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The evaluation of single Bessel function sums

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Abstract

We examine convergent representations for the sums of Bessel functions

$$\sum_{n=1}^{\infty}\frac{J_\nu(nx)}{n^\alpha}\quad (x>0)\quad\text{and}\quad \sum_{n=1}^{\infty}\frac{K_\nu(nz)}{n^\alpha}\quad (\Re(z)>0),$$

together with their alternating versions, by a Mellin transform approach. We take α to be a real parameter with $\nu>-\frac{1}{2}$ for the first sum and $\nu\geq 0$ for the second sum. Such representations enable easy computation of the series in the limit x or $z\to 0+$. Particular attention is given to logarithmic cases that occur for certain values of α and ν.

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1. Introduction

We consider the sums

$$S_{\alpha,\nu}^{(1)}(x)=\sum_{n=1}^{\infty}\frac{J_\nu(nx)}{n^\alpha},\quad\text{and}\quad S_{\alpha,\nu}^{(2)}(z)=\sum_{n=1}^{\infty}\frac{K_\nu(nz)}{n^\alpha},\tag{1.1}$$

where $J_\nu(z)$ is the Bessel function of the first kind and $K_\nu(z)$ the modified Bessel function, together with their alternating versions when the additional factor $(-)^{n-1}$ is present. In the first sum it is supposed that the order $\nu>-\frac{1}{2}$ and that α is a real parameter. The sum $S_{\alpha,\nu}^{(1)}(x)$ converges absolutely for $\alpha>\frac{1}{2}$, although convergence (non-absolute) is assured when $\alpha>-\frac{1}{2}$. The sum $S_{\alpha,\nu}^{(2)}(z)$ converges without restriction on the parameter α on account of the exponential decay of $K_\nu(nz)$ as $n\to\infty$.

The sums in (1.1) become difficult to compute in the limit x or $z\to 0$ due to the resulting slow convergence of the series. The sum $S_{\alpha,\nu}^{(1)}(x)$ has been considered by Tričković *et al.* in [7], where approaches using Poisson's summation formula and Bessel's integral were employed to derive convergent expansions suitable for computation when x is small. Here, we use a possibly more straightforward method based on Mellin transforms; see, for example [5, Section 4.1.1]. Such an approach basically reduces the problem to routine residue evaluation and leads to an easier understanding of the logarithmic cases that can arise when α and ν assume certain values.

Sums involving the product of m J-Bessel functions have been termed m-dimensional Schlömilch-type series by Miller [3]. The sum involving a product of two J-Bessel functions has been considered by Williamson [9] and by Dominici *et al.* [1]. A discussion of series involving two Bessel functions (involving $J_\nu(z)$ and both modified Bessel functions) has been given recently by the author in [6] also using a Mellin transform approach.

2. The series $S^{(1)}_{\alpha,\nu}(x)$ We consider the sum

$$S^{(1)}_{\alpha,\nu}(x)=\sum_{n=1}^\infty \frac{J_\nu(nx)}{n^\alpha}=x^\alpha\sum_{n=1}^\infty f(nx),\qquad f(\tau)=\frac{J_\nu(\tau)}{\tau^\alpha},\tag{2.1}$$

where $\alpha>\frac{1}{2}$, $\nu>-\frac{1}{2}$ and $x>0$. From the elementary properties of the Bessel function it is seen that $f(\tau)=O(\tau^{\nu-\alpha})$ as $\tau\to 0+$ and $f(\tau)=O(\tau^{-\alpha-1/2})$ as $\tau\to+\infty$.

We introduce the Mellin transform of $f(\tau)$ by [4, p. 243]

$$\begin{aligned}F(s)=\int_0^\infty \tau^{s-1}f(\tau)\,d\tau &= \int_0^\infty \frac{J_\nu(\tau)}{\tau^\lambda}\,d\tau,\qquad \lambda:=\alpha+1-s\\ &= 2^{-\lambda}\,\frac{\Gamma(\frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}\lambda+\frac{1}{2})}\end{aligned}\tag{2.2}$$

valid in the strip of analyticity $\alpha-\nu<\Re(s)<\alpha+\frac{1}{2}$. Then we have by the Mellin inversion theorem (see, for example, [5, p. 80])

$$f(\tau)=\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} F(s)\,\tau^{-s}ds,\qquad \alpha-\nu<c<\alpha+\tfrac{1}{2}.\tag{2.3}$$

It may be observed, from the convergence conditions following (2.2), that the right-hand boundary of the strip of analyticity $\Re(s)=\alpha+\frac{1}{2}>1$ and the left-hand boundary is $\Re(s)<\alpha+\frac{1}{2}$. Then we obtain [5, p. 118]

$$\begin{aligned}S^{(1)}_{\alpha,\nu}(x)&=\frac{x^\alpha}{2\pi i}\int_{c-\infty i}^{c+\infty i} F(s)\zeta(s)x^{-s}ds\\ &=\frac{(\frac{1}{2}x)^\alpha}{4\pi i}\int_{c-\infty i}^{c+\infty i}\frac{\Gamma(\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}s)}{\Gamma(1+\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}\,\zeta(s)\,(\tfrac{1}{2}x)^{-s}ds,\end{aligned}\tag{2.4}$$

where $\max\{1,\alpha-\nu\}<c<\alpha+\frac{1}{2}$ and $\zeta(s)$ denotes the Riemann zeta function.

