Evaluations of some terminating hypergeometric $_2F_1(2)$ series with applications

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Abstract: Explicit expressions for the hypergeometric series $_2F_1(-n, a; 2a \pm j; 2)$ and $_2F_1(-n, a; -2n \pm j; 2)$ for positive integer n and arbitrary integer j are obtained with the help of generalizations of Kummer's second and third summation theorems obtained earlier by Rakha and Rathie. Results for $|j| \leq 5$ derived previously using different methods are also obtained as special cases. Two applications are considered, where the first summation formula is applied to a terminating $_3F_2(2)$ series and the confluent hypergeometric function $_1F_1(x)$.

Key words: Terminating hypergeometric series, generalized Kummer's second and third summation theorems

1. Introduction

In a problem arising in a model of a biological problem, Samoletov [14] obtained by means of a mathematical induction argument the following sum containing factorials:

$$\sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!} \frac{(2k+1)!!}{k!(k+1)!} = \frac{(-1)^n}{\sqrt{n!(n+1)!}} \left(\sqrt{n+1} \, \frac{(n-1)!!}{n!!} \right)^{(-1)^n},$$

where throughout n denotes a positive integer and, as usual,

$$(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!, \qquad (2n+1)!! = 1 \cdot 3 \cdot 5 \cdots (2n+1) = \frac{(2n+1)!}{2^n n!}.$$

Samoletov [14] also expressed the above sum in the equivalent hypergeometric form:

$$_2F_1 \left[\begin{array}{c} -n, \frac{3}{2} \\ 2 \end{array} ; 2 \right] = \begin{cases} \frac{\Gamma(\frac{1}{2}n + \frac{1}{2})}{\sqrt{\pi}\,\Gamma(\frac{1}{2}n + 1)}, & (n \text{ even}) \\ \frac{-\Gamma(\frac{1}{2}n + 1)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}n + \frac{3}{2})}, & (n \text{ odd}). \end{cases}$$

Subsequently, Srivastava [17] pointed out that this result could be easily derived from a known hypergeometric summation formula [11, Vol. 2, p. 493] for $_2F_1(-n, a; 2a - 1; 2)$ with $a = \frac{3}{2}$, which is a contiguous result to the

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well-known summation

$$
{}_2F_1\left[\begin{array}{c} -n, a \\ 2a \end{array}; 2\right] = \frac{2^n\sqrt{\pi}\,\Gamma(1-a)}{(2a)_n\Gamma(\frac{1}{2}-\frac{1}{2}n)\Gamma(1-a-\frac{1}{2}n)} \qquad (n=0,1,2,\ldots). \tag{1}
$$

The aim in this note is to obtain explicit expressions for

$$
{}_2F_1\left[\begin{array}{c} -n, a \\ 2a \pm j \end{array}; 2\right] \qquad \text{and} \qquad {}_2F_1\left[\begin{array}{c} -n, a \\ -2n \pm j \end{array}; 2\right] \tag{2}
$$

for arbitrary integer j. We shall employ the following generalizations of Kummer's second and third summation theorems given in [13] (we correct a misprint in Theorem 6 of this reference). These are respectively

$$
{}_2F_1\left[\begin{array}{c} \alpha, \beta \\ \frac{1}{2}(\alpha+\beta\pm j+1) \end{array}; \frac{1}{2}\right] = \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\pm\frac{1}{2}j)}{\Gamma(\frac{1}{2}\alpha+\frac{1}{2})\Gamma(\frac{1}{2}\beta+\frac{1}{2})} \frac{\Gamma(\frac{1}{2}\alpha-\frac{1}{2}\beta+\frac{1}{2}\mp\frac{1}{2}j)}{\Gamma(\frac{1}{2}\alpha-\frac{1}{2}\beta+\frac{1}{2}+\frac{1}{2}j)}
$$

$$
\times \sum_{r=0}^{j}(\mp1)^r \binom{j}{r} \frac{(\frac{1}{2}\beta)_{r/2}}{(\frac{1}{2}\alpha+\frac{1}{2})_{(r-j)/2}} \tag{3}
$$

and

$$
{}_2F_1\left[\begin{array}{c} \alpha, 1-\alpha\pm j \\ \gamma \end{array}; \frac{1}{2}\right] = \frac{2^{\pm j}\Gamma(\frac{1}{2}\gamma)\Gamma(\frac{1}{2}\gamma+\frac{1}{2})}{\Gamma(\frac{1}{2}\gamma+\frac{1}{2}\alpha)\Gamma(\frac{1}{2}\gamma-\frac{1}{2}\alpha+\frac{1}{2})} \frac{\Gamma(\alpha\mp j)}{\Gamma(\alpha+\epsilon_j)}
$$

$$
\times \sum_{r=0}^{j}(\mp1)^r \binom{j}{r} \frac{(\frac{1}{2}\gamma-\frac{1}{2}\alpha)_{r/2}}{(\frac{1}{2}\gamma+\frac{1}{2}\alpha)_{r/2-\delta_j}} \tag{4}
$$

for $j=0,1,2,\ldots$, where $\epsilon_j = 0$ (resp. j), $\delta_j = j$ (resp. 0) when the upper (resp. lower) signs are taken and $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol defined for *arbitrary* index k. The formulas (3) (for $j>0$) and (4) (separately for $j>0$ and $j<0$) are given in a different form in [1, p. 582, (130)–(132)]. When $j=0$, the summations (3) and (4) reduce to the well-known second and third summation theorems due to Kummer [16, p. 243]:

$$
{}_2F_1\left[\begin{array}{c} \alpha, \beta \\ \frac{1}{2}(\alpha+\beta+1) \end{array}; \frac{1}{2}\right] = \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2})}{\Gamma(\frac{1}{2}\alpha+\frac{1}{2})\Gamma(\frac{1}{2}\beta+\frac{1}{2})}
$$

and*

$$
{}_2F_1\left[\begin{array}{c} \alpha, 1-\alpha \\ \gamma \end{array}; \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2}\gamma)\Gamma(\frac{1}{2}\gamma+\frac{1}{2})}{\Gamma(\frac{1}{2}\gamma+\frac{1}{2}\alpha)\Gamma(\frac{1}{2}\gamma-\frac{1}{2}\alpha+\frac{1}{2})}.
$$

