably fine degree of discretisation can give rise to astronomical (though finite) numbers of embedded sets. For instance, when a prototype type-2 FIS was invoked using a primary and secondary degree of discretisation of 0.02, the number of embedded sets generated was of the order of 2.9×10^{63} [21].

Figures 4 to 6 are in accord with Mendel's probability analysis; they show that the lower strata are, generally, denser than the higher ones. Nevertheless, as more embedded sets are sampled, new strata come into being, frequently at greater heights than the existing strata. Table 1 shows the increasing height of the maximum TRS tuple. If all the embedded sets were to be sampled, would a strata be produced at the height of the highest secondary membership grade of the originating type-2 fuzzy set? This question is answered in the next subsection.

Figure	Number of TRS Tuples Sampled	Maximum Stratum Height
4	50	0.184
5	500	0.270
6	5000	0.575

Table 1: Maximum strata heights for different embedded sets sample sizes.

3.4. Structure of the TRS of a Continuous Generalised Type-2 Fuzzy Set

If the simplification stage (Lines 6–8) of Algorithm 1 is omitted, then Algorithm 2 results. We term the type-1 fuzzy set resulting from this algorithm the *Unsimplified Type-Reduced Set (UTRS).*

We will now look at how the TRS strata relate to the originating type-2 fuzzy set.

Definition 6 (Minimum Secondary Membership Function)**.** *Let $\tilde{A}$ be a type-2 fuzzy set. Given $x \in X$, $\mu_{\tilde{A}}(x)$ its secondary membership function, and*

$$\mu_{\tilde{A}}(x)(\overline{u}) = \max_{u \in J_x} \mu_{\tilde{A}}(x)(u)$$

Input: a discretised generalised type-2 fuzzy set
Output: a discrete type-1 fuzzy set (the TRS)

forall *embedded sets* **do**
    find the minimum secondary membership grade (z) ;
    calculate the primary domain value (x) of the type-1 centroid of the embedded type-2 fuzzy set ;
    pair the secondary grade (z) with the primary domain value (x) to give set of ordered pairs (x,z) $\{$some values of x may correspond to more than one value of $z\}$;
end

Algorithm 2: Algorithm 1, omitting the simplification stage (Lines 6–8), so creating the UTRS.

the maximum secondary membership grade of x. A minimum secondary membership function *of a type-2 fuzzy set $\tilde{A}$ is a secondary membership function, $\mu_{\tilde{A}}(\underline{x})$, whose maximum membership grade is the least of all the maximum membership grades of the vertical slices comprising the set, i.e.*

$$\mu_{\tilde{A}}(\underline{x})(\overline{u}) = \min_{x \in X} \mu_{\tilde{A}}(x)(\overline{u}).$$

When the degree of discretisation $d_u \to 0$, the discrete secondary membership function of the type-2 fuzzy set tends to the original assumed convex secondary membership function, and therefore the discrete secondary membership functions becomes continuous. By definition, the minimum secondary membership function will become continuous and will take all values between its corresponding minimum and maximum values. Each secondary membership grade gives rise to a stratum. Therefore, in a continuous generalised type-2 fuzzy set, there is a stratum at every secondary membership grade that lies within the minimum secondary membership function. We have already shown (Section 3.1) that the TRS of non-discretised interval type-2 fuzzy set is a continuous horizontal line. Taking these two observations together, we conclude that the UTRS in the continuous case is a continuous planar surface. On simplification (at which the UTRS is converted into the TRS), all the z-values apart from the highest are eliminated, so forming a continuous type-1 membership function.

In 2008 Liu [27, 32] proposed the *α-Planes Representation* (Subsection 6.2). Via this technique a generalised type-2 fuzzy set is decomposed into a set of α-planes, which are horizontal slices akin to interval type-2 fuzzy sets. According to the assumptions in Subsubsection 1.1.2, all the secondary membership functions of the originating type-2 fuzzy set are convex. Therefore any given α-plane must fit within the contours of a lower α-plane. It follows that the stratum corresponding to a lower α-plane must occupy an interval that includes the interval associated with a higher α-plane. A higher α-plane cannot overhang a lower α-plane; the TRS of a continuous type-2 fuzzy set is convex, rising to a maximum and then decreasing. From this it follows that the TRS of a continuous generalised[5] type-2 fuzzy set comprises a convex membership function.

4. The Truncated Type-2 Fuzzy Set

In this section the concept of the *truncated type-2 fuzzy set* is introduced; the observations which follow apply to both generalised and interval type-2 fuzzy sets. Figure 7 depicts a minimum secondary membership function of a type-2 fuzzy set, behind which can be seen the rest of the set. Were the originating type-2 fuzzy set to be truncated horizontally at the level of the maximum grade of the minimum secondary membership function (Figure 8), the set's TRS would be unchanged, since no strata exist at grades higher than this level.

