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The expansion of a finite number of terms of the Gauss hypergeometric function of unit argument and the Landau constants

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Abstract

We obtain convergent inverse factorial expansions for the sum $S_n(a,b;c)$ of the first $n \geq 1$ terms of the Gauss hypergeometric function ${}_2F_1(a,b;c;1)$ of unit argument. The form of these expansions depends on the location of the parametric excess $s := c-a-b$ in the complex s-plane. The leading behaviour as $n \to \infty$ agrees with previous results in the literature. The case $a = b = \frac{1}{2}$, $c = 1$ corresponds to the Landau contants for which an expansion is obtained.

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1 Introduction

If $f(z) = \sum_{k=0}^\infty a_k z^k$ is analytic inside the unit disc and satisfies $|f(z)| < 1$ when $|z| < 1$, then $|\sum_{k=0}^n a_k| \leq G_n$, where G_n are the Landau constants defined by

$$G_n = \sum_{k=0}^n 2^{-4k} \binom{2k}{k}^2 = \frac{1}{\pi} \sum_{k=0}^n \frac{\Gamma^2(k+\frac{1}{2})}{(k!)^2}, \qquad n \geq 1. \tag{1.1}$$

It was shown by Landau [7] that $G_n \sim \pi^{-1} \log n$ as $n \to \infty$.

Subsequently, it was established by Watson [16] that G_n is given by the convergent expansion

$$G_n = \frac{\Gamma^2(n+\frac{3}{2})}{\pi\Gamma(n+1)\Gamma(n+2)} \sum_{k=0}^\infty \frac{(\frac{1}{2})_k(\frac{1}{2})_k}{(n+2)_k k!} \{\psi(k+n+2) + \psi(k+1) - 2\psi(k+\tfrac{1}{2})\} \tag{1.2}$$

by writing G_n as an integral over $[0,1]$ that involves the complete elliptic integral. Here and throughout $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the gamma function, also known as the ψ-function, and $(a)_k = \Gamma(a+k)/\Gamma(a)$ denotes the Pochhammer symbol. From this result combined with the basic property $\psi(z+1) = \psi(z) + 1/z$ and the asymptotic expansion [13, p. 140]

$$\psi(z) \sim \log z - \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kz^{2k}} \qquad (z \to +\infty) \tag{1.3}$$

in terms of the Bernoulli numbers B_{2k}, Watson deduced the asymptotic expansion

$$G_n \sim \frac{1}{\pi}(\log(n+1) + \gamma + 4\log 2) - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} + \cdots \tag{1.4}$$

as $n \to \infty$, where $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant.

A different expansion for G_n was given by Cvijović and Klinowski [3] in the form

$$G_n = \frac{1}{\pi}(\psi(n+\tfrac{3}{2}) + \gamma + 4\log 2) - \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(\frac{1}{2})_k(\frac{1}{2})_k}{k\,k!(n+\frac{3}{2})_k}. \tag{1.5}$$

This elegant result is a convergent inverse factorial expansion that is directly amenable to computation for large n. Its proof is particularly simple and relies on writing the coefficients in (1.1) in terms of the Gauss hypergeometric series ${}_2F_1(\frac{1}{2}, \frac{1}{2}; k+\frac{3}{2}; 1)$ by application of the well-known Gauss summation theorem

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \Re(c-a-b) > 0 \tag{1.6}$$

followed by expansion of the ${}_2F_1(1)$ as a convergent series.

Based on (1.5), Nemes [12] established the expansion

$$G_n \sim \frac{1}{\pi}(\log(n+h) + \gamma + 4\log 2) - \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{g_k(h)}{(n+h)^k} \qquad (0 < h < \tfrac{3}{2}) \tag{1.7}$$

as $n \to \infty$, where the coefficients $g_k(h)$ are computable constants that depend on the Bernoulli polynomials and the Stirling numbers of the second kind. These coefficients are polynomials of degree k in h and satisfy the symmetry relation $g_k(h) = (-)^k g_k(\frac{3}{2} - h)$ for $k \geq 1$. The first few $g_k(h)$ are

$$g_1(h) = \frac{1}{4}(4h-3), \quad g_2(h) = \frac{1}{192}(96h^2 - 144h + 43),$$

$$g_3(h) = \frac{1}{384}(128h^3 - 288h^2 + 172h - 21).$$

Various authors have discussed the problem of determining upper and lower bounds for G_n; see, for example, [2, 3, 4, 11, 17].

