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# BRANCHING PARTICLE SYSTEMS AND COMPOUND POISSON PROCESSES RELATED TO PÓLYA-AEPPLI DISTRIBUTIONS

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ABSTRACT. We establish numerous new refined local limit theorems for a class of compound Poisson processes with Pólya-Aeppli marginals, and for a particular family of the branching particle systems which undergo critical binary branching and can be approximated by the backshifted Feller diffusion. To this end, we also derive new results for the families of Pólya–Aeppli and Poisson–exponential distributions. We relate a few of them to properties of certain special functions some of which were previously unknown.

## 1. Introduction

This paper is devoted to the derivation of various asymptotics for the probabilities associated with specific *branching particle systems* (or *BPS*'s) which undergo critical binary branching and constitute a family of time-homogeneous Markov processes with discontinuous paths, and with the following class of *compound Poisson-geometric* processes (which are also frequently referred to as *Pólya-Aeppli* Lévy processes):

$$\mathcal{R}^{(\rho,\gamma)}(t) := \begin{cases} 0 & \text{if} \quad \Pi_\rho(t) = 0; \\ \mathcal{Q}_1 + ... + \mathcal{Q}_{\Pi_\rho(t)} & \text{if} \quad \Pi_\rho(t) \geq 1. \end{cases} \tag{1.1}$$

Hereinafter, $\{\mathcal{Q}_n^{(\gamma)}, n \geq 1\}$ is a sequence of geometrically distributed i.i.d. random variables (or *r.v.*'s) whose range is $\mathbf{N}$, and which are characterized by the probability of success $\gamma \in (0,1)$. In addition, they are assumed to be independent of the Poisson counting process $\{\Pi_\rho(t),\ t \geq 0\}$ with intensity $\rho > 0$. In view of [9], [26], [28], the marginals of both these classes of stochastic processes belong to the two-parameter *Pólya-Aeppli* family of distributions described in Definition 3.1 of Section 3 (see also formulas (5.3)–(5.5) of Section 5). This is an important common feature of these two classes of stochastic processes.

The BPS's, whose new properties are derived in Section 4, are identical to those dealt with in [9]. Since [9, Introduction] already contains a comprehensive bibliography on relevant BPS's as well as on their applications and limits, we will now provide a few references on the compound Poisson-geometric Lévy processes. They were considered as early as in [5, p. 96]. More recently, their realizations in specific stochastic models,

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characterizations, properties and applications were discussed in [3], [4], [15], [16], [22], [32].

Thus, [4] gives two natural realizations of the Lévy process $\{\mathcal{R}^{(\rho,\gamma)}(t),\ t \geq 0\}$, which is defined above by formula (1.1). One of them, presented as [4, Proposition 3.1], specifies each such process as a *delayed renewal process* with the first inter-arrival time being exponentially distributed, and all the subsequent inter-arrival times being i.i.d.r.v.'s with common zero-modified exponential law. (We refer to [27, Sections 2–3] for a comprehensive description of the class of zero-modified exponential distributions.) The other realization characterizes a Pólya-Aeppli Lévy process as a particular *pure birth* process (see [4, Proposition 3.2]). On the other hand, [3, pp. 19, 27–28] employs the fact that the marginals of a generic Pólya-Aeppli process coincide with those of a particular mixed Poisson process with the mixing process being a specific compound *Poisson-exponential* process. We extend this observation in Proposition 5.1 of Section 5 (see also its proof). But even at this stage, it is still relevant to refer the reader to our article [20] where numerous properties of compound Poisson-exponential processes were derived, as well as to [14] and to [29], where those of a more general class of *Hougaard* processes were investigated. (Note in passing that the subclass of Hougaard processes which corresponds to the value of the *power parameter* $p = 3/2$ comprises the entire family of compound Poisson-exponential processes.)

Some applications in Risk Theory for more general classes of Lévy processes, but which all contain Pólya-Aeppli processes as their components associated with the structure of the jumps of these processes, were addressed in [15], [22], [32].

It is known (see, for example, [26, Section 3], [9, Section 3]) that the totality of the class of Pólya-Aeppli distributions can be parameterized in a manner that this family would constitute an additive *exponential dispersion model* (or *EDM*) on the set $\mathbf{Z}_+ := \{0, 1, 2, ...\}$ (compare [9, Section 3]). We refer to [12, Chapter 3] for a comprehensive consideration of such structures. By [26, formula (3.4)], each Pólya-Aeppli distribution is infinitely divisible. It turns out that the theory of weak convergence of the EDMs to members of the *power-variance family* (see, for example, [12, Theorem 4.5] and its variants) is quite relevant to our studies of the BPS's which are pursued in Section 4. In contrast, a natural bijection between additive EDMs comprised of specific infinitely divisible distributions and the marginals of the corresponding exponential families of Lévy processes, which is discussed in [12, Subsection 3.2.3], is of importance in Section 5.

Our investigation of the BPS's with Pólya-Aeppli marginals and of the compound Poisson-geometric processes, which is undertaken in Sections 4 and 5, respectively, necessitated rather thorough studies of both the family of Pólya-Aeppli distributions per se and a closely related class of Poisson-exponential probability laws. There are some properties of the class of Pólya-Aeppli distributions already available, and they occurred to us to be useful for the derivation of new subtle results on the two classes of stochastic processes mentioned above. In addition, our studies of these processes led to the derivation of a variety of new distribution theory results on the Pólya-Aeppli and Poisson-exponential probability laws. Since these results take up a substantial amount of space, it was natural to isolate them into a separate Section 3.

We stress that our interest in this family of distributions, whose members take values in $\mathbf{Z}_+$, is three-fold. First, it is interesting in its own right, since this class represents a toy example for which numerous approximations can be derived in closed form. Secondly, members of the Pólya-Aeppli class frequently emerge in applications (see, for example, [9, Introduction] and the references therein). Recall that one of them is their use for describing the stochastic evolution of some BPS's. This point of view has already been developed in [26], [28], and [9, Introduction].

Thirdly, this article is closely related to numerous assertions of the Theory of Special Functions, some of which were previously unknown. We regard these connections between Probability Theory and the Theory of Stochastic Processes on one side and the Theory of Special Functions on the other one to be a two-way road (compare to [30] and [31]). For instance, the well-known result on the asymptotics of the *confluent hypergeometric function* in the third argument implies the local version of the limit theorem on Poisson convergence (see formula (2.6), Theorem 3.13, Corollary 4.3, and Remark 4.4). In contrast, it appears that sometimes, special cases of previously unknown results of Analysis which concern special functions, are present implicitly in certain general results of Probability Theory. Thus, a special case of Corollary 3.8 which can be recovered from [17, Theorem 2] and is discussed in Remark 3.9.i led us to conjecture the validity of the purely analytical assertion of Theorem 2.3.i, which we were fortunate to establish by applying very recent results of Analysis (see [24, formula (27.4.62)]). We reckon that revealing such interplays between two separate branches of Mathematics is particularly important.

The other illustration of this interplay is the fact that the Poisson-exponential approximation for the Pólya-Aeppli family as well as its stochastic processes counterpart, which involves the Feller-diffusion approximation for a family of BPS's, is parallel to some properties of confluent hypergeometric and Bessel functions. For instance, we derive new asymptotic properties of particular BPS's by specifying our more general results on the Pólya-Aeppli family. See Theorems 3.10–3.11 and Corollary 4.1, which are of a *probabilistic* character, but rely on the *analytical* results given by representations (3.9)–(3.10) and (3.31).

Some results summarized in Section 3 were already known (see the references therein). At the same time, our studies of the stochastic processes mentioned above necessitate the derivation of a variety of new subtle results on the Pólya-Aeppli probability laws, which are also presented in Section 3. In turn, since the probability function of a generic Pólya-Aeppli distribution is expressed in terms of the *confluent hypergeometric function* (which is introduced in Definition 2.1 of Section 2), we had to derive new and also modify specific properties of this class of special functions which were already available.

In view of a large number of such technicalities which pertain to the Theory of Special Functions and which are employed in this paper, it was natural to isolate them into a separate Section 2. Hence, that section has a primarily analytic character.

We believe that both the properties of members of the Pólya-Aeppli family and relevant results on the confluent hypergeometric function are interesting in their own right. However, they only play an auxiliary role in this paper and are employed in Sections 4–5.