[5]Here the term 'generalised type-2 fuzzy set' is used in a way which excludes interval type-2 fuzzy sets.

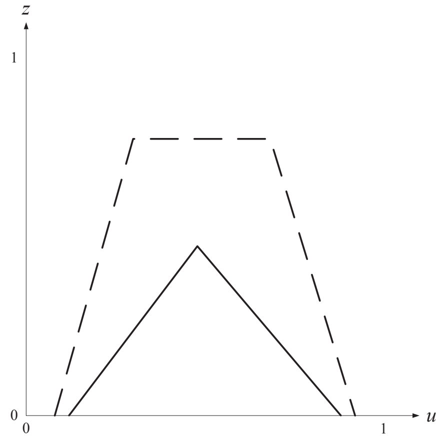


Figure 7: A generalised type-2 fuzzy set viewed from the $u-z$ plane. The minimum secondary membership function is shown in bold. The dashed line shows a non-minimum secondary membership function.

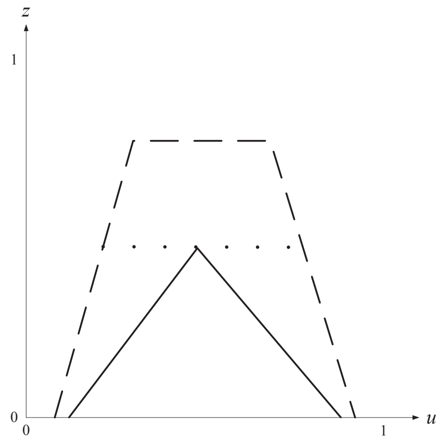


Figure 8: The generalised type-2 fuzzy set depicted in Figure 7. The dashed line shows a non-minimum secondary membership function; the dotted line indicates where it has been truncated to the height of the maximum grade of the minimum secondary membership function (i.e. the truncation grade).

Definition 7 (Truncation Grade)**.** *The* truncation grade *of a type-2 fuzzy set is the maximum membership grade of the minimum secondary membership function.*

Definition 8 (Truncated Type-2 Fuzzy Set [11])**.** *A* truncated type-2 fuzzy set *is a type-2 fuzzy set for which all secondary membership grades greater than the truncation grade have been reduced to the truncation grade.*

Recall that the TRS of a type-2 fuzzy set $\tilde{A}$ is the type-2 fuzzy set derived from the application of expression (7), and that this is a type-1 fuzzy set with elements the centroid of all type-2 fuzzy embedded sets of $\tilde{A}$ and membership function that associates each element a membership degree equal to the minimum of all secondary membership grades of the corresponding type-2 fuzzy embedded set. It is therefore clear that the membership function of the TRS will have a maximum value equal to the truncation grade of the type-2 fuzzy set $\tilde{A}$, because the minimum t-norm will discard any other value above it. Thus, the TRS of a continuous generalised type-2 fuzzy set $\tilde{A}$ with convex secondary membership functions is characterised by a continuous membership function with domain a closed interval subset of the unit interval U with maximum value the truncation grade of $\tilde{A}$. Consequently, we have that the TRS of a type-2 fuzzy set $\tilde{A}$ will be identical to the TRS of its truncated type-2 fuzzy set. This is summarised in the following result.

Theorem 1 (Type-2 Truncation Theorem)**.** *The TRS of a truncated type-2 fuzzy set is identical to that of the original type-2 fuzzy set, and consequently the defuzzified value of a truncated type-2 fuzzy set is equal to that of the original type-2 fuzzy set.*

Any number of type-2 fuzzy sets may be truncatable to the same type-2 fuzzy set; there is clearly an equivalence between these fuzzy sets.

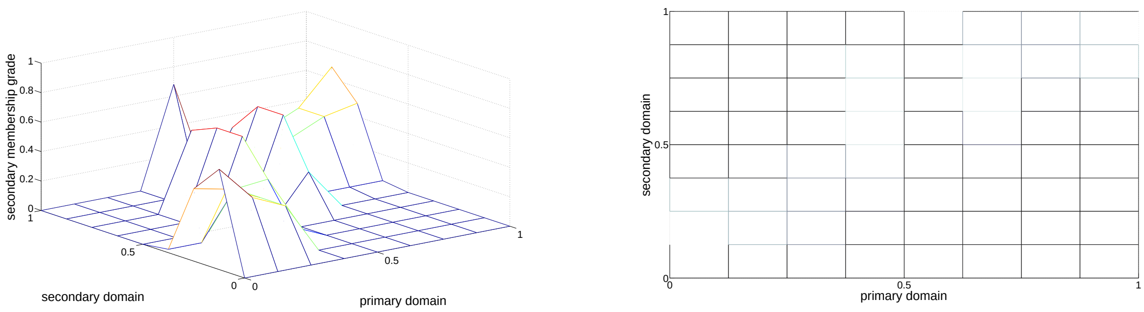
Definition 9 (Equivalence Under Truncation)**.** *Two type-2 fuzzy sets are* equivalent under truncation*, if, when truncated, they give rise to the same truncated type-2 fuzzy set.*