In addition, we shall make use of the transformation [10, (15.8.6)]

$$
{}_2F_1\left[\begin{array}{c} -n, \beta \\ \gamma \end{array}; 2\right] = \frac{(-2)^n(\beta)_n}{(\gamma)_n}\, {}_2F_1\left[\begin{array}{c} -n, 1-\gamma-n \\ 1-\beta-n \end{array}; \frac{1}{2}\right]. \tag{5}
$$

Expressions for the series in (2) for arbitrary integer j were recently obtained by Chu [3] using a different approach. This involved expressing the series for $j\neq0$ as finite sums of ${}_2F_1(2)$ series in (2) with $j=0$. The

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*In [16, p. 243], this summation formula is referred to as Bailey's theorem. However, it has been pointed out in [2] that this theorem was originally found by Kummer.

cases with $|j| \leq 5$ were also given by Kim and Rathie [6] and Kim et al. [5]. An application of the first series in (2) for $j = 0, 1, \ldots, 5$ was discussed in [7]. In Sections 4 and 5, we give two additional applications, the first to the evaluation of a terminating $_3F_2(2)$ series and the second to the confluent hypergeometric function $_1F_1(x)$.

2. Statement of the results

Our principal results are stated in the following two theorems.

Theorem 1 *Let n be a positive integer, let a be a complex parameter, and define $j_0 = \lfloor \frac{1}{2} j \rfloor$. Then we have*

$$_2F_1 \left[\begin{array}{c} -2n, a \\ 2a \pm j \end{array} ; 2 \right] = \frac{2^{2n} (\frac{1}{2})_n}{(2a \pm j)_{2n}} \sum_{r=0}^{j_0} (-1)^r \binom{j}{2r} (-n)_r (a + \delta_j)_{n-r} \tag{6}$$

and

$$_2F_1 \left[\begin{array}{c} -2n - 1, a \\ 2a \pm j \end{array} ; 2 \right] = \frac{\pm 2^{2n} (\frac{3}{2})_n}{(2a \pm j)_{2n+1}} \sum_{r=0}^{j_0} (-1)^r \binom{j}{2r + 1} (-n)_r (a + \delta_j)_{n-r} \tag{7}$$

for $j = 0, 1, 2, \ldots$, where $\delta_j = j$ (resp. 0) when the upper (resp. lower) signs are taken.

Proof From the result (5) we have

$$_2F_1 \left[\begin{array}{c} -n, a \\ 2a \pm j \end{array} ; 2 \right] = \frac{(-2)^n (a)_n}{(2a \pm j)_n} \, _2F_1 \left[\begin{array}{c} -n, 1 - 2a \mp j - n \\ 1 - a - n \end{array} ; \frac{1}{2} \right].$$

The hypergeometric function on the right-hand side can be summed by the generalized second Kummer summation theorem (3). Proceeding first with the function whose denominator parameter is $2a + j$, we put $\alpha = 1 - 2a - j - n$ and $\beta = -n$ in (3) to find, after some straightforward algebra,

$$_2F_1 \left[\begin{array}{c} -n, a \\ 2a + j \end{array} ; 2 \right] = \frac{(-2)^n (a)_n}{(2a + j)_n} \frac{\sqrt{\pi} \Gamma(1 - a - n)}{\Gamma(-\frac{1}{2}n + \frac{1}{2}) \Gamma(1 - a)} \sum_{r=0}^{j} (-1)^r \binom{j}{r} \frac{(-\frac{1}{2}n)_{r/2}}{(1 - a - j)_{(r-n)/2}}$$

$$= \frac{2^n}{(2a + j)_n} \frac{\sqrt{\pi}}{\Gamma(-\frac{1}{2}n + \frac{1}{2})} \sum_{r=0}^{j} (-1)^r \binom{j}{r} \frac{(-\frac{1}{2}n)_{r/2}}{(1 - a - j)_{(r-n)/2}} . \tag{8}$$

Now replace n by $2n$ and use the reflection formula for the gamma function to yield

$$_2F_1 \left[\begin{array}{c} -2n, a \\ 2a + j \end{array} ; 2 \right] = \frac{(-1)^n 2^{2n} (\frac{1}{2})_n}{(2a + j)_{2n}} \sum_{r=0}^{j} (-1)^r \binom{j}{r} \frac{(-n)_{r/2}}{(1 - a - j)_{r/2 - n}} .$$

Since $(-n)_{r/2}$ vanishes for odd r and positive integer n, we finally obtain

$$_2F_1 \left[\begin{array}{c} -2n, a \\ 2a + j \end{array} ; 2 \right] = \frac{2^{2n} (\frac{1}{2})_n}{(2a + j)_{2n}} \sum_{r=0}^{j_0} (-1)^r \binom{j}{2r} (-n)_r (a + j)_{n-r} , \tag{9}$$

where $j_0 = \lfloor \frac{1}{2} j \rfloor$ and we have used the fact that $(1 - a - j)_{r-n} = (-1)^{n-r}/(a + j)_{n-r}$.