From the second series in (1.1), it is seen that G_n can be related to a finite number of terms of the series expansion of the Gauss hypergeometric function of unit argument, viz.

$$G_n = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1)_{\rfloor n+1},$$

where the symbol $\rfloor_n$ signifies that only the first n terms are taken. The problem of the determination of

$$S_n(a, b; c) := {}_2F_1(a, b; c; 1)_{\rfloor n}$$

as $n \to \infty$ goes back to the papers of Hill [5, 6] and Bromwich [1] in the early years of the last century. By means of some lengthy algebraic manipulation and induction arguments, Hill established the leading behaviour of $S_n(a, b; c)$ for large n. Bromwich provided an alternative approach by use of Jensen's lemma applied to the generalised hypergeometric series of unit argument.

In this paper, we derive convergent inverse factorial expansions for $S_n(a, b; c)$ valid for all positive integer n. We achieve this by means of a Mellin-Barnes integral representation, which involves routine path displacement and evaluation of residues. The type of expansion obtained depends on the location of the parameter measuring the parametric excess $s := c - a - b$ of the hypergeometric series in the complex s-plane. From these results, we can obtain the asymptotic expansion (in inverse powers of n) of $S_n(a, b; c)$; the leading behaviour as $n \to \infty$ is easily recovered and is found to agree with the values obtained by Hill. In the special case $a = b = \frac{1}{2}$, $c = 1$, this leads to an alternative and simpler derivation of the inverse factorial expansion for the Landau constants G_n stated in (1.2).

2 The expansion for $S_n(a, b; c)$

We consider the sum to n terms of the hypergeometric series ${}_2F_1(a, b; c; 1)$ of unit argument

$$S_n(a, b; c) = \sum_{k=0}^{n-1} \frac{(a)_k (b)_k}{(c)_k k!} \tag{2.1}$$

and define the associated quantities

$$s := c - a - b, \quad \omega_n := \frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(n)\Gamma(n+c)}, \quad \lambda_n := \frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(n)\Gamma(n+a+b)}. \tag{2.2}$$

The parameters a, b and c are arbitrary complex constants and it will be supposed throughout that none of them equals zero or a negative integer. The quantity s is known as the parametric excess; if $\Re s > 0$ the series $S_n(a,b;c)$ converges to a finite limit as $n \to \infty$ (given by Gauss' summation theorem (1.6)), whereas if $\Re s \leq 0$ the series diverges in this limit.

From [15, p. 81], the finite sum $S_n(a,b;c)$ can be expressed as a ${}_3F_2$ series of unit argument in the form

$$S_n(a,b;c) = \frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(n)\Gamma(n+a+b)}\, {}_3F_2\left[\begin{array}{c} a, b, c+n-1 \\ c, n+a+b \end{array}; 1\right]. \tag{2.3}$$

The Mellin-Barnes integral representation for the ${}_3F_2(1)$ series is

$${}_3F_2\left[\begin{array}{c} a, b, c \\ d, e \end{array}; 1\right] = \frac{\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(b)\Gamma(e-c)\Gamma(d-a)\Gamma(d-b)}$$

$$\times \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \frac{\Gamma(a+\tau)\Gamma(b+\tau)}{\Gamma(e+\tau)}\Gamma(d-a-b-\tau)\Gamma(e-c+\tau)\Gamma(-\tau)\,d\tau, \tag{2.4}$$

where the integration path lies to the left of the poles of $\Gamma(-\tau)$ and $\Gamma(d-a-b-\tau)$. This result is given in [15, p. 112] in the derivation of Barnes' second lemma.

We present the evaluation of $S_n(a,b;c)$ for real or complex values of s in the following theorems.

Theorem 1. *Let n be a positive integer and $s = c-a-b$, with the quantities ω_n and λ_n as defined in (2.2). Then, for finite values of s such that $s \neq 0, \pm 1, \pm 2, \ldots$ and $c-a \neq 0, -1, -2, \ldots$, $c-b \neq 0, -1, -2, \ldots$ we have*

$$S_n(a,b;c) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - \frac{\omega_n \Gamma(c)}{s\Gamma(a)\Gamma(b)}\, {}_3F_2\left[\begin{array}{c} c-a, c-b, 1 \\ n+c, 1+s \end{array}; 1\right]. \tag{2.5}$$