From this it immediately follows that,

Theorem 2 (Type-2 Equivalence Theorem)**.** *The defuzzified values of type-2 fuzzy sets that are equivalent under truncation are equal.*

5. Experimental Confirmation of Equivalence of the Truncated Set

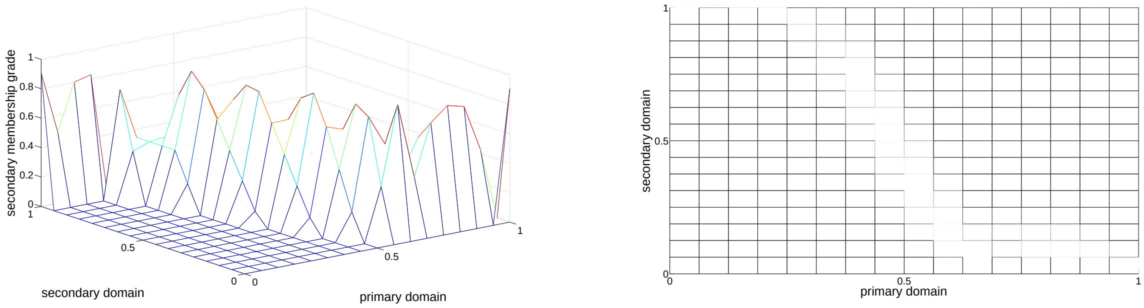
In the previous section it was demonstrated theoretically that "the defuzzified value of a truncated type-2 fuzzy set is equal to that of the original type-2 fuzzy set" (Theorem 1, the Type-2 Truncation Theorem). In this section the result is experimentally corroborated, to determine whether discretisation choices have any bearing on the results. Six generalised type-2 fuzzy test sets were created, depicted in Figures 9 to 14. These are aggregated sets output by the inferencing stage of a prototype type-2 FIS, coded in Matlab^{TM}. For each inference the degree of discretisation adopted was sufficiently coarse to allow Exhaustive Defuzzification. Three rule sets were used. For each rule set the FIS was run with two distinct sets of parameters[6]. Table 2 summarises the features of the test sets. More information about the rule sets may be found in [12]. The FIS generated test sets were selected because of the complexity and lack of symmetry evident in their graphs; their benchmark defuzzified values were found by Exhaustive Defuzzification. The defuzzification methods were coded in Matlab^{TM} R2014a and tested on a PC with a Pentium 4 CPU and a 0.99 GB RAM, with a clock speed of 3.00 GHz. The MS Windows XP Professional operating system was used. Each test program was run as a process with a higher priority than that of the operating system, so as to as far as possible eliminate timing errors resulting from other operating system processes. The main purpose of the experiments was to confirm that the defuzzified value of the truncated type-2 fuzzy set equals that of the non-truncated type-2 fuzzy set. Defuzzification time was also investigated experimentally. Timings were taken for each type-2 fuzzy set in its non-truncated, truncated and pre-truncated (i.e. truncated, but with the time for truncation not counted) form.



(a) 3-D representation. (b) FOU.

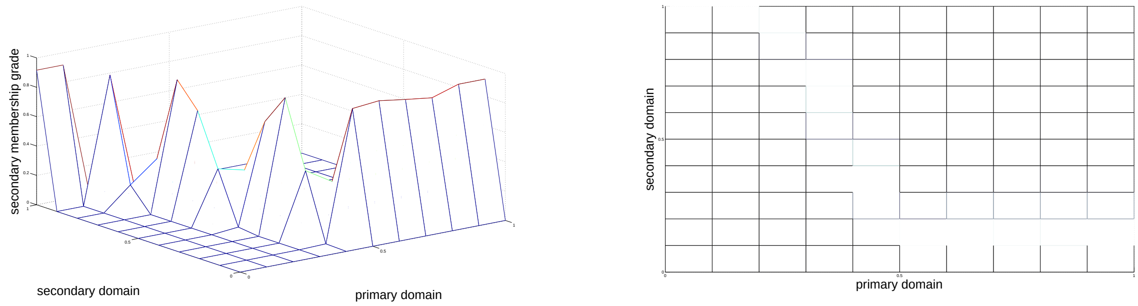
Figure 9: Heater0.125 — Heater FIS generated generalised test set, domain and co-domain degree of discretisation 0.125.