If we replace n by $2n+1$ in (8), we find

$$
{}_2F_1\left[\begin{array}{c} -2n-1, a \\ 2a+j \end{array} ; 2\right] = \frac{2^{2n+1}}{(2a+j)_{2n+1}} \frac{\sqrt{\pi}}{\Gamma(-n-\frac{1}{2})} \sum_{r=0}^{j_0} (-1)^r \binom{j}{r} \frac{(-n)_{(r-1)/2}}{(1-a-j)_{(r/2-1/2-n)}},
$$

where we have used

$$
\frac{(-n-\frac{1}{2})_{r/2}}{\Gamma(-n)} = \frac{(-n)_{(r-1)/2}}{\Gamma(-n-\frac{1}{2})}.
$$

Since $(-n)_{(r-1)/2}$ vanishes for even r and positive integer n, we find

$$
{}_2F_1\left[\begin{array}{c} -2n-1, a \\ 2a+j \end{array} ; 2\right] = \frac{2^{2n}(\frac{3}{2})_n}{(2a+j)_{2n+1}} \sum_{r=0}^{j_0} (-1)^r \binom{j}{2r+1} (-n)_r (a+j)_{n-r}. \tag{10}
$$

Proceeding in a similar manner for the function whose denominator parameter is $2a-j$, we have upon letting $\alpha = 1-a+j-n$ and $\beta = -n$ in (3)

$$
{}_2F_1\left[\begin{array}{c} -n, a \\ 2a-j \end{array} ; 2\right] = \frac{2^n}{(2a-j)_n} \frac{\sqrt{\pi}}{\Gamma(-\frac{1}{2}n+\frac{1}{2})} \sum_{r=0}^{j} \binom{j}{r} \frac{(-\frac{1}{2}n)_{r/2}}{(1-a)_{(r-n)/2}}. \tag{11}
$$

Then we have

$$
{}_2F_1\left[\begin{array}{c} -2n, a \\ 2a-j \end{array} ; 2\right] = \frac{2^{2n}(\frac{1}{2})_n}{(2a-j)_{2n}} \sum_{r=0}^{j_0} (-1)^r \binom{j}{2r} (-n)_r (a)_{n-r} \tag{12}
$$

and

$$
{}_2F_1\left[\begin{array}{c} -2n-1, a \\ 2a-j \end{array} ; 2\right] = \frac{-2^{2n}(\frac{3}{2})_n}{(2a-j)_{2n+1}} \sum_{r=0}^{j_0} (-1)^r \binom{j}{2r+1} (-n)_r (a)_{n-r}. \tag{13}
$$

The summations in (9), (10) and (12), (13) then correspond to the results stated in the theorem. □

We remark that in (7) the upper limit of summation can be replaced by $\lfloor \frac{1}{2}j \rfloor - 1$ when j is even. Also, since $(-n)_r$ vanishes when $r > n$, it is possible to replace the upper summation limit in both (6) and (7) by n whenever $n > \lfloor \frac{1}{2}j \rfloor$.

Theorem 2 *Let n be a positive integer and a be a complex parameter. Then we have*

$$
{}_2F_1\left[\begin{array}{c} -n, a \\ -2n+j \end{array} ; 2\right] = \frac{2^{2n-j}(n-j)!}{(2n-j)!} \sum_{r=0}^{j} \binom{j}{r} (\tfrac{1}{2}a + \tfrac{1}{2} - \tfrac{1}{2}r)_n \tag{14}
$$

provided that j does not lie in the interval $[n+1, 2n]$ (where the hypergeometric function on the left-hand side of (14) is, in general, not defined), and

$$
{}_2F_1\left[\begin{array}{c} -n, a \\ -2n-j \end{array} ; 2\right] = \frac{2^{2n+j}n!}{(2n+j)!} \sum_{r=0}^{j} (-1)^r \binom{j}{r} (\tfrac{1}{2}a + \tfrac{1}{2} - \tfrac{1}{2}r)_{n+j} \tag{15}
$$

for $j = 0, 1, 2, \ldots$. When $j \geq 2n+1$ in (14), the ratio of factorials $(n-j)!/(2n-j)!$ can be replaced by $(-1)^n(j-2n-1)!/(j-n-1)!$.

Proof From the result (5) we have

$$
{}_2F_1\left[\begin{array}{c} -n, a \\ -2n \mp j \end{array} ; 2\right] = \frac{2^n (a)_n (n \pm j)!}{(2n \pm j)!} \, {}_2F_1\left[\begin{array}{c} -n, 1+n \pm j \\ 1-a-n \end{array} ; \frac{1}{2}\right]
$$

when the parameters are such that the hypergeometric functions make sense. The hypergeometric function on the right-hand side can be summed by the generalized third Kummer theorem (4), where we put $\alpha = -n$ and $\gamma = 1 - a - n$. Then we obtain

$$
\begin{aligned}
{}_2F_1\left[\begin{array}{c} -n, a \\ -2n+j \end{array} ; 2\right] &= \frac{2^{n-j}(a)_n(n-j)!}{(2n-j)!} \frac{\Gamma(\frac{1}{2}\gamma)\Gamma(\frac{1}{2}\gamma+\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}a)\Gamma(1-\frac{1}{2}a)} \sum_{r=0}^{j} \binom{j}{r} \frac{\Gamma(\frac{1}{2}-\frac{1}{2}a+\frac{1}{2}r)}{\Gamma(\frac{1}{2}-\frac{1}{2}a+\frac{1}{2}r-n)} \\
&= \frac{(-1)^n 2^{2n-j}(a)_n(n-j)!}{(2n-j)!} \frac{\Gamma(1-a-n)}{\Gamma(1-a)} \sum_{r=0}^{j} \binom{j}{r} (\tfrac{1}{2}-\tfrac{1}{2}a+\tfrac{1}{2}r)_n \\
&= \frac{2^{2n-j}(n-j)!}{(2n-j)!} \sum_{r=0}^{j} \binom{j}{r} (\tfrac{1}{2}-\tfrac{1}{2}a+\tfrac{1}{2}r)_n,
\end{aligned}
$$

