Equivalent point-source modeling of small obstacles for electromagnetic waves

Justine Labat, Victor Péron, Sébastien Tordeux

PhD student in Applied Mathematics

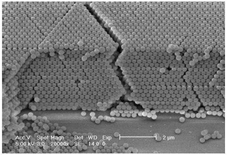
EPC Magique 3D — Université de Pau et des Pays de l'Adour, INRIA Bordeaux Sud-Ouest, UPPA-E2S, LMAP UMR CNRS 5142

14th International Conference on
Mathematical and Numerical Aspects of Wave Propagation

Vienna, Austria, 25-30 August 2019

Outline

Electromagnetic scattering problem by small obstacles

Atmospheric particles	Cosmic dust	Photonic crystal
		
	(by Allan Dyer)	(by Koen Clays)

Atmospheric particles
- λ from 400 to 800nm
- δ from 10 to 400nm

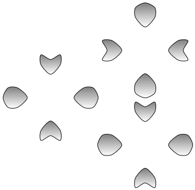
Cosmic dust
- λ from 400 to 700nm
- δ from 1 to 100nm

Photonic crystal
- λ from 400 to 800nm
- δ from 10 to 200nm

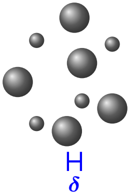
δ size of particle, λ wavelength

Electromagnetic scattering problem by small obstacles

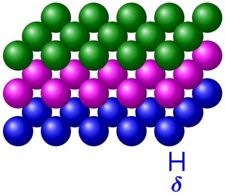
Arbitrary structure



Spherical scatterers

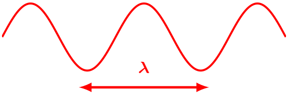


Periodic structure



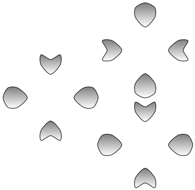
Electromagnetic source

- Time-harmonic $\propto \exp(-\mathrm{i}\omega t)$
- Wavelength $\lambda = \frac{2\pi c}{\omega}$
- Homogeneous and isotropic

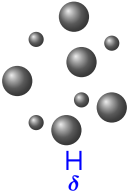


Electromagnetic scattering problem by small obstacles

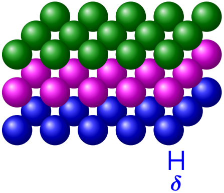
Arbitrary structure



Spherical scatterers

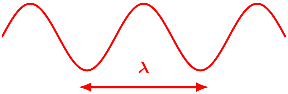


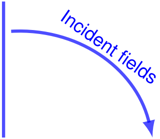
Periodic structure



Electromagnetic source

- Time-harmonic $\propto \exp(-\mathrm{i}\omega t)$
- Wavelength $\lambda = \frac{2\pi c}{\omega}$
- Homogeneous and isotropic





Reduced Maxwell equations

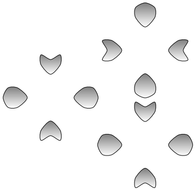
$\kappa = \frac{\omega}{c}$ constant

$$\textbf{curl}\ \mathbf{E}^{\mathrm{i}} - \mathrm{i}\kappa\, \mathbf{H}^{\mathrm{i}} = 0 \quad \text{in } \mathbb{R}^3$$

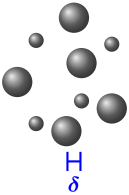
$$\textbf{curl}\ \mathbf{H}^{\mathrm{i}} + \mathrm{i}\kappa\, \mathbf{E}^{\mathrm{i}} = \mathbf{J} \quad \text{in } \mathbb{R}^3$$

Electromagnetic scattering problem by small obstacles

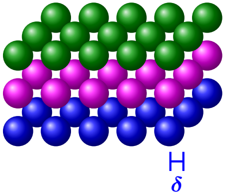
Arbitrary structure



Spherical scatterers

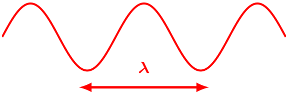


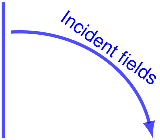
Periodic structure



Electromagnetic source

- Time-harmonic $\propto \exp(-\mathrm{i}\omega t)$
- Wavelength $\lambda = \frac{2\pi c}{\omega}$
- Homogeneous and isotropic





Asymptotic asumption

$\delta \ll \lambda$

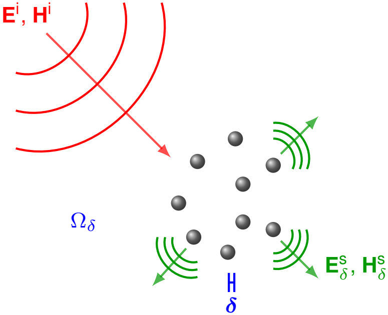
Reduced Maxwell equations

$\kappa = \frac{\omega}{c}$ constant

$$\mathbf{curl}\ \mathbf{E}^{\mathrm{i}} - \mathrm{i}\kappa\, \mathbf{H}^{\mathrm{i}} = 0 \quad \text{in } \mathbb{R}^3$$

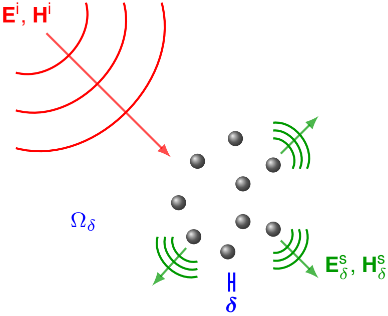
$$\mathbf{curl}\ \mathbf{H}^{\mathrm{i}} + \mathrm{i}\kappa\, \mathbf{E}^{\mathrm{i}} = \mathbf{J} \quad \text{in } \mathbb{R}^3$$

Model problem



- Obstacles $\mathcal{B}_\delta^k = \mathcal{B}(\boldsymbol{c}_k, \boldsymbol{\delta})$
- Exterior domain $\Omega_\delta = \mathbb{R}^3 \setminus \bigcup \overline{\mathcal{B}_\delta^k}$
- Boundary $\Gamma_\delta = \bigcup \Gamma_\delta^k$

Model problem



- Obstacles $\mathcal{B}^k_\delta = \mathcal{B}(\boldsymbol{c}_k, \boldsymbol{\delta})$
- Exterior domain $\Omega_\delta = \mathbb{R}^3 \setminus \bigcup \overline{\mathcal{B}^k_\delta}$
- Boundary $\Gamma_\delta = \bigcup \Gamma^k_\delta$

Total electromagnetic fields

$$\mathbf{E}_\delta = \mathbf{E}^{\mathsf{i}} + \mathbf{E}^{\mathsf{s}}_\delta \qquad \mathbf{H}_\delta = \mathbf{H}^{\mathsf{i}} + \mathbf{H}^{\mathsf{s}}_\delta$$

Time-harmonic Maxwell equations

$$\begin{cases} \mathbf{curl}\, \mathbf{E}^{\mathsf{s}}_\delta - \mathrm{i}\boldsymbol{\kappa}\, \mathbf{H}^{\mathsf{s}}_\delta = 0 & \text{in } \Omega_\delta \\ \mathbf{curl}\, \mathbf{H}^{\mathsf{s}}_\delta + \mathrm{i}\boldsymbol{\kappa}\, \mathbf{E}^{\mathsf{s}}_\delta = 0 & \text{in } \Omega_\delta \end{cases}$$

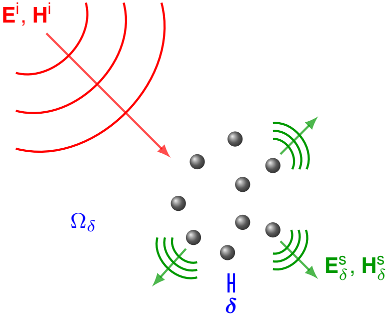
Silver-Müller condition

$$r\left(\mathbf{H}_\delta \times \frac{\mathbf{x}}{r} - \mathbf{E}_\delta\right) \xrightarrow[r\to\infty]{} 0 \quad \text{unif. in } \frac{\mathbf{x}}{r}$$

Perfect conductor condition

$$\mathbf{n} \times \mathbf{E}^{\mathsf{s}}_\delta = -\mathbf{n} \times \mathbf{E}^{\mathsf{i}} \qquad \text{on } \Gamma_\delta$$

Model problem



- Obstacles $\mathcal{B}^k_\delta = \mathcal{B}(\boldsymbol{c}_k, \boldsymbol{\delta})$
- Exterior domain $\Omega_\delta = \mathbb{R}^3 \setminus \bigcup \overline{\mathcal{B}^k_\delta}$
- Boundary $\Gamma_\delta = \bigcup \Gamma^k_\delta$

Total electromagnetic fields

$$\mathbf{E}_\delta = \mathbf{E}^{\mathrm{i}} + \mathbf{E}^{\mathrm{s}}_\delta \qquad \mathbf{H}_\delta = \mathbf{H}^{\mathrm{i}} + \mathbf{H}^{\mathrm{s}}_\delta$$

Time-harmonic Maxwell equations

$$\begin{cases} \mathbf{curl}\, \mathbf{E}^{\mathrm{s}}_\delta - \mathrm{i}\boldsymbol{\kappa}\, \mathbf{H}^{\mathrm{s}}_\delta = 0 & \text{in } \Omega_\delta \\ \mathbf{curl}\, \mathbf{H}^{\mathrm{s}}_\delta + \mathrm{i}\boldsymbol{\kappa}\, \mathbf{E}^{\mathrm{s}}_\delta = 0 & \text{in } \Omega_\delta \end{cases}$$

Silver-Müller condition

$$r\left(\mathbf{H}_\delta \times \frac{\mathbf{x}}{r} - \mathbf{E}_\delta\right) \underset{r\to\infty}{\longrightarrow} 0 \quad \text{unif. in } \frac{\mathbf{x}}{r}$$

Perfect conductor condition

$$\mathbf{n} \times \mathbf{E}^{\mathrm{s}}_\delta = -\mathbf{n} \times \mathbf{E}^{\mathrm{i}} \qquad \text{on } \Gamma_\delta$$

Well-posedness

For $\mathbf{E}^{\mathrm{i}}, \mathbf{H}^{\mathrm{i}} \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \Omega_\delta)$ there is a unique solution $\mathbf{E}^{\mathrm{s}}_\delta, \mathbf{H}^{\mathrm{s}}_\delta \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \Omega_\delta)$

Numerical strategies

Mesh-dependent methods

- Finite differences
- Finite element method
- Discontinuous Galerkin
- Boundary element method