Section 4 concerns the derivation of numerous local approximations of various degrees of accuracy for the same class of the BPS's as those dealt with in [9]. Also, we correct and refine [9, Theorem 3], where the second-order local approximation for these BPS's was constructed (see Corollary 4.1). The proof of this result involves the derivation of a refinement of the local *Poisson-exponential approximation* for the Pólya-Aeppli family, which is given as Theorem 3.10 of Section 3. The necessary background on the Poisson-exponential class is provided in Section 3.

It is relevant that the assertions of Theorems 3.10–3.11 are closely related to the *Feller-diffusion* approximation for some BPS's. This important member of the class of the real-valued diffusion stochastic processes with continuous trajectories, which is hereinafter denoted by $\{\widetilde{F}(t),\ t \geq 0\}$, is characterized by formulas (1.2)–(1.4) below. In particular, it constitutes a continuous martingale, which satisfies the following stochastic differential equation that describes the time dynamics:

$$d\widetilde{\mathcal{F}}(t) = \sqrt{\widetilde{\mathcal{F}}(t)} \cdot d\mathcal{W}(t). \tag{1.2}$$

Here, $\mathcal{W}(t)$ denotes the standard univariate Brownian motion (with zero drift). In addition, we impose the following initial condition:

$$\widetilde{\mathcal{F}}(0) = 1. \tag{1.3}$$

The solution to (1.2)–(1.3) is hereinafter called the *Feller diffusion* with zero drift, and which starts from a point source. Also, the stochastic process $\widetilde{\mathcal{F}}(t)$ is a time-homogeneous Markov process with generator $\mathcal{V}$ such that for an arbitrary twice-continuously differentiable function $\tau(x)$,

$$\mathcal{V}\tau(x) = (x/2) \cdot \tau''(x). \tag{1.4}$$

See [6] or [21] for more detail.

Also in Section 4, we present Corollary 4.3 to Theorem 3.13 of Section 3. This corollary specifies the second-order-term approximation in the *local* limit theorem on Poisson convergence for a specific subclass of Pólya-Aeppli distributions which are associated with the stochastic backward evolution of the BPS dealt with in Section 4. The *integral* limit theorem on Poisson convergence, which is analogous to this corollary, was presented in [9, Theorem 2.iv and formula (4.30)]. In turn, that result serves as an excellent illustration to [12, Theorem 4.5]; see [9, Proposition 2] for more details. In short, these results stipulate that particular BPS's which undergo critical binary branching and start from a random, Pólya-Aeppli distributed number of particles, must have originated from a Poisson field (compare [9, p. 257]).

We reckon that the results of such kind deserve being discussed in detail. This is because such assertions on the Poisson convergence do not appear to be related to the Poisson law of small numbers. See Remark 4.4 of Section 4 for a relevant discussion.

At the same time, our results which pertain to the Poisson-exponential approximation for the Pólya-Aeppli family are parallel to the classical Gnedenko-Kolmogorov theory on weak convergence to infinitely divisible distributions. This is partly because of several invariance properties of this class, which are presented in [26, formulas (3.1) and (3.3)], [9, Theorem 1]. Namely, the Poisson sum (4.3) (with a bounded mean) of strictly positive geometrically distributed i.i.d.r.v.'s can also be represented as a different Poisson sum (with an increasing mean) of certain i.i.d.r.v.'s with common zero-modified geometric distribution. Recall that the resulting Poisson-exponential limit is closely related to the Feller diffusion, which is described by formulas (1.2)–(1.4). Other characterization properties of members of the Pólya-Aeppli class are given in [4, Section 5].

Note that in contrast to [9], where emphasis was made primarily on the use of the cumulant-generating function (or *c.g.f.*), the unit variance function (or *u.v.f.*), and the Poisson-mixture representation (see formula (3.8)), here we also employ representation

(3.3) for the probability function of a generic member of the Pólya-Aeppli family in terms of the *confluent hypergeometric function* defined by formula (2.2). We relate some of our results to those discussed in [26], [28], and [9, Introduction].

This article is not self-contained. Therefore, we refer to [6] or [9, Introduction] for more detail on BPS's (including their properties and limits), to [12] or [30]–[31] for a comprehensive description and important examples of EDMs and *natural exponential families* (or *NEF*s), and to [23] for more information on the relevant special functions.

## 2. Auxiliary Definitions and Relevant Properties of Special Functions

First, we summarize some relevant notation and terminology. We will follow the custom of formulating various statements of distribution theory in terms of the properties of r.v.'s, even when such results pertain only to their *distributions*. Hereinafter, $\mathbf{R}^1_+$ stands for the set of all positive reals. In what follows, the sign "$\stackrel{\mathrm{d}}{=}$" will denote the fact that the distributions of (univariate) r.v.'s coincide, whereas the symbol "$\stackrel{d}{\to}$" will stand for *weak convergence*. Given $a \in \mathbf{R}^1$, we denote by $\mathbf{D}[a,\infty)$ the càdlàg space of functions on the time interval $[a,\infty)$ that are right continuous and possess left-hand limits, which is equipped with the Skorohod topology. The sign "$\stackrel{\mathbf{D}[\mathrm{a},\infty)}{=\!=\!=}$" is understood as the fact that the laws of two stochastic processes coincide in this space, and the symbol "$\stackrel{\mathbf{D}[\mathrm{a},\infty)}{\Longrightarrow}$" will denote convergence in the càdlàg space $\mathbf{D}[a,\infty)$. An empty sum is interpreted as zero. In the sequel, we will denote a sequence of i.i.d.r.v.'s which possess the same distribution as a generic r.v. $\mathcal{Y}$ by $\{\mathcal{Y}^{(n)},\ n \geq 1\}$.

Given $k \in \mathbf{Z}_+$, denote the *Pochhammer symbol* by

$$(w)_k := \frac{\Gamma(w+k)}{\Gamma(w)} = w(w+1)...(w+k-1)$$

with the convention that $(w)_0 := 1$.

Hereinafter, $I_\upsilon(\cdot)$, $\chi(\cdot)$ and log stand for the *modified Bessel function of the first kind* of order $\upsilon$, the indicator function, and the *natural* logarithm, respectively. Given Poisson r.v. $\mathcal{P}oiss(\rho)$ with mean $\rho \in \mathbf{R}^1_+$ and $\ell \in \mathbf{Z}_+$, set

$$\pi_\rho(\ell) := \mathbf{P}\{\mathcal{P}oiss(\rho) = \ell\} = e^{-\rho}\rho^\ell/\ell!. \tag{2.1}$$

**Definition 2.1.** For arbitrary complex values of $a$, $b$ and $z$ such that $b \notin \{0,-1,-2,\ldots\}$, set

$${}_1F_1(a;b;z) := \sum_{\ell=0}^{\infty} \frac{(a)_\ell}{(b)_\ell} \cdot \frac{z^\ell}{\ell!}. \tag{2.2}$$

Following [23, formula (1.1.8)], we refer to ${}_1F_1(a;b;z)$ as the confluent hypergeometric function.

In this work, we will concentrate primarily on the case where the second argument $b$ of the function ${}_1F_1(a;b;z)$ equals 2.

The following Poincaré series is derived with some effort from [23, formula (3.8.3)]. For an arbitrary fixed $z \in \mathbf{R}^1_+$ and as real $u \to \infty$,

$${}_1F_1(u+1;2;z) \sim \frac{(zu)^{-3/4}e^{z/2}}{2\sqrt{\pi}} \cdot e^{2\sqrt{zu}} \cdot \sum_{\ell=0}^{\infty}(-1)^\ell \cdot \mathcal{B}_\ell(z) \cdot (zu)^{-\ell/2}. \tag{2.3}$$

Here, $\{\mathcal{B}_\ell(x),\ \ell \geq 0\}$ are certain polynomials such that $\mathcal{B}_0(x) = 1$, $\mathcal{B}_1(x) = (3-(4/3)x^2)/16$, $\mathcal{B}_2(x) = (-15/16 - 5x^2/2 + x^4/9)/32, \ldots$ . Next, from the expansion of the ratio of two gamma functions given in [2, formula (5.11.13)], we note that for an arbitrary fixed $r \in \mathbf{N}$,

$$(w)_r \sim w^r(1 + r(r-1)/(2w) + \cdots) \text{ as } w \to +\infty.$$

Subsequently, the above asymptotic result implies the following refinement of [23, formula (4.4.1)]. Given $u \in \mathbf{R}^1_+$, $y \in \mathbf{R}^1_+$, and as real $a \to \infty$,

$$\begin{aligned} {}_1F_1(au; 2; y/a) &\sim \sum_{r=0}^{\infty} \frac{(uy)^r}{r!(r+1)!} \cdot \left(1 + \frac{r(r-1)}{2ua} + \cdots\right) \\ &\sim I_1(2\sqrt{uy})/\sqrt{uy} + \frac{\sqrt{y/u}}{2a} \cdot I_3(2\sqrt{uy}) + \cdots . \end{aligned} \tag{2.4}$$

We were not able to find a reference to the following assertion in the literature on Analysis. At the same time, its version has already been known to probabilists (see the proof of the lemma below). It will be employed for the derivation of the asymptotic representation (2.6), which is of particular value in the studies of branching populations.

