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On the asymptotics of products related to generalizations of the Wilf and Mortini problems

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Abstract

In 1997, Wilf posed the following elegant infinite product formula as a problem:

$$\prod_{j=1}^{\infty}\left\{e^{-1/j}\left(1+\frac{1}{j}+\frac{1}{2j^2}\right)\right\}=\frac{e^{\pi/2}+e^{-\pi/2}}{\pi e^{\gamma}},$$

which contains the most important mathematical constants π, e and the Euler-Mascheroni constant γ. In 2009, Mortini posed the following problem to determine the limit as $n\to\infty$ of the product

$$n\prod_{j=1}^{n}\left(1-\frac{1}{j}+\frac{5}{4j^2}\right).$$

In this paper, we shall establish the connection between generalized versions involving m parameters of Wilf's and Mortini's problems. We also consider the asymptotic expansion of these generalized products with several parameters for large values of the index n.

1. Introduction

In 1997, Wilf [9] posed the following elegant infinite product formula as a problem:

$$\prod_{j=1}^{\infty}\left\{e^{-1/j}\left(1+\frac{1}{j}+\frac{1}{2j^2}\right)\right\}=\frac{e^{\pi/2}+e^{-\pi/2}}{\pi e^{\gamma}}, \tag{1.1}$$

which contains the most important mathematical constants π, e and the Euler-Mascheroni constant γ. Subsequently, Choi and Seo [4] proved (1.1) together with three other similar product formulas by making use of well-known infinite product formulas for the circular and hyperbolic functions and the familiar Stirling formula for the factorial function. In 2009 a closely related example of the determination of an infinite product was posed by Mortini [7] also as a problem in the form

$$\lim_{n\to\infty} n\prod_{j=1}^{n}\left(1-\frac{1}{j}+\frac{5}{4j^2}\right). \tag{1.2}$$

A solution to (1.2) was given in [6].

In 2003, Choi et al. [5] extended these results and obtained the infinite products

$$\prod_{j=1}^{\infty}\left\{e^{-1/j}\left(1+\frac{1}{j}+\frac{\alpha^2+1/4}{j^2}\right)\right\}=\frac{2(e^{\pi\alpha}+e^{-\pi\alpha})}{(4\alpha^2+1)\pi e^{\gamma}}\qquad (\alpha\in C;\ \alpha\neq\pm\frac{1}{2}i) \tag{1.3}$$

and

$$\prod_{j=1}^{\infty}\left\{e^{-2/j}\left(1+\frac{2}{j}+\frac{\beta^2+1}{j^2}\right)\right\}=\frac{e^{\pi\beta}-e^{-\pi\beta}}{2\beta(\beta^2+1)\pi e^{2\gamma}}\qquad (\beta\in C\setminus\{0\};\ \beta\neq\pm i)\,, \tag{1.4}$$

where C denotes the set of complex numbers and $i = \sqrt{-1}$. Subsequently, Chen and Choi [1] presented a more general infinite product formula that includes (1.3) and (1.4) as special cases:

$$\prod_{j=1}^{\infty} \left\{ e^{-p/j} \left(1 + \frac{p}{j} + \frac{q}{j^2} \right) \right\} = \frac{e^{-p\gamma}}{\Gamma\left(1 + \frac{1}{2}p + \frac{1}{2}\Delta\right) \Gamma\left(1 + \frac{1}{2}p - \frac{1}{2}\Delta\right)}, \tag{1.5}$$

where p, $q \in C$ and $\Delta := \sqrt{p^2 - 4q}$. In the same paper, the authors presented another infinite product formula as follows:

$$\prod_{j=1}^{\infty} \left\{ e^{-p/(2j-1)} \left(1 + \frac{p}{2j-1} + \frac{q}{(2j-1)^2} \right) \right\} = \frac{2^{-p} \pi \, e^{-p\gamma/2}}{\Gamma\left(\frac{1}{2} + \frac{1}{4}p + \frac{1}{4}\Delta\right) \Gamma\left(\frac{1}{2} + \frac{1}{4}p - \frac{1}{4}\Delta\right)}. \tag{1.6}$$