} Too expensive

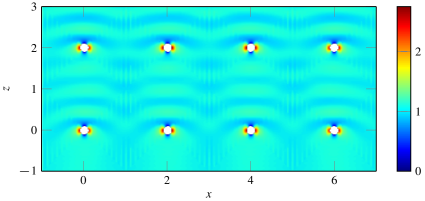
Numerical strategies

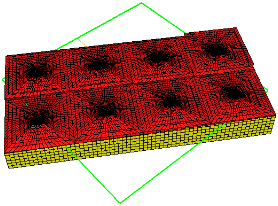
Mesh-dependent methods

- Finite differences
- Finite element method
- Discontinuous Galerkin
- Boundary element method

Montjoie on Plafrim

- Order 2
- $2\,436\,832$ dof
- Memory ≈ 40 GB
- Time ≈ 2h on 4 cores





Numerical strategies

Mesh-dependent methods

- Finite differences
- Finite element method
- Discontinuous Galerkin
- Boundary element method

Mesh-less methods

- Asymptotic methods
- Spectral-based algorithms

Numerical strategies

Mesh-dependent methods

- Finite differences
- Finite element method
- Discontinuous Galerkin
- Boundary element method

Mesh-less methods

- Asymptotic methods
- Spectral-based algorithms

Asymptotic expansions

- Single scattering
- Restricted to small obstacles
- Low computational cost

Foldy-Lax model

- Multiple scattering
- Restricted to small obstacles
- Low computational cost

Numerical strategies

Mesh-dependent methods

- Finite differences
- Finite element method
- Discontinuous Galerkin
- Boundary element method

Mesh-less methods

- Asymptotic methods
- Spectral-based algorithms

Asymptotic expansions

- Single scattering
- Restricted to small obstacles
- Low computational cost

Foldy-Lax model

- Multiple scattering
- Restricted to small obstacles
- Low computational cost

Spectral method

- Single and multiple scattering
- All sizes of obstacles
- Analytical for spheres

(Non-exhaustive) List of references

- Historic references
  Rayleigh (1884), Foldy (1945), Lax (1951)
- Small defect theory
  Il'In (1992), Maz'ya *et al.* (2000)

- Acoustic obstacle
  Ammari and Kang (2003), Ramm (2005), Claeys (2008)
- Time-dependent domain
  Mattesi (2014), Korikov (2015), Marmorat (2015)
- Electromagnetic obstacle
  Vogelius and Volkov (2000), Ammari *et al.* (2001), Korikov and Plamenevskii (2017)

- Foldy theory
  Martin (2004), Cassier and Hazard (2013), Bendali *et al.* (2014), Challa *et al.* (2014), Bouzekri and Sini (2019)
- High-order spectral algorithms
  Xu (1995), Ganesh and Hawkins (2009), Thierry (2011), Ammari *et al.* (2013), Barucq *et al.* (2017), Egel *et al.* (2017)
- Inverse problem
  Volkov (2001), Ammari and Kang (2004), Challa and Sini (2012)

(Non-exhaustive) List of references

- Historic references

  Rayleigh (1884), Foldy (1945), Lax (1951)
- Small defect theory

  Il'In (1992), Maz'ya *et al.* (2000)

- Acoustic obstacle

  Ammari and Kang (2003), Ramm (2005), Claeys (2008)
- Time-dependent domain

  Mattesi (2014), Korikov (2015), Marmorat (2015)

- Electromagnetic obstacle

  Vogelius and Volkov (2000), Ammari *et al.* (2001), Korikov and Plamenevskii (2017)

- Foldy theory

  Martin (2004), Cassier and Hazard (2013), Bendali *et al.* (2014), Challa *et al.* (2014), Bouzekri and Sini (2019)
- High-order spectral algorithms

  Xu (1995), Ganesh and Hawkins (2009), Thierry (2011), Ammari *et al.* (2013), Barucq *et al.* (2017), Egel *et al.* (2017)

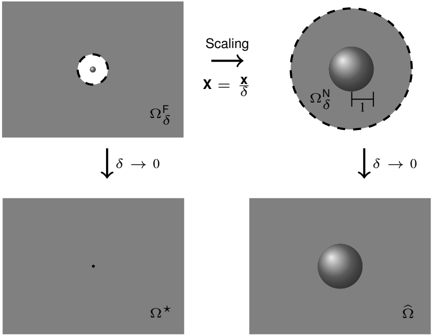
- Inverse problem

  Volkov (2001), Ammari-Kang (2004), Challa-Sini (2012)

Outline

Approximate solution to single scattering by a sphere

- Asymptotic method: the matched asymptotic expansions



- Domain decomposition
- Local approximations
- Matching procedure

Approximate solution to single scattering by a sphere

- Far from the obstacle, the solution is approximated by

$$\mathbf{E}^{\mathsf{s}}_{\delta} \approx \delta^3\, \widetilde{\mathbf{E}}_3 + \delta^5\, \widetilde{\mathbf{E}}_5 + \dots \qquad \mathbf{H}^{\mathsf{s}}_{\delta} \approx \delta^3\, \widetilde{\mathbf{H}}_3 + \delta^5\, \widetilde{\mathbf{H}}_5 + \dots$$

Approximate solution to single scattering by a sphere

- Far from the obstacle, the solution is approximated by

$$\mathbf{E}^{\mathsf{s}}_{\delta} \approx \delta^3\, \widetilde{\mathbf{E}}_3 + \delta^5\, \widetilde{\mathbf{E}}_5 + \dots \qquad \mathbf{H}^{\mathsf{s}}_{\delta} \approx \delta^3\, \widetilde{\mathbf{H}}_3 + \delta^5\, \widetilde{\mathbf{H}}_5 + \dots$$

$$\widetilde{\mathbf{E}}_3(x) = -\kappa^3\left(\widetilde{h}_1^{(1)}(\kappa r)\gamma_{\mathsf{t}}[\mathbf{d}_{\mathsf{e}}] + 2\frac{h_1^{(1)}(\kappa r)}{\mathrm{i}\kappa r}\gamma_{\mathsf{n}}[\mathbf{d}_{\mathsf{e}}]\frac{x}{r} - h_1^{(1)}(\kappa r)\gamma_{\times}[\mathbf{d}_{\mathsf{h}}]\right)$$

$$\widetilde{\mathbf{H}}_3(x) = -\kappa^3\left(\widetilde{h}_1^{(1)}(\kappa r)\gamma_{\mathsf{t}}[\mathbf{d}_{\mathsf{h}}] + 2\frac{h_1^{(1)}(\kappa r)}{\mathrm{i}\kappa r}\gamma_{\mathsf{n}}[\mathbf{d}_{\mathsf{h}}]\frac{x}{r} + h_1^{(1)}(\kappa r)\gamma_{\times}[\mathbf{d}_{\mathsf{e}}]\right)$$

$$\widetilde{\mathbf{E}}_5(x) = \dots + \frac{\kappa^4}{4}\left(\widetilde{h}_2^{(1)}(\kappa r)\gamma_{\mathsf{t}}[\mathbf{Q}_{\mathsf{e}}x] + 3\frac{h_2^{(1)}(\kappa r)}{\mathrm{i}\kappa r}\gamma_{\mathsf{n}}[\mathbf{Q}_{\mathsf{e}}x]\frac{x}{r} + h_2^{(1)}(\kappa r)\gamma_{\times}[\mathbf{Q}_{\mathsf{h}}x]\right)$$

$$\widetilde{\mathbf{H}}_5(x) = \dots + \frac{\kappa^4}{4}\left(\widetilde{h}_2^{(1)}(\kappa r)\gamma_{\mathsf{t}}[\mathbf{Q}_{\mathsf{h}}x] + 3\frac{h_2^{(1)}(\kappa r)}{\mathrm{i}\kappa r}\gamma_{\mathsf{n}}[\mathbf{Q}_{\mathsf{h}}x]\frac{x}{r} - h_2^{(1)}(\kappa r)\gamma_{\times}[\mathbf{Q}_{\mathsf{e}}x]\right)$$

where

$$\mathbf{d}_{\mathsf{e}} = \mathbf{E}^{\mathsf{i}}(0) \qquad \mathbf{d}_{\mathsf{h}} = \mathbf{H}^{\mathsf{i}}(0) \qquad \mathbf{Q}_{\mathsf{e}} = -\frac{2}{3}\mathbf{J}^{\mathsf{sym}}_{\mathbf{E}^{\mathsf{i}}}(0) \qquad \mathbf{Q}_{\mathsf{h}} = \frac{4}{3}\mathbf{J}^{\mathsf{sym}}_{\mathbf{H}^{\mathsf{i}}}(0)$$

Approximate solution to single scattering by a sphere

- Far from the obstacle, the solution is approximated by

$$\mathbf{E}^{\mathsf{s}}_{\delta} \approx \delta^3\, \widetilde{\mathbf{E}}_3 + \delta^5\, \widetilde{\mathbf{E}}_5 + \dots \qquad \mathbf{H}^{\mathsf{s}}_{\delta} \approx \delta^3\, \widetilde{\mathbf{H}}_3 + \delta^5\, \widetilde{\mathbf{H}}_5 + \dots$$

$$\widetilde{\mathbf{E}}_3(x) = -\kappa^3 \Big(\underbrace{\widetilde{h}_1^{(1)}(\kappa r)\gamma_{\mathsf{t}}[\mathbf{d}_{\mathsf{e}}] + 2\frac{h_1^{(1)}(\kappa r)}{\mathrm{i}\kappa r}\gamma_{\mathsf{n}}[\mathbf{d}_{\mathsf{e}}]\frac{x}{r}}_{\text{electric dipole}} \underbrace{- h_1^{(1)}(\kappa r)\gamma_{\times}[\mathbf{d}_{\mathsf{h}}]}_{\text{magnetic dipole}} \Big)$$

$$\widetilde{\mathbf{H}}_3(x) = -\kappa^3 \Big(\widetilde{h}_1^{(1)}(\kappa r)\gamma_{\mathsf{t}}[\mathbf{d}_{\mathsf{h}}] + 2\frac{h_1^{(1)}(\kappa r)}{\mathrm{i}\kappa r}\gamma_{\mathsf{n}}[\mathbf{d}_{\mathsf{h}}]\frac{x}{r} + h_1^{(1)}(\kappa r)\gamma_{\times}[\mathbf{d}_{\mathsf{e}}] \Big)$$

$$\widetilde{\mathbf{E}}_5(x) = \text{dipole} + \frac{\kappa^4}{4} \Big(\underbrace{\widetilde{h}_2^{(1)}(\kappa r)\gamma_{\mathsf{t}}[\mathbf{Q}_{\mathsf{e}}x] + 3\frac{h_2^{(1)}(\kappa r)}{\mathrm{i}\kappa r}\gamma_{\mathsf{n}}[\mathbf{Q}_{\mathsf{e}}x]\frac{x}{r}}_{\text{electric quadrupole}} \underbrace{+ h_2^{(1)}(\kappa r)\gamma_{\times}[\mathbf{Q}_{\mathsf{h}}x]}_{\text{magnetic quadrupole}} \Big)$$

$$\widetilde{\mathbf{H}}_5(x) = \text{dipole} + \frac{\kappa^4}{4} \Big(\widetilde{h}_2^{(1)}(\kappa r)\gamma_{\mathsf{t}}[\mathbf{Q}_{\mathsf{h}}x] + 3\frac{h_2^{(1)}(\kappa r)}{\mathrm{i}\kappa r}\gamma_{\mathsf{n}}[\mathbf{Q}_{\mathsf{h}}x]\frac{x}{r} - h_2^{(1)}(\kappa r)\gamma_{\times}[\mathbf{Q}_{\mathsf{e}}x] \Big)$$