**Lemma 2.2.** *For fixed integer $z \geq 2$ and $s \in \mathbb{C} \setminus \{0\}$,*

$${}_1F_1(z; 2; s) = \frac{e^s}{\Gamma(z)} \cdot s^{z-2} \cdot \sum_{k=0}^{z-2} \frac{(2-z)_k (1-z)_k}{k!} \cdot s^{-k}. \tag{2.5}$$

*Proof.* It can be derived by application of Kummer's transformation with a subsequent reversion of the terminating hypergeometric series (compare to [11, formulas (9.138)–(9.139)]). □

In turn, (2.5) leads to the asymptotic result that for a fixed *integer* $z \geq 2$ and as real $s \to \infty$,

$${}_1F_1(z; 2; s) = \frac{e^s}{\Gamma(z)} \cdot s^{z-2} \cdot (1 + (z-2) \cdot (z-1)/s + \mathcal{O}(s^{-2})) \tag{2.6}$$

(compare [23, formula (4.1.6)]).

We conclude this section with an important analytical result and its two corollaries. Thus, the following theorem can be regarded as the *analytical* counterpart of the *probabilistic* local large deviation limit theorem for the lattice family of the Pólya-Aeppli distributions (see Theorem 3.7 of Section 3). This is because Theorem 2.3 below is consistent with the *probabilistic* local limit Theorem 3.7. The latter assertion concerns the case where there is no upper bound on the magnitude of large deviations imposed. Specifically, the following statement stresses the *analytical* reason behind the asymptotic representation (3.21), which is valid because of the *double asymptotic* result on the behavior of function ${}_1F_1$.

**Theorem 2.3.** *Suppose that $y$ and $\nu$ are real-valued parameters, and that $y \to \infty$.*
**(i)** *In addition, assume that $y \cdot \nu \to \infty$. Then*

$$\begin{aligned} {}_1F_1(1+y; 2; \nu) \sim &\frac{1}{\nu\sqrt{2\pi y}(1 + 4y/\nu)^{1/4}} \\ &\times \exp\{\nu(1 + \sqrt{1 + 4y/\nu})/2\} \cdot (\sqrt{1 + 4y/\nu} + 1)/(\sqrt{1 + 4y/\nu} - 1))^y. \end{aligned}$$

**(ii)** *Assume that* $0 < \nu \le Const/y$. *Then*

$$ {}_1F_1(1+y;2;\nu) \sim I_1(2\sqrt{\nu y})/\sqrt{\nu y}. \tag{2.7}$$

*Proof.* **(i)** From the expansion in terms of modified Bessel functions $I_c$ given in [24, formula (27.4.64)] we have the leading term

$$\frac{1}{\Gamma(c)} \cdot {}_1F_1(a;c;ax) \sim \beta^{1-c}\frac{\Gamma(1+a-c)}{\Gamma(a)} \times \exp\{ax/2\}\{\mathfrak{A}_0(c)I_{c-1}(2a\beta) - \mathfrak{B}_0(c)I_c(2a\beta)\}$$

for $a \to +\infty$ uniformly in $x \in [0,\infty)$ with the parameter $c$ bounded. The coefficients $\mathfrak{A}_0(c)$ and $\mathfrak{B}_0(c)$ are as follows:

$$\begin{cases} \mathfrak{A}_0(c) = \frac{2^{1/2}\beta^{c-1/2}x^{-c/2}}{(1+4/x)^{1/4}}\cosh(\frac{1}{2}cw_0), \\ \mathfrak{B}_0(c) = \frac{2^{1/2}\beta^{c-1/2}x^{-c/2}}{(1+4/x)^{1/4}}\sinh(\frac{1}{2}cw_0). \end{cases} \tag{2.8}$$

Here,

$$\beta = \tfrac{1}{2}(w_0 + \sinh w_0), \qquad w_0 = 2\,\mathrm{arcsinh}(\tfrac{1}{2}\sqrt{x}). \tag{2.9}$$

The following recurrence relation can be easily derived from [23, formula (2.2.3)]:

$${}_1F_1(1+a;2;z) = \frac{(a-1)}{a}\cdot {}_1F_1(a;2;z) + \frac{1}{a}\cdot {}_1F_1(a;1;z).$$

Therefore, we find that, with $\zeta := 2a\beta$,

$$\begin{aligned} &{}_1F_1(1+a;2;ax) \\ &\sim \frac{\exp\{\frac{1}{2}ax\}}{a\beta}\{\mathfrak{A}_0(2)I_1(\zeta) - \mathfrak{B}_0(2)I_2(\zeta) + \beta(\mathfrak{A}_0(1)I_0(\zeta) - \mathfrak{B}_0(1)I_1(\zeta))\}. \end{aligned} \tag{2.10}$$

In the case where $x$ is such that $a\beta \to \infty$, we can employ the well-known asymptotic approximation $I_\nu(z) \sim e^z/\sqrt{2\pi z}$ for $z \to +\infty$. Observing from formula (2.8) that

$$\mathfrak{A}_0(2) - \mathfrak{B}_0(2) = \frac{2^{1/2}\beta^{3/2}}{(1+4/x)^{1/4}x}\,(1+\omega(x)),$$

$$\mathfrak{A}_0(1) - \mathfrak{B}_0(1) = -\frac{2^{1/2}\beta^{1/2}\omega(x)}{(1+4/x)^{1/4}x}\ ,$$

with $\omega(x) := \frac{1}{2}x(1-\sqrt{1+4/x})$, we then obtain

$$\begin{aligned} {}_1F_1(1+a;2;ax) &\sim \frac{\exp\{\frac{1}{2}ax + 2a\beta\}}{2\sqrt{\pi}(a\beta)^{3/2}}\{\mathfrak{A}_0(2) - \mathfrak{B}_0(2) + \beta(\mathfrak{A}_0(1) - \mathfrak{B}_0(1))\} \\ &= \frac{\exp\{\frac{1}{2}ax + 2a\beta\}}{\sqrt{2\pi}a^{3/2}x(1+4/x)^{1/4}} \\ &= \frac{\exp\{\frac{1}{2}ax(1+\sqrt{1+4/x})\}}{\sqrt{2\pi}\cdot a^{3/2}x(1+4/x)^{1/4}}\left(\frac{\sqrt{1+4/x}+1}{\sqrt{1+4/x}-1}\right)^a \end{aligned} \tag{2.11}$$

upon expressing the arcsinh appearing in the quantity $\beta$ in formula (2.9) in its standard logarithmic form.

For the function ${}_1F_1(1+y;2;\nu) \equiv {}_1F_1(1+y;2;xy)$, with $x = \nu/y \to 0$, we have from formula (2.9) that $\beta \sim \sqrt{x} = (\nu/y)^{1/2}$ and hence, the argument of the Bessel

functions $\zeta \sim 2(\nu y)^{1/2}$. The result stated in Theorem 2.3.i then immediately follows from formula (2.11) as $y \to \infty$ with $\nu$ such that $\nu y \to \infty$.
**(ii)** When $\nu \le C/y$, $x = \nu/y \le C/y^2$, then $\zeta \sim 2(\nu y)^{1/2} = \mathcal{O}(1)$ and we can no longer approximate the Bessel functions which emerge in formula (2.10) by their asymptotic form. Since $\beta \sim x^{1/2}$ as $x \to 0$ and, from formula (2.8), $\mathfrak{A}_0(c) \sim 1$, $\mathfrak{B}_0(c) \sim \frac{1}{2}c\sqrt{x}$ in this limit, we obtain from formula (2.10) the validity of the asymptotic relation (2.7). This result can also be obtained from [23, formula (3.8.3)]. □

The following corollary to Theorem 2.3.i is of particular value. Namely, we will employ it in the proof of Corollary 5.5 of the concluding Section 5, when studying asymptotic properties of the *average process* $\{\mathfrak{N}_{\mu,\lambda}(t),\ t \ge 0\}$. It is constructed starting from the Pólya-Aeppli Lévy process $\mathcal{R}^{(\rho,\gamma)}(t)$, which is defined by formula (1.1) with the values of $\mu$ and $\lambda$ specified by formulas (5.4)– (5.5).