[6]For example Heater0.0625 is not a finer version of Heater0.125; it uses different parameters for the input rules. That these two test sets differ can be clearly seen from their 3-D representations.



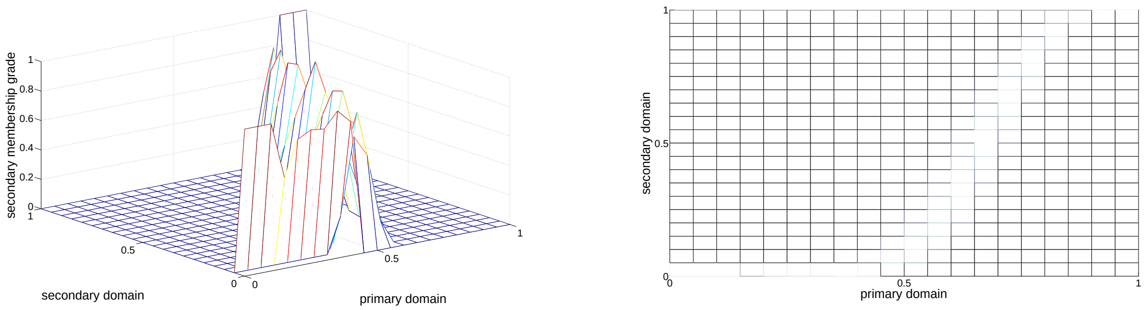
(a) 3-D representation. (b) FOU.

Figure 10: Heater0.0625 — Heater FIS generated generalised test set, domain and co-domain degree of discretisation 0.0625.



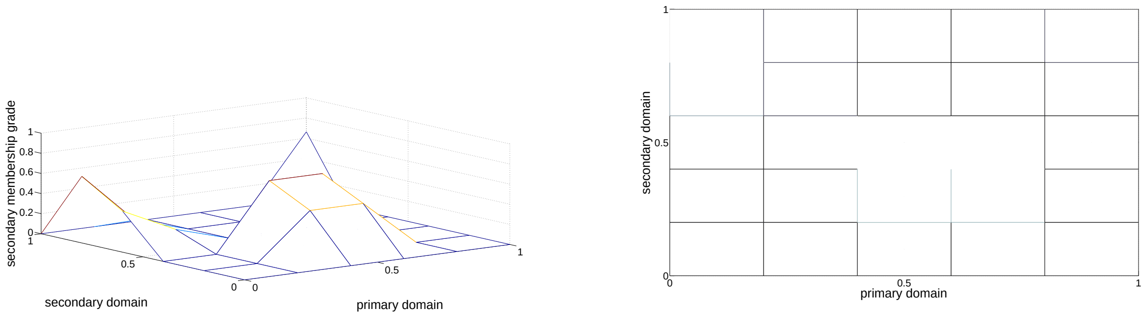
(a) 3-D representation. (b) FOU.

Figure 11: Powder0.1 — Powder FIS generated generalised test set, domain and co-domain degree of discretisation 0.1.



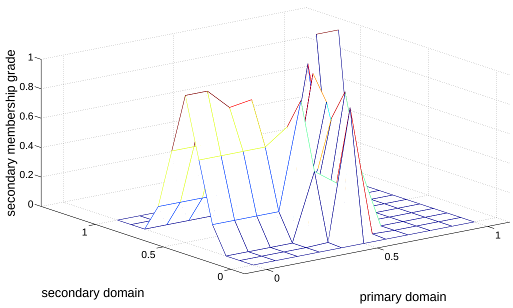
(a) 3-D representation. (b) FOU.

Figure 12: Powder0.05 — Powder FIS generated generalised test set, domain and co-domain degree of discretisation 0.05.

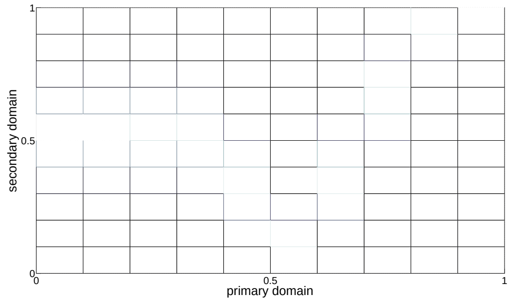


(a) 3-D representation. (b) FOU.

Figure 13: Shopping0.2 — Shopping FIS generated generalised test set, domain and co-domain degree of discretisation 0.2.



(a) 3-D representation.



(b) FOU.

Figure 14: Shopping0.1 — Shopping FIS generated generalised test set, domain and co-domain degree of discretisation 0.1.