Approximate solution to single scattering by a sphere

- Far from the obstacle, the solution is approximated by

$$\mathbf{E}^{s}_{\delta} \approx \delta^3\, \widetilde{\mathbf{E}}_3 + \delta^5\, \widetilde{\mathbf{E}}_5 + \dots \qquad \mathbf{H}^{s}_{\delta} \approx \delta^3\, \widetilde{\mathbf{H}}_3 + \delta^5\, \widetilde{\mathbf{H}}_5 + \dots$$

$$\widetilde{\mathbf{E}}_3(x) = \boldsymbol{\mathcal{E}}^{\text{dip}}_{\text{elec}}[\mathbf{d}_{\text{e}}](x) + \boldsymbol{\mathcal{E}}^{\text{dip}}_{\text{mag}}[\mathbf{d}_{\text{h}}](x)$$

$$\widetilde{\mathbf{H}}_3(x) = \boldsymbol{\mathcal{H}}^{\text{dip}}_{\text{elec}}[\mathbf{d}_{\text{e}}](x) + \boldsymbol{\mathcal{H}}^{\text{dip}}_{\text{mag}}[\mathbf{d}_{\text{h}}](x)$$

$$\widetilde{\mathbf{E}}_5(x) = \frac{3\kappa^2}{10}\boldsymbol{\mathcal{E}}^{\text{dip}}_{\text{elec}}[\mathbf{d}_{\text{e}}](x) - \frac{3\kappa^2}{5}\boldsymbol{\mathcal{E}}^{\text{dip}}_{\text{mag}}[\mathbf{d}_{\text{h}}](x) + \boldsymbol{\mathcal{E}}^{\text{quad}}_{\text{elec}}[\mathbf{Q}_{\text{e}}](x) + \boldsymbol{\mathcal{E}}^{\text{quad}}_{\text{mag}}[\mathbf{Q}_{\text{h}}](x)$$

$$\widetilde{\mathbf{H}}_5(x) = \frac{3\kappa^2}{10}\boldsymbol{\mathcal{H}}^{\text{dip}}_{\text{elec}}[\mathbf{d}_{\text{e}}](x) - \frac{3\kappa^2}{5}\boldsymbol{\mathcal{H}}^{\text{dip}}_{\text{mag}}[\mathbf{d}_{\text{h}}](x) + \boldsymbol{\mathcal{H}}^{\text{quad}}_{\text{elec}}[\mathbf{Q}_{\text{e}}](x) + \boldsymbol{\mathcal{H}}^{\text{quad}}_{\text{mag}}[\mathbf{Q}_{\text{h}}](x)$$

Approximate solution to single scattering by a sphere

- Far from the obstacle, the solution is approximated by

$$\mathbf{E}^{\mathsf{s}}_{\delta} \approx \delta^3\, \widetilde{\mathbf{E}}_3 + \delta^5\, \widetilde{\mathbf{E}}_5 + \dots \qquad \mathbf{H}^{\mathsf{s}}_{\delta} \approx \delta^3\, \widetilde{\mathbf{H}}_3 + \delta^5\, \widetilde{\mathbf{H}}_5 + \dots$$

- By using multipole expansion

$$\mathbf{E}^{\mathsf{s}}_{\delta} \approx \alpha_{\mathsf{e}}(\delta)\boldsymbol{\mathcal{E}}^{\mathsf{dip}}_{\mathsf{elec}}[\mathbf{d}_{\mathsf{e}}] + \alpha_{\mathsf{h}}(\delta)\boldsymbol{\mathcal{E}}^{\mathsf{dip}}_{\mathsf{mag}}[\mathbf{d}_{\mathsf{h}}] + \beta_{\mathsf{e}}(\delta)\boldsymbol{\mathcal{E}}^{\mathsf{quad}}_{\mathsf{elec}}[\mathbf{Q}_{\mathsf{e}}] + \beta_{\mathsf{h}}(\delta)\boldsymbol{\mathcal{E}}^{\mathsf{quad}}_{\mathsf{mag}}[\mathbf{Q}_{\mathsf{h}}] + \dots$$

$$\mathbf{H}^{\mathsf{s}}_{\delta} \approx \alpha_{\mathsf{e}}(\delta)\boldsymbol{\mathcal{H}}^{\mathsf{dip}}_{\mathsf{elec}}[\mathbf{d}_{\mathsf{e}}] + \alpha_{\mathsf{h}}(\delta)\boldsymbol{\mathcal{H}}^{\mathsf{dip}}_{\mathsf{mag}}[\mathbf{d}_{\mathsf{h}}] + \beta_{\mathsf{e}}(\delta)\boldsymbol{\mathcal{H}}^{\mathsf{quad}}_{\mathsf{elec}}[\mathbf{Q}_{\mathsf{e}}] + \beta_{\mathsf{h}}(\delta)\boldsymbol{\mathcal{H}}^{\mathsf{quad}}_{\mathsf{mag}}[\mathbf{Q}_{\mathsf{h}}] + \dots$$

Multipole moments

- $\mathbf{d}_{\mathsf{e}}$, $\mathbf{d}_{\mathsf{h}}$: dipole moment
- $\mathbf{Q}_{\mathsf{e}}$, $\mathbf{Q}_{\mathsf{h}}$: quadrupole moment tensor

depend on

- the incident fields
- the center of the obstacle
- the shape of the obstacle

$$\mathbf{d}_{\mathsf{e}} = \mathbf{E}^{\mathsf{i}}(\mathbf{0}) \qquad \mathbf{d}_{\mathsf{h}} = -\frac{1}{2}\mathbf{H}^{\mathsf{i}}(\mathbf{0})$$

$$\mathbf{Q}_{\mathsf{e}} = -\frac{2}{3}\mathbb{J}^{\mathsf{sym}}_{\mathbf{E}^{\mathsf{i}}}(\mathbf{0}) \qquad \mathbf{Q}_{\mathsf{h}} = \frac{4}{3}\mathbb{J}^{\mathsf{sym}}_{\mathbf{H}^{\mathsf{i}}}(\mathbf{0})$$

Approximate solution to single scattering by a sphere

- Far from the obstacle, the solution is approximated by

$$\mathbf{E}^{\mathsf{s}}_{\delta} \approx \delta^3\, \widetilde{\mathbf{E}}_3 + \delta^5\, \widetilde{\mathbf{E}}_5 + \ldots \qquad \mathbf{H}^{\mathsf{s}}_{\delta} \approx \delta^3\, \widetilde{\mathbf{H}}_3 + \delta^5\, \widetilde{\mathbf{H}}_5 + \ldots$$

- By using multipole expansion

$$\mathbf{E}^{\mathsf{s}}_{\delta} \approx \alpha_{\mathsf{e}}(\delta)\boldsymbol{\mathcal{E}}^{\mathsf{dip}}_{\mathsf{elec}}[\mathbf{d}_{\mathsf{e}}] + \alpha_{\mathsf{h}}(\delta)\boldsymbol{\mathcal{E}}^{\mathsf{dip}}_{\mathsf{mag}}[\mathbf{d}_{\mathsf{h}}] + \beta_{\mathsf{e}}(\delta)\boldsymbol{\mathcal{E}}^{\mathsf{quad}}_{\mathsf{elec}}[\mathbf{Q}_{\mathsf{e}}] + \beta_{\mathsf{h}}(\delta)\boldsymbol{\mathcal{E}}^{\mathsf{quad}}_{\mathsf{mag}}[\mathbf{Q}_{\mathsf{h}}] + \ldots$$

$$\mathbf{H}^{\mathsf{s}}_{\delta} \approx \alpha_{\mathsf{e}}(\delta)\boldsymbol{\mathcal{H}}^{\mathsf{dip}}_{\mathsf{elec}}[\mathbf{d}_{\mathsf{e}}] + \alpha_{\mathsf{h}}(\delta)\boldsymbol{\mathcal{H}}^{\mathsf{dip}}_{\mathsf{mag}}[\mathbf{d}_{\mathsf{h}}] + \beta_{\mathsf{e}}(\delta)\boldsymbol{\mathcal{H}}^{\mathsf{quad}}_{\mathsf{elec}}[\mathbf{Q}_{\mathsf{e}}] + \beta_{\mathsf{h}}(\delta)\boldsymbol{\mathcal{H}}^{\mathsf{quad}}_{\mathsf{mag}}[\mathbf{Q}_{\mathsf{h}}] + \ldots$$

Multipole moments

- $\mathbf{d}_{\mathsf{e}}$, $\mathbf{d}_{\mathsf{h}}$: dipole moment
- $\mathbf{Q}_{\mathsf{e}}$, $\mathbf{Q}_{\mathsf{h}}$: quadrupole moment tensor

Order of approximation

- Order 3: $\beta_{\mathsf{e}} = \beta_{\mathsf{h}} = 0$ and $\alpha_{\mathsf{e}} = \alpha_{\mathsf{h}} = \delta^3$
- Order 5: $\beta_{\mathsf{e}} = \beta_{\mathsf{h}} = \delta^5$, $\alpha_{\mathsf{e}} = \delta^3(1 + \frac{3(\kappa\delta)^2}{10})$
  $\alpha_{\mathsf{h}} = \delta^3(1 - \frac{3(\kappa\delta)^2}{5})$

$$\mathbf{d}_{\mathsf{e}} = \mathbf{E}^{\mathsf{i}}(\mathbf{0}) \qquad \mathbf{d}_{\mathsf{h}} = -\frac{1}{2}\mathbf{H}^{\mathsf{i}}(\mathbf{0})$$

$$\mathbf{Q}_{\mathsf{e}} = -\frac{2}{3}\mathbb{J}^{\mathsf{sym}}_{\mathbf{E}^{\mathsf{i}}}(\mathbf{0}) \qquad \mathbf{Q}_{\mathsf{h}} = \frac{4}{3}\mathbb{J}^{\mathsf{sym}}_{\mathbf{H}^{\mathsf{i}}}(\mathbf{0})$$