Very recently, Chen and Paris [3] generalized the formulas (1.5) and (1.6) and obtained the following results valid for any positive integer m:

$$P_n' := \prod_{j=1}^{\infty} \left\{ e^{-p_1/j} \left(1 + \frac{p_1}{j} + \frac{p_2}{j^2} + \cdots + \frac{p_m}{j^m} \right) \right\} = \frac{e^{-p_1\gamma}}{\prod_{j=1}^{m} \Gamma(1 + \rho_j)} \tag{1.7}$$

and

$$Q_n' := \prod_{j=1}^{\infty} \left\{ e^{-p_1/(2j-1)} \left(1 + \frac{p_1}{2j-1} + \frac{p_2}{(2j-1)^2} + \cdots + \frac{p_m}{(2j-1)^m} \right) \right\} = \frac{2^{-p_1} \pi^{m/2} e^{-p_1\gamma/2}}{\prod_{j=1}^{m} \Gamma(\frac{1}{2} + \frac{1}{2}\rho_j)}, \tag{1.8}$$

where $p_j \in C$ and ρ_j $(j = 1, 2, \ldots, m)$ satisfy the following expressions

$$\begin{aligned} &\sum_{1 \le i \le m} \rho_i = p_1, \qquad \sum_{1 \le i < j \le m} \rho_i \rho_j = p_2, \qquad \sum_{1 \le i < j < k \le m} \rho_i \rho_j \rho_k = p_3, \\ &\ldots, \\ &\rho_1 \rho_2 \ldots \rho_m = p_m. \end{aligned} \tag{1.9}$$

The choice $m = 2$ in (1.7) and (1.8) with $\rho_{1,2} = \frac{1}{2}p_1 \pm \frac{1}{2}\sqrt{p_1^2 - 4p_2}$ yields (1.5) and (1.6), respectively.

In this paper, we shall determine the asymptotic expansion as $n \to \infty$ of the products defined by

$$P_n \equiv P_n(p_1, p_2, \ldots, p_m) := \prod_{j=1}^{n} \left(1 + \frac{p_1}{j} + \frac{p_2}{j^2} + \cdots + \frac{p_m}{j} \right) \tag{1.10}$$

and

$$Q_n := \prod_{j=1}^{n} \left(1 + \frac{p_1}{2j-1} + \frac{p_2}{(2j-1)^2} + \cdots + \frac{p_m}{(2j-1)^m} \right), \tag{1.11}$$

and present the connection between the generalized Wilf and Mortini problems.

2. The asymptotic expansion of the products P_n and Q_n

We determine the asymptotic expansions as $n \to \infty$ of the products P_n and Q_n defined in (1.10) and (1.11). In terms of the Pochhammer symbol $(\lambda)_n$ defined by

$$(\lambda)_n = \lambda(\lambda + 1) \ldots (\lambda + n - 1) = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)}, \qquad (\lambda)_0 = 1,$$

it is easily seen that

$$\prod_{j=1}^{n}\left(1+\frac{\rho}{j}\right)=\frac{(1+\rho)_n}{n!}=\frac{\Gamma(n+1+\rho)}{\Gamma(1+\rho)\Gamma(n+1)}.$$

If we write

$$1+\frac{p_1}{j}+\frac{p_2}{j^2}+\cdots+\frac{p_m}{j^m}=\prod_{j=1}^{m}\left(1+\frac{\rho_j}{j}\right),$$

where the ρ_j $(1\leq j\leq m)$ satisfy (1.9), then it follows that

$$P_n=\prod_{j=1}^{n}\left(1+\frac{\rho_1}{j}\right)\ldots\left(1+\frac{\rho_m}{j}\right)=\frac{1}{\prod_{j=1}^{n}\Gamma(1+\rho_j)}\prod_{j=1}^{m}\frac{\Gamma(n+1+\rho_j)}{\Gamma(n+1)}.$$