In the case $s = 0$ we have

$$S_n(a,b;a+b) = \frac{\lambda_n \Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(n+a+b)_k k!}$$

$$\times\{\psi(n+a+b+k) + \psi(1+k) - \psi(a+k) - \psi(b+k)\}. \tag{2.6}$$

Proof. We employ the Mellin-Barnes integral representation in (2.4) to find from (2.3) that

$$S_n(a,b;c) = \frac{\lambda_n \Gamma(c)\Gamma(n+a+b)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)\Gamma(1-s)}$$

$$\times \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \frac{\Gamma(a+\tau)\Gamma(b+\tau)}{\Gamma(n+a+b+\tau)}\Gamma(1-s+\tau)\Gamma(s-\tau)\Gamma(-\tau)\,d\tau, \tag{2.7}$$

where the integration path may be suitably indented (if necessary) to separate the poles of $\Gamma(-\tau)$ and $\Gamma(s-\tau)$ from those of $\Gamma(1-s+\tau)$, $\Gamma(a+\tau)$ and $\Gamma(b+\tau)$. The above separation of the sequences of poles in the τ-plane is possible if $s \neq 1, 2, \ldots$, and $c-a$, $c-b \neq 0, -1, -2, \ldots$ and a, $b \neq 0, -1, -2, \ldots$.

Provided s is not an integer, the poles on the right of the path are all simple. We denote the integral appearing in (2.7) (including the factor $(2\pi i)^{-1}$) by I. Displacement of the integration path in the usual manner to the right over the poles at $\tau = k$, $0 \leq k \leq N$, and $\tau = s+k$, $0 \leq k \leq M = N - \lfloor \Re(s) \rfloor$, then yields

$$I = \frac{\pi}{\sin \pi s}\left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(n+a+b)}\sum_{k=0}^{N}\frac{(a)_k(b)_k}{(n+a+b)_k k!}\right.$$
$$\left.-\frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(n+c)\Gamma(1+s)}\sum_{k=0}^{M}\frac{(c-a)_k(c-b)_k}{(n+c)_k(1+s)_k}\right) + I_N,$$

where

$$I_N = \frac{1}{2\pi i}\int_{-\infty i+N+\delta}^{\infty i+N+\delta}\frac{\Gamma(a+\tau)\Gamma(b+\tau)}{\Gamma(n+a+b+\tau)\Gamma(1+\tau)}\,\frac{\pi^2 d\tau}{\sin \pi\tau\, \sin \pi(s-\tau)}$$

with $0 < \delta < 1$ chosen such that the integration path is a straight line that does not pass through a pole. From the well-known fact that

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b} \qquad (|z| \to \infty,\ |\arg z| < \pi),$$

it follows that the ratio of gamma functions appearing in the integrand of I_N is $O(|\tau|^{-n-1})$ as $|\tau| \to \infty$ on the displaced integration path. It is then readily seen that $I_N = O(N^{-n-1})$ and hence that $I_N \to 0$ as $N \to \infty$; the upper limits of summation in the above two series may therefore be replaced by ∞. Using the Gauss summation formula (1.6) and expressing the second sum as a ${}_3F_2(1)$ series, we then obtain the expansion in (2.5). This result can also be obtained directly from the relation between two ${}_3F_2(1)$ series given in [15, Eq. (4.3.4.3)].

When $s = 0$ ($c = a+b$), all the poles situated at $\tau = k$ are double poles. Putting $\tau = k+\epsilon$, we find that the behaviour of the integrand in (2.7) as $\epsilon \to 0$ is

$$\frac{1}{\epsilon^2}\,\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(n+a+b+k)k!}\{1+\epsilon[\psi(a+k)+\psi(b+k)-\psi(n+a+b+k)-\psi(1+k)]+\cdots\},$$

with the residue

$$-\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(n+a+b+k)k!}\{\psi(n+a+b+k)+\psi(1+k)-\psi(a+k)-\psi(b+k)\}.$$

Then displacement of the integration path over the infinite set of double poles yields the expansion in (2.6).

Let m be a positive integer. The case $s = m$ requires separate treatment since it is no longer possible to separate the poles at $\tau = 0, 1, \ldots, m-1$ in the integrand in (2.7), whereas the case $s = -m$ involves both simple and double poles in the integrand.