Approximate solution to single scattering by a sphere

- Far from the obstacle, the solution is approximated by

$$\mathbf{E}^{\mathsf{s}}_{\delta} \approx \delta^3\,\widetilde{\mathbf{E}}_3 + \delta^5\,\widetilde{\mathbf{E}}_5 + \dots \qquad \mathbf{H}^{\mathsf{s}}_{\delta} \approx \delta^3\,\widetilde{\mathbf{H}}_3 + \delta^5\,\widetilde{\mathbf{H}}_5 + \dots$$

- By using multipole expansion

$$\mathbf{E}^{\mathsf{s}}_{\delta} \approx \alpha_{\mathsf{e}}(\delta)\boldsymbol{\mathcal{E}}^{\mathsf{dip}}_{\mathsf{elec}}[\mathbf{d}_{\mathsf{e}}] + \alpha_{\mathsf{h}}(\delta)\boldsymbol{\mathcal{E}}^{\mathsf{dip}}_{\mathsf{mag}}[\mathbf{d}_{\mathsf{h}}] + \beta_{\mathsf{e}}(\delta)\boldsymbol{\mathcal{E}}^{\mathsf{quad}}_{\mathsf{elec}}[\mathbf{Q}_{\mathsf{e}}] + \beta_{\mathsf{h}}(\delta)\boldsymbol{\mathcal{E}}^{\mathsf{quad}}_{\mathsf{mag}}[\mathbf{Q}_{\mathsf{h}}] + \dots$$

$$\mathbf{H}^{\mathsf{s}}_{\delta} \approx \alpha_{\mathsf{e}}(\delta)\boldsymbol{\mathcal{H}}^{\mathsf{dip}}_{\mathsf{elec}}[\mathbf{d}_{\mathsf{e}}] + \alpha_{\mathsf{h}}(\delta)\boldsymbol{\mathcal{H}}^{\mathsf{dip}}_{\mathsf{mag}}[\mathbf{d}_{\mathsf{h}}] + \beta_{\mathsf{e}}(\delta)\boldsymbol{\mathcal{H}}^{\mathsf{quad}}_{\mathsf{elec}}[\mathbf{Q}_{\mathsf{e}}] + \beta_{\mathsf{h}}(\delta)\boldsymbol{\mathcal{H}}^{\mathsf{quad}}_{\mathsf{mag}}[\mathbf{Q}_{\mathsf{h}}] + \dots$$

Multipole moments

- $\mathbf{d}_{\mathsf{e}}$, $\mathbf{d}_{\mathsf{h}}$: dipole moment
- $\mathbf{Q}_{\mathsf{e}}$, $\mathbf{Q}_{\mathsf{h}}$: quadrupole moment tensor

$$\mathbf{d}_{\mathsf{e}} = \mathbf{E}^{\mathsf{i}}(\mathbf{0}) \qquad \mathbf{d}_{\mathsf{h}} = -\frac{1}{2}\mathbf{H}^{\mathsf{i}}(\mathbf{0})$$

$$\mathbf{Q}_{\mathsf{e}} = -\frac{2}{3}\mathbb{J}^{\mathsf{sym}}_{\mathbf{E}^{\mathsf{i}}}(\mathbf{0}) \qquad \mathbf{Q}_{\mathsf{h}} = \frac{4}{3}\mathbb{J}^{\mathsf{sym}}_{\mathbf{H}^{\mathsf{i}}}(\mathbf{0})$$

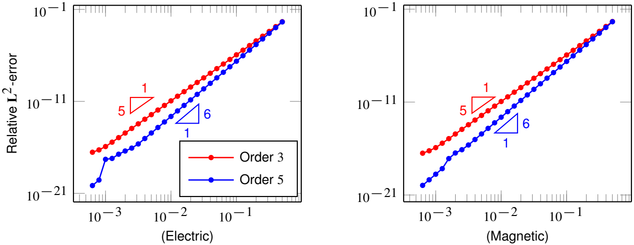
Order of approximation

- Order 3: $\beta_{\mathsf{e}} = \beta_{\mathsf{h}} = 0$ and $\alpha_{\mathsf{e}} = \alpha_{\mathsf{h}} = \delta^3$
- Order 5: $\beta_{\mathsf{e}} = \beta_{\mathsf{h}} = \delta^5$, $\alpha_{\mathsf{e}} = \delta^3(1 + \frac{3(\kappa\delta)^2}{10})$
  $\alpha_{\mathsf{h}} = \delta^3(1 - \frac{3(\kappa\delta)^2}{5})$

intermediate approximations

- Collected dipole
- Modified approximation

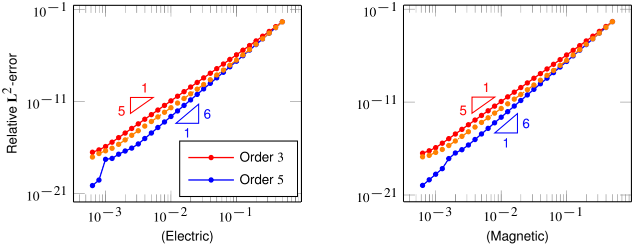
Complements on single scattering



Data

- $\lambda = 5.0$
- Plane wave
- δ varies

Complements on single scattering



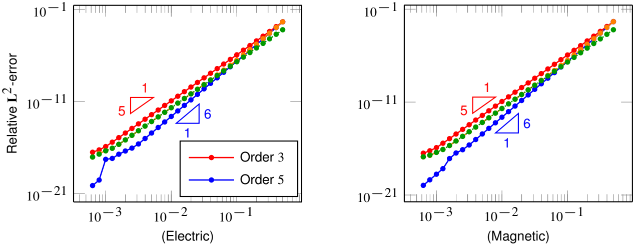
Data

- $\lambda = 5.0$
- Plane wave
- δ varies

$$\mathbf{E}^{\mathsf{s}}_{\delta} \approx \alpha_{\mathsf{e}}(\delta)\boldsymbol{\mathcal{E}}^{\mathsf{elec}}_{\mathsf{dip}}[\mathbf{d}_{\mathsf{e}}] + \alpha_{\mathsf{h}}(\delta)\boldsymbol{\mathcal{E}}^{\mathsf{mag}}_{\mathsf{dip}}[\mathbf{d}_{\mathsf{h}}] \qquad \mathbf{H}^{\mathsf{s}}_{\delta} \approx \alpha_{\mathsf{e}}(\delta)\boldsymbol{\mathcal{H}}^{\mathsf{elec}}_{\mathsf{dip}}[\mathbf{d}_{\mathsf{e}}] + \alpha_{\mathsf{h}}(\delta)\boldsymbol{\mathcal{H}}^{\mathsf{mag}}_{\mathsf{dip}}[\mathbf{d}_{\mathsf{h}}]$$

- Collected dipole: $\beta_{\mathsf{e}} = \beta_{\mathsf{h}} = 0$, $\alpha_{\mathsf{e}} = \delta^3(1 + \frac{3(\kappa\delta)^2}{10})$ and $\alpha_{\mathsf{h}} = \delta^3(1 - \frac{3(\kappa\delta)^2}{5})$

Complements on single scattering



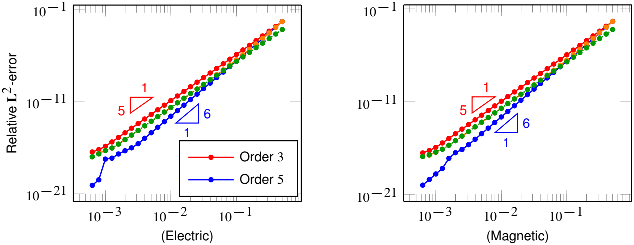
Data

- $\lambda = 5.0$
- Plane wave
- δ varies

$$\mathbf{E}^{\mathsf{s}}_{\delta} \approx \alpha_{\mathsf{e}}(\delta)\boldsymbol{\mathcal{E}}^{\mathsf{elec}}_{\mathsf{dip}}[\mathbf{d}_{\mathsf{e}}] + \alpha_{\mathsf{h}}(\delta)\boldsymbol{\mathcal{E}}^{\mathsf{mag}}_{\mathsf{dip}}[\mathbf{d}_{\mathsf{h}}] \qquad \mathbf{H}^{\mathsf{s}}_{\delta} \approx \alpha_{\mathsf{e}}(\delta)\boldsymbol{\mathcal{H}}^{\mathsf{elec}}_{\mathsf{dip}}[\mathbf{d}_{\mathsf{e}}] + \alpha_{\mathsf{h}}(\delta)\boldsymbol{\mathcal{H}}^{\mathsf{mag}}_{\mathsf{dip}}[\mathbf{d}_{\mathsf{h}}]$$

- Collected dipole: $\beta_{\mathsf{e}} = \beta_{\mathsf{h}} = 0,\ \alpha_{\mathsf{e}} = \delta^3(1 + \frac{3(\kappa\delta)^2}{10})$ and $\alpha_{\mathsf{h}} = \delta^3(1 - \frac{3(\kappa\delta)^2}{5})$
- Modified approximation: $\beta_{\mathsf{e}} = \beta_{\mathsf{h}} = 0,\ \alpha_{\mathsf{e}} = \frac{3\mathrm{i}}{2(\kappa\delta)^3}\frac{j_1(\kappa\delta)}{h_1(\kappa\delta)}$ and $\alpha_{\mathsf{h}} = -\frac{3\mathrm{i}}{(\kappa\delta)^3}\frac{\widetilde{j}_1(\kappa\delta)}{\widetilde{h}_1(\kappa\delta)}$

Complements on single scattering



Data

- $\lambda = 5.0$
- Plane wave
- δ varies

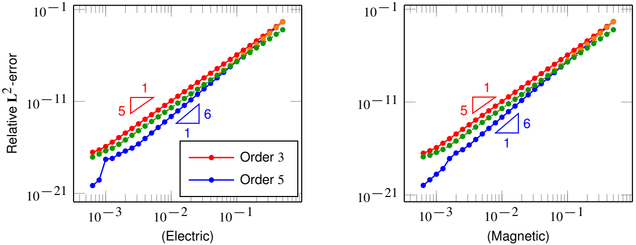
$$\mathbf{E}^{\mathsf{s}}_{\delta} \approx \alpha_{\mathsf{e}}(\delta)\boldsymbol{\mathcal{E}}^{\mathsf{elec}}_{\mathsf{dip}}[\mathbf{d}_{\mathsf{e}}] + \alpha_{\mathsf{h}}(\delta)\boldsymbol{\mathcal{E}}^{\mathsf{mag}}_{\mathsf{dip}}[\mathbf{d}_{\mathsf{h}}] \qquad \mathbf{H}^{\mathsf{s}}_{\delta} \approx \alpha_{\mathsf{e}}(\delta)\boldsymbol{\mathcal{H}}^{\mathsf{elec}}_{\mathsf{dip}}[\mathbf{d}_{\mathsf{e}}] + \alpha_{\mathsf{h}}(\delta)\boldsymbol{\mathcal{H}}^{\mathsf{mag}}_{\mathsf{dip}}[\mathbf{d}_{\mathsf{h}}]$$