TEST SET	NORMAL FOU	NORMAL SEC. MF	NARROW FOU	NO. OF EMB. SETS
Heater0.125	yes	no	no	14580
Heater0.0625	yes	no	yes	13778100
Powder0.1	yes	no	yes	24300
Powder0.05	yes	yes	yes	3840000
Shopping0.2	no	no	no	16
Shopping0.1	yes	yes	no	312500

Table 2: Features of the generalised test sets.

Test Set	Exhaustive Defuzzified Value	Truncated Exhaustive Defuzz. Value	Maximum Secondary Memb. Grd.	Truncation Threshold
Heater0.125	0.6313618377	0.6313618377	0.6806	0.3438
Heater0.0625	0.2621587894	0.2621587894	0.9096	0.3875
Powder0.1	0.2806983775	0.2806983775	0.9167	0.8594
Powder0.05	0.8180632180	0.8180632180	1.0000	0.6191
Shopping0.2	0.5481044441	0.5481044441	0.6625	0.1563
Shopping0.1	0.5954109472	0.5954109472	1.0000	0.8594

Table 3: Defuzzified values for Exhaustive Defuzzification and Truncated Exhaustive Defuzzification for the six test sets.

Test Set	Exhaustive Defuzz. Time	Truncated Exhaustive Defuzz. Time	Pre-Truncated Exhaustive Defuzz. Time	Number of Embedded Sets
Heater0.125	0.8706 s	0.8499 s	0.9758 s	14580
Heater0.0625	880.1 s	881.9 s	881.7 s	13778100
Powder0.1	1.45 s	1.45 s	1.44 s	24300
Powder0.05	268.08 s	268.98 s	268.39 s	3840000
Shopping0.2	0.0013 s	0.0014 s	0.0012 s	16
Shopping0.1	19.40 s	19.26 s	19.26 s	312500

Table 4: Defuzzification times for Exhaustive Defuzzification, truncated exhaustive defuzzification and Pre-Truncated Exhaustive Defuzzification for the six test sets.

The results shown in Table 3 confirm that there is no difference whatsoever between exhaustive defuzzification and truncated exhaustive defuzzification in relation to the defuzzified values obtained. Discretisation choices are of no consequence. The timings recorded in Table 4 show that it matters very little whether the type-2 fuzzy set is truncated or not, and if it is truncated, whether the timing for the truncation itself is taken into account. These results are to be expected, as the truncation process is much computationally simpler than exhaustive defuzzification.

6. Implications of Truncation

The Type-2 Truncation Theorem (Theorem 1) states that truncation does not alter the defuzzified value of a generalised type-2 fuzzy set. The experiments described above show that truncation does not appreciably alter the defuzzification time either. So is the notion of truncation merely of theoretical interest? In this section, two practical applications of the concept of truncation are considered.

6.1. Truncation and Uncertainty

Zadeh's 1975 innovation of the *type-2 fuzzy set* [39–41] intuitively models uncertainty. A type-2 fuzzy set (defined in Subsection 1.1.1) may be thought of as an adaptation of a type-1 fuzzy set [31, page 118]:

> "Imagine blurring the type-1 membership function ...Then, at a specific value of x, say x', there no longer is a single value for the membership function (u'); instead the membership function takes on values wherever the vertical line intersects the blur. Those values need not all be weighted the same; hence, we can assign an amplitude distribution to all of those points. Doing this for all $x \in X$, we create a three-dimensional membership function — a type-2 membership function — that characterizes a type-2 fuzzy set."

How type-2 fuzzy sets model uncertainty is the subject of [17]. In this 2009 book chapter it is proposed that the type-2 fuzzy set's third dimension (z) reflects the uncertainty arising out of a deficit in information. From this premise it is argued that the volume under the surface of the type-2 fuzzy set is a measure of the uncertainty relating to the set. On this measure, the minimum amount of uncertainty possible is 0 (no uncertainty), and the maximum 1 (total uncertainty).

According to the Type-2 Equivalence Theorem (Theorem 2), the defuzzified values of type-2 fuzzy sets that are equivalent under truncation are equal. Yet two type-2 fuzzy sets may be equivalent but have different uncertainty measures. Within an FIS (Section 1, Figure 1), the volume uncertainty measure may be applied to the aggregated type-2 fuzzy set to give a quantification of the uncertainty relating to the fuzzy inference. Thus two values may be obtain from the aggregated type-2 fuzzy set:

1. The defuzzified value (i.e. the result of the fuzzy inference), obtained by defuzzification;
2. A quantification of the uncertainty relating to the fuzzy inference, obtained via the volume measure of uncertainty.