Theorem 2. *Let m and n be positive integers and $s = c - a - b$, with the quantities ω_n and λ_n as defined in (2.2). Then, when $s = m$, we have the finite inverse factorial series*

$$S_n(a, b; c) = \lambda_n \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{m-1} \frac{(a)_k(b)_k}{(n+a+b)_k k!} \tag{2.8}$$

and when $s = -m$ and either a or $b \neq 1, 2, \ldots, m$ we have

$$S_n(a, b; c) = \frac{\omega_n \Gamma(c)}{m\Gamma(a)\Gamma(b)} \sum_{k=0}^{m-1} \frac{(c-a)_k(c-b)_k}{(n+c)_k(1-m)_k} + \frac{(-)^m \lambda_n \Gamma(c)}{m!\,\Gamma(c-a)\Gamma(c-b)}$$

$$\times \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(n+a+b)_k k!} \{\psi(n+a+b+k) + \psi(1+k) - \psi(a+k) - \psi(b+k)\}. \tag{2.9}$$

Proof. From (2.3), we obtain when $s = c - a - b = m$

$$S_n(a, b; c) = \frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(n)\Gamma(n+a+b)} {}_3F_2\left[\begin{matrix} a, b, f+m-1 \\ c, f \end{matrix}; 1\right], \qquad f := n + a + b.$$

The above ${}_3F_2(1)$ series can be evaluated by the generalised Karlsson-Minton summation theorem [10], [8, Thm. 6] to find

$${}_3F_2\left[\begin{matrix} a, b, f+m-1 \\ c, f \end{matrix}; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{m-1} (-)^k \mathcal{A}_k \frac{(a)_k(b)_k}{(1-m)_k},$$

where the coefficients $\mathcal{A}_k$ are given by [9]

$$\mathcal{A}_k = \frac{(-)^k}{k!} {}_2F_1\left[\begin{matrix} -k, f+m-1 \\ f \end{matrix}; 1\right] = \frac{(-)^k(1-m)_k}{k!(f)_k}.$$

The ${}_2F_1(1)$ evaluation in the above expression follows from Vandermonde's theorem [15, p. 243]. Hence we obtain the evaluation when $s = m$ given in (2.8). We note that $\lambda_n = 1 + O(n^{-1})$ as $n \to \infty$ and the limiting value of the right-hand side of (2.8) then correctly reduces to the Gauss summation formula in (1.6). The case $m = 1$ is seen to agree with the expression given in [15, p. 84] when n is replaced by $n + 1$.

Finally, when $s = -m$, and provided $a,\ b \neq 1, 2, \ldots, m$, there are simple poles at $\tau = -1, -2, \ldots, -m$ and double poles at $\tau = k$, $k \geq 0$ situated on the right of the indented integration path in (2.7). Straightforward evaluation of the residues as in Theorem 1 then yields the expansion in (2.9).

Remark. We observe that in the case $s = -m$ when either a or $b = 1, 2, \ldots, m$, then $c - b$ or $c - a = 0, -1, -2, \ldots, 1 - m$, respectively, and the second sum in (2.9) accordingly vanishes. Although the integral representation (2.7) fails in these cases (since the integration path cannot be made to separate the sequences of poles), we conjecture from (2.9) when $s = -m$ and a or $b = p$, where $p = 1, 2, \ldots, m$, that

$$S_n(a, b; c) = \frac{\omega_n \Gamma(c)}{m\Gamma(a)\Gamma(b)} \sum_{k=0}^{m-p} \frac{(a-m)_k (b-m)_k}{(n+c)_k (1-m)_k} \qquad (s = -m,\ \ p = 1, 2, \ldots, m). \tag{2.10}$$

This assertion is supported by numerical evidence.

The results in (2.5), (2.6) and (2.9) involve absolutely convergent series of inverse factorial type in the summation index n. This makes these formulas suitable for calculation when n is large. Using the fact that $\omega_n \sim n^{a+b-c}$ as $n \to \infty$, we find from (2.5), (2.8) and (2.9) the leading large-n behaviour given by,

$$S_n(a, b; c) \sim \begin{cases} \dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} & \Re(s) > 0 \\ \dfrac{\Gamma(c)\, n^{a+b-c}}{(-\sigma)\Gamma(a)\Gamma(b)} & \Re(s) < 0 \end{cases} \tag{2.11}$$

and, when $s = 0$, we have from (2.6) and the fact that $\psi(n + a + b) \sim \log n$ from (1.3)

$$S_n(a, b; a + b) \sim \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \log n. \tag{2.12}$$

The result (2.11) when $\Re(s) > 0$ corresponds to the case where $S_n(a, b; c)$ converges to a finite sum as $n \to \infty$ given by the well-known Gauss summation formula (1.6). The leading behaviour when $\Re(s) < 0$ and $s = 0$, where the sum $S_n(a, b; c)$ diverges as $n \to \infty$, agrees with that obtained by Hill [5, 6], who derived only the leading terms in the expansions (2.5), (2.6) and (2.9) by means of elaborate algebraic manipulation and induction arguments.