where we have used the reflection formula for the gamma function and

$$
(2a)_{2n} = 2^{2n}(a)_n(a+\tfrac{1}{2})_n.
$$

Similarly, we find

$$
\begin{aligned}
{}_2F_1\left[\begin{array}{c} -n, a \\ -2n-j \end{array} ; 2\right] &= \frac{(-1)^j 2^{n+j}(a)_n n!}{(2n+j)!} \frac{\Gamma(\frac{1}{2}\gamma)\Gamma(\frac{1}{2}\gamma+\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}a)\Gamma(1-\frac{1}{2}a)} \sum_{r=0}^{j}(-1)^r \binom{j}{r} \frac{\Gamma(\frac{1}{2}-\frac{1}{2}a+\frac{1}{2}r)}{\Gamma(\frac{1}{2}-\frac{1}{2}a+\frac{1}{2}r-n-j)} \\
&= \frac{(-1)^n 2^{2n+j}(a)_n n!}{(2n+j)!} \frac{\Gamma(1-a-n)}{\Gamma(1-a)} \sum_{r=0}^{j}(-1)^r \binom{j}{r} (\tfrac{1}{2}-\tfrac{1}{2}a+\tfrac{1}{2}r)_{n+j} \\
&= \frac{2^{2n+j}n!}{(2n+j)!} \sum_{r=0}^{j}(-1)^r \binom{j}{r} (\tfrac{1}{2}-\tfrac{1}{2}a+\tfrac{1}{2}r)_{n+j},
\end{aligned}
$$

which establishes the theorem. $\qquad\square$

The sums on the right-hand sides of (14) and (15) can be written in an alternative form involving just two Pochhammer symbols containing the index n by making use of the result

$$
(\alpha - r)_n = \frac{(\alpha)_n (1-\alpha)_r}{(1-\alpha-n)_r}
$$

for positive integers r and n. Then we find, with $j_0 = \lfloor \frac{1}{2}j \rfloor$,

$$
{}_2F_1\left[\begin{array}{c} -n, a \\ -2n+j \end{array} ; 2\right]
$$

$$
= \frac{2^{2n-j}(n-j)!}{(2n-j)!}\left\{ (\tfrac{1}{2}a+\tfrac{1}{2})_n \sum_{r=0}^{j_0} \binom{j}{2r} A_r(n,0) + (\tfrac{1}{2}a)_n \sum_{r=0}^{j_0} \binom{j}{2r+1} B_r(n,0) \right\} \qquad (16)
$$

and

$$_2F_1\left[\begin{array}{c} -n, a \\ -2n-j \end{array};2\right]$$

$$= \frac{2^{2n+j}n!}{(2n+j)!}\left\{(\tfrac{1}{2}a+\tfrac{1}{2})_{n+j}\sum_{r=0}^{j_0}\binom{j}{2r}A_r(n,j)-(\tfrac{1}{2}a)_{n+j}\sum_{r=0}^{j_0}\binom{j}{2r+1}B_r(n,j)\right\},\tag{17}$$

where

$$A_r(n,j):=\frac{(\tfrac{1}{2}-\tfrac{1}{2}a)_r}{(\tfrac{1}{2}-\tfrac{1}{2}a-n-j)_r},\qquad B_r(n,j):=\frac{(1-\tfrac{1}{2}a)_r}{(1-\tfrac{1}{2}a-n-j)_r}.$$

Again, when j is even, the upper summation limit in the second sums in (16) and (17) can be replaced by j_0-1, if so desired.

3. Special cases

If we set $0\le j\le 5$ in (6) we obtain the following summations:

$$_2F_1\left[\begin{array}{c} -2n, a \\ 2a \end{array};2\right]=\frac{(\tfrac{1}{2})_n}{(a+\tfrac{1}{2})_n}=\,_2F_1\left[\begin{array}{c} -2n, a \\ 2a+1 \end{array};2\right],\tag{18}$$

$$_2F_1\left[\begin{array}{c} -2n, a \\ 2a+2 \end{array};2\right]=\frac{(\tfrac{1}{2})_n}{(a+\tfrac{3}{2})_n}\left(1+\frac{2n}{a+1}\right),\tag{19}$$

$$_2F_1\left[\begin{array}{c} -2n, a \\ 2a+3 \end{array};2\right]=\frac{(\tfrac{1}{2})_n}{(a+\tfrac{3}{2})_n}\left(1+\frac{4n}{a+2}\right),\tag{20}$$

$$_2F_1\left[\begin{array}{c} -2n, a \\ 2a+4 \end{array};2\right]=\frac{(\tfrac{1}{2})_n}{(a+\tfrac{5}{2})_n}\left(1+\frac{8n}{a+2}+\frac{8n(n-1)}{(a+2)(a+3)}\right),\tag{21}$$

$$_2F_1\left[\begin{array}{c} -2n, a \\ 2a+5 \end{array};2\right]=\frac{(\tfrac{1}{2})_n}{(a+\tfrac{5}{2})_n}\left(1+\frac{12n}{a+3}+\frac{16n(n-1)}{(a+3)(a+4)}\right)\tag{22}$$

and

$$_2F_1\left[\begin{array}{c} -2n, a \\ 2a-1 \end{array};2\right]=\frac{(\tfrac{1}{2})_n}{(a-\tfrac{1}{2})_n},\tag{23}$$

$$_2F_1\left[\begin{array}{c} -2n, a \\ 2a-2 \end{array};2\right]=\frac{(\tfrac{1}{2})_n}{(a-\tfrac{1}{2})_n}\left(1+\frac{2n}{a-1}\right),\tag{24}$$