The integrand in (2.4) has simple poles at $s = 1$ resulting from $\zeta(s)$ and at

$$s_m = \alpha - \nu - 2m, \qquad m = 0, 1, 2, \ldots \tag{2.5}$$

resulting from the numerator gamma function. If, however, $\alpha - \nu$ is a positive odd integer then the pole at $s = 1$ is double; and if $\alpha - \nu$ is an even integer then some, or all, of these poles are regular points on account of the trivial zeros of $\zeta(s)$ at $s = -2, -4, \ldots$. The procedure now consists of displacement of the integration path to the left over the poles at $s = 1$, where $\zeta(s)$ has residue 1, and the first M poles of the sequence $\{s_m\}$ to find (in the case when all poles are simple)

$$S^{(1)}_{\alpha,\nu}(x) = x^{\alpha-1}F(1) + \sum_{m=0}^{M-1} \frac{(-)^m \zeta(s_m)}{m!\Gamma(1+\nu+m)} \left(\frac{x}{2}\right)^{2m+\nu} + R_M(x), \tag{2.6}$$

where $R_M(x)$ is the remainder. The justification of this process is considered in the appendix, where it is established that $R_M(x) = O((x/2\pi)^{2M})$. It then follows that $R_M(x) \to 0$ as $M \to \infty$ when $0 < x < 2\pi$.

The domain of convergence of the series in (2.6) as $M \to \infty$ can be determined by examining the large-m behaviour of the terms. If we use the functional relation for $\zeta(s)$ given by [4, p. 603]

$$\zeta(s) = 2^s \pi^{s-1} \zeta(1-s)\Gamma(1-s) \sin \tfrac{1}{2}\pi s, \tag{2.7}$$

this behaviour is essentially controlled by

$$(-)^m (2\pi)^{-2m} \frac{\zeta(1-s_m)\Gamma(1-s_m)}{m!\Gamma(1+\nu+m)} \sin(\tfrac{1}{2}\pi s_m)(\tfrac{1}{2}x)^{2m}$$

$$= \left(\frac{x}{2\pi}\right)^{2m} O\left(\frac{\Gamma(m+\frac{1}{2}(1+\nu-\alpha))\Gamma(m+\frac{1}{2}(2+\nu-\alpha))}{m!\Gamma(1+\nu+m)}\right) = \left(\frac{x}{2\pi}\right)^{2m} O(m^{-\alpha-1/2})$$

as $m \to \infty$, provided $\alpha - \nu$ is not an even integer. Here we have made use of the duplication formula for the gamma function, the well-known result $\Gamma(m+a)/\Gamma(m+b) \sim m^{a-b}$ as $m \to \infty$ and the fact that $\zeta(2m+1+\nu-\alpha) = O(1)$ for large m. This shows that the sum in (2.6) converges as $m \to \infty$ in the wider domain $0 < x \le 2\pi$ when $\alpha > \frac{1}{2}$ (and $\alpha - \nu$ is not an even integer).

Then we have the expansion:

Theorem 1. *Let $\alpha - \nu$ be non-integer with $\nu > -\frac{1}{2}$. Then we have the convergent expansion*

$$S^{(1)}_{\alpha,\nu}(x) = \left(\frac{x}{2}\right)^{\alpha-1} \frac{\Gamma(\frac{\nu-\alpha+1}{2})}{2\Gamma(\frac{\nu+\alpha+1}{2})} + \sum_{m=0}^{\infty} \frac{(-)^m \zeta(\alpha-\nu-2m)}{m!\Gamma(1+\nu+m)} \left(\frac{x}{2}\right)^{2m+\nu} \tag{2.8}$$

valid[1] *for $0 < x \le 2\pi$ when $\alpha > \frac{1}{2}$.*

[1]The domain of convergence of (2.8) is $0 < x < 2\pi$ when $-\frac{1}{2} < \alpha \le \frac{1}{2}$.

This result agrees with [7, Eq. (10)] who obtained the expansion (2.6) by other means in the domain $0 < x < 2\pi$ when $\alpha > -\frac{1}{2}$.

The expansion (2.6) has to be modified when $\alpha - \nu$ is an integer. If $\alpha - \nu = 2N$, $N = 0, 1, 2 \ldots$ then $s_m = 2N - 2m$. The sum in (2.8) terminates with the summation index satisfying $0 \leq m \leq N$ on account of the trivial zeros of $\zeta(s)$ at $s = -2, -4, \ldots$. If $\alpha - \nu = -2, -4, \ldots$ the sum in (2.8) is absent. Thus we have

$$S_{\alpha,\nu}^{(1)}(x) = \begin{cases} \left(\dfrac{x}{2}\right)^{\alpha-1} \dfrac{\Gamma(-N+\frac{1}{2})}{2\Gamma(-N+\alpha+\frac{1}{2})} + \displaystyle\sum_{m=0}^{N} \dfrac{(-)^m \zeta(2N-2m)}{m!\Gamma(1+\nu+m)} \left(\dfrac{x}{2}\right)^{2m+\nu} \\ \qquad\qquad\qquad\qquad\qquad\qquad (\alpha-\nu = 2N,\ N = 0, 1, 2, \ldots) \\ \left(\dfrac{x}{2}\right)^{\alpha-1} \dfrac{\Gamma(N+\frac{1}{2})}{2\Gamma(N+\alpha+\frac{1}{2})} \qquad\qquad (\alpha-\nu = -2N,\ N = 1, 2, \ldots). \end{cases} \tag{2.9}$$