With the help of (1.6) and the properties of the ψ-function, the case $s = 0$ $(c = a + b)$ in (2.6) can be written in the alternative form

$$S_n(a, b; a + b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \psi(n + a + b) + c_0(a, b)$$
$$+ \frac{\lambda_n \Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(n+a+b)_k k!} \left\{ \sum_{r=0}^{k-1} \frac{1}{n+a+b+r} - \sigma_k(a, b) \right\}, \tag{2.13}$$

where

$$c_0(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\{\psi(1)-\psi(a)-\psi(b)\} \tag{2.14}$$

and the coefficients

$$\sigma_k(a,b) = \sum_{r=0}^{k-1}\left(\frac{1}{a+r}+\frac{1}{b+r}-\frac{1}{r+1}\right). \tag{2.15}$$

3 The Landau constants

From (1.1), it is seen that the Landau constants G_n are given by

$$G_{n-1} = S_n(\tfrac{1}{2},\tfrac{1}{2};1);$$

this corresponds to the logarithmic case in Theorem 1 with the parametric excess $s=0$. From (2.6) and (2.13), we therefore obtain

$$\begin{aligned}\mathcal{S}_n \equiv S_n(\tfrac{1}{2},\tfrac{1}{2};1) &= \frac{\lambda_n}{\pi}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k(\frac{1}{2})_k}{(n+1)_k k!}\{\psi(n+1+k)+\psi(1+k)-2\psi(\tfrac{1}{2}+k)\}\\ &= \frac{1}{\pi}\psi(n+1)+c_0+\frac{\lambda_n}{\pi}\sum_{k=1}^{\infty}\frac{(\frac{1}{2})_k(\frac{1}{2})_k}{(n+1)_k k!}\left\{\sum_{r=1}^{k}\frac{1}{n+r}-\sigma_k\right\},\end{aligned} \tag{3.1}$$

where

$$\lambda_n = \frac{\Gamma^2(n+\frac{1}{2})}{\Gamma(n)\Gamma(n+1)}, \qquad \sigma_k \equiv \sigma_k(\tfrac{1}{2},\tfrac{1}{2}) = \sum_{r=1}^{k}\frac{2r+1}{2r-1}\cdot\frac{1}{r}$$

and

$$c_0 \equiv c_0(\tfrac{1}{2},\tfrac{1}{2}) = \frac{1}{\pi}\{\psi(1)-2\psi(\tfrac{1}{2})\} = \frac{1}{\pi}(\gamma+4\log 2)$$

with γ being the Euler-Mascheroni constant. The first expansion in (3.1) was given in an equivalent form by Watson [16, §3], who obtained it by expressing G_n as an integral over $[0,1]$ involving the complete elliptic integral.

Let M be a fixed positive integer. Then, since $\sum_{r=1}^{k}(n+r)^{-1} < k/(n+1)$ and $\sigma_k < 3k$, the remainder R_M after $M-1$ terms in the sum appearing in (3.1) satisfies the bound[1]

$$|R_M| = \frac{\lambda_n}{\pi}\sum_{k=M}^{\infty}\frac{(\frac{1}{2})_k(\frac{1}{2})_k}{(n+1)_k k!}\left|\left\{\sum_{r=1}^{k}\frac{1}{n+r}-\sigma_k\right\}\right|$$

[1]Watson [16] obtained $|R_M| = O(n^{1-M})$ but this resulted from his use of the crude bound $\psi(x) < x$ for $x > 0$.

$$
\begin{aligned}
&\leq \frac{\lambda_n}{\pi}(3+(n+1)^{-1})\sum_{k=M}^{\infty}\frac{(\frac{1}{2})_k(\frac{1}{2})_k}{(n+1)_k(k-1)!}\\
&\leq \frac{4\lambda_n}{\pi}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_{k+M}(\frac{1}{2})_{k+M}}{(n+1)_{k+M}(k+M-1)!}\\
&\leq \frac{4\lambda_n}{\pi}\frac{(\frac{1}{2})_M(\frac{1}{2})_M}{(n+1)_M}\sum_{k=0}^{\infty}\frac{(\frac{1}{2}+M)_k(\frac{1}{2}+M)_k}{(n+M+1)_k k!}\\
&= \frac{4\lambda_n}{\pi^2}\frac{\Gamma(n)\Gamma^2(M+\frac{1}{2})}{\Gamma(n-M)} = O(n^{-M}) \qquad (3.2)
\end{aligned}
$$