**Corollary 2.4.** *Suppose that* $\mathfrak{a} \in \mathbf{R}^1_+$, $\mathfrak{b} \in \mathbf{R}^1_+$ *are fixed, and that the real-valued parameter* $\nu \to \infty$. *Then*

$$
\begin{aligned}
{}_1F_1(1+\mathfrak{a}\nu; 2; \mathfrak{b}\nu) \sim \frac{1}{\mathfrak{b}\sqrt{2\pi\mathfrak{a}}} \cdot \frac{1}{\nu^{3/2}(1+4\mathfrak{a}/\mathfrak{b})^{1/4}} \\
\times \exp\left\{\nu\Big[\frac{\mathfrak{b}(1+\sqrt{1+4\mathfrak{a}/\mathfrak{b}})}{2} - \mathfrak{a}\cdot\log\frac{\sqrt{1+4\mathfrak{a}/\mathfrak{b}}-1}{\sqrt{1+4\mathfrak{a}/\mathfrak{b}}+1}\Big]\right\}.
\end{aligned}
\tag{2.12}
$$

*Proof.* It easily follows from Theorem 2.3.i when $y/\nu = \mathcal{O}(1)$. In addition, it can also be derived by a simple saddle-point calculation using the contour integral representation, which can be found in [23, formula (3.1.27)]. □

Finally, we will utilize the following modifications of formula (2.4) in Theorems 3.7.ii and 3.11 of Section 3, which in turn are employed for the derivation of Corollary 4.1.ii and Theorem 5.3 of Sections 4 and 5, respectively.

**Corollary 2.5.** *Fix* $\mathcal{K} \in \mathbf{R}^1_+$ *and* $\mathcal{L} \in \mathbf{R}^1_+$. *Then*
(**i**) *Suppose that the positive real-valued parameter* $\mathcal{C} \to \infty$, *and* $u \in \mathbf{R}^1_+$ *(which might depend on* $\mathcal{C}$*) is such that* $u\mathcal{C} \to \infty$. *Then*

$$
{}_1F_1(u\mathcal{C}+1; 2; \mathcal{K}/(\mathcal{L}+\mathcal{C})) \sim I_1(2\sqrt{\mathcal{K}u})/\sqrt{\mathcal{K}u}. \tag{2.13}
$$

(**ii**) *In the case where all the conditions of part (i) are fulfilled and* $u \to \infty$, *the expressions on both sides of (2.13) are equivalent to* $e^{2\sqrt{\mathcal{K}u}}/(2\sqrt{\pi}(\mathcal{K}u)^{3/4})$.

*Proof.* The result (2.13) follows from the leading term of the expansion given in [23, formula (3.8.3)]. □

## 3. Background and New Results for Pólya-Aeppli and Poisson-exponential Laws

Throughout the remainder of this paper, we will index two related families of Pólya-Aeppli and Poisson-exponential distributions $\{\mathcal{X}_{\mu,\lambda};\ \mu \in \mathbf{R}^1_+, \lambda \in \mathbf{R}^1_+\}$ and $\{\mathcal{U}_{\mu,\lambda};\ \mu \in \mathbf{R}^1_+, \lambda \in \mathbf{R}^1_+\}$, whose members take values in $\mathbf{Z}_+$ and $[0,\infty)$, respectively, with two parameters $\mu$ and $\lambda$. Both $\mu$ and $\lambda$ take values in $\mathbf{R}^1_+$. For each such admissible pair $(\mu,\lambda)$, set

$$
\mathcal{M} := 2\lambda\sqrt{\mu};\ \theta := 2\lambda/\sqrt{\mu};\ \mathcal{Z} := \mathcal{M}\theta/(\theta+1). \tag{3.1}
$$

**Definition 3.1.** (Pólya-Aeppli family). Given arbitrary $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$, a generic Pólya-Aeppli r.v. $\mathcal{X}$ $(= \mathcal{X}_{\mu,\lambda})$ is determined by the following probability function on $\mathbf{Z}_+$:

$$\mathbf{P}\{\mathcal{X} = 0\} = e^{\mathcal{Z}-\mathcal{M}}; \tag{3.2}$$

$$\mathbf{P}\{\mathcal{X} = k\} = e^{-\mathcal{M}}(\theta+1)^{-k}\mathcal{Z} \cdot {}_1F_1(k+1; 2; \mathcal{Z}) \text{ for } k \in \mathbf{N}. \tag{3.3}$$

By [26, formula (3.2)], the c.g.f. $\xi_{\mu,\lambda}(s)$ of the r.v. $\mathcal{X}_{\mu,\lambda}$ is such that for $s < \log(1+\theta)$,

$$\begin{aligned}\xi_{\mu,\lambda}(s) &:= \log \mathbf{E}\exp\{s\mathcal{X}_{\mu,\lambda}\} \\ &= \mathcal{M}(e^s - 1)/(1+\theta-e^s) = \mathcal{M}\theta/(1+\theta-e^s) - \mathcal{M}.\end{aligned} \tag{3.4}$$

It easily follows from formula (3.4) that

$$\mathbf{E}\mathcal{X}_{\mu,\lambda} = \mu. \tag{3.5}$$

The variance of this r.v. is given in [9, formula (3.6)], where a slightly different notation is used. See also the variance-to-mean relationship (3.15) below.

The algorithms for computing the *cumulative* distribution function of a generic member of the Pólya-Aeppli class as well as some interesting applications can be found in [18]. Also, we recall that a connection between a generic compound Poisson-geometric process defined by formula (1.1) and the above class of the Pólya-Aeppli probability laws is specified by formulas (5.3)–(5.5) of Section 5.

**Definition 3.2.** (Poisson-exponential family). Given $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$, a generic Poisson-exponential r.v. $\mathcal{U}$ $(= \mathcal{U}_{\mu,\lambda})$ is characterized by the following c.g.f.:

$$\zeta_{\mu,\lambda}(s) := \log \mathbf{E}\exp\{s\mathcal{U}_{\mu,\lambda}\} = \mathcal{M}s/(\theta - s), \text{ where the argument } s < \theta. \tag{3.6}$$

The r.v. $\mathcal{U}_{\mu,\lambda}$ has a *mixed* probability law. Thus, since

$$\mathbf{P}\{\mathcal{U}_{\mu,\lambda} = 0\} = \lim_{s\to-\infty} \exp\{\zeta_{\mu,\lambda}(s)\} = \exp\{-\mathcal{M}\},$$

it has a positive mass at zero. Also, it has an absolutely continuous component in $\mathbf{R}^1_+$ with the following density:

$$\begin{aligned}f_{\mu,\lambda}(x) &= 2\lambda x^{-1/2} \cdot \exp\{-\theta(x+\mu)\} \cdot I_1(4\lambda \cdot \sqrt{x}) \\ &= \sqrt{\theta\mathcal{M}} \cdot x^{-1/2} \cdot \exp\{-(\theta x + \mathcal{M})\} \cdot I_1(2\sqrt{\theta\mathcal{M} \cdot x}).\end{aligned} \tag{3.7}$$

It is well known (see, for example, [26, formulas (2.3)–(2.6)]) that for arbitrary fixed values of $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$, the r.v. $\mathcal{U}$ $(= \mathcal{U}_{\mu,\lambda})$ admits a compound Poisson-exponential representation with the Poisson parameter $\mathcal{M}$ and common mean $1/\theta$ of the corresponding i.i.d. exponentially distributed summands. An analogous, compound Poisson-geometric representation for a generic member of the Pólya-Aeppli class can be found in [26, formula (3.3)] or [9, formulas (4.1)–(4.8)].