- Collected dipole: $\beta_{\mathsf{e}} = \beta_{\mathsf{h}} = 0,\ \alpha_{\mathsf{e}} = \delta^3(1 + \frac{3(\kappa\delta)^2}{10})$ and $\alpha_{\mathsf{h}} = \delta^3(1 - \frac{3(\kappa\delta)^2}{5})$
- Modified approximation: $\beta_{\mathsf{e}} = \beta_{\mathsf{h}} = 0,\ \alpha_{\mathsf{e}} = \frac{3\mathrm{i}}{2(\kappa\delta)^3}\frac{j_1(\kappa\delta)}{h_1(\kappa\delta)}$ and $\alpha_{\mathsf{h}} = -\frac{3\mathrm{i}}{(\kappa\delta)^3}\frac{\widetilde{j}_1(\kappa\delta)}{\widetilde{h}_1(\kappa\delta)}$

Obstacle of arbitrary shape

- Approximation of order 3: $\mathbf{d}_{\mathsf{e}} = \mathbb{M}_{\mathsf{e}}\, \mathbf{E}^{\mathsf{i}}(\boldsymbol{c})$ and $\mathbf{d}_{\mathsf{h}} = -\dfrac{1}{2}\mathbb{M}_{\mathsf{h}}\, \mathbf{H}^{\mathsf{i}}(\boldsymbol{c})$

Complements on single scattering



Data

- $\lambda = 5.0$
- Plane wave
- δ varies

$$\mathbf{E}^{\mathrm{s}}_{\delta} \approx \alpha_{\mathrm{e}}(\delta)\boldsymbol{\mathcal{E}}^{\mathrm{elec}}_{\mathrm{dip}}[\mathbf{d}_{\mathrm{e}}] + \alpha_{\mathrm{h}}(\delta)\boldsymbol{\mathcal{E}}^{\mathrm{mag}}_{\mathrm{dip}}[\mathbf{d}_{\mathrm{h}}] \qquad \mathbf{H}^{\mathrm{s}}_{\delta} \approx \alpha_{\mathrm{e}}(\delta)\boldsymbol{\mathcal{H}}^{\mathrm{elec}}_{\mathrm{dip}}[\mathbf{d}_{\mathrm{e}}] + \alpha_{\mathrm{h}}(\delta)\boldsymbol{\mathcal{H}}^{\mathrm{mag}}_{\mathrm{dip}}[\mathbf{d}_{\mathrm{h}}]$$

- Collected dipole: $\beta_{\mathrm{e}} = \beta_{\mathrm{h}} = 0$, $\alpha_{\mathrm{e}} = \delta^3(1 + \frac{3(\kappa\delta)^2}{10})$ and $\alpha_{\mathrm{h}} = \delta^3(1 - \frac{3(\kappa\delta)^2}{5})$
- Modified approximation: $\beta_{\mathrm{e}} = \beta_{\mathrm{h}} = 0$, $\alpha_{\mathrm{e}} = \frac{3\mathrm{i}}{2(\kappa\delta)^3}\frac{j_1(\kappa\delta)}{h_1(\kappa\delta)}$ and $\alpha_{\mathrm{h}} = -\frac{3\mathrm{i}}{(\kappa\delta)^3}\frac{\widetilde{j}_1(\kappa\delta)}{\widetilde{h}_1(\kappa\delta)}$

Obstacle of arbitrary shape

- Approximation of order 3: $\mathbf{d}_{\mathrm{e}} = \mathbb{M}_{\mathrm{e}}\,\mathbf{E}^{\mathrm{i}}(\boldsymbol{c})$ and $\mathbf{d}_{\mathrm{h}} = -\frac{1}{2}\mathbb{M}_{\mathrm{h}}\,\mathbf{H}^{\mathrm{i}}(\boldsymbol{c})$
- High-order approximation:

$$\widetilde{\mathbf{E}}_4 = \text{dipole} + \text{quadrupole} \qquad \widetilde{\mathbf{E}}_5 = \text{dipole} + \text{quadrupole} + \text{octupole}$$

Approximate solution to multiple scattering by small spheres

- The electromagnetic fields are decomposed by superposition principle

$$\mathbf{E}^{\mathsf{s}}_{\delta}(x) = \sum_{k=1}^{N_{\mathsf{obs}}} \mathbf{E}^{\mathsf{s}}_{\delta,k}(x - \boldsymbol{c}_k) \qquad \mathbf{H}^{\mathsf{s}}_{\delta}(x) = \sum_{k=1}^{N_{\mathsf{obs}}} \mathbf{H}^{\mathsf{s}}_{\delta,k}(x - \boldsymbol{c}_k)$$

- Each obstacle is modeled by a dipolar source around $\boldsymbol{c}_k$

$$\mathbf{E}^{\mathsf{s}}_{\delta,k} \approx \boldsymbol{\mathcal{E}}^{\mathsf{dip}}_{\mathsf{elec}}[\mathbf{d}^{k}_{\delta,\mathsf{e}}] + \boldsymbol{\mathcal{E}}^{\mathsf{dip}}_{\mathsf{mag}}[\mathbf{d}^{k}_{\delta,\mathsf{h}}] \qquad \mathbf{H}^{\mathsf{s}}_{\delta,k} \approx \boldsymbol{\mathcal{H}}^{\mathsf{dip}}_{\mathsf{elec}}[\mathbf{d}^{k}_{\delta,\mathsf{e}}] + \boldsymbol{\mathcal{H}}^{\mathsf{dip}}_{\mathsf{mag}}[\mathbf{d}^{k}_{\delta,\mathsf{h}}]$$

Approximate solution to multiple scattering by small spheres

- The electromagnetic fields are decomposed by superposition principle

$$\mathbf{E}^{\mathsf{s}}_{\delta}(x) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{E}^{\mathsf{s}}_{\delta,k}(x - \boldsymbol{c}_k) \qquad \mathbf{H}^{\mathsf{s}}_{\delta}(x) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{H}^{\mathsf{s}}_{\delta,k}(x - \boldsymbol{c}_k)$$

- Each obstacle is modeled by a dipolar source around $\boldsymbol{c}_k$

$$\mathbf{E}^{\mathsf{s}}_{\delta,k} \approx \boldsymbol{\mathcal{E}}^{\text{dip}}_{\text{elec}}[\mathbf{d}^k_{\delta,\mathsf{e}}] + \boldsymbol{\mathcal{E}}^{\text{dip}}_{\text{mag}}[\mathbf{d}^k_{\delta,\mathsf{h}}] \qquad \mathbf{H}^{\mathsf{s}}_{\delta,k} \approx \boldsymbol{\mathcal{H}}^{\text{dip}}_{\text{elec}}[\mathbf{d}^k_{\delta,\mathsf{e}}] + \boldsymbol{\mathcal{H}}^{\text{dip}}_{\text{mag}}[\mathbf{d}^k_{\delta,\mathsf{h}}]$$

Born approximation

$$\mathbf{d}^k_{\delta,\mathsf{e}} = \alpha_{\mathsf{e}}(\delta)\,\mathbf{E}^{\mathsf{i}}(\boldsymbol{c}_k)$$

$$\mathbf{d}^k_{\delta,\mathsf{h}} = -\frac{1}{2}\alpha_{\mathsf{h}}(\delta)\,\mathbf{H}^{\mathsf{i}}(\boldsymbol{c}_k)$$

✓ Explicit formulation

✗ Interactions are not taken into account

Approximate solution to multiple scattering by small spheres

- The electromagnetic fields are decomposed by superposition principle

$$\mathbf{E}^{\mathsf{s}}_{\delta}(x) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{E}^{\mathsf{s}}_{\delta,k}(x - \boldsymbol{c}_k) \qquad \mathbf{H}^{\mathsf{s}}_{\delta}(x) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{H}^{\mathsf{s}}_{\delta,k}(x - \boldsymbol{c}_k)$$

- Each obstacle is modeled by a dipolar source around $\boldsymbol{c}_k$

$$\mathbf{E}^{\mathsf{s}}_{\delta,k} \approx \boldsymbol{\mathcal{E}}^{\text{dip}}_{\text{elec}}[\mathbf{d}^k_{\delta,\mathsf{e}}] + \boldsymbol{\mathcal{E}}^{\text{dip}}_{\text{mag}}[\mathbf{d}^k_{\delta,\mathsf{h}}] \qquad \mathbf{H}^{\mathsf{s}}_{\delta,k} \approx \boldsymbol{\mathcal{H}}^{\text{dip}}_{\text{elec}}[\mathbf{d}^k_{\delta,\mathsf{e}}] + \boldsymbol{\mathcal{H}}^{\text{dip}}_{\text{mag}}[\mathbf{d}^k_{\delta,\mathsf{h}}]$$

Born approximation

$$\mathbf{d}^k_{\delta,\mathsf{e}} = \alpha_{\mathsf{e}}(\delta)\, \mathbf{E}^{\mathsf{i}}(\boldsymbol{c}_k)$$

$$\mathbf{d}^k_{\delta,\mathsf{h}} = -\frac{1}{2}\alpha_{\mathsf{h}}(\delta)\, \mathbf{H}^{\mathsf{i}}(\boldsymbol{c}_k)$$

- ✓ Explicit formulation
- ✗ Interactions are not taken into account

Foldy-Lax approximation

$$\mathbf{d}^k_{\delta,\mathsf{e}} = \alpha_{\mathsf{e}}(\delta) \left(\mathbf{E}^{\mathsf{i}}(\boldsymbol{c}_k) + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_{\text{obs}}} \mathbf{E}^{\mathsf{s}}_{\delta,\ell}(\boldsymbol{c}_k) \right)$$

$$\mathbf{d}^k_{\delta,\mathsf{h}} = -\frac{1}{2}\alpha_{\mathsf{h}}(\delta) \left(\mathbf{H}^{\mathsf{i}}(\boldsymbol{c}_k) + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_{\text{obs}}} \mathbf{H}^{\mathsf{s}}_{\delta,\ell}(\boldsymbol{c}_k) \right)$$