as $n \to \infty$, where we have made use of the facts that $(a)_{k+M} = (a+M)_k(a)_M$ and $\lambda_n = 1 + O(n^{-1})$. Hence we have

$$
\mathcal{S}_n = \frac{1}{\pi}\psi(n+1) + c_0 + \frac{\lambda_n}{\pi}\sum_{k=1}^{M-1}\frac{(\frac{1}{2})_k(\frac{1}{2})_k}{(n+1)_k k!}\left\{\sum_{r=1}^{k}\frac{1}{n+r} - \sigma_k\right\} + O(n^{-M}). \quad (3.3)
$$

3.1 Alternative expression for the coefficients in (3.3)

We consider the double sum appearing in (3.3), namely

$$
F \equiv \frac{\lambda_n}{\pi}\sum_{k=1}^{M-1}\frac{(\frac{1}{2})_k(\frac{1}{2})_k}{(n+1)_k k!}\sum_{p=1}^{k}\frac{1}{n+p},
$$

which we shall rearrange into a single sum. We express the quantity $(n+p)^{-1}$, where p denotes a positive integer, as an inverse factorial series in the form

$$
\frac{1}{n+p} = \sum_{r=1}^{p}\frac{(-)^{r-1}}{(n+1)_r}\prod_{m=1}^{r-1}(p-m),
$$

where an empty product is interpreted as unity; compare [14, p. 177]. Then we have

$$
\begin{aligned}
\sum_{p=1}^{k}\frac{1}{n+p} &= \sum_{p=1}^{k}\sum_{r=1}^{p}\frac{(-)^{r-1}}{(n+1)_r}\prod_{m=1}^{r-1}(p-m) = \sum_{r=1}^{k}\frac{(-)^{r-1}}{(n+1)_r}\sum_{p=r}^{k}\prod_{m-1}^{r-1}(p-m)\\
&= \sum_{r=1}^{k}\frac{(-)^{r-1}}{r(n+1)_r}\,k(k-1)\ldots(k-r+1),
\end{aligned}
$$

since

$$
\sum_{p=r}^{k}\prod_{m=1}^{r-1}(p-m) = \frac{1}{r}\,k(k-1)\ldots(k-r+1).
$$

Substitution into F then yields

$$\begin{aligned} F &= \frac{\lambda_n}{\pi}\sum_{k=1}^{M-1}\frac{(\frac{1}{2})_k(\frac{1}{2})_k}{(n+1)_k}\sum_{r=1}^{k}\frac{(-)^{r-1}}{r(n+1)_r(k-r)!} \\ &= \frac{\lambda_n}{\pi}\sum_{r=1}^{M-1}\frac{(-)^{r-1}}{r(n+1)_r}\sum_{k=r}^{M-1}\frac{(\frac{1}{2})_k(\frac{1}{2})_k}{(n+1)_k(k-r)!} \end{aligned}$$

upon reversal of the order of summation.

The upper limit in the inner sum may be replaced by ∞, which by the same argument employed to obtain (3.2) is easily seen to introduce an error term of $O(n^{-M})$ as $n\to\infty$ with M fixed. Then we find

$$\begin{aligned} F &= \frac{\lambda_n}{\pi}\sum_{r=1}^{M-1}\frac{(-)^{r-1}}{r(n+1)_r}\sum_{k=r}^{\infty}\frac{(\frac{1}{2})_k(\frac{1}{2})_k}{(n+1)_k(k-r)!}+O(n^{-M-1}) \\ &= \frac{\lambda_n}{\pi}\sum_{r=1}^{M-1}\frac{(-)^{r-1}}{r(n+1)_r}\frac{(\frac{1}{2})_r(\frac{1}{2})_r}{(n+1)_r}\sum_{k=0}^{\infty}\frac{(\frac{1}{2}+r)_k(\frac{1}{2}+r)_k}{(n+r+1)_k k!}+O(n^{-M-1}) \\ &= \frac{1}{\pi}\sum_{r=1}^{M-1}\frac{(-)^{r-1}}{r(n+1)_r}\frac{(\frac{1}{2})_r(\frac{1}{2})_r\Gamma(n-r)}{\Gamma(n)}+O(n^{-M-1}) \\ &= \frac{1}{\pi}\sum_{r=1}^{M-1}\frac{(-)^{r-1}(\frac{1}{2})_r(\frac{1}{2})_r}{r(n^2-1^2)\ldots(n^2-r^2)}+O(n^{-M-1}) \qquad (3.4) \end{aligned}$$

upon use of the Gauss summation formula in (1.6) to evaluate the inner infinite sum.