A combination of these representations with the Yaglom theorem on the exponential limit for a scaled geometric family (see, for example, [9, formula (4.12)]) justifies the validity of the Poisson-exponential approximation for a scaled Pólya-Aeppli family on a heuristic level. Its local version is closely ralated to Theorem 3.10 below.

The following assertion, which stipulates that the Pólya-Aeppli r.v. $\mathcal{X}_{\mu,\lambda}$ can be represented as the Poisson mixture (with unit value of the Poisson parameter) of the Poisson-exponential r.v. $\mathcal{U}_{\mu,\lambda}$ is well known (compare [28, formula (3.8)] or [9, formula (3.28)]). However, we elected to present its alternative, but still simple proof here because of its originality, since it relies on representation (3.9) of the Theory of Special Functions and thus, emphasizes a connection between two branches of Mathematics.

**Lemma 3.3.** *For arbitrary fixed $\mu \in \mathbf{R}^1_+$, $\lambda \in \mathbf{R}^1_+$, and integer $k \in \mathbf{Z}_+$,*

$$\mathbf{P}\{\mathcal{X}_{\mu,\lambda} = k\} = \int_0^\infty e^{-u} \cdot \frac{u^k}{k!} \cdot d\mathbf{P}\{\mathcal{U}_{\mu,\lambda} \le u\}. \tag{3.8}$$

*Proof.* It follows from [23, p. 44, formula (3.2.27)] that for each integer $k \in \mathbf{Z}_+$,

$$\int_0^\infty e^{-t} t^{k-1/2} I_1(2\sqrt{xt})dt = k!\sqrt{x} \cdot {}_1F_1(k+1;2;x). \tag{3.9}$$

It then follows from formulas (3.7) and (3.9) that for an arbitrary fixed integer $k \in \mathbf{Z}_+$,

$$\int_0^\infty e^{-u} \cdot \frac{u^k}{k!} \cdot f_{\mu,\lambda}(u) \cdot du = e^{-\mathcal{M}}(\theta+1)^{-k}\mathcal{Z} \cdot {}_1F_1(k+1,2,\mathcal{Z}). \tag{3.10}$$

A subsequent combination of formulas (3.2), (3.3), (3.10) with the trivial identity $z \cdot {}_1F_1(1,2,z) \equiv e^z - 1$ implies the validity of representation (3.8). □

The following definition was introduced in [17, Definition 2] in order to describe the class of univariate lattice distributions for which it turned out to be possible to derive numerous results on the Cramér-type asymptotics up to equivalence for the probabilities of large deviations for the partial sums of the corresponding i.i.d.r.v.'s. The subsequent technical Lemma 3.5 stipulates that each Pólya-Aeppli probability law belongs to this class. Moreover, it is interesting that our Theorem 3.7 and Corollary 3.8 below generalize [17, Theorem 2] in the case where the common distribution of these i.i.d.r.v.'s belongs to the Pólya-Aeppli family (see Remark 3.9.i).

**Definition 3.4.** A generic r.v. $\mathcal{N}$ which takes values on the lattice $\{f + nh\}$ (with real $f \ge 0$, span $h \in \mathbf{R}^1_+$, and $n \in \mathbf{Z}$) is said to belong to the class $(\mathcal{S})$ if there exists a fixed $\kappa \in \mathbf{R}^1_+$ such that for $\ell \in \{f + nh\}$, and as $\ell \to \infty$,

$$\mathbf{P}\{\mathcal{N} = \ell\} \sim \exp\big\{-\kappa\ell + \int_{x_0}^\ell g(u)du\big\}. \tag{3.11}$$

Also, it is assumed that the function $g(\cdot) : \mathbf{R}^1_+ \to \mathbf{R}^1$ is such that **(i)** there exists $x_0 \in \mathbf{R}^1_+$ such that $\forall x \ge x_0 > 0$, $g(x) > 0$; **(ii)** $g(\infty) = 0$; **(iii)** $g''(x) \downarrow$; **(iv)** the product $x \cdot g(x) \to +\infty$ as $x \to +\infty$, and **(v)** $\forall x \ge x_0$, $0 \le -g''(x)/g'(x) \le 2/x$.

**Lemma 3.5.** *Each member $\mathcal{X}_{\mu,\lambda}$ of the Pólya-Aeppli family belongs to the class $(\mathcal{S})$ with $f = 0$, $h = 1$, and integer $\ell \in \mathbf{Z}_+$.*

*Proof.* The first step is similar to [26, Proposition 3.1]. Given a generic Pólya-Aeppli r.v. $\mathcal{X}_{\mu,\lambda}$, a combination of formulas (2.3) and (3.3) yields that this r.v. admits representation (3.11) with $\kappa = \log(\theta+1) > 0$. Namely, as integer $\ell \to \infty$,

$$\mathbf{P}\{\mathcal{X}_{\mu,\lambda} = \ell\} \sim \exp\big\{-\ell \cdot \log(\theta+1) + (2\sqrt{\mathcal{Z}\ell} - \frac{3}{4} \cdot \log\ \ell + \mathbf{K})\big\}. \tag{3.12}$$

Here, the constant $\mathbf{K} = \log(e^{-\mathcal{M}}\mathcal{Z}^{1/4}/(2\sqrt{\pi}))$ (compare [26, Proposition 3.1]). In view of formula (3.12), representation (3.11) is valid with

$$g(x) := \frac{d}{dx}\Big(2\sqrt{\mathcal{Z}x} - \frac{3}{4} \cdot \log\ x + \mathbf{K}\Big) = \sqrt{\mathcal{Z}/x} - 3/(4x). \tag{3.13}$$

The first two derivatives of this function are as follows:

$$g'(x) = -\sqrt{\mathcal{Z}}/(2x^{3/2}) + 3/(4x^2);\ g''(x) = 3\sqrt{\mathcal{Z}}/(4x^{5/2}) - 3/(2x^3). \tag{3.14}$$

The verification of the fact that the function $g(x)$ defined by formula (3.13) satisfies all the conditions imposed in Definition 3.4 is straightforward. Thus, formula (3.14) yields that as $x \to \infty$,

$$\begin{aligned} 0 \le -g''(x)/g'(x) &= (2/x) \cdot (3\sqrt{\mathcal{Z}}/4 - 3/(2\sqrt{x})/(\sqrt{\mathcal{Z}} - 3/(2\sqrt{x})) \\ &< 2/x \text{ for } x > x_0 := 4/\mathcal{Z} > 0. \end{aligned}$$

□

*Remark* 3.6. **(i)** Recall that the totality of the class of Pólya-Aeppli distributions can be parameterized in such a way that it constitutes an additive EDM on $\mathbf{Z}_+$. It is relevant that the invariant $\alpha$ of the exponential tilting transformation of this family, which is defined by [9, formula (3.7)], coincides with the third argument $\mathcal{Z}$ of the function ${}_1F_1$ (see formulas (3.1) and (3.3)). A combination of this observation with [9, formula (3.10)] implies that the u.v.f. $V_{\mathcal{Z}}(\mu)$ of the NEF comprised of the members of the Pólya-Aeppli family with this value of invariant $\mathcal{Z}$ is as follows:

$$V_{\mathcal{Z}}(\mu) = \mu \cdot \sqrt{4\mathcal{Z}^{-1} \cdot \mu + 1}\,. \tag{3.15}$$

**(ii)** A combination of formulas (3.2)–(3.3) stipulates that a probabilistic interpretation for the reciprocal $1/\mathcal{Z}$ of the invariant of the exponential tilting transformation of the Pólya-Aeppli class, which is given below [9, proof of Lemma 3], is equivalent to the following representation for the confluent hypergeometric function, which is easily derived from its basic properties:

$$(1 + z/2) \cdot {}_1F_1(2; 2; z)^2 \equiv e^z \cdot {}_1F_1(3; 2; z),\ \text{where the argument z} \in \mathbb{C}.$$