From the expansion of the ratio of two gamma function [8, §5.11(iii)] we have

$$\frac{\Gamma(n+1+\rho_j)}{\Gamma(n+1)}=n^{\rho_j}\left\{1+\frac{\alpha_j}{n}+\frac{\beta_j}{n^2}+O(n^{-3})\right\}\qquad(n\to\infty),$$

where

$$\alpha_j=\frac{1}{2}\rho_j(1+\rho_j),\qquad\beta_j=\frac{1}{24}\rho_j(\rho_j^2-1)(3\rho_j+2).$$

Then some straightforward algebra shows that

$$\begin{aligned}P_n&=\frac{n^{p_1}}{\prod_{j=1}^{m}\Gamma(1+\rho_j)}\prod_{j=1}^{m}\left\{1+\frac{\alpha_j}{n}+\frac{\beta_j}{n^2}+O(n^{-3})\right\}\\&=\frac{n^{p_1}}{\prod_{j=1}^{m}\Gamma(1+\rho_j)}\left\{1+\frac{C_1}{n}+\frac{C_2}{n^2}+O(n^{-3})\right\}\end{aligned}\tag{2.1}$$

as $n\to\infty$, where

$$C_1\equiv C_1(\vec{\rho})=\sum_{j=1}^{m}\alpha_j=\frac{1}{2}p_1+\frac{1}{2}\sum_{j=1}^{m}\rho_j^2,\qquad C_2\equiv C_2(\vec{\rho})=\sum_{j=1}^{m}\beta_j+\sum_{1\leq j<k\leq m}\alpha_j\alpha_k$$

with $\vec{\rho}=\{\rho_1,\rho_2,\ldots,\rho_m\}$.

For the product Q_n in (1.11) we can determine the expansion in a similar manner using the fact that

$$Q_n=\frac{\pi^{1/2}}{\prod_{j=1}^{m}\Gamma(\frac{1}{2}+\frac{1}{2}\rho_j)}\prod_{j=1}^{m}\frac{\Gamma(n+\frac{1}{2}+\rho_j)}{\Gamma(n+\frac{1}{2})}.$$

Alternatively, we can proceed as follows making use of the expansion for P_n obtained in (2.1) above. We find

$$\begin{aligned}Q_n&=\prod_{j=1}^{n}\left(1+\frac{p_1}{2j-1}+\cdots+\frac{p_m}{(2j-1)^m}\right)=\frac{\prod_{j=1}^{2n}\left(1+\frac{p_1}{j}+\cdots+\frac{p_m}{j^m}\right)}{\prod_{j=1}^{n}\left(1+\frac{p_1}{2j}+\cdots+\frac{p_m}{(2j)^m}\right)}\\&=\frac{P_{2n}(p_1,p_2,\ldots,p_m)}{P_n(p_1/2,p_2/2^2,\ldots,p_m/2^m)}\\&=2^{p_1}n^{p_1/2}\prod_{j=1}^{m}\frac{\Gamma(1+\frac{1}{2}\rho_j)}{\Gamma(1+\rho_j)}\frac{\{1+C_1(\vec{\rho})/(2n)+C_2(\vec{\rho})/(2n)^2+O(n^{-3})\}}{\{1+C_1(\frac{1}{2}\vec{\rho})/n+C_2(\frac{1}{2}\vec{\rho})/n^2+O(n^{-3})\}}\\&=\frac{\pi^{m/2}n^{p_1/2}}{\prod_{j=1}^{m}\Gamma(\frac{1}{2}+\frac{1}{2}\rho_j)}\left(1+\frac{D_1}{n}+\frac{D_2}{n^2}+O(n^{-3})\right),\end{aligned}\tag{2.2}$$

where

$$D_1 = \frac{1}{2}C_1(\vec{\rho}) - C_1(\tfrac{1}{2}\vec{\rho}) = \frac{1}{8}\sum_{j=1}^{m}\rho_j^2, \qquad D_2 = \frac{1}{4}C_2(\vec{\rho}) - C_2(\tfrac{1}{2}\vec{\rho}) - C_1(\tfrac{1}{2}\vec{\rho})D_1,$$

upon use of the duplication formula for the gamma function (see, e.g. [8, (5.5.5)])

$$\Gamma(2z) = (2\pi)^{-\frac{1}{2}}2^{2z-\frac{1}{2}}\Gamma(z)\Gamma(z+\tfrac{1}{2}).$$