$$_2F_1\left[\begin{array}{c} -2n, a \\ 2a-3 \end{array};2\right]=\frac{(\tfrac{1}{2})_n}{(a-\tfrac{3}{2})_n}\left(1+\frac{4n}{a-1}\right),\tag{25}$$

$$_2F_1\left[\begin{array}{c} -2n, a \\ 2a-4 \end{array};2\right]=\frac{(\tfrac{1}{2})_n}{(a-\tfrac{3}{2})_n}\left(1+\frac{8n}{a-2}+\frac{8n(n-1)}{(a-1)(a-2)}\right),\tag{26}$$

$$_2F_1\left[\begin{array}{c} -2n, a \\ 2a-5 \end{array};2\right] = \frac{(\frac{1}{2})_n}{(a-\frac{5}{2})_n}\left(1+\frac{12n}{a-2}+\frac{16n(n-1)}{(a-1)(a-2)}\right). \tag{27}$$

Similarly, if we set $0 \le j \le 5$ in (7) we obtain the following summations:

$$_2F_1\left[\begin{array}{c} -2n-1, a \\ 2a \end{array};2\right] = 0, \tag{28}$$

$$_2F_1\left[\begin{array}{c} -2n-1, a \\ 2a+1 \end{array};2\right] = \frac{(\frac{3}{2})_n}{(2a+1)(a+\frac{3}{2})_n}, \tag{29}$$

$$_2F_1\left[\begin{array}{c} -2n-1, a \\ 2a+2 \end{array};2\right] = \frac{2(\frac{3}{2})_n}{(2a+2)(a+\frac{3}{2})_n}, \tag{30}$$

$$_2F_1\left[\begin{array}{c} -2n-1, a \\ 2a+3 \end{array};2\right] = \frac{(\frac{3}{2})_n}{(2a+3)(a+\frac{5}{2})_n}\left(3+\frac{4n}{a+2}\right), \tag{31}$$

$$_2F_1\left[\begin{array}{c} -2n-1, a \\ 2a+4 \end{array};2\right] = \frac{(\frac{3}{2})_n}{(2a+4)(a+\frac{5}{2})_n}\left(4+\frac{8n}{a+3}\right), \tag{32}$$

$$_2F_1\left[\begin{array}{c} -2n-1, a \\ 2a+5 \end{array};2\right] = \frac{(\frac{3}{2})_n}{(2a+5)(a+\frac{7}{2})_n}\left(5+\frac{20n}{a+3}+\frac{16n(n-1)}{(a+3)(a+4)}\right) \tag{33}$$

and

$$_2F_1\left[\begin{array}{c} -2n-1, a \\ 2a-1 \end{array};2\right] = -\frac{(\frac{3}{2})_n}{(2a-1)(a+\frac{1}{2})_n}, \tag{34}$$

$$_2F_1\left[\begin{array}{c} -2n-1, a \\ 2a-2 \end{array};2\right] = -\frac{2(\frac{3}{2})_n}{(2a-2)(a-\frac{1}{2})_n}, \tag{35}$$

$$_2F_1\left[\begin{array}{c} -2n-1, a \\ 2a-3 \end{array};2\right] = -\frac{(\frac{3}{2})_n}{(2a-3)(a-\frac{1}{2})_n}\left(3+\frac{4n}{a-1}\right), \tag{36}$$

$$_2F_1\left[\begin{array}{c} -2n-1, a \\ 2a-4 \end{array};2\right] = -\frac{(\frac{3}{2})_n}{(2a-4)(a-\frac{3}{2})_n}\left(4+\frac{8n}{a-1}\right), \tag{37}$$

$$_2F_1\left[\begin{array}{c} -2n-1, a \\ 2a-5 \end{array};2\right] = -\frac{(\frac{3}{2})_n}{(2a-5)(a-\frac{3}{2})_n}\left(5+\frac{20n}{a-2}+\frac{16n(n-1)}{(a-1)(a-2)}\right). \tag{38}$$

Finally, from (16) and (17) we obtain:

$$_2F_1\left[\begin{array}{c} -n, a \\ -2n \end{array};2\right] = \frac{2^{2n}n!}{(2n)!}(\tfrac{1}{2}a+\tfrac{1}{2})_n = \frac{(\frac{1}{2}a+\frac{1}{2})_n}{(\frac{1}{2})_n}, \tag{39}$$

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$$_2F_1 \left[\begin{array}{c} -n, a \\ -2n+1 \end{array} ; 2 \right] = \frac{2^{2n-1}(n-1)!}{(2n-1)!} \{ (\tfrac{1}{2}a + \tfrac{1}{2})_n + (\tfrac{1}{2}a)_n \}, \tag{40}$$

$$_2F_1 \left[\begin{array}{c} -n, a \\ -2n-1 \end{array} ; 2 \right] = \frac{2^{2n+1}n!}{(2n+1)!} (\tfrac{1}{2}a + \tfrac{1}{2})_{n+1}, \tag{41}$$

$$_2F_1 \left[\begin{array}{c} -n, a \\ -2n+2 \end{array} ; 2 \right] = \frac{2^{2n-1}(n-2)!}{(2n-2)!} \left\{ \frac{1-a-n}{1-a-2n} (\tfrac{1}{2}a + \tfrac{1}{2})_n + (\tfrac{1}{2}a)_n \right\}, \tag{42}$$

$$_2F_1 \left[\begin{array}{c} -n, a \\ -2n-2 \end{array} ; 2 \right] = \frac{2^{2n+3}n!}{(2n+2)!} \left\{ \frac{(1-a-n-j)}{1-a-2n-2j} (\tfrac{1}{2}a + \tfrac{1}{2})_{n+2} - (\tfrac{1}{2}a)_{n+2} \right\} \tag{43}$$

and so on.