Next, we present the *unit deviance function* (or *u.d.f.*) $d_{\mathcal{Z}}(w, \mu)$ of the Pólya-Aeppli EDM, which describes the rate of the exponential decay (compare to formula (3.21)). Thus, a combination of [12, p. 68, Exercise 2.25], formula (3.15) and some calculus stipulates that for real $w \ge \mu$,

$$\begin{aligned} \frac{1}{2}\, d_{\mathcal{Z}}(w, \mu) &= \int_\mu^w \frac{w - t}{\mathbf{V}_{\mathcal{Z}}(t)}\, dt \\ &= \Big(w \cdot \log \frac{\sqrt{1 + (4/\mathcal{Z})t} - 1}{\sqrt{1 + (4/\mathcal{Z})t} + 1} - \frac{\mathcal{Z}}{2} \cdot \sqrt{1 + (4/\mathcal{Z})t}\Big)\Big|_\mu^w \\ &= w \cdot \Big(\log \frac{\sqrt{1 + (4/\mathcal{Z})w} - 1}{\sqrt{1 + (4/\mathcal{Z})w} + 1} + \log(\theta + 1)\Big) - \frac{\mathcal{Z}}{2} \cdot (1 + \sqrt{1 + (4/\mathcal{Z})w}). \end{aligned} \tag{3.16}$$

At this stage, we proceed with the presentation of a series of new results of Probability Theory which pertain to the asymptotic behavior of various probabilities for certain Pólya-Aeppli distributed r.v.'s. They are geared towards our further studies in the Theory of Stochastic Processes presented in Sections 4-5, and involve two different transformations of the pair of parameters $(\mu, \lambda)$ of the Pólya-Aeppli family. Specifically, the results needed for the studies of branching particle systems, which are undertaken in Section 4, involve either the following scaling of the parameters:

$$\begin{cases} \mu \to \mathcal{C}\mu; \\ \lambda \to \lambda/\sqrt{\mathcal{C}} \end{cases} \tag{3.17}$$

or no scaling of the parameter $\mu$ at all. It is easy to check that the transformation (3.17) does not change the value of $\mathcal{M}$, whereas $\theta$ is divided by the scaling factor $\mathcal{C}$ (compare with formulas (3.28)–(3.29)). In contrast, our new results geared towards the studies of compound Poisson-geometric processes with Pólya-Aeppli marginals, which are conducted in Section 5, rely on a *different* scaling of these parameters of the Pólya-Aeppli distributions. Namely,

$$\begin{cases} \mu \to \mathcal{C}\mu; \\ \lambda \to \lambda\sqrt{\mathcal{C}}. \end{cases} \tag{3.18}$$

It easily follows from formula (3.1) that the transformation (3.18) leaves the value of $\theta$ unchanged, whereas $\mathcal{M}$ is multiplied by the scaling factor $\mathcal{C}$. A subsequent application of formula (3.4) implies that this transformation corresponds to the addition of specific independent Pólya-Aeppli r.v.'s. In particular, the scaling transformation (3.18) which is consistent with summation of i.i.d.r.v.'s with common Pólya-Aeppli distribution, implies that this class constitutes an additive EDM for counts, and leads to consideration of Lévy processes with Pólya-Aeppli marginals.

The first group of our results pertains primarily to the probabilities of large deviations and saddlepoint-type approximations. In addition, part (ii) of Theorem 3.7 below addresses a version of the Poisson-exponential approximation. Recall that the density component $f_{\mu,\lambda}(x)$ of a generic Pólya-Aeppli distribution, which is employed on the right-hand side of formula (3.20) below, is given by formula (3.7).

**Theorem 3.7.** *Fix $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$, and assume that the parameter $\nu$ takes on positive real values. Consider a family of the Pólya-Aeppli distributed r.v.'s $\{\mathcal{X}_{\nu\mu,\sqrt{\nu}\lambda},\ \nu \in \mathbf{R}^1_+\}$ and suppose that the integer-valued $y \to +\infty$.*

**(i)** *Suppose that $\nu$ varies in such way that $y \cdot \nu \to +\infty$. Then*

$$\begin{aligned} \mathbf{P}\{\mathcal{X}_{\nu\mu,\sqrt{\nu}\lambda} = y\} &\sim \frac{1}{\sqrt{2\pi\nu \mathbf{V}_{\mathcal{Z}}(y/\nu)}} \exp\Big\{ -\frac{\nu}{2} \cdot d_{\mathcal{Z}}(y/\nu, \mu)\Big\} \\ &= \frac{1}{\sqrt{2\pi y}(1+4\mathcal{Z}^{-1}y/\nu)^{1/4}} \cdot (\theta+1)^{-y} \cdot \exp\{-\nu\mathcal{M}\} \\ &\times \left( \frac{\sqrt{1+4\mathcal{Z}^{-1}y/\nu}+1}{\sqrt{1+4\mathcal{Z}^{-1}y/\nu}-1}\right)^{y} \exp\Big\{\frac{\mathcal{Z}}{2}\nu\big(1+\sqrt{1+4\mathcal{Z}^{-1}y/\nu}\big)\Big\}. \end{aligned} \tag{3.19}$$

**(ii)** *Suppose that $\nu$ varies in such way that $\nu \le Const/y$. Then*

$$\mathbf{P}\{\mathcal{X}_{\nu\mu,\sqrt{\nu}\lambda} = y\} \sim \frac{1}{\theta+1} \exp\big\{y\big(\frac{\theta}{\theta+1} - \log(\theta+1)\big)\big\} \cdot f_{\nu\mu,\sqrt{\nu}\lambda}(y/(\theta+1)). \tag{3.20}$$

*Proof.* It follows with some effort from a combination of formulas (3.1), (3.3), (3.15), (3.16) with Theorem 2.3.i-ii. □

The following corollary to Theorem 3.7 stipulates the validity of both a *local limit theorem* which takes into account small, normal and large deviations for a sequence of the partial sums of Pólya-Aeppli distributed i.i.d.r.v.'s with an increasing number of terms, and also the tail asymptotics up to equivalence of such sums in the case of a *fixed* number of the Pólya-Aeppli distributed summands.

**Corollary 3.8.** *Fix $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$. Consider the sequence $\mathcal{X}^{(1)}_{\mu,\lambda} + ... + \mathcal{X}^{(n)}_{\mu,\lambda}$ of the partial sums of Pólya-Aeppli distributed i.i.d.r.v.'s. Suppose that $y \in \mathbf{N}$ and $n \in \mathbf{N}$ vary in such way that $y \to \infty$ and $y \cdot n \to \infty$. Then*

$$\begin{aligned} \mathbf{P}\{\mathcal{X}^{(1)}_{\mu,\lambda} + ... + \mathcal{X}^{(n)}_{\mu,\lambda} = y\} &\sim \frac{1}{\sqrt{2\pi n \mathbf{V}_{\mathcal{Z}}(y/n)}} \exp\Big\{ -\frac{n}{2} \cdot d_{\mathcal{Z}}(y/n, \mu)\Big\} \\ = \frac{1}{\sqrt{2\pi y}(1+4\mathcal{Z}^{-1}y/n)^{1/4}} &\cdot (\theta+1)^{-y} \cdot \exp\{-n\mathcal{M}\} \\ \times \left( \frac{\sqrt{1+4\mathcal{Z}^{-1}y/n}+1}{\sqrt{1+4\mathcal{Z}^{-1}y/n}-1} \right)^y &\exp\Big\{ \frac{\mathcal{Z}}{2} n\big(1+\sqrt{1+4\mathcal{Z}^{-1}y/n}\big)\Big\}. \end{aligned} \tag{3.21}$$

*Proof.* It follows from formula (3.4) that for each $n \in \mathbf{N}$,

$$\mathcal{X}_{n\mu,\sqrt{n}\lambda} \overset{\mathrm{d}}{=} \mathcal{X}^{(1)}_{\mu,\lambda} + ...\mathcal{X}^{(n)}_{\mu,\lambda}, \tag{3.22}$$

where $\{\mathcal{X}^{(\ell)}_{\mu,\lambda}\ 1 \le \ell \le n\}$ are the i.i.d.r.v.'s which have the same distribution as the Pólya-Aeppli r.v. $\mathcal{X}_{\mu,\lambda}$. The rest follows from formula (3.19). □

*Remark* 3.9. **(i)** Suppose that $\epsilon > 0$ is an arbitrary fixed real. Then in the special case where integer $y = y(n)$ varies in such a way that $y \ge (\mu+\epsilon)n$ as $n \to \infty$, the validity of representation (3.21) can be derived with some effort from [17, Theorem 2]. Indeed, it requires straightforward calculus to verify that formula (3.21) is consistent with [17, Theorem 2]. In fact, it is [17, Theorem 2] which was our driving force for the derivation of Theorems 2.3.i and 3.7, as well as Corollaries 2.4 and 3.8.
**(ii)** In the case where $y \sim Const \cdot n$, i.e., when the magnitude of large deviations is proportional to the number of summands, Corollary 3.8 becomes the local version of the classical Cramér–Petrov theorem for the partial sums of i.i.d.r.v.'s with common *lattice* distribution. Moreover, it appears that one can refine this theorem with some effort. To this end, one would need to derive the closed-form expression for the terms of the Poincaré series which emerge in [10, Theorem 5] or [7, Theorem 2.4.1] in our special case of Pólya-Aeppli distributed i.i.d.r.v.'s. We conjecture that such results would be consistent with those that can be obtained by modifying the terms of the Poincaré series which emerges in [24, formula (27.4.62)].
**(iii)** The middle expressions in formulas (3.19) and (3.21), which involve the functions introduced by formulas (3.15) and (3.16), can be regarded as the *saddlepoint-type* approximations for the Pólya-Aeppli additive EDM and for the sample mean, respectively, in the case where there are specific constraints imposed on the parameter values.