If $\alpha - \nu = 2N + 1$, $N = 0, 1, 2, \ldots$ then $s_m = 2N + 1 - 2m$. When $m = N$ we now have a double pole at $s = 1$. The residue at the double pole is given by the ϵ^{-1} coefficient in the expansion of the integrand in (2.4) about the point $s = 1 + \epsilon$ as $\epsilon \to 0$. Thus

$$\frac{(\frac{1}{2}x)^{\alpha-1-\epsilon}\Gamma(-N+\frac{1}{2}\epsilon)\zeta(1+\epsilon)}{2\Gamma(1+\nu+N-\frac{1}{2}\epsilon)} = \frac{(-)^N(\frac{1}{2}x)^{\alpha-1-\epsilon}\zeta(1+\epsilon)}{2\Gamma(1+\nu+N-\frac{1}{2}\epsilon)\Gamma(N+1-\frac{1}{2}\epsilon)} \frac{\pi}{\sin\frac{1}{2}\pi\epsilon}$$

$$= \frac{(-)^N(\frac{1}{2}x)^{\alpha-1}}{\epsilon^2 N!\Gamma(1+\nu+N)} \left\{1 + \epsilon\left(\gamma - \log\tfrac{1}{2}x + \tfrac{1}{2}\psi(N+1) + \tfrac{1}{2}\psi(1+\nu+N)\right) + O(\epsilon^2)\right\},$$

where $\gamma = 0.55721\ldots$ is the Euler-Mascheroni constant. Here we have employed the expansions $\zeta(1+\epsilon) = \epsilon^{-1}\{1 + \epsilon\gamma + O(\epsilon^2)\}$ and $\Gamma(a+\epsilon) = \Gamma(a)\{1 + \epsilon\psi(a) + O(\epsilon^2)\}$, where $\psi(a)$ is the digamma function. Thus we obtain the expansion

$$S_{\alpha,\nu}^{(1)}(x) = \frac{(-)^N(\frac{1}{2}x)^{2N+\nu}}{N!\Gamma(1+\nu+N)} \left\{\gamma - \log\tfrac{1}{2}x + \tfrac{1}{2}\psi(N+1) + \tfrac{1}{2}\psi(1+\nu+N)\right\}$$
$$+ \sum_{\substack{m=0 \\ m\neq N}}^{\infty} \frac{(-)^m\zeta(2N+1-2m)}{m!\Gamma(1+\nu+m)} \left(\frac{x}{2}\right)^{2m+\nu} \qquad (\alpha-\nu = 2N+1,\ N = 0, 1, 2, \ldots), \tag{2.10}$$

which is the result obtained in [7, Eq. (12)].

2.1. The alternating case

The alternating version of $S_{\alpha,\nu}^{(1)}(x)$ is given by

$$\hat{S}_{\alpha,\nu}^{(1)}(x) = \sum_{n=1}^{\infty} (-)^{n-1} \frac{J_\nu(nx)}{n^\alpha}.$$

It is easy to see that

$$\hat{S}^{(1)}_{\alpha,\nu}(x) = S^{(1)}_{\alpha,\nu}(x) - 2^{1-\alpha} S^{(1)}_{\alpha,\nu}(2x),$$

so that from Theorem 1 we obtain

Theorem 2. *Let $\nu > -\frac{1}{2}$ and $x > 0$. Then we have the expansion for the alternating series given by*

$$\hat{S}^{(1)}_{\alpha,\nu}(x) = 2^{1-\alpha} \sum_{m=0}^{\infty} \frac{(-)^{m-1}\hat{\zeta}(\alpha-\nu-2m)}{m!\Gamma(1+\nu+m)}\, x^{2m+\nu}, \tag{2.11}$$

where $\hat{\zeta}(s) := (1-2^{s-1})\zeta(s)$, valid for $0 < x \leq \pi$ when $\alpha > \frac{1}{2}$ (and $0 < x < \pi$ when $-\frac{1}{2} < \alpha \leq \frac{1}{2}$).

3. The series $S^{(2)}_{\alpha,\nu}(z)$ when $|\arg z| < \frac{1}{2}\pi$ We consider the sum involving the modified Bessel function

$$S^{(2)}_{\alpha,\nu}(z) = \sum_{n=1}^{\infty} \frac{K_\nu(nz)}{n^\alpha} \qquad (|\arg z| < \tfrac{1}{2}\pi,\ \nu \geq 0), \tag{3.1}$$

where we can take $\nu \geq 0$ since $K_\nu(z) = K_{-\nu}(z)$. The parameter α is unrestricted on account of the exponential decay of $K_\nu(z)$ in the sector $|\arg z| < \frac{1}{2}\pi$. From the Mellin transform [4, p. 243]

$$F(s) = \int_0^\infty \frac{K_\nu(\tau)}{\tau^\lambda}\, d\tau = 2^{-\lambda-1}\Gamma(\tfrac{1}{2}\nu-\tfrac{1}{2}\lambda+\tfrac{1}{2})\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\nu-\tfrac{1}{2}\lambda)$$

valid for $\Re(\lambda) \pm \nu < 1$, we obtain the integral representation

$$S^{(2)}_{\alpha,\nu}(z) = \frac{1}{8\pi i}\int_{c-\infty i}^{c+\infty i} \Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\alpha+\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\alpha-\tfrac{1}{2}\nu)\zeta(s)(\tfrac{1}{2}z)^{\alpha-s} ds \quad (|\arg z| < \tfrac{1}{2}\pi), \tag{3.2}$$

where $c > \max\{1, \alpha \pm \nu\}$.