The above evaluations agree with those given in [3, 6], although presented in a different format; the results (18) and (23), together with (28), (29), and (34), are also recorded in [11] in another form.

4. An application of Theorem 1 to a terminating $_3F_2(2)$ series

From the results in Section 2 we shall establish a summation formula for the terminating $_3F_2(2)$ series

$$_3F_2 \left[\begin{array}{cc} -n, a, & d+m \\ 2a+p, & d \end{array} ; 2 \right],$$

where m, n are positive integers and p is an arbitrary integer such that $2a + p \neq 0, -1, -2, \ldots, -n+1$. The cases $m = 1$ and $m = 2$ (when $m = p$) have been obtained previously by different means in [4, 12]. When $m = p = 0$ the above series reduces to that in (1).

We employ a general result proved in [9, Lemma 4] expressing an $_{r+2}F_{r+1}(x)$ hypergeometric function, with r pairs of numeratorial and denominatorial parameters differing by positive integers, as a finite sum of $_2F_1(x)$ functions. In the particular case $r = 1$, we obtain for positive integer m

$$_3F_2 \left[\begin{array}{cc} a, b & d+m \\ c & d \end{array} ; x \right] = \frac{1}{(d)_m} \sum_{j=0}^{m} A_j \frac{(a)_j (b)_j}{(c)_j} x^j {}_2F_1 \left[\begin{array}{c} a+j, b+j \\ c+j \end{array} ; x \right]. \tag{44}$$

The coefficients A_j are defined by

$$A_j = \sum_{k=j}^{m} \sigma_{m-k} \mathbf{S}_k^{(j)},$$

where $\mathbf{S}_k^{(j)}$ is the Stirling number of the second kind and the σ_j $(0 \leq j \leq m)$ are generated by the relation

$$(d+x)_m = \sum_{j=0}^{m} \sigma_{m-j} x^j.$$

It was shown in [8] that

$$A_j = \frac{(-1)^j (d)_m}{j!} {}_2F_1 \left[\begin{array}{c} -j, d+m \\ d \end{array} ; 1 \right] = \frac{(-1)^j (-m)_j (d)_m}{j! \, (d)_j}, \tag{45}$$

the second result following by application of Vandermonde's theorem [16, p. 243].

In order to simplify the presentation of our summation theorem we introduce the coefficients $C_r(n, j)$ given by

$$C_r(n, j) = \frac{\Gamma(-\frac{1}{2}n + \frac{1}{2}j + \frac{1}{2}r)}{\Gamma(-\frac{1}{2}n + \frac{1}{2}j)\Gamma(-\frac{1}{2}n + \frac{1}{2}j + \frac{1}{2})},$$

which yield the even- and odd-order values

$$C_{2r}(n, j) = \frac{(-\frac{1}{2}n + \frac{1}{2}j)_r}{\Gamma(-\frac{1}{2}n + \frac{1}{2}j + \frac{1}{2})}, \qquad C_{2r+1}(n, j) = \frac{(-\frac{1}{2}n + \frac{1}{2}j + \frac{1}{2})_r}{\Gamma(-\frac{1}{2}n + \frac{1}{2}j)}.$$

We further introduce the finite sums for $0 \le j \le m$:

$$S_j^{(1)}(n, p) = \sum_{r=0}^{p-j} \binom{p-j}{r} \frac{(-1)^r C_r(n, j)}{(1 - a - p)_{(r-n+j)/2}} \quad (p \ge j),$$

$$S_j^{(2)}(n, p) = \sum_{r=0}^{j-p} \binom{j-p}{r} \frac{C_r(n, j)}{(1 - a - j)_{(r-n+j)/2}} \quad (p \le j).$$

Then we obtain the following theorem.

Theorem 3 *Let m, n be positive integers and p be an arbitrary integer such that $2a+p \ne 0, -1, -2, \ldots, -n+1$. Then*

$${}_3F_2\left[\begin{array}{cc} -n, a, & d+m \\ 2a+p, & d \end{array}; 2\right] = \begin{cases} \dfrac{2^n \sqrt{\pi}}{(2a+p)_n} \sum\limits_{j=0}^{m} \dfrac{(-1)^j(-m)_j(-n)_j(a)_j}{j!\,(d)_j} S_j^{(1)}(n, p) & (p \ge m > 0) \\[4mm] \dfrac{2^n \sqrt{\pi}}{(2a+p)_n}\Big\{ \sum\limits_{j=0}^{p} \dfrac{(-1)^j(-m)_j(-n)_j(a)_j}{j!\,(d)_j} S_j^{(1)}(n, p) \\[4mm] \quad + \sum\limits_{j=p+1}^{m} \dfrac{(-1)^j(-m)_j(-n)_j(a)_j}{j!\,(d)_j} S_j^{(2)}(n, p)\Big\} & (0 \le p < m) \\[4mm] \dfrac{2^n \sqrt{\pi}}{(2a+p)_n} \sum\limits_{j=0}^{m} \dfrac{(-1)^j(-m)_j(-n)_j(a)_j}{j!\,(d)_j} S_j^{(2)}(n, p) & (p \le 0). \end{cases}$$

Proof If we put $b = -n$, $c = 2a + p$ and $x = 2$ in (44), combined with (45), we find

$${}_3F_2\left[\begin{array}{cc} -n, a & d+m \\ 2a+p & d \end{array}; 2\right] = \sum_{j=0}^{m} \frac{(-2)^j(-m)_j(-n)_j(a)_j}{j!\,(2a+p)_j(d)_j} {}_2F_1\left[\begin{array}{c} -n+j, a+j \\ 2a+p+j \end{array}; 2\right].$$

Substitution of (8) and (11) with $n \to n - j$, $a \to a + j$ and the parameter in the denominator replaced by $2(a+j) + p - j$, followed by some straightforward algebra to distinguish between the cases with $p - j \ge 0$ and $p - j < 0$, leads to the results stated in the theorem. □