The second group of our results pertains to the *local Poisson-exponential* approximations for the Pólya-Aeppli family. Note in passing that the corresponding result on weak convergence in an important special case which pertains to branching particle systems is given by formula (4.8) of Section 4.

Next, for arbitrary fixed $\mu \in \mathbf{R}^1_+$, $\lambda \in \mathbf{R}^1_+$, and values of the argument $u \in \mathbf{R}^1_+$, define

$$\mathcal{D}_{\mu,\lambda}(u) := 4\lambda^2 + \theta^2 u - 4\lambda\sqrt{u}\theta I_0(4\lambda\sqrt{u})/I_1(4\lambda\sqrt{u}); \tag{3.23}$$

$$\mathcal{E}_{\mu,\lambda}(u) := \theta^3 u(3\theta u - 8)/4 + 6\lambda^2\theta(3\theta u - 4) + 12\lambda^4 I_3(4\lambda\sqrt{u})/I_1(4\lambda\sqrt{u}) \\ + 8\lambda^3(2 - 3\theta u) \cdot \frac{I_2(4\lambda\sqrt{u})}{\sqrt{u}\cdot I_1(4\lambda\sqrt{u})} + 6\lambda\sqrt{u}\theta^2(2 - \theta u) \cdot \frac{I_0(4\lambda\sqrt{u})}{I_1(4\lambda\sqrt{u})}\,. \tag{3.24}$$

Since $\forall n \in \mathbf{Z}_+$, $I_n(z)/I_1(z) \sim (z/2)^{n-1}/n!$ as $z \to 0$, it is natural to define these functions at the origin by continuity:

$$\mathcal{D}_{\mu,\lambda}(0) := \lim_{u\downarrow 0} \mathcal{D}_{\mu,\lambda}(u) = 4\lambda^2 - 2\theta = (\mathcal{M} - 2)\theta; \tag{3.25}$$

$$\mathcal{E}_{\mu,\lambda}(0) := \lim_{u\downarrow 0} \mathcal{E}_{\mu,\lambda}(u) = 2(8\lambda^4 - 12\lambda^2\theta + 3\theta^2) = (\mathcal{M}^2 - 6\mathcal{M} + 6)\theta^2. \tag{3.26}$$

**Theorem 3.10.** *Fix $\mu \in \mathbf{R}^1_+$, $\lambda \in \mathbf{R}^1_+$, and the value of argument $u \in [0, +\infty)$. Suppose that the positive real-valued parameter $\mathcal{C}$ is such that $u\mathcal{C}$ is an integer as $\mathcal{C} \to \infty$. Then*

$$\mathbf{P}\{\mathcal{X}_{\mathcal{C}\mu,\lambda/\sqrt{\mathcal{C}}} = u\cdot\mathcal{C}\} = e^{-\mathcal{M}}\chi(\{u = 0\}) \\ + \frac{f_{\mu,\lambda}(u)}{\mathcal{C}} \cdot \Big\{1 + \frac{\mathcal{D}_{\mu,\lambda}(u)}{2\mathcal{C}} + \frac{\mathcal{E}_{\mu,\lambda}(u)}{6\mathcal{C}^2} + \mathcal{O}(1/\mathcal{C}^3)\Big\}. \tag{3.27}$$

*Proof.* First, a combination of formula (3.1) with the fact that $\mu_{\mathcal{C}} := \mathcal{C}\mu$ and $\lambda_{\mathcal{C}} := \lambda/\sqrt{\mathcal{C}}$ yields that

$$\mathcal{M}(\mathcal{C}) \equiv \mathcal{M}(1) := \mathcal{M}; \tag{3.28}$$

$$\theta(\mathcal{C}) \equiv \theta/\mathcal{C}. \tag{3.29}$$

A subsequent combination of formulas (3.7) and (3.8) yields that the probability which emerges on the left-hand side of formula (3.27) equals

$$\frac{e^{-\mathcal{M}}}{(u\mathcal{C})!} \cdot \int_0^\infty e^{-(\theta(\mathcal{C})+1)v} \cdot v^{u\mathcal{C}-1} \cdot (4\lambda^2 \cdot v/\mathcal{C})^{1/2} \cdot I_1(4\lambda \cdot \sqrt{v/\mathcal{C}})dv \\ = \frac{e^{-\mathcal{M}} \cdot \mathcal{C}^{u\mathcal{C}}}{(u\mathcal{C})!} \cdot \int_0^\infty e^{-\mathcal{C}z} \cdot z^{u\mathcal{C}} f_{\mu,\lambda}(z)dz, \tag{3.30}$$

where $f_{\mu,\lambda}(z)$ is defined by formula (3.7). Here, we made the change of variables $z = v/\mathcal{C}$.

The rest of the proof is similar to that of [28, Theorem 2.9.i]. We rewrite the integrand on the right-hand side of formula (3.30) and consider the following integral

$$I := \int_0^\infty e^{-\mathcal{C}(z-u\log\, z)} \cdot f_{\mu,\lambda}(z)dz \tag{3.31}$$

as $\mathcal{C} \to +\infty$. The asymptotics of the integral (3.31) is easily derived by an application of Laplace's method. Thus, we find that as $\mathcal{C} \to +\infty$

$$I = u^{u\mathcal{C}}e^{-u\mathcal{C}} \cdot \sqrt{\frac{2\pi u}{\mathcal{C}}}\, f_{\mu,\lambda}(u)\left\{1 + \frac{d_1}{\mathcal{C}} + \frac{d_2}{\mathcal{C}^2} + \mathcal{O}(\mathcal{C}^{-3})\right\}, \tag{3.32}$$

where (see, for example, [19, p. 13])

$$d_1 := \frac{1}{2}uf_2 + f_1 + \frac{1}{12u}, \qquad d_2 := \frac{1}{8}u^2 f_4 + \frac{5}{6}uf_3 + \frac{25}{24}f_2 + \frac{f_1}{12u} + \frac{1}{288u^2}$$

and we have defined $f_k := f^{(k)}_{\mu,\lambda}(u)/f_{\mu,\lambda}(u)$ with $k \geq 1$.

Combining formulas (3.30)–(3.32) with the Stirling expansion

$$(u\mathcal{C})! = \sqrt{2\pi}(u\mathcal{C})^{u\mathcal{C}+1/2}e^{-u\mathcal{C}}\left\{1 + \frac{1}{12u\mathcal{C}} + \frac{1}{288(u\mathcal{C})^2} + \mathcal{O}(1/\mathcal{C}^3)\right\}, \tag{3.33}$$

we find

$$\frac{e^{-\mathcal{M}}\mathcal{C}^{u\mathcal{C}}\cdot I}{(u\mathcal{C})!} = \frac{e^{-\mathcal{M}}f_{\mu,\lambda}(u)}{\mathcal{C}}\left\{1+\frac{\mathcal{D}_{\mu,\lambda}(u)}{2\mathcal{C}}+\frac{\mathcal{E}_{\mu,\lambda}(u)}{6\mathcal{C}^2}+\mathcal{O}(1/\mathcal{C}^3)\right\},$$

where

$$\mathcal{D}_{\mu,\lambda}(u) = d_1 - \frac{1}{12u}\,, \qquad \mathcal{E}_{\mu,\lambda}(u) = d_2 - \frac{d_1}{12u} + \frac{1}{288u^2}\,.$$

After some lengthy algebra, the coefficients $\mathcal{D}_{\mu,\lambda}(u)$ and $\mathcal{E}_{\mu,\lambda}(u)$ are found to be expressible in terms of ratios of modified Bessel functions as given in formulas (3.23) and (3.24). □

The following assertion is analogous to limit theorems which *act on the whole axis* for a sequence of partial sums of i.i.d.r.v.'s with a common not necessarily lattice distribution. We refer to [25] for a comprehensive description of numerous limit theorems which *act on the whole axis* in the context of partial sums of i.i.d.r.v.'s.