The integrand in (3.2) has poles at $s = 1$ and at the two infinite sequences

$$s^{\pm}_m = \alpha \pm \nu - 2m, \qquad m = 0, 1, 2, \ldots . \tag{3.3}$$

The integration path is displaced to the left over the poles (we omit the details justifying this process, which are analogous to those presented in the appendix for the sum $S^{(1)}_{\alpha,\nu}(x)$), where the residues at $s = s^{\pm}_m$ (when all poles are simple) are given by

$$\frac{\pi}{2\sin\pi\nu}\, \frac{\zeta(s^{\pm}_m)}{m!\Gamma(1\mp\nu+m)} \left(\frac{z}{2}\right)^{2m\mp\nu}.$$

Using (2.7) we see that the large-m behaviour of these residues is $(|z|/2\pi)^{2m}$ $O(m^{-\alpha-1/2})$, so that the infinite residue sums will converge when $0 < |z| \leq 2\pi$ for $\alpha > \frac{1}{2}$. Then we obtain the following result:

Theorem 3. *Let $\nu>0$ and α be an unrestricted parameter. Then, provided $\nu\neq 1,2,\ldots,$ we have the following expansion (assuming the pole at $s=1$ to be simple)*

$$S^{(2)}_{\alpha,\nu}(z)=\frac{1}{4}\left(\frac{z}{2}\right)^{\alpha-1}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\alpha)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\nu-\tfrac{1}{2}\alpha)$$

$$+\frac{\pi}{2\sin\pi\nu}\left\{\sum_{m=0}^\infty\frac{\zeta(\alpha+\nu-2m)}{m!\Gamma(1-\nu+m)}\left(\frac{z}{2}\right)^{2m-\nu}-\sum_{m=0}^\infty\frac{\zeta(\alpha-\nu-2m)}{m!\Gamma(1+\nu+m)}\left(\frac{z}{2}\right)^{2m+\nu}\right\} \quad (3.4)$$

valid for $|\arg z|<\frac{1}{2}\pi$ and $0<|z|\leq 2\pi$ when $\alpha>\frac{1}{2}$ (and $0<|z|<\pi$ when $-\frac{1}{2}<\alpha\leq\frac{1}{2}$).

3.1. Examples

The expansion in (3.4) assumes all the poles to be simple. Double poles will arise when either (i) one of the poles of the sequences $\{s_m^\pm\}$ coincides with the pole of $\zeta(s)$ at $s=1$ or (ii) when $\nu=N$, $N=0,1,2,\ldots$. In this last case the poles $\{s_m^+\}$ are simple for $0\leq m\leq N-1$, with double poles for $m\geq N$ (the order of some of these poles will be reduced if $\alpha+N$ is an even integer due to the trivial zeros of $\zeta(s)$ at $s=-2,-4,\ldots$). If $\nu=N$ and a double pole from the sequences $\{s_m^\pm\}$ with $m\geq N$ coincides with $s=1$, then there will be a treble pole.

We do not consider all the above special cases in generality, but confine our attention to particular examples. The procedure involves the routine evaluation of residues of the integrand in (3.2).

Example 1. Let $\nu=N$, $N=0,1,2,\ldots$ with α non-integer; then we have $s_m^+=\alpha+N-2m$ and $s_m^-=\alpha-N-2m$. In this case the poles at $s=1$ and $s=s_m^+$ $(0\leq m\leq N-1)$ are simple, with those at $s=s_m^-$ $(m\geq 0)$ being double. The residues at the double poles are obtained as the coefficient of ϵ^{-1} in the expansion of the integrand

$$\tfrac{1}{4}\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\alpha+\tfrac{1}{2}N)\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\alpha-\tfrac{1}{2}N)\zeta(s)(\tfrac{1}{2}z)^{\alpha-s}$$

about $s=s_m^-+\epsilon$ to yield

$$\frac{(-)^N(\frac{1}{2}z)^{N+2m}\zeta(s_m^-)}{m!(m+N)!}\left\{\frac{\zeta(s_m^-)}{\zeta(s_m^-)}+\tfrac{1}{2}\psi(m+1)+\tfrac{1}{2}\psi(m+N+1)-\log\tfrac{1}{2}z\right\}.$$

Consequently we have the expansion (provided α is non-integer)

$$S^{(2)}_{\alpha,N}(z)=\frac{1}{4}\left(\frac{z}{2}\right)^{\alpha-1}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\alpha+\tfrac{1}{2}N)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\alpha-\tfrac{1}{2}N)$$

$$+\frac{1}{2}\sum_{m=0}^{N-1}\frac{(-)^m}{m!}\,\Gamma(N-m)\zeta(s_m^+)\left(\frac{z}{2}\right)^{2m-N}$$

$$+\sum_{m=0}^\infty \frac{\zeta(s_m^-)(-\frac{1}{2}z)^{2m+N}}{m!(m+N)!}\biggl\{\frac{\zeta'(s_m^-)}{\zeta(s_m^-)}+\tfrac{1}{2}\psi(m+1)+\tfrac{1}{2}\psi(m+N+1)-\log\tfrac{1}{2}z\biggr\}. \quad (3.5)$$