Combining (3.4) with (3.3), we then obtain the following result.

Theorem 3. *Let M be a fixed positive integer. Then, with $S_n\equiv S_n(\frac{1}{2},\frac{1}{2};1)$, we have as $n\to\infty$ the expansion*

$$\mathcal{S}_n=\frac{1}{\pi}\psi(n+1)+c_0+\frac{1}{\pi}\sum_{r=1}^{K-1}\frac{(-)^{r-1}(\frac{1}{2})_r(\frac{1}{2})_r}{r(n^2-1^2)\ldots(n^2-r^2)}-\frac{\lambda_n}{\pi}\sum_{k=1}^{M-1}\frac{(\frac{1}{2})_k(\frac{1}{2})_k\sigma_k}{(n+1)_k k!}+O(n^{-M}), \qquad (3.5)$$

where $K=\lfloor\frac{1}{2}(M+1)\rfloor$ and the coefficients σ_k satisfy the recurrence

$$\sigma_k=\sigma_{k-1}+\frac{2k+1}{2k-1}\cdot\frac{1}{k}\qquad (k\geq 2),\qquad \sigma_1=3.$$

3.2 Asymptotic expansion for $\mathcal{S}_n$ as $n\to\infty$

Although the expansion (3.5) is suitable for computation when n is large, we can obtain the asymptotic expansion of $\mathcal{S}_n$ in inverse powers of n by routine algebra. By making use of the facts that

$$\sigma_1=3,\quad \sigma_2=\tfrac{23}{6},\quad \sigma_3=\tfrac{43}{10},\quad \sigma_4=\tfrac{647}{140},\quad \sigma_5=\tfrac{6131}{1260},\quad \sigma_6=\tfrac{70171}{13860},\ldots$$

and from the expansion of the ratio of two gamma functions [13, p. 141] that

$$\lambda_n = 1 - \frac{1}{4n} + \frac{1}{32n^2} + \frac{1}{128n^3} - \frac{5}{2048n^4} - \frac{23}{8192n^5} + O(n^{-6}) \qquad (n \to \infty),$$

we find with the help of *Mathematica* that

$$\mathcal{S}_n \sim \frac{1}{\pi}\psi(n+1) + c_0 + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-)^k C_k}{n^k} \tag{3.6}$$

as $n \to \infty$, where the first few coefficients are

$$C_1 = \tfrac{3}{4}, \quad C_2 = \tfrac{7}{64}, \quad C_3 = -\tfrac{3}{128}, \quad C_4 = -\tfrac{91}{8192}, \quad C_5 = \tfrac{75}{8192}, \quad C_6 = \tfrac{641}{131072}.$$

If n is replaced by $n+1$ in (3.6) and use made of the result $\psi(n+2) \sim \log(n+1) + \frac{1}{2}(n+1)^{-1} + \frac{1}{12}(n+1)^{-2} + \cdots$, then Watson's expansion in (1.4) is recovered. Similarly, if we put $h = 1$ and replace n by $n-1$ in (1.7) and make use of (1.3), we find agreement with the expansion obtained by Nemes in (1.7).

4 The general case of $S_n(a, b; c)$ when $s = 0$ or a negative integer

The same procedure can be brought to bear on the general logarithmic case $s = 0$. From (2.13), we have

$$S_n(a, b; a+b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\,\psi(n+a+b) + c_0(a, b) + \frac{\lambda_n \Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(n+a+b)_k k!} \left\{ \sum_{r=0}^{k-1} \frac{1}{n+a+b+r} - \sigma_k(a, b) \right\}, \tag{4.1}$$

where

$$\lambda_n = \frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(n)\Gamma(n+a+b)} = 1 - \frac{ab}{n} + \frac{ab}{2n^2}(a+b-1+ab) + O(n^{-3})$$

as $n \to \infty$ and $c_0(a, b)$ and the coefficients $\sigma_k(a, b)$ are defined in (2.14) and (2.15). The double sum appearing in (4.1) can be rearranged, if so desired, following the method used in Section 3 to find

$$\begin{aligned} \mathcal{F} &\equiv \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(n+a+b)_k k!} \sum_{r=0}^{k-1} \frac{1}{n+a+b+r} \\ &= \sum_{r=1}^{K-1} \frac{(-)^{r-1} (a)_r (b)_r}{r(n+a+b)_r (n-1)\ldots(n-r)} + O(n^{-M-1}), \end{aligned} \tag{4.2}$$

where $K = \lfloor\frac{1}{2}(M+1)\rfloor$.