We conclude this section by giving some specific examples of the summation in Theorem 3. Alternatively, these results may be obtained from (44), (45), and the ${}_2F_1(2)$ summations listed in Section 3. For convenience in presentation we define

$$\wp_k := \frac{(a)_k}{(d)_k} \qquad (k = 1, 2, \ldots).$$

When $m = p$, we have for $1 \leq m \leq 4$ and positive integer n

$$
{}_3F_2\left[\begin{array}{cc} -2n, a, & d+1 \\ 2a+1, & d \end{array}; 2\right] = \frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n},
\tag{46}
$$

$$
{}_3F_2\left[\begin{array}{cc} -2n-1, a, & d+1 \\ 2a+1, & d \end{array}; 2\right] = \frac{(\frac{3}{2})_n}{(2a+1)(a+\frac{3}{2})_n}\left(1 - 2\wp_1\right),
\tag{47}
$$

$$
{}_3F_2\left[\begin{array}{cc} -2n, a, & d+2 \\ 2a+2, & d \end{array}; 2\right] = \frac{(\frac{1}{2})_n}{(a+\frac{3}{2})_n}\left\{1 + \frac{2n}{a+1}\left[1 - 2\wp_1 + 2\wp_2\right]\right\},
\tag{48}
$$

$$
{}_3F_2\left[\begin{array}{cc} -2n-1, a, & d+2 \\ 2a+2, & d \end{array}; 2\right] = \frac{(\frac{3}{2})_n}{(a+1)(a+\frac{3}{2})_n}\left\{1 - 2\wp_1\right\},
\tag{49}
$$

$$
{}_3F_2\left[\begin{array}{cc} -2n, a, & d+3 \\ 2a+3, & d \end{array}; 2\right] = \frac{(\frac{1}{2})_n}{(a+\frac{3}{2})_n}\left\{1 + \frac{4n}{a+2}(1 - 3\wp_1 + 3\wp_2)\right\},
\tag{50}
$$

$$
{}_3F_2\left[\begin{array}{cc} -2n-1, a, & d+3 \\ 2a+3, & d \end{array}; 2\right] = \frac{(\frac{3}{2})_n}{(2a+3)(a+\frac{5}{2})_n}\left\{3 - 6\wp_1 + \frac{4n}{a+2}(1 - 3\wp_1 + 3\wp_2 - 2\wp_3)\right\},
\tag{51}
$$

$$
{}_3F_2\left[\begin{array}{cc} -2n, a, & d+4 \\ 2a+4, & d \end{array}; 2\right]
$$

$$
= \frac{(\frac{1}{2})_n}{(a+\frac{5}{2})_n}\left\{1 + \frac{8n}{a+2}(1 - 3\wp_1 + 3\wp_2) + \frac{8n(n-1)}{(a+2)(a+3)}(1 - 4\wp_1 + 6\wp_2 - 4\wp_3 + 2\wp_4)\right\},
\tag{52}
$$

and

$$
{}_3F_2\left[\begin{array}{cc} -2n-1, a, & d+4 \\ 2a+4, & d \end{array}; 2\right] = \frac{2(\frac{3}{2})_n}{(a+2)(a+\frac{5}{2})_n}\left\{1 - 2\wp_1 + \frac{2n}{a+3}(1 - 4\wp_1 + 6\wp_2 - 4\wp_3)\right\}.
\tag{53}
$$

Finally, we give some evaluations when $m \neq p$. When $m = 2$ and $p = 0, \pm 1$, for example, we have

$$
{}_3F_2\left[\begin{array}{cc} -2n, a, & d+2 \\ 2a, & d \end{array}; 2\right] = \frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n}\left\{1 + \frac{4n}{a}(\wp_1 + \wp_2) + \frac{8n(n-1)}{a(a+1)}\wp_2\right\},
\tag{54}
$$

$$
{}_3F_2\left[\begin{array}{cc} -2n-1, a, & d+2 \\ 2a, & d \end{array}; 2\right] = -\frac{2(\frac{3}{2})_n}{a(a+\frac{1}{2})_n}\left\{\wp_1 + \frac{2n\wp_2}{a+1}\right\},
\tag{55}
$$

$$
{}_3F_2\left[\begin{array}{cc} -2n, a, & d+2 \\ 2a+1, & d \end{array}; 2\right] = \frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n}\left\{1 + \frac{4n\wp_2}{a+1}\right\},
\tag{56}
$$

$$
{}_3F_2\left[\begin{array}{cc} -2n-1, a, & d+2 \\ 2a+1, & d \end{array}; 2\right] = \frac{(\frac{3}{2})_n}{(2a+1)(a+\frac{3}{2})_n}\left\{1 - 4\wp_1 - \frac{4n\wp_2}{a+1}\right\},
\tag{57}
$$

$$_3F_2\left[\begin{array}{cc} -2n, a, & d+2 \\ 2a-1, & d \end{array}; 2\right] = \frac{(\frac{1}{2})_n}{(a-\frac{1}{2})_n}\left\{1 + \frac{4n}{a}(2\wp_1 + \wp_2) + \frac{16n(n-1)}{a(a+1)}\wp_2\right\}, \tag{58}$$

and

$$_3F_2\left[\begin{array}{cc} -2n-1, a, & d+2 \\ 2a-1, & d \end{array}; 2\right] = -\frac{(\frac{3}{2})_n}{(2a-1)(a+\frac{1}{2})_n}\left\{1 + 4\wp_1 + \frac{4n}{a}(2\wp_1 + 3\wp_2) + \frac{16n(n-1)}{a(a+1)}\wp_2\right\}. \tag{59}$$