Example 2. Let $\alpha=\frac{2}{3}$, $\nu=\frac{4}{3}$ so that $s_m^+=2-2m$ and $s_m^-=\frac{2}{3}-2m$. The poles at $s=0,1,2$ and $s=s_m^-$ $(m\geq 0)$ are simple, with the poles belonging to the sequence $\{s_m^+\}$ for $m\geq 2$ being regular points on account of the trivial zeros of $\zeta(s)$. Hence we obtain

$$S^{(2)}_{\frac{2}{3},\frac{4}{3}}(z)=-\tfrac{1}{2}\sqrt{\pi}\Gamma(\tfrac{1}{6})(\tfrac{1}{2}z)^{1/3}+\tfrac{1}{2}\Gamma(\tfrac{1}{3})(\tfrac{1}{2}z)^{-2/3}\biggl(\frac{\pi^2}{6}-\frac{3z^2}{8}\biggr)$$
$$-\frac{\pi}{\sqrt{3}}\sum_{m=0}^\infty \frac{\zeta(\frac{2}{3}-2m)}{m!\Gamma(m+\frac{5}{3})}\biggl(\frac{z}{2}\biggr)^{2m+2/3}. \quad (3.6)$$

Example 3. Let $\alpha=\frac{7}{3}$, $\nu=\frac{2}{3}$ so that $s_m^+=3-2m$ and $s_m^-=\frac{5}{3}-2m$. In this case, all the poles are simple except for the double pole at $s=1$. The residue at $s=1$ is found by expanding the integrand about the point $s=1+\epsilon$ and so we obtain

$$S^{(2)}_{\frac{7}{3},\frac{2}{3}}(z)=-\tfrac{1}{2}\Gamma(-\tfrac{1}{3})(\tfrac{1}{2}z)^{4/3}\Bigl\{\tfrac{1}{2}+\tfrac{1}{2}\gamma+\tfrac{1}{2}\psi(-\tfrac{1}{3})-\log\tfrac{1}{2}z\Bigr\}$$
$$+\frac{\pi}{\sqrt{3}}\biggl\{\sum_{\substack{m=0\\m\neq 1}}^\infty \frac{\zeta(s_m^+)(\frac{1}{2}z)^{2m-2/3}}{m!\Gamma(m+\frac{1}{3})}-\sum_{m=0}^\infty \frac{\zeta(s_m^-)(\frac{1}{2}z)^{2m+2/3}}{m!\Gamma(m+\frac{5}{3})}\biggr\}. \quad (3.7)$$

Example 4. Let $\alpha=\nu=0$ so that $s_m^\pm=-2m$. In this case the integrand possesses simple poles at $s=1$ and at $s=-2,-4,\ldots$ on account of the trivial zeros of $\zeta(s)$; the pole at $s=0$ is double with residue $\frac{1}{2}(\gamma+\log(z/4\pi))$. The residues at $s=-2m$ $(m\geq 1)$ are obtained by writing the integrand in the form using (2.8)

$$\frac{1}{4\pi}\biggl(\frac{z}{4\pi}\biggr)^{-s}\Gamma^2(\tfrac{1}{2}s)\zeta(1-s)\Gamma(1-s)\sin\tfrac{1}{2}\pi s=\frac{1}{4}\biggl(\frac{z}{4\pi}\biggr)^{-s}\Gamma(\tfrac{1}{2}s)\,\frac{\zeta(1-s)\Gamma(1-s)}{\Gamma(1-\frac{1}{2}s)},$$

which possesses the residues

$$\frac{1}{2}\biggl(\frac{z}{4\pi}\biggr)^{2m}\frac{(-)^m}{(m!)^2}\,\zeta(2m+1)\Gamma(2m+1)=\frac{1}{2\sqrt{\pi}}\,\frac{(-)^m}{m!}\Gamma(m+\tfrac{1}{2})\zeta(2m+1)\biggl(\frac{z}{2\pi}\biggr)^{2m}.$$

Hence we have the expansion

$$S^{(2)}_{0,0}(z)=\frac{\pi}{2z}+\frac{1}{2}\biggl(\gamma+\log\frac{z}{4\pi}\biggr)+\frac{1}{2\sqrt{\pi}}\sum_{m=1}^\infty \frac{(-)^m}{m!}\Gamma(m+\tfrac{1}{2})\zeta(2m+1)\biggl(\frac{z}{2\pi}\biggr)^{2m} \quad (3.8)$$

valid for $0 < |z| \le 2\pi$.

If we replace $\zeta(2m+1)$ by its series expansion the infinite sum in (3.8) can be written as

$$\frac{1}{2}\sum_{n=1}^\infty \frac{1}{n}\sum_{m=1}^\infty \frac{(-)^m(\frac{1}{2})_m}{m!}\Big(\frac{z}{2\pi n}\Big)^{2m} = \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n}\Big\{{}_1F_0\Big(\frac{1}{2};-\frac{z^2}{(2\pi n)^2}\Big)-1\Big\}$$

$$= \pi\sum_{n=1}^\infty\Big\{\frac{1}{\sqrt{z^2+(2\pi n)^2}}-\frac{1}{2\pi n}\Big\}, \qquad \Re(z)>0,$$

where we have used the fact that ${}_1F_0(\frac{1}{2};-w^2) = (1+w^2)^{-1/2}$ for $\Re(w) > 0$. This is the result obtained by Watson [9].