6.2. Type-Reduction via the α-Plane Representation

In this section the impact of truncation on the α-Planes Method of type-2 defuzzification, (alluded to in Section 3.4), is examined. This recognised technique for the defuzzification of generalised type-2 fuzzy sets employs the α-Planes Representation, proposed by Liu in 2008, [27, 32][7]. In this strategy a generalised type-2 fuzzy set is decomposed into a set of α-planes, which are horizontal slices equivalent to interval type-2 fuzzy sets. Each α-plane is then defuzzified via the Karnik-Mendel Iterative Procedure [27], so forming an approximation to the TRS. By defuzzifying the resultant type-1 fuzzy set, the defuzzified value for the generalised type-2 fuzzy set is obtained. Below this method is presented algorithmically (Algorithm 3), and diagrammatically (Figure 15).

Input: a discretised generalised type-2 fuzzy set
Output: a discrete type-1 fuzzy set

decompose the type-2 fuzzy set into α-planes ;
forall *α-planes* **do**
  find the left and right endpoints using the KMIP ;
  pair each endpoint with the α-plane height to give set of ordered pairs (i.e. a type-1 fuzzy set) {each α-plane is paired with two endpoints } ;
end

Algorithm 3: Type-reduction of a type-2 fuzzy set to a type-1 fuzzy set using the α-plane method.

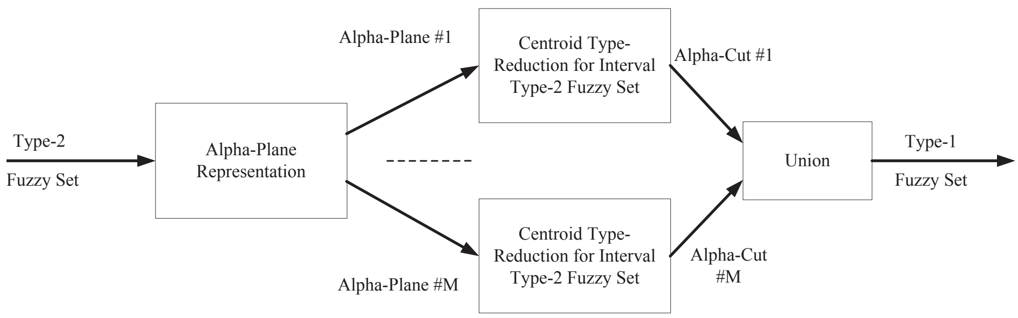


Figure 15: Defuzzification using the α-Planes Representation (from Liu [27]).

Though the α-plane representation was envisaged by Liu as being used in conjunction with the KMIP [27], any interval defuzzification method may be used. Any variation on the KMIP, such as the Enhanced Iterative Algorithm with Stop Condition (EIASC) [36] will locate the endpoints of the TRS interval. Other interval methods, such as the Greenfield-Chiclana Collapsing Defuzzifier [19, 20], or the Nie-Tan Method [33], will defuzzify the α-plane [10]; their defuzzified values (located in the vicinity of the centre of the interval) may then be formed into a type-1 fuzzy set equivalent to the TRS.

[7] Independently of Liu, and at about the same time, Wagner and Hagras introduced the notion of zSlices [34], a concept very similar to that of α-planes

In [13] the α-Planes Method has been shown to be inferior to the Sampling Defuzzifier [21] and Vertical Slice Centroid Type Reduction (VSCTR) [28] in relation to both accuracy and efficiency. It is assumed that applying the α-Planes strategy to the truncated type-2 fuzzy set would make for more efficient defuzzification, as there would be fewer α-planes to process. More importantly, it is reasonable to suppose that accuracy would also be improved, as irrelevant α-planes (between the maximum secondary membership grade and the truncation grade) would be eliminated and therefore not be able to distort the defuzzified value.

Experiments were performed using the test sets described in Section 5, with the exception of test set Powder0.05, which was not used since the defuzzification time required is exorbitant. These test sets, whose exhaustive defuzzified values had already been determined, were defuzzified using α-Planes/Interval Exhaustive Defuzzification, and then using α-Planes/Truncated Interval Exhaustive Defuzzification. The number of α-planes actually defuzzified was recorded, for both the non-truncated and truncated test sets. Five groups of experiments were performed using different potential numbers of α-planes ranging from 11 to 201; the timings and defuzzified values for α-planes/interval exhaustive defuzzification and α-Planes/Truncated Interval Exhaustive Defuzzification are recorded in Tables 5 to 9. For the truncated test runs, the times for the truncation process itself were included in the timings. The errors are calculated by subtracting the benchmark exhaustive defuzzified value from the α-planes defuzzification value.