- ✗ Implicit formulation
- ✓ Interactions are taken into account

Approximate solution to multiple scattering by small spheres

- The electromagnetic fields are decomposed by superposition principle

$$\mathbf{E}^{\mathsf{s}}_{\delta}(x) = \sum_{k=1}^{N_{\mathsf{obs}}} \mathbf{E}^{\mathsf{s}}_{\delta,k}(x - \boldsymbol{c}_k) \qquad \mathbf{H}^{\mathsf{s}}_{\delta}(x) = \sum_{k=1}^{N_{\mathsf{obs}}} \mathbf{H}^{\mathsf{s}}_{\delta,k}(x - \boldsymbol{c}_k)$$

- Each obstacle is modeled by a dipolar source around $\boldsymbol{c}_k$

$$\mathbf{E}^{\mathsf{s}}_{\delta,k} \approx \boldsymbol{\mathcal{E}}^{\mathsf{dip}}_{\mathsf{elec}}[\mathbf{d}^{k}_{\delta,\mathsf{e}}] + \boldsymbol{\mathcal{E}}^{\mathsf{dip}}_{\mathsf{mag}}[\mathbf{d}^{k}_{\delta,\mathsf{h}}] \qquad \mathbf{H}^{\mathsf{s}}_{\delta,k} \approx \boldsymbol{\mathcal{H}}^{\mathsf{dip}}_{\mathsf{elec}}[\mathbf{d}^{k}_{\delta,\mathsf{e}}] + \boldsymbol{\mathcal{H}}^{\mathsf{dip}}_{\mathsf{mag}}[\mathbf{d}^{k}_{\delta,\mathsf{h}}]$$

Vectorial formulation

$$\mathbf{d}_\delta = (\mathbf{d}^1_{\delta,\mathsf{e}}, \dots \mathbf{d}^{N_{\mathsf{obs}}}_{\delta,\mathsf{e}}, \mathbf{d}^1_{\delta,\mathsf{h}}, \dots \mathbf{d}^{N_{\mathsf{obs}}}_{\delta,\mathsf{h}}) \in \mathbb{C}^{6N_{\mathsf{obs}}}$$

$$(\mathbb{I} - \alpha(\delta)\mathbb{A})\,\mathbf{d}_\delta = \alpha(\delta)\mathbf{f}$$

where $\mathbb{A}$ is the "interaction" matrix

- Order 5: $\alpha(\delta) = \delta^3$
- Collected and Modified:

$$\alpha(\delta) = \begin{pmatrix} \alpha_{\mathsf{e}}(\delta)\mathbb{I} & \mathbf{0} \\ \mathbf{0} & \alpha_{\mathsf{h}}(\delta)\mathbb{I} \end{pmatrix}$$

Foldy-Lax approximation

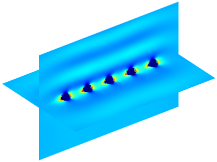
$$\mathbf{d}^{k}_{\delta,\mathsf{e}} = \alpha_{\mathsf{e}}(\delta)\left(\mathbf{E}^{\mathsf{i}}(\boldsymbol{c}_k) + \sum_{\substack{\ell=1\\ \ell\neq k}}^{N_{\mathsf{obs}}} \mathbf{E}^{\mathsf{s}}_{\delta,\ell}(\boldsymbol{c}_k)\right)$$

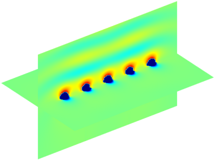
$$\mathbf{d}^{k}_{\delta,\mathsf{h}} = -\frac{1}{2}\alpha_{\mathsf{h}}(\delta)\left(\mathbf{H}^{\mathsf{i}}(\boldsymbol{c}_k) + \sum_{\substack{\ell=1\\ \ell\neq k}}^{N_{\mathsf{obs}}} \mathbf{H}^{\mathsf{s}}_{\delta,\ell}(\boldsymbol{c}_k)\right)$$

✗ Implicit formulation

✓ Interactions are taken into account

Validation of asymptotic models

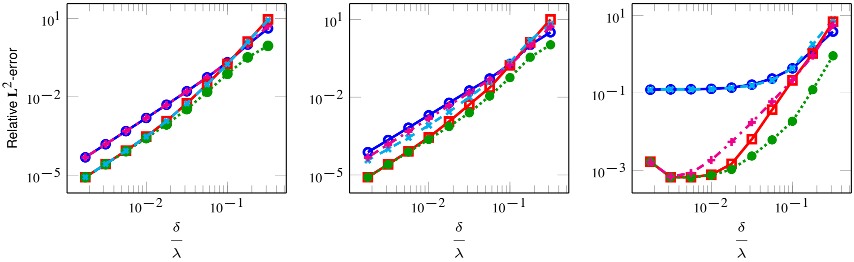




Data

- $\lambda = 1.0$
- Plane wave
- δ varies

- Born
- Foldy
- Collected Born
- Collected Foldy
- Modified Foldy



(a) Distance $\propto \lambda$ (b) Distance $\propto \sqrt{\lambda\delta}$ (c) Distance $\propto \delta$

Outline

Principle of the spectral method

- The electromagnetic fields are represented by the Stratton-Chu formula for $x \in \Omega_\delta$

$$\mathbf{E}^{\mathsf{s}}_\delta(x) = \sum_{k=1}^{N_{\text{obs}}} \textbf{curl} \int_{\Gamma^k_\delta} \Phi(x - \boldsymbol{c}_k, y - \boldsymbol{c}_k)\mathbf{p}_k(y)\, ds_y$$

where $\Phi(x,y) = \frac{\exp(i\kappa|x-y|)}{4\pi|x-y|}$

Principle of the spectral method

- The electromagnetic fields are represented by the Stratton-Chu formula for $x \in \Omega_\delta$

$$\mathbf{E}^{\mathsf{s}}_\delta(x) = \sum_{k=1}^{N_{\mathsf{obs}}} \mathbf{curl} \int_{\Gamma^k_\delta} \Phi(x - \boldsymbol{c}_k, y - \boldsymbol{c}_k)\mathbf{p}_k(y)\, ds_y$$

- Each $\mathbf{p}_\ell \in \mathbf{H}_t^{-\frac{1}{2}}(\mathrm{div}_{\Gamma^\ell_\delta}, \Gamma^\ell_\delta)$ satisfies the boundary integral equation

$$\sum_{\ell=1}^{N_{\mathsf{obs}}} \left\langle \mathcal{M}^{k\ell}_\Gamma \mathbf{p}_\ell, \mathbf{v}_k \right\rangle_{\Gamma^k_\delta} = -\left\langle \mathbf{n} \times \mathbf{E}^{\mathsf{inc}}, \mathbf{v}_k \right\rangle_{\Gamma^k_\delta} \qquad \forall\, \mathbf{v}_k \in \mathbf{H}_{\mathsf{t}}^{-\frac{1}{2}}(\mathrm{curl}_{\Gamma^k_\delta}, \Gamma^k_\delta)$$

where $\mathcal{M}^{k\ell}_\Gamma : \mathbf{H}_t^{-\frac{1}{2}}(\mathrm{div}_{\Gamma^\ell_\delta}, \Gamma^\ell_\delta) \longrightarrow \mathbf{H}_t^{-\frac{1}{2}}(\mathrm{div}_{\Gamma^k_\delta}, \Gamma^k_\delta)$ is the extension of

$$\mathcal{M}^{k\ell}_\Gamma \boldsymbol{\lambda}(x_\Gamma) = \mathbf{n}(x_\Gamma) \times \lim_{x \to x_\Gamma} \mathbf{curl} \int_{\Gamma^\ell_\delta} \Phi(x - \boldsymbol{c}_\ell, y - \boldsymbol{c}_\ell)\boldsymbol{\lambda}(y)\, ds_y \qquad \boldsymbol{\lambda} \in \mathscr{C}^\infty(\Gamma^\ell_\delta) \qquad x_\Gamma \in \Gamma^k_\delta$$

Principle of the spectral method

- The electromagnetic fields are represented by the Stratton-Chu formula for $x \in \Omega_\delta$

$$\mathbf{E}^{\mathsf{s}}_{\delta}(x) = \sum_{k=1}^{N_{\mathsf{obs}}} \mathbf{curl} \int_{\Gamma^k_\delta} \Phi(x - \boldsymbol{c}_k, y - \boldsymbol{c}_k)\mathbf{p}_k(y)\, ds_y$$

- Each $\mathbf{p}_\ell \in \mathbf{H}_t^{-\frac{1}{2}}(\mathrm{div}_{\Gamma^\ell_\delta}, \Gamma^\ell_\delta)$ satisfies the boundary integral equation

$$\sum_{\ell=1}^{N_{\mathsf{obs}}} \left\langle \mathcal{M}^{k\ell}_{\Gamma} \mathbf{p}_\ell, \mathbf{v}_k \right\rangle_{\Gamma^k_\delta} = -\left\langle \mathbf{n} \times \mathbf{E}^{\mathsf{inc}}, \mathbf{v}_k \right\rangle_{\Gamma^k_\delta} \qquad \forall\, \mathbf{v}_k \in \mathbf{H}_{\mathrm{t}}^{-\frac{1}{2}}(\mathrm{curl}_{\Gamma^k_\delta}, \Gamma^k_\delta)$$

- Galerkin discretization of the BIE on local spectral basis with N_{mod} modes

$$\mathbf{p}_\ell(x) = \sum_{n=1}^{N_{\mathsf{mod}}} \sum_{m=-n}^{n} p^{\ell,\perp}_{n,m}\, a_n(\delta)\, \nabla_{\mathbb{S}^2} Y_{n,m}(\widehat{x}_\ell) + p^{\ell,\times}_{n,m}\, b_n(\delta)\, \mathbf{curl}_{\mathbb{S}^2} Y_{n,m}(\widehat{x}_\ell) \qquad x \in \Gamma^\ell_\delta$$

with $\widehat{x}_\ell = \frac{x - c_\ell}{|x - c_\ell|}$ and $\nabla_{\mathbb{S}^2} Y_{n,m}, \mathbf{curl}_{\mathbb{S}^2} Y_{n,m}$: complex-valued vector spherical harmonics

Vectorial formulation

The variational formulation: Find $(\mathbf{p}_k) \in \mathbf{H}_t^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$ such that

$$\langle \mathcal{M}_\Gamma^{kk} \mathbf{p}_k, \mathbf{v}_k \rangle_{\Gamma_\delta^k} + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_{\text{obs}}} \langle \mathcal{M}_\Gamma^{k\ell} \mathbf{p}_\ell, \mathbf{v}_k \rangle_{\Gamma_\delta^k} = -\langle \mathbf{n} \times \mathbf{E}^{\text{inc}}, \mathbf{v}_k \rangle_{\Gamma_\delta^k} \qquad \forall\, \mathbf{v}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\mathrm{curl}_{\Gamma_\delta^k}, \Gamma_\delta^k)$$