Example 5. Let $\alpha = \nu = 3$ so that $s_m^+ = 6-2m$ and $s_m^- = -2m$. The poles at $s = 1, 2, 4, 6$ and at $s = s_m^-$ $(m \ge 1)$ are simple, with the pole at $s = 0$ being double possessing the residue

$$-\frac{z^3}{96}\Big(\log\Big(\frac{z}{4\pi}\Big)+\gamma-\frac{11}{12}\Big).$$

The residues at $s = s_m^-$ $(m \ge 1)$ are given by[2] $-\zeta'(-2m)(\frac{1}{2}z)^{2m+3}/(m!(m+3)!)$. Hence

$$S_{3,3}^{(2)}(z) = -\frac{\pi z^2}{30}-\frac{z^3}{96}\Big(\log\Big(\frac{z}{4\pi}\Big)+\gamma-\frac{11}{12}\Big)+\frac{1}{2}\sum_{m=0}^{2}\frac{(-)^m}{m!}\Gamma(3-m)\zeta(6-2m)(\tfrac{1}{2}z)^{2m-3}$$
$$-\sum_{m=1}^\infty \frac{\zeta'(-2m)(\frac{1}{2}z)^{2m+3}}{m!(m+3)!}. \qquad (3.9)$$

Example 6. Our final example has $\alpha = 3$, $\nu = 2$ so that $s_m^+ = 5-2m$ and $s_m^- = 1-2m$. The poles at $s = 3$ and 5 are simple, with those at $s = s_m^-$ $(m \ge 1)$ being double poles. The point $s = 1$ is a treble pole.

The residues at the double poles $s = 1-2m$, $m \ge 1$ are

$$\frac{\zeta(1-2m)(\frac{1}{2}z)^{2m+2}}{m!(m+2)!}\Big\{\frac{\zeta'(1-2m)}{\zeta(1-2m)}+\frac{1}{2}\psi(m+1)+\frac{1}{2}\psi(m+3)-\log\tfrac{1}{2}z\Big\}.$$

The residue at $s = 1$ is given by the coefficient of ϵ^{-2} in the expansion about $s = 1+\epsilon$ of

$$\frac{1}{4}\Gamma(\tfrac{1}{2}\epsilon)\Gamma(-2+\tfrac{1}{2}\epsilon)\zeta(1+\epsilon)(\tfrac{1}{2}z)^{2-\epsilon} = \frac{\Gamma^2(1+\frac{1}{2}\epsilon)\zeta(1+\epsilon)}{\epsilon^2(1-\frac{1}{2}\epsilon)(2-\frac{1}{2}\epsilon)}\,(\tfrac{1}{2}z)^{2-\epsilon}$$

[2]If desired, we can use the relation $\zeta'(-2m) = \frac{1}{2}(-)^m\zeta(2m+1)\Gamma(2m+1)(2\pi)^{-2m}$ for integer $m \ge 1$.

$$= \frac{z^2}{8\epsilon^3}\Big\{1 + (\frac{3}{4} - L)\epsilon + \Big(\frac{7}{16} - \frac{1}{2}\gamma^2 - \frac{3}{4}L + \frac{1}{2}L^2 - \gamma_1 + \frac{\pi^2}{24}\Big)\epsilon^2 + O(\epsilon^3)\Big\},$$

where we have used the result $\zeta(1+\epsilon) = \epsilon^{-1}(1 + \epsilon\gamma - \epsilon^2\gamma_1 + O(\epsilon^3))$, with $\gamma_1 = -0.0728158\ldots$ being the first Stieltjes constant, and $L := \log \frac{1}{2}z$. Hence the residue at $s = 1$ is

$$\frac{z^2}{8}\Big(\frac{7}{16} - \frac{1}{2}\gamma^2 - \frac{3}{4}L + \frac{1}{2}L^2 - \gamma_1 + \frac{\pi^2}{24}\Big).$$

We therefore obtain the expansion

$$S_{3,3}^{(2)}(z) = \frac{2}{z^2}\Big(\zeta(5) - \frac{1}{4}z^2\zeta(3)\Big) + \frac{z^2}{8}\Big(\frac{7}{16} - \frac{1}{2}\gamma^2 - \frac{3}{4}\log \tfrac{1}{2}z + \frac{1}{2}(\log \tfrac{1}{2}z)^2 - \gamma_1 + \frac{\pi^2}{24}\Big)$$
$$+ \sum_{m=1}^{\infty} \frac{\zeta(1-2m)(\frac{1}{2}z)^{2m+2}}{m!(m+2)!}\Big\{\frac{\zeta'(1-2m)}{\zeta(1-2m)} + \tfrac{1}{2}\psi(m+1) + \tfrac{1}{2}\psi(m+3) - \log \tfrac{1}{2}z\Big\}. \tag{3.10}$$