5. A second application of Theorem 1 to $_1F_1(x)$

Kummer's second theorem applied to the confluent hypergeometric function $_1F_1$ is [15, p. 12]

$$e^{-x/2}\,_1F_1\left[\begin{array}{c} a \\ 2a \end{array}; x\right] = \,_0F_1\left[\begin{array}{c} - \\ a+\frac{1}{2} \end{array}; \frac{x^2}{16}\right] = (\tfrac{1}{4}x)^{\frac{1}{2}-a}\Gamma(a+\tfrac{1}{2})I_{a-\frac{1}{2}}(\tfrac{1}{2}x), \tag{60}$$

where $I_\nu(z)$ denotes a modified Bessel function of the first kind. We now show how the result in Theorem 1 can be used to derive a generalization of the above theorem for the functions

$$e^{-x/2}\,_1F_1\left[\begin{array}{c} a \\ 2a\pm j \end{array}; x\right]$$

for arbitrary integer j.

We have upon series expansion

$$e^{-x/2}\,_1F_1\left[\begin{array}{c} a \\ 2a\pm j \end{array}; x\right] = \sum_{n=0}^\infty \frac{(-\frac{1}{2}x)^n}{n!}\sum_{m=0}^\infty \frac{(a)_m}{(2a\pm j)_m}\frac{x^m}{m!} = \sum_{n=0}^\infty\sum_{m=0}^\infty \frac{(-1)^n(a)_m x^{m+n}}{2^n(2a\pm j)_m m! n!}.$$

Making the change of summation index $n \to n-m$ and using the fact that $(n-m)! = (-1)^m m!/(-n)_m$, we find

$$e^{-x/2}\,_1F_1\left[\begin{array}{c} a \\ 2a\pm j \end{array}; x\right] = \sum_{n=0}^\infty\sum_{m=0}^n \frac{(-1)^n(a)_m(-n)_m x^n}{2^{n-m}(2a\pm j)_m m! n!}$$

$$= \sum_{n=0}^\infty \frac{(-\frac{1}{2}x)^n}{n!}\,_2F_1\left[\begin{array}{c} -n, a \\ 2a\pm j \end{array}; 2\right].$$

Separation of the above sum into even and odd n and use of the evaluation of the $_2F_1(2)$ series given in (6) and (7) followed by inversion of the order of summation leads to the result in the following theorem.

Theorem 4 *For integer j we have*

$$e^{-x/2}\,_1F_1\left[\begin{array}{c} a \\ 2a\pm j \end{array}; x\right] = \sum_{r=0}^{j_0}(-1)^r\binom{j}{2r}\sum_{n=0}^\infty \frac{(-n)_r(a+\delta_j)_{n-r}x^{2n}}{2^{2n}(2a\pm j)_{2n}n!}$$

$$\mp\sum_{r=0}^{j_0}(-1)^r\binom{j}{2r+1}\sum_{n=0}^\infty \frac{(-n)_r(a+\delta_j)_{n-r}x^{2n+1}}{2^{2n+1}(2a\pm j)_{2n+1}n!}, \tag{61}$$

where j_0 and δ_j are defined in Theorem 1.

When $j = 0$ it is easily seen that (61) reduces to (60).

The result (61) is given in a different form in terms of modified Bessel functions in [11, Vol. 3, p. 579]. For some interesting and general expressions for $_2F_1(z)$ we refer to [1, pp. 563–564, Eq. (1)–(7)].

An alternative method of proof of this relation can be obtained from the following generalization of a result due to Ramanujan, considered in [7] for $j = 0, \pm 1, \ldots, \pm 5$.

Theorem 5 *Let $\phi(t)$ be analytic for $|t - 1| < R$, where $R > 1$. Suppose that a and $\phi(t)$ are such that the order of summation in*

$$\sum_{n=0}^{\infty} \frac{2^n (a)_n}{(2a \pm j)_n n!} \sum_{k=n}^{\infty} \frac{(-1)^k}{k!} (-k)_n \phi^{(k)}(1)$$

may be inverted. Then

$$\sum_{n=0}^{\infty} \frac{2^n (a)_n \phi^{(n)}(0)}{(2a \pm j)_n n!} = \sum_{n=0}^{\infty} \frac{\phi^{(2n)}(1)}{(2a \pm j)_{2n} n!} \sum_{r=0}^{j_0} (-1)^r \binom{j}{2r} (-n)_r (a + \delta_j)_{n-r}$$

$$\mp \sum_{n=0}^{\infty} \frac{\phi^{(2n+1)}(1)}{(2a \pm j)_{2n+1} n!} \sum_{r=0}^{j_0} (-1)^r \binom{j}{2r+1} (-n)_r (a + \delta_j)_{n-r} \tag{62}$$

for $j = 0, 1, 2, \ldots$, where j_0 and δ_j are defined in Theorem 1.

Proof We employ the identity obtained in [7, §2] relating the sum of derivatives of $\phi(t)$ evaluated at $t = 0$ to a sum of derivatives evaluated at $t = 1$ given by

$$\sum_{n=0}^{\infty} \frac{2^n (a)_n \phi^{(n)}(0)}{(2a \pm j)_n n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{(n)}(1)}{n!} \, _2F_1 \left[\begin{array}{c} -n, a \\ 2a \pm j \end{array} ; 2 \right].$$

Separation of the sum on the right-hand side into even and odd n and insertion of the $_2F_1(2)$ series evaluations in (6) and (7) then establishes the theorem. □

If we take $\phi(t) = e^{xt/2}$ in (62), where x is an arbitrary variable, we immediately obtain (61) after reversal of the order of summation. Other choices for $\phi(t)$ in (62) were considered in [7] but with the integer j restricted to the values $j = 0, \pm 1, \ldots, \pm 5$.

6. Acknowledgment

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