3. The expansion of the associated products P'_n and Q'_n

The expansion of the associated products P'_n and Q'_n in (1.7) and (1.8) follows from

$$P'_n = P_n\prod_{j=1}^{n}e^{-p_1/j} = n^{-p_1}P_n\mathcal{E}_1, \qquad Q'_n = Q_n\prod_{j=1}^{n}e^{-p_1/(2j-1)} = n^{-p_1/2}Q_n\mathcal{E}_2, \tag{3.1}$$

where

$$\mathcal{E}_1 := e^{-p_1(\sum_{k=1}^{n}1/k-\ln n)}, \qquad \mathcal{E}_2 := e^{-p_1\left(\sum_{k=1}^{n}1/(2k-1)-\frac{1}{2}\ln n\right)}.$$

From [8, (5.4.14), (5.11.2)] we obtain that

$$\sum_{k=1}^{n}1/k - \ln n = \gamma + \frac{1}{2n} - \frac{1}{12n^2} + O(n^{-4}),$$

$$\sum_{k=1}^{n}1/(2k-1) - \tfrac{1}{2}\ln n = \tfrac{1}{2}\gamma + \ln 2 + \frac{1}{48n^2} + O(n^{-4})$$

as $n\to\infty$, and hence that

$$\mathcal{E}_1 = e^{-p_1\gamma}\left(1 - \frac{p_1}{2n} + \frac{p_1(2+3p_1)}{24n^2} + O(n^{-3})\right),$$

$$\mathcal{E}_2 = 2^{-p_1}e^{-p_1\gamma/2}\left(1 - \frac{p_1}{48n^2} + O(n^{-4})\right).$$

Substitution of these last results in (3.1), combined with (2.1) and (2.2), then yields the expansions

$$P'_n = \frac{e^{-p_1\gamma}}{\prod_{j=1}^{m}\Gamma(1+\rho_j)}\left\{1 + \frac{C'_1}{n} + \frac{C'_2}{n^2} + O(n^{-3})\right\}, \tag{3.2}$$

$$Q'_n = \frac{2^{-p_1}\pi^{m/2}e^{-p_1\gamma/2}}{\prod_{j=1}^{m}\Gamma(\frac{1}{2}+\frac{1}{2}\rho_j)}\left\{1 + \frac{D'_1}{n} + \frac{D'_2}{n^2} + O(n^{-3})\right\} \tag{3.3}$$

as $n\to\infty$, where

$$C'_1 = \frac{1}{2}\sum_{j=1}^{m}\rho_j^2, \qquad C'_2 = C_2 - \frac{1}{4}p_1\sum_{j=1}^{m}\rho_j^2 + \frac{1}{24}p_1(2-3p_1)$$

and

$$D'_1 = D_1, \qquad D'_2 = D_2 - \frac{p_1}{48n^2}.$$

4. Concluding remarks

The limiting values of the products P_n and Q_n immediately follow from the results in (2.1) and (2.2) to yield the connection between the generalized Wilf and Mortini problems given by

Theorem 1 *For positive integer m, we have*

$$\lim_{n\to\infty} n^{-p_1}P_n = \frac{1}{\prod_{j=1}^m \Gamma(1+\rho_j)}, \qquad \lim_{n\to\infty} n^{-p_1/2}Q_n = \frac{\pi^{m/2}}{\prod_{j=1}^m \Gamma(\frac{1}{2}+\frac{1}{2}\rho_j)},$$

where $p_j \in C$ and ρ_j $(1 \le j \le m)$ satisfy (1.9).