3.2. The alternating case

The alternating version of the modified Bessel function sum is

$$\hat{S}_{\alpha,\nu}^{(2)}(z) = \sum_{n=1}^{\infty} (-)^{n-1}\frac{K_\nu(nz)}{n^\alpha} \qquad (|\arg z| < \tfrac{1}{2}\pi).$$

This function satisfies the relation

$$\hat{S}_{\alpha,\nu}^{(2)}(z) = S_{\alpha,\nu}^{(2)}(z) - 2^{1-\alpha}S_{\alpha,\nu}^{(2)}(2z);$$

hence, from (3.4) and (3.5) we obtain the following:

Theorem 4. *When $\nu \neq 0, 1, 2, \ldots$ we have the expansion*

$$\hat{S}_{\alpha,\nu}^{(2)}(z) = \frac{\pi}{2^\alpha \sin \pi\nu}\Big\{\sum_{m=0}^{\infty} \frac{\hat{\zeta}(\alpha-\nu-2m)}{m!\Gamma(1+\nu+m)}\, z^{2m+\nu} - \sum_{m=0}^{\infty} \frac{\hat{\zeta}(\alpha+\nu-2m)}{m!\Gamma(1-\nu+m)}\, z^{2m-\nu}\Big\} \tag{3.11}$$

and when $\nu = N$, $N = 0, 1, 2, \ldots$

$$\hat{S}_{\alpha,N}^{(2)}(z) = 2^{-\alpha}\sum_{m=0}^{N-1} \frac{(-)^{m-1}}{m!}\,\Gamma(N-m)\hat{\zeta}(s_m^+)z^{2m-N}$$
$$-2^{1-\alpha}\sum_{m=0}^{\infty} \frac{\hat{\zeta}(s_m^-)(-z)^{2m+N}}{m!(m+N)!}\Big\{\frac{\zeta'(s_m^-)}{\zeta(s_m^-)} - \frac{\log 2}{\Lambda_m} + \tfrac{1}{2}\psi(m+1) + \tfrac{1}{2}\psi(m+N+1) - \log \tfrac{1}{2}z\Big\} \tag{3.12}$$

valid for $|\arg z| < \frac{1}{2}\pi$ and $0 < |z| \leq \pi$ when $\alpha > \frac{1}{2}$ (and $0 < |z| < \pi$ when $-\frac{1}{2} < \alpha \leq \frac{1}{2}$). In these expressions $s_m^\pm = \alpha \pm \nu - 2m$, $\Lambda_m := 1 - 2^{\alpha-\nu-2m-1}$ and $\hat{\zeta}(s)$ is defined in Theorem 2.

4. Concluding remarks All the expansions presented have been verified numerically with the aid of *Mathematica*. In the computation of the expansions in (3.5) and (3.12) the term $\zeta'(s_m^-)/\zeta(s_m^-)$ can be computed using the command `Zeta'[s]/Zeta[s]` for the derivative. Alternatively, if so desired, we can express this ratio in terms of zeta functions of positive argument for large m by the result

$$\frac{\zeta(a-2m)}{\zeta(a-2m)} = \log 2\pi + \tfrac{1}{2}\pi \cot \tfrac{1}{2}\pi a - \psi(2m+1-a) - \frac{\zeta'(2m+1-a)}{\zeta(2m+1-a)}.$$

In particular, the large-m behaviour of the terms in the various expansions was examined to verify the domains of convergence given in Theorems 1–4.

Finally, we remark that combination of the sums $S_{\alpha,\nu}^{(1)}(x)$ and $S_{\alpha,\nu}^{(2)}(z)$ with their alternating versions enables the evaluation of the sums with odd summation index, namely

$$\sum_{n=1}^{\infty} \frac{J_\nu((2n-1)x)}{(2n-1)^\alpha} = \tfrac{1}{2}\{S_{\alpha,\nu}^{(1)}(x) + \hat{S}_{\alpha,\nu}^{(1)}(x)\}$$

and

$$\sum_{n=1}^{\infty} \frac{K_\nu((2n-1)z)}{(2n-1)^\alpha} = \tfrac{1}{2}\{S_{\alpha,\nu}^{(2)}(z) + \hat{S}_{\alpha,\nu}^{(2)}(z)\}.$$

Appendix: Justification of the path displacement argument We consider the integral in (2.4) taken round the rectangular contour with vertices at $c \pm iT$, $-d \pm iT$, where $d = 2M-1-\alpha+\nu > 0$ so that the side parallel to the imaginary axis passes midway between the poles at $s = \alpha - \nu - 2M + 2$ and $s = \alpha - \nu - 2M$. The contribution from the upper and lower sides $s = \sigma \pm iT$, $-d \leq \sigma \leq c$ as $T \to \infty$ can be estimated by use of the standard results

$$|\Gamma(\sigma \pm it)| \sim \sqrt{2\pi} t^{\sigma - \frac{1}{2}} e^{-\frac{1}{2}\pi t} \qquad (t \to +\infty), \tag{A.1}$$

which follows from Stirling's formula for the gamma function, and [2, p. 25]

$$|\zeta(\sigma \pm it)| = O(t^{\hat{\mu}(\sigma)} \log^\beta t) \qquad (t \to +\infty)$$

where $\hat{\mu}(\sigma) = 0$ $(\sigma > 1)$, $\frac{1}{2} - \frac{1}{2}\sigma$ $(0 \leq \sigma \leq 1)$, $\frac{1}{2} - \sigma$ $(\sigma \leq 0)$ and $\beta = 0$ $(\sigma > 1)$, 1 $(\sigma \leq 1)$. Then it follows that the function $F(s)$ in (2.2) satisfies

$$|F(\sigma \pm it)| = O\left(\frac{\Gamma(\frac{\nu-\alpha+\sigma}{2} \pm \frac{1}{2}it)}{\Gamma(\frac{\nu+\alpha-\sigma}{2} \mp \frac{1}{2}it)}\right) = O(t^{\sigma-\alpha-1}) \qquad (t \to \infty). \tag{A.2}$$