Can be put under vectorial form $\mathbf{p} = \left((p_{n,m}^{1,\perp}), \ldots, (p_{n,m}^{N_{\text{obs}},\perp}), (p_{n,m}^{1,\times}), \ldots, (p_{n,m}^{N_{\text{obs}},\times})\right)^\top \in \mathbb{C}^{\boldsymbol{N}}$

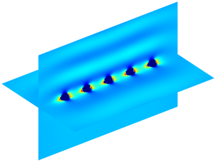
$$\mathbb{M}\,\mathbf{p} = \mathbf{f}$$

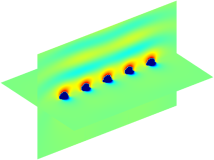
with $\boldsymbol{N} = 2N_{\text{mod}}(N_{\text{mod}}+2)N_{\text{obs}}$

$$\mathbb{M} = \begin{pmatrix} \mathbb{M}_{\perp\perp} & \mathbb{M}_{\perp\times} \\ \mathbb{M}_{\times\perp} & \mathbb{M}_{\times\times} \end{pmatrix} \qquad \text{with} \qquad \mathbb{M}_{\alpha\beta} = \begin{pmatrix} \mathbb{M}_{\alpha\beta}^{11} & \mathbb{M}_{\alpha\beta}^{12} & \cdots & \mathbb{M}_{\alpha\beta}^{1N_{\text{obs}}} \\ \mathbb{M}_{\alpha\beta}^{21} & \mathbb{M}_{\alpha\beta}^{22} & \cdots & \mathbb{M}_{\alpha\beta}^{2N_{\text{obs}}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{M}_{\alpha\beta}^{N_{\text{obs}}1} & \mathbb{M}_{\alpha\beta}^{N_{\text{obs}}2} & \cdots & \mathbb{M}_{\alpha\beta}^{N_{\text{obs}}N_{\text{obs}}} \end{pmatrix}$$

The analytical blocks $\mathbb{M}_{\alpha\beta}^{k\ell}$ depend on δ and $(\boldsymbol{c}_k - \boldsymbol{c}_\ell)$

Validation of spectral models

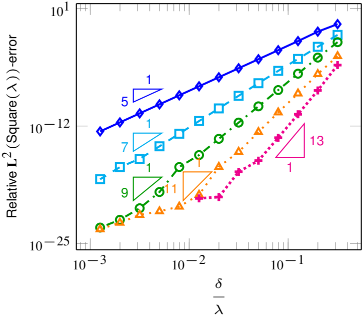




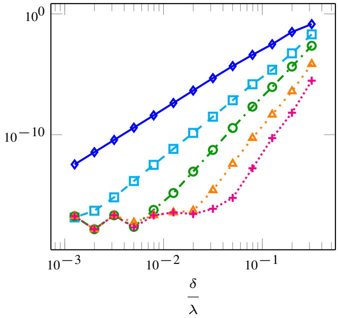
Data

- $\lambda = 1.0$
- Plane wave
- δ varies

Spectral 1
Spectral 2
Spectral 3
Spectral 4
Spectral 5

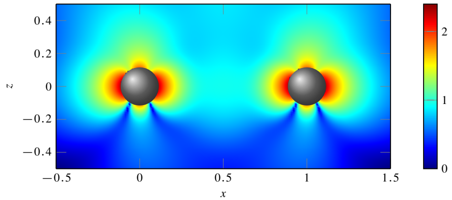


(d) Distance $\propto \lambda$



(e) Distance $\propto \sqrt{\lambda\delta}$

Validation of spectral models



Data

- $\lambda = 1.0$
- Plane wave
- h varies

- Finite Element 1
- Finite Element 2
- Finite Element 3

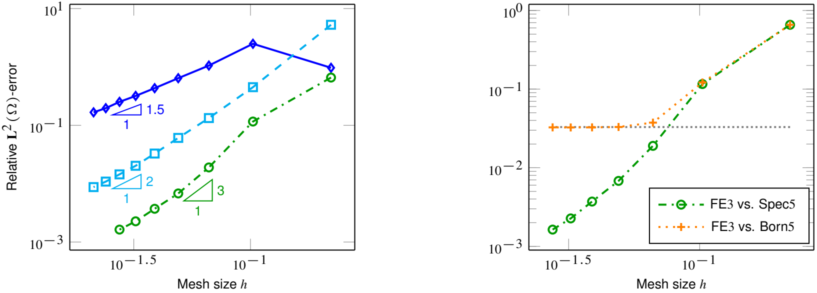
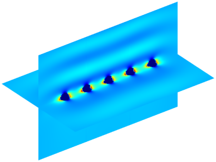
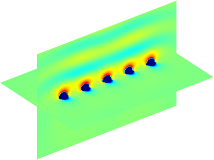


Figure: Spectral solutions vs. Finite element solutions (Montjoie)

Validation of spectral models



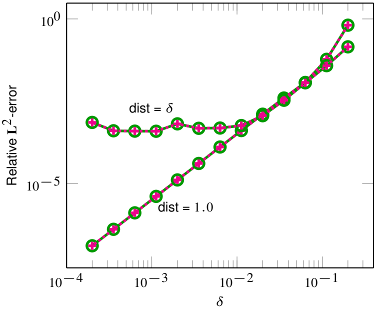


Data

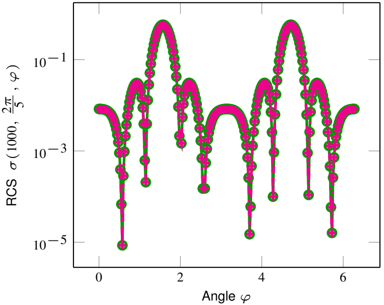
- $\lambda = 1.0$
- Plane wave
- δ varies

Modified Foldy

Spectral 1



(a) Foldy vs Spectral 1



(b) Radar cross section

However . . .

The spectral method has some disadvantages . . .

- Dense matrix inherited from BIEs
- Increasing in number of unknowns as the number of obstacles grows
- Quickly limited with memory resources

- Ill-conditionned system requiring preconditionning
- Smart storage and assembling in specific configurations
- Implementation of an iterative resolution

. . . and that can be applied on the Foldy systems too

However . . .

The spectral method has some disadvantages . . .

- Dense matrix inherited from BIEs
- Increasing in number of unknowns as the number of obstacles grows
- Quickly limited with memory resources

- Ill-conditionned system requiring preconditionning
- Smart storage and assembling in specific configurations
- Implementation of an iterative resolution

. . . and that can be applied on the Foldy systems too

Outline

Iterative solver

- Simple calculations improve condition number associated with the matrix

$$\mathbb{M}\,\mathbf{p} = \mathbf{f}$$

- The matrix is decomposed as

$$\mathbb{M} = \mathbb{I} + \mathbb{A} + \mathbb{B}$$

where $(\mathbb{I} + \mathbb{A})$ invertible contains the *main* interactions

$$A_{ij} = M_{ij} \qquad \text{if } |M_{ij}| \geq \text{tolerance}$$

We solve iteratively

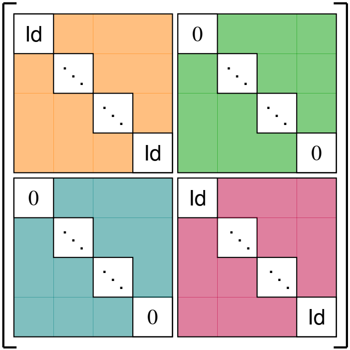
$$\begin{cases} \mathbf{p}^{(0)} = (\mathbb{I} + \mathbb{A})^{-1}\,\mathbf{f} \\ (\mathbb{I} + \mathbb{A})\,\mathbf{p}^{(n+1)} = \mathbf{f} - \mathbb{B}\,\mathbf{p}^{(n)} \end{cases}$$

- $\mathbb{I}$ is the identity matrix
- $\mathbb{A}$ is a sparse matrix
- $\mathbb{B}$ is dense and never assembled

Smart storage and assembling

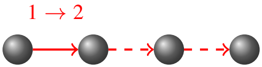
- $\mathbb{M}^{k\ell}_{\alpha\beta}$ depends only on δ and $(\boldsymbol{c}_k - \boldsymbol{c}_\ell)$
- Example: 4 aligned obstacles

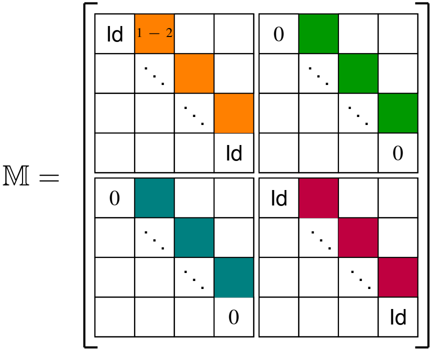
$\mathbb{M} =$



Smart storage and assembling

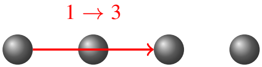
- $\mathbb{M}^{k\ell}_{\alpha\beta}$ depends only on δ and $(\boldsymbol{c}_k - \boldsymbol{c}_\ell)$
- Example: 4 aligned obstacles



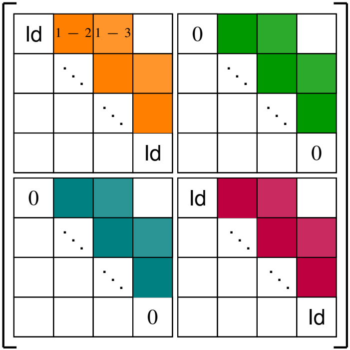


Smart storage and assembling

- $\mathbb{M}^{k\ell}_{\alpha\beta}$ depends only on δ and $(\boldsymbol{c}_k - \boldsymbol{c}_\ell)$
- Example: 4 aligned obstacles



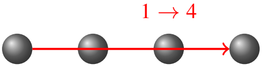
$\mathbb{M} =$

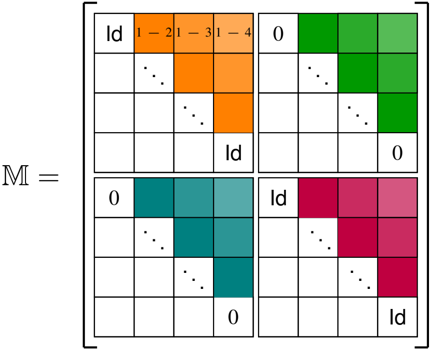


Smart storage and assembling

- $\mathbb{M}^{k\ell}_{\alpha\beta}$ depends only on δ and $(\boldsymbol{c}_k - \boldsymbol{c}_\ell)$