The choice $m = 2$, with $p_1 = p$, $p_2 = q$ and $\rho_{1,2} = \frac{1}{2}p + \frac{1}{2}\Delta$, $\Delta = \sqrt{p^2 - 4q}$ in the expansions (2.1) and (2.2) yields the following results:

$$\prod_{j=1}^n \left(1 + \frac{p}{j} + \frac{q}{j^2}\right) = \frac{n^p}{\Gamma(1+\frac{1}{2}p+\frac{1}{2}\Delta)\Gamma(1+\frac{1}{2}p-\frac{1}{2}\Delta)}$$

$$\times \left\{1 + \frac{p(1+p)-2q}{2n} + \frac{p(3p^3+2p^2-2)+12q(1+q)-3p^2(1+4q)}{24n^2} + O(n^{-3})\right\} \tag{4.1}$$

and

$$\prod_{j=1}^n \left(1 + \frac{p}{2j-1} + \frac{q}{(2j-1)^2}\right) = \frac{\pi n^{p/2}}{\Gamma(\frac{1}{2}+\frac{1}{4}p+\frac{1}{4}\Delta)\Gamma(\frac{1}{2}+\frac{1}{4}p-\frac{1}{4}\Delta)}$$

$$\times \left\{1 + \frac{p^2-2q}{8n} + \frac{p^3(3p-8)-12q(p^2-q)+8p(1+3q)}{384n^2} + O(n^{-3})\right\} \tag{4.2}$$

as $n \to \infty$.

In particular, setting $(p, q) = (-1, 5/4)$, so that $\rho_{1,2} = -\frac{1}{2} \pm i$, we have from (4.1)

$$\prod_{j=1}^n \left(1 - \frac{1}{j} + \frac{5}{4j^2}\right) = \frac{\cosh \pi}{\pi n}\left\{1 - \frac{5}{4n} + \frac{25}{32n^2} + O(n^{-3})\right\}$$

as $n \to \infty$, where we have employed the result [8, (5.4.4)]

$$\Gamma(\tfrac{1}{2}+iy)\Gamma(\tfrac{1}{2}-iy) = |\Gamma(\tfrac{1}{2}+iy)|^2 = \frac{\pi}{\cosh \pi y}.$$

Similarly, from (4.2) we obtain

$$\prod_{j=1}^n \left(1 - \frac{1}{2j-1} + \frac{5}{4(2j-1)^2}\right) = \frac{\pi n^{-1/2}}{|\Gamma(\frac{1}{4}+\frac{1}{2}i)|^2}\left\{1 - \frac{3}{16n} - \frac{31}{512n^2} + O(n^{-3})\right\}.$$

Finally, the determination of the quantities ρ_j from the set of coefficients $p_1, \ldots, p_m$ requires the computation of the zeros of an mth degree polynomial. Apart from the cases with $m = 2$ and $m = 3$, this would necessitate, in general, a numerical approach to determine the zeros. If, on the other hand, the ρ_j are specified the coefficients p_j can be simply determined by (1.9).

References

[1] C.-P. Chen and J. Choi, *Two infinite product formulas with two parameters*, Integral Transforms Spec. Funct. **24** (2013), 357–363.

[2] C.-P. Chen and C. Mortici, *Limits and inequalities associated with the psi function*, Appl. Math. Comput. **219** (2013), 9755–9761.

[3] C.-P. Chen and R.B. Paris, *Generalizations of two infinite product formulas*, Integral Transforms Spec. Funct. (2014), http://dx.doi.org/10.1080/10652469.2014.885965

[4] J. Choi and T. Y. Seo, *Identities involving series of the Riemann Zeta function*, Indian J. Pure Appl. Math. **30** (1999), 649–652.

[5] J. Choi, J. Lee and H. M. Srivastava, *A generalization of Wilf's formula*, Kodai Math. J. **26** (2003), 44–48.

[6] O. Geupel, *Asymptotics of a Product*, Amer. Math. Monthly **118** (2011), 185. http://users.uoa.gr/ dcheliotis/Problems/Problems$2_2$011.*pdf*

[7] R. Mortini, Problems: 11456, Amer. Math. Monthly 116 (2009), no. 8, 747.

[8] F. W. J. Olver, D. W. Lozier, F. Boisvert, C. W. Clark (eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge, 2010.

[9] H. S. Wilf, *Problem 10588, Problems and Solutions*, Amer. Math. Monthly **104** (1997), 456.