The expansion in inverse powers of n then follows by observing that[2]

$$\mathcal{F} = \frac{ab}{n^2} - \frac{a+b-1}{n^3} + O(n^{-4})$$

and

$$\lambda_n \sum_{k=1}^{\infty} \frac{(a)_k(b)_k\sigma_k(a,b)}{(n+a+b)_k k!} = \frac{1}{n}(a+b-ab) + \frac{1}{4n^2}(a-1)(b-1)(2a+2b+ab)$$

$$-\frac{1}{36n^3}(a-1)(b-1)\{6(2a^2+2b^2-a-b)+ab(8a+8b+2ab+5))\} + O(n^{-4}).$$

Then we obtain the following expansion in the logarithmic case $s = 0$

$$S_n(a,b;a+b) \sim \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\psi(n+a+b) + c_0(a,b) + \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\sum_{k=1}^{\infty}\frac{(-)^{k-1}A_k}{n^k} \qquad (4.3)$$

as $n \to \infty$, where the first three coefficients are given by

$$A_1 = ab - a - b, \qquad A_2 = \frac{1}{4}\Big\{(a-1)(b-1)(2a+2b+ab) - 4ab\Big\},$$

$$A_3 = \frac{1}{36}\Big\{(a-1)(b-1)\{6(2a^2+2b^2-a-b)+ab(8a+8b+2ab+5)\} - 36ab(a+b-1)\Big\}.$$

The above expansion in the case $a = b = \frac{1}{2}$, $c = 1$ is seen to reduce correctly to the first three terms in (3.6).

In the case $s = -m$, where m is a positive integer and either a or $b \neq 1, 2, \ldots, m$, we find after a similar rearrangement of (2.9) that

$$S_n(a,b;c) \sim \frac{\omega_n\Gamma(c)}{m\Gamma(a)\Gamma(b)}\sum_{k=0}^{m-1}\frac{(c-a)_k(c-b)_k}{(n+c)_k(1-m)_k} + \frac{(-)^m\Gamma(c)}{m!\,\Gamma(c-a)\Gamma(c-b)}\Big\{\psi(n+a+b)$$

$$+\frac{\Gamma(a)\Gamma(b)}{\Gamma(c-a)\Gamma(c-b)}c_0(a,b) + \sum_{k=1}^{\infty}\frac{(-)^{k-1}A_k}{n^k}\Big\} \qquad (c = a+b-m) \qquad (4.4)$$

as $n \to \infty$.

To conclude, we present the results of numerical calculations to illustrate the accuracy of the expansions in (4.3) and (4.4). In Table 1 below we show the absolute error in the computation of $S_n(a,b;c)$ using the expansion (4.3) truncated after k terms for selected values of a, b and n.

[2] This expansion of $\mathcal{F}$ can be obtained from either side of (4.2).

Table 1: Values of the absolute error in the computation of $S_n(a,b;c)$ by (4.3) and (4.4) for different truncation index k. The upper half of the table corresponds to $s=0$ ($c=a+b$) and the lower half to $s=-m$.

k	$a=\frac{1}{3},\ b=\frac{2}{3}$ $n=40$	$a=\frac{3}{2},\ b=\frac{1}{2}$ $n=50$	$a=\frac{1}{2}+i,\ b=\frac{1}{4}$ $n=100$
1	1.711×10^{-5}	2.616×10^{-4}	1.545×10^{-5}
2	9.618×10^{-8}	4.954×10^{-6}	1.291×10^{-7}
3	9.845×10^{-10}	9.922×10^{-8}	1.227×10^{-9}
k	$a=\frac{4}{3},\ b=\frac{1}{3}$ $c=-\frac{7}{3},\ n=40$	$a=\frac{3}{2},\ b=-\frac{1}{4}$ $c=\frac{1}{4},\ n=50$	$a=\frac{3}{4}+i,\ b=\frac{1}{4}+i$ $c=-2+2i,\ n=100$
1	9.820×10^{-5}	9.654×10^{-6}	6.556×10^{-5}
2	1.601×10^{-6}	6.888×10^{-7}	9.752×10^{-7}
3	2.812×10^{-8}	4.141×10^{-11}	1.520×10^{-8}

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