- Example: 4 aligned obstacles

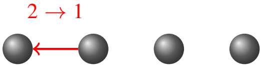




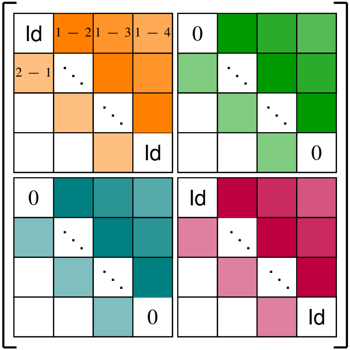
Smart storage and assembling

- $\mathbb{M}^{k\ell}_{\alpha\beta}$ depends only on δ and $(\boldsymbol{c}_k - \boldsymbol{c}_\ell)$

- Example: 4 aligned obstacles



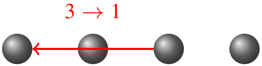
$\mathbb{M} =$



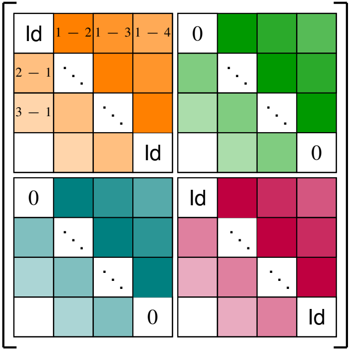
Smart storage and assembling

- $\mathbb{M}^{k\ell}_{\alpha\beta}$ depends only on δ and $(\boldsymbol{c}_k - \boldsymbol{c}_\ell)$

- Example: 4 aligned obstacles



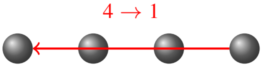
$\mathbb{M} =$



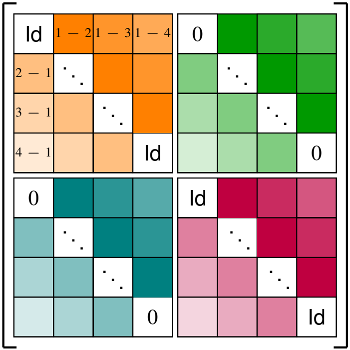
Smart storage and assembling

- $\mathbb{M}^{k\ell}_{\alpha\beta}$ depends only on δ and $(\boldsymbol{c}_k - \boldsymbol{c}_\ell)$

- Example: 4 aligned obstacles



$\mathbb{M} =$

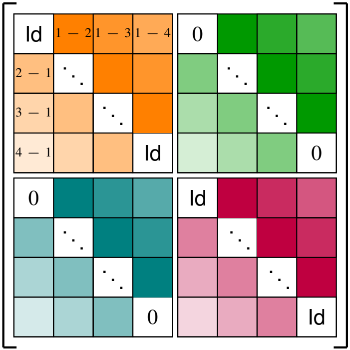


Smart storage and assembling

- $\mathbb{M}^{k\ell}_{\alpha\beta}$ depends only on δ and $(\boldsymbol{c}_k - \boldsymbol{c}_\ell)$

Instead of storing wholly $\mathbb{M}$, we only keep the (two-by-two) different blocks $\mathbb{M}^{k\ell}_{\perp\perp}$, $\mathbb{M}^{k\ell}_{\perp\times}$, $\mathbb{M}^{k\ell}_{\times\perp}$ and $\mathbb{M}^{k\ell}_{\times\times}$

$\mathbb{M} =$



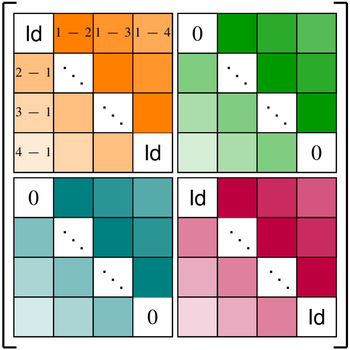
$\mathbb{M}_{\text{Block}} =$

Smart storage and assembling

- $\mathbb{M}^{k\ell}_{\alpha\beta}$ depends only on δ and $(\boldsymbol{c}_k - \boldsymbol{c}_\ell)$

Instead of storing wholly $\mathbb{M}$, we only keep the (two-by-two) different blocks $\mathbb{M}^{k\ell}_{\perp\perp}$, $\mathbb{M}^{k\ell}_{\perp\times}$, $\mathbb{M}^{k\ell}_{\times\perp}$ and $\mathbb{M}^{k\ell}_{\times\times}$

$\mathbb{M} =$



$\mathbb{M}_{\text{Block}} =$

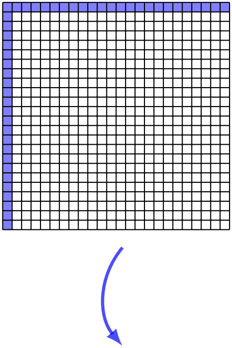
- The decomposition becomes $\mathbb{M}_{\text{Block}} = \mathbb{A}_{\text{Block}} + \mathbb{B}_{\text{Block}}$
- Assembling of $\mathbb{A}_{\text{Block}}$ in sparse matrix
- Define the action of $\mathbb{B}_{\text{Block}}$

$$\mathcal{A}(\mathbb{B}_{\text{Block}}, \mathbf{p}) = \mathbb{B}\,\mathbf{p}$$

Uniformly distributed configurations of obstacles

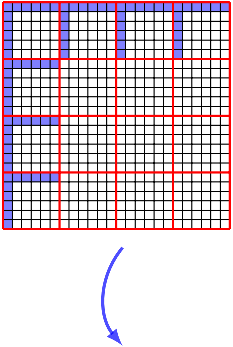
Example with 24 obstacles. For **each** part

On a line



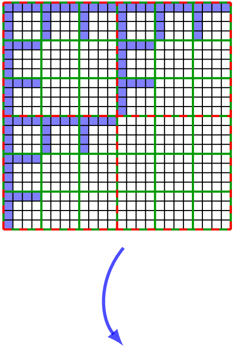
$2N_{\mathsf{obs}} - 1$ blocks

On a plane (4×6)



$(2N_{\mathsf{l}} - 1)(2N_{\mathsf{c}} - 1)$ blocks

Into a volume $(2 \times 3 \times 4)$



$(2N_{\mathsf{h}} - 1)(2N_{\mathsf{l}} - 1)(2N_{\mathsf{c}} - 1)$ blocks

instead of $N_{\mathsf{obs}}^2 = N_{\mathsf{l}}^2 N_{\mathsf{c}}^2 = N_{\mathsf{l}}^2\, N_{\mathsf{c}}^2\, N_{\mathsf{h}}^2$ blocks per part

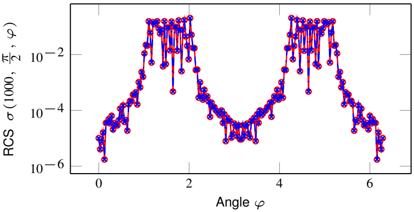
Numerical results

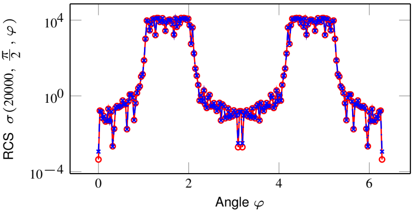
- Test n°1: 1 000 aligned obstacles
  $\delta = 0.1, d = 1.0, \lambda = 5.0$

- Test n°2: 10 000 aligned obstacles
  $\delta = 0.5, d = 2.0, \lambda = 5.0$

	Modified Foldy	Spectral 1
Solver	Direct	
Density	100%	
Linear system	21.25s	44.59s
Post-processing	16.12s	21.19s
Total time	37.37s	65.78s

	Modified Foldy	Spectral 1
Solver	Iterative	
Density	4.95%	
Linear system	29.74s	163.25s
Post-processing	76.64s	262.97s
Total time	106.46s	426.34s

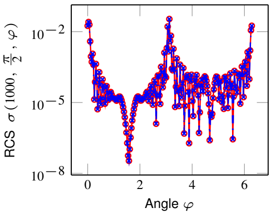




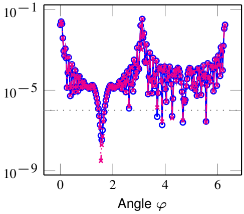
Numerical results

- Test n°3: $50 \times 50 = 2\,500$ obstacles uniformly distributed on a plane
  $\delta = 0.1$, $d = 1.0$, $\lambda = 5.0$

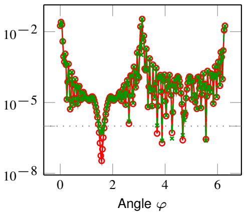
	Modified Foldy		Spectral 1	
Solver	Direct	Iterative	Direct	Iterative
Density	100%	29.66%	100%	29.66%
Linear system	595.10s	38.32s	613.10s	234.84s
Post-processing	22.96s	19.00s	62.69s	58.47s
Total time	618.48s	57.32s	676.09s	293.31s



(f) Foldy vs Spectral: direct solver



(g) Foldy Iterative vs Direct



(h) Spectral Iterative vs Direct

Outline

Conclusion and Perspectives

Conclusion

- ✓ High-order asymptotic expansions to single scattering (Labat, Péron and Tordeux, In revision 2019)
- ✓ Low-order Born and Foldy-Lax models to multiple scattering
- ✓ High-order spectral models equivalent to the *Generalized Multiparticle Mie-solution* theory (Xu, 1995)
- ✓ Fast resolution using few memory to the multiple scattering problem by a large number of spheres

On-going work

- ∼ Comparison of preconditionners and iterative solvers
- ∼ Smart assembling for obstacles uniformly distributed into a volume

Perspectives

- ✗ Definition of high-order asymptotic models to multiple scattering
- ✗ Extension to obstacles of arbitrary shape
- ✗ Extension to time-dependent domain

Thank you for your attention

Conclusion and Perspectives

Conclusion

- ✓ High-order asymptotic expansions to single scattering (Labat, Péron and Tordeux, In revision 2019)
- ✓ Low-order Born and Foldy-Lax models to multiple scattering
- ✓ High-order spectral models equivalent to the *Generalized Multiparticle Mie-solution* theory (Xu, 1995)
- ✓ Fast resolution using few memory to the multiple scattering problem by a large number of spheres

On-going work

- ∼ Comparison of preconditionners and iterative solvers
- ∼ Smart assembling for obstacles uniformly distributed into a volume

Perspectives

- ✗ Definition of high-order asymptotic models to multiple scattering
- ✗ Extension to obstacles of arbitrary shape
- ✗ Extension to time-dependent domain

Thank you for your attention