Test Set	Exhaustive Defuzzified Value	α-planes/ Interval Exh. DV	Error	No. of α-P. Defuzz.	Time	α-planes/ Trunc. Int. Exh. DV	Error	No. of α-P. Defuzz.	Time
Heater0.125	0.6313618377	0.6202019617	-0.0111598760	8	0.11 s	0.6100619324	-0.0212999053	4	0.06 s
Heater0.0625	0.2621587894	0.2791839651	0.0170251757	11	34.7 s	0.2901775808	0.0280187914	4	13.9 s
Powder0.1	0.2806983775	0.2860677799	0.0053694024	11	0.60 s	0.2974284749	0.0167300974	9	0.53 s
Shopping0.2	0.5481044441	0.5381656623	-0.0099387818	8	0.018 s	0.5363791187	-0.0117253254	2	0.006 s
Shopping0.1	0.5954109472	0.5946460014	-0.0007649458	11	0.67 s	0.5810476375	-0.0143633097	9	0.54 s

Table 5: Defuzzified values and timings for α-Planes/Interval Exhaustive Defuzzification and α-Planes/Truncated Interval Exhaustive Defuzzification. Each test set was decomposed into 11 α-planes.

Test Set	Exhaustive Defuzzified Value	α-planes/ Interval Exh. DV	Error	No. of α-P. Defuzz.	Time	α-planes/ Trunc. Int. Exh. DV	Error	No. of α-P. Defuzz.	Time
Heater0.125	0.6313618377	0.6176441546	-0.0137176831	15	0.18 s	0.6099705337	-0.0213913040	7	0.09 s
Heater0.0625	0.2621587894	0.2839784863	0.0218196969	20	65.7 s	0.2884665979	0.0263078085	8	27.6 s
Powder0.1	0.2806983775	0.2903173044	0.0096189269	20	1.12 s	0.2961634579	0.0154650804	18	1.07 s
Shopping0.2	0.5481044441	0.5365375672	-0.0115668769	15	0.028 s	0.5363791187	-0.0117253254	4	0.009 s
Shopping0.1	0.5954109472	0.5929838018	-0.0024271454	21	1.24 s	0.5812827597	-0.0141281875	18	1.05 s

Table 6: Defuzzified values and timings for α-Planes/Interval Exhaustive Defuzzification and α-Planes/Truncated Interval Exhaustive Defuzzification. Each test set was decomposed into 21 α-planes.

Test Set	Exhaustive Defuzzified Value	α-planes/ Interval Exh. DV	Error	No. of α-P. Defuzz.	Time	α-planes/ Trunc. Int. Exh. DV	Error	No. of α-P. Defuzz.	Time
Heater0.125	0.6313618377	0.6149552604	-0.0164065773	36	0.45 s	0.6097558940	-0.0216059437	18	0.24 s
Heater0.0625	0.2621587894	0.2845118383	0.0223530489	47	159 s	0.2876980149	0.0255392255	20	69 s
Powder0.1	0.2806983775	0.2928844669	0.0121860894	47	2.69 s	0.2963359791	0.0156376016	43	2.51 s
Shopping0.2	0.5481044441	0.5362804955	-0.0118239486	35	0.07 s	0.5363791187	**-0.0117253254**	8	0.02 s
Shopping0.1	0.5954109472	0.5920110566	-0.0033998906	51	2.99 s	0.5812602023	-0.0141507449	43	2.51 s

Table 7: Defuzzified values and timings for α-Planes/Interval Exhaustive Defuzzification and α-Planes/Truncated Interval Exhaustive Defuzzification. Each test set was decomposed into 51 α-planes.

6.2.1. Discussion of Results

The timings show that, as expected, the defuzzification time is roughly proportional to the number of α-planes defuzzified. Truncation, therefore, is a device which may be used to improve the efficiency of the α-Planes Method.

Test Set	Exhaustive Defuzzified Value	α-planes/ Interval Exh. DV	Error	No. of α-P. Defuzz.	Time	α-planes/ Trunc. Int. Exh. DV	Error	No. of α-P. Defuzz.	Time
Heater0.125	0.6313618377	0.6146732228	-0.0166886149	70	0.88 s	0.6097638065	-0.0215980312	35	0.44 s
Heater0.0625	0.2621587894	0.2857559640	0.0235971746	92	315 s	0.2876829504	0.0255241610	39	135 s
Powder0.1	0.2806983775	0.2909086286	0.0102102511	93	5.37 s	0.2960372128	0.0153388353	86	5.02 s
Shopping0.2	0.5481044441	0.5361962986	-0.0119081455	68	0.12 s	0.5363791187	**-0.0117253254**	16	0.03 s
Shopping0.1	0.5954109472	0.5919454769	-0.0034654703	101	5.92 s	0.5812947933	-0.0141161539	86	5.02 s

Table 8: Defuzzified values and timings for α-Planes/Interval Exhaustive Defuzzification and α-Planes/Truncated Interval Exhaustive Defuzzification. Each test set was decomposed into 101 α-planes.