Hence the modulus of the integrand on these horizontal paths is $O(T^\xi \log T)$ as $T \to \infty$, where $\xi = \sigma + \hat{\mu}(\sigma) - \alpha - 1$. Taking into account the different forms of $\hat{\mu}(\sigma)$ and the fact that $\alpha > 0$, we obtain the order estimate $O(T^{-\frac{1}{2}} \log T)$ so that the contribution from these paths vanishes as $T \to \infty$.

Displacement of the integration path over the pole at $s = 1$, where $\zeta(s)$ has residue 1, and the first M poles of the sequence $\{s_m\}$ in (2.5) we find (provided $\alpha - \nu$ is not an integer) the expression given in (2.6), where the remainder $R_M(x)$ is given by

$$R_M(x) = \frac{x^\alpha}{2\pi i} \int_{-d-\infty i}^{-d+\infty i} F(s)\,\zeta(s)\,x^{-s} ds$$

$$= \frac{(\frac{1}{2}x)^\alpha}{4\pi} \int_{-\infty}^{\infty} \tilde{F}(-d+it)\zeta(-d+it)(\tfrac{1}{2}x)^{d-it} dt, \quad \text{(A.3)}$$

where we have set $\tilde{F}(s) = 2^\lambda F(s)$. Use of the functional relation (2.7) together with the fact that $|\zeta(\sigma \pm it)| \leq \zeta(\sigma)$ when $\sigma > 1$ shows that for real t

$$\begin{aligned} |\zeta(-d+it)| &\leq \frac{(2\pi)^{-d}}{\pi} |\zeta(1+d-it)|\,|\Gamma(1+d-it)| \cosh(\tfrac{1}{2}\pi|t|) \\ &= \pi^{-2M} \zeta(2M-\alpha+\nu)\, g(t), \end{aligned}$$

where

$$g(t) = \pi^{\alpha-\nu-\frac{1}{2}} |\Gamma(\tfrac{1}{2}d+\tfrac{1}{2}-\tfrac{1}{2}it)\Gamma(1+\tfrac{1}{2}d-\tfrac{1}{2}it)|\, \cosh(\tfrac{1}{2}\pi|t|)$$

$$= O(|t|^{2M-\alpha+\nu-\frac{1}{2}}) \qquad (t \to \pm\infty)$$

and we have used the duplication formula for the gamma function and (A.1). Then, since $\zeta(2M - \alpha + \nu) = O(1)$ for large M,

$$|R_M(x)| = O\Big(\Big(\frac{x}{2\pi}\Big)^{2M} \int_{=\infty}^{\infty} |\tilde{F}(-d+it)|\, g(t)\, dt\Big).$$

From (A.2) the integrand is $O(|t|^{-2M-\nu}|t|^{2M-\alpha+\nu-\frac{1}{2}}) = O(|t|^{-\alpha-\frac{1}{2}})$ as $t \to \pm\infty$. Hence the integral converges for $\alpha > \frac{1}{2}$ and is independent of x. Consequently we find that $|R_M(x)| = O((x/2\pi)^{2M})$, and hence $R_M(x) \to 0$ as $M \to \infty$ provided $0 < x < 2\pi$.

It is conjectured that a more refined treatment of the remainder integral $R_M(x)$, which takes into account the oscillatory nature of the integrand, would produce a more precise estimate that included, in addition to the basic order term $(x/2\pi)^{2M}$, a negative power of M. This would yield $R_M(x) \to 0$ as $M \to \infty$ for $0 < x \leq 2\pi$ when $\alpha > \frac{1}{2}$.

References

[1] D.E. Dominici, P.M.W. Gill and T. Limpanuparb, A remarkable identity involving Bessel functions, Proc. Roy. Soc. London **468A** (2012) 2667–2681.

[2] A. Ivić, *The Riemann Zeta-Function*, Wiley, New York, 1985.

[3] A.R. Miller, On certain Schlömilch-type series, J. Comp. Appl. Math. **80** (1997) 83–95.

[4] F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.

[5] R.B. Paris and D. Kaminski, *Asymptotics and Mellin-Barnes Integrals*, Cambridge University Press, Cambridge, 2001.

[6] R.B. Paris, The evaluation of infinite sums of products of two Bessel functions. (2018) arXiv:1803.02757.

[7] S.B. Tričković, M.V. Vidanović and M.S. Stanković, On the summation of series in terms of Bessel functions, J. Anal. Appl. **25** (2006) 393–406.

[8] G.N. Watson, Some self-reciprocal functions, Qu. J. Math. (Oxford) **2** (1931) 298.

[9] A.G. Williamson, A note on the summation of an infinite series involving a hypergeometric function, Bull. Austral. Math. Soc. **10** (1974) 305–309.

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