Test Set	Exhaustive Defuzzified Value	α-planes/ Interval Exh. DV	Error	No. of α-P. Defuzz.	Time	α-planes/ Trunc. Int. Exh. DV	Error	No. of α-P. Defuzz.	Time
Heater0.125	0.6313618377	0.6149272030	-0.0164346347	138	1.73 s	0.6097715004	-0.0215903373	69	0.87 s
Heater0.0625	0.2621587894	0.2843776382	0.0222188488	183	630 s	0.2875788268	0.0254200374	78	270 s
Powder0.1	0.2806983775	0.2903453151	0.0096469376	185	10.8 s	0.2959017026	0.0152033251	172	10.1 s
Shopping0.2	0.5481044441	0.5363170602	-0.0117873839	134	0.24 s	0.5363791187	**-0.0117253254**	32	0.06 s
Shopping0.1	0.5954109472	0.5917184848	-0.0036924624	201	11.8 s	0.5812980253	-0.0141129219	172	10.1 s

Table 9: Defuzzified values and timings for α-Planes/Interval Exhaustive Defuzzification and α-Planes/Truncated Interval Exhaustive Defuzzification. Each test set was decomposed into 201 α-planes.

Regarding accuracy, truncation worsened the defuzzification errors in 22 out of 25 cases. Those cases in which it was helpful all relate to the Shopping0.2 test set (with 51, 101 and 201 α-planes employed). In Tables 7 to 9 the reduced errors are shown in bold. Rather than being a remedy for the inaccuracies of the α-Planes Method, truncation exacerbates the problem, pointing to issues with the accuracy of the technique that are deeper than those noted in [12]. These unexpected, counterintuitive results warrant further investigation.

7. Conclusions

This paper contributes to the theory of type-2 fuzzy logic, particularly in relation to defuzzification. The structure of the TRS of the continuous type-2 fuzzy set in both its interval and generalised forms has been investigated by first looking into the structures of the discretised sets. The TRS of a continuous interval type-2 fuzzy set has been shown to be a continuous straight line, specifically an interval of the line $u = 1$ with least domain value ≥ 0 and greatest domain value ≤ 1, i.e. i.e. the TRS of a continuous interval type-2 fuzzy set is a closed interval subset of the unit interval U with membership function identically equal to 1. The TRS of a continuous generalised type-2 fuzzy set $\tilde{A}$ with convex secondary membership functions is characterised by a continuous membership function with domain a closed interval subset of the unit interval U with maximum value the truncation grade of $\tilde{A}$.

The innovative concept of the truncated type-2 fuzzy set has been introduced. Its characteristic property is that its TRS is identical to that of the originating non-truncated type-2 fuzzy set. From this it immediately follows that its defuzzified value equals that of the non-truncated type-2 fuzzy set. Experiments have corroborated the equivalence of the truncated to the non-truncated type-2 fuzzy set as regards defuzzified values (the Type-2 Truncation Theorem), and shown that there is no time to be saved by defuzzifying the truncated set in place of the non-truncated set. A corollary of the Type-2 Truncation Theorem is the Type-2 Equivalence Theorem, which states that the defuzzified values of type-2 fuzzy sets that are equivalent under truncation are equal.

Two type-2 fuzzy sets may be equivalent under truncation, yet be associated with different amounts of uncertainty as quantified by the volume measure of uncertainty as applied to the originating type-2 fuzzy set.

Unsurprisingly, experiments have shown that truncating the generalised type-2 fuzzy set improves the efficiency of the α-Planes Method. Surprisingly, these experiments have shown that truncating the generalised type-2 fuzzy set does not improve the accuracy of the α-Planes Method; on the contrary, accuracy is diminished.

7.1. Further Work

7.1.1. The Uncertainty Represented by a Type-2 Fuzzy Set

In relation to Subsection 6.1, it would be interesting to explore further how the volume measure of uncertainty for a type-2 fuzzy set [17] relates to the concept of truncation.

7.1.2. Issues with the α-Planes Method

As observed in Subsection 6.2, there are unexpected, intriguing, experimental results in relation to the α-Planes Method. These experiments have shown that truncating the generalised type-2 fuzzy set diminishes, rather than improves, the accuracy of the α-Planes Method. Further investigation into this matter is desirable.

7.1.3. More Specific Characterisation of the Shape of the TRS

At the end of Section 3 the shape of the TRS of a generalised type-2 fuzzy set is outlined. A more specific characterisation of the TRS shape is desirable. This would be interesting in its own right, and hopefully open the way for the development of another defuzzification strategy for generalised type-2 fuzzy sets. An algorithm that computes the TRS is likely to be amenable to parallelisation.

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