**Theorem 3.11.** *Fix $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$. Suppose that the positive real-valued parameter $\mathcal{C} \to \infty$. Let the argument $u$ (which might depend on $\mathcal{C}$) take on non-negative real values, but $u\mathcal{C}$ be an integer. Then*

$$\mathbf{P}\{\mathcal{X}_{\mathcal{C}\mu,\lambda/\sqrt{\mathcal{C}}} = u\mathcal{C}\} = e^{-\mathcal{M}}\chi(\{u=0\}) + \frac{f_{\mu,\lambda}(u)}{\mathcal{C}}\cdot(1+o(1)). \tag{3.34}$$

*Proof.* In the case where non-negative integer $u$ is bounded from above as $\mathcal{C} \to \infty$, the validity of (3.34) easily follows from formula (3.27). In the case where $u \to \infty$ as $\mathcal{C} \to \infty$ (and even as just $u \cdot \mathcal{C} \to \infty$ as $\mathcal{C} \to \infty$), the validity of (3.34) is obtained from a combination of formulas (3.1), (3.3), (3.28) and (3.29). This ascertains the applicability of Corollary 2.5 with $\mathcal{K} = 4\lambda^2$ and $\mathcal{L} = 2\lambda/\sqrt{\mu}$. The rest is simple algebra. □

*Remark* 3.12. It is relatively easy to extend Theorem 3.11 to the case when the value of $u\mathcal{C}$, which emerges inside the sign of probability on the left-hand side of formula (3.34), is replaced by a more general integer-valued function $k(\mathcal{C})$ which varies in such a manner that $k(\mathcal{C})/\mathcal{C} \to u$ as $\mathcal{C} \to \infty$. The details are left to the reader.

The third "group" of our results presented in this section is comprised of Theorem 3.13, which pertains to the Poisson approximation for the Pólya-Aeppli family. Thus, it is relevant that formula (2.6) implies the following second-order-term approximation in the local limit theorem on Poisson convergence for the Pólya-Aeppli EDM in the case where $\mu \in \mathbf{R}^1_+$ is fixed and real $\lambda \to \infty$. This result is closely related to the stochastic backward evolution problem investigated in [9, Theorem 2.iv and formula (4.30)]. Recall that $\pi_\mu(\ell)$ is defined by formula (2.1).

**Theorem 3.13.** *Fix $\mu \in \mathbf{R}^1_+$, and assume that the real-valued parameter $\lambda \to \infty$. Then for an arbitrary fixed $\ell \in \mathbf{Z}_+$,*

$$\mathbf{P}\{\mathcal{X}_{\mu,\lambda} = \ell\} = \pi_\mu(\ell)\cdot\Big(1+\frac{\ell^2-(1+2\mu)\ell+\mu^2}{2\sqrt{\mu}\lambda}+\mathcal{O}(\lambda^{-2})\Big). \tag{3.35}$$

*Proof.* It follows with some effort from a combination of formulas (2.6) and (3.2)–(3.3). Thus, in the case where $\ell = 0$, the verification of the validity of formula (3.35) is straightforward. Also, in view of formula (3.3), in the case where $\ell \in \mathbf{N}$ representation (3.35) is equivalent to the following formula:

$$\begin{aligned} &e^{-2\lambda\sqrt{\mu}} \cdot 4\lambda^2 \cdot (1+2\lambda/\sqrt{\mu})^{-(\ell+1)} \cdot {}_1F_1(\ell+1;2;4\lambda^2/(1+2\lambda/\sqrt{\mu})) \\ &\quad = \frac{e^{-\mu}\mu^\ell}{\ell!} \cdot \Big(1 + \frac{\ell^2-(1+2\mu)\ell+\mu^2}{2\sqrt{\mu}\lambda} + \mathcal{O}(\lambda^{-2})\Big) \text{ as } \lambda \to \infty. \end{aligned} \tag{3.36}$$

The validity of formula (3.36) then follows by combining formula (2.6) with some algebra. □

*Remark* 3.14. It is not difficult to derive the asymptotics up to equivalence for the probabilities of "large deviations" under fulfillment of the assumptions of Theorem 3.13 which would complement this theorem. For instance, one can consider the case where integer $\ell \to \infty$ with the same rate as $\mathcal{Z}$. For simplicity, set $\mu = 1$. Then $\mathcal{Z} = \mathcal{Z}(\lambda) = 4\lambda^2/(2\lambda+1) \sim 2\lambda \to \infty$. Assume that $\ell := \ell(\lambda) = 4\lambda$ takes on *integer* values. Then it relatively easily follows from formula (2.12) that as $\lambda \to \infty$,

$$\begin{aligned} \mathbf{P}\{\mathcal{X}_{1,\lambda} = 4\lambda\} &= e^{-2\lambda} \cdot 4\lambda^2 \cdot (1+2\lambda)^{-(4\lambda+1)} \cdot {}_1F_1(4\lambda+1;2;4\lambda^2/(1+2\lambda)) \\ &\sim \sqrt{\lambda/(6\pi)} \cdot \frac{e^{2\lambda}2^{4\lambda}}{(1+2\lambda)^{4\lambda+1}} \,. \end{aligned} \tag{3.37}$$

At the same time, Stirling's formula (compare representation (3.33)) implies that in the case where $\mu = 1$ and as $\lambda \to \infty$, the Poisson probability

$$\pi_1(4\lambda) = e^{-1}/\Gamma(4\lambda+1) \sim \frac{e^{4\lambda-1}}{\sqrt{8\lambda\pi}\cdot(4\lambda)^{4\lambda}} \,. \tag{3.38}$$

However, it is easily seen that the right-hand sides of formulas (3.37) and (3.38) are not equivalent to each other as $\lambda \to \infty$.

Note that [10, Theorems 1 and 5] are quite general providing asymptotic expansions in local limit theorems for lattice distributions, which take into account large deviations, in the case of convergence to various limiting distributions. However, these results do not seem to be applicable under the fulfillment of the conditions of the above Theorems 3.10–3.11, since [10, condition (1)] is not met. In addition, it would be of interest to investigate possible relations of the *local* large deviation limit theorems of the same type as representatiom (3.37) above with the *integral* large deviation limit theorems in the case of approximation by the Poisson law, which can be found in [1] (see also the references therein). However, this is beyond the scope of this paper.

## 4. Local Limit Theorems for Branching Particle Systems

This section pertains to applications of some of our general results on Pólya-Aeppli distributions to the derivation of new asymptotic expansions in the local approximations for specific BPS's. They are presented as Corollaries 4.1 and 4.3.

Here, we provide a rather limited description of this class of the BPS's, which undergo critical binary branching, as well as of the limiting diffusion process, which is closely related to that defined by formulas (1.2)–(1.4). We refer to [6], [21] or [9] for a more comprehensive consideration of these topics. First, recall that there is an important result of Probability Theory which concerns the Poisson-exponential approximation for scaled members of the Pólya-Aeppli family. (This assertion is presented below in the relevant special case as formula (4.8).) Its interesting counterparts in the context of the stochastic evolution of the BPS's are also widely available. This group of results of the Theory of Stochastic Processes is usually termed the *Feller-diffusion* approximation (see, for example, [6] or [21]). Its version is presented below as formula (4.6). Note that although the compound Poisson representations for specific members of the Pólya-Aeppli and Poisson-exponential classes of distributions along with the Yaglom theorem on the exponential convergence provide a simple *probabilistic* explanation for the validity of this approximation in our special case, but in fact, it is representation (3.8) which is quite suitable for the derivation of rather accurate asymptotic representations for specific probabilities associated with members of the Pólya-Aeppli and Poisson-exponential classes. They are given by formulas (3.27) and (4.13).

It is relevant that our major driving force behind the studies of subtle properties of the class of Pólya-Aeppli distributions is the fact that members of this family emerge as the total number of particles of the BPS considered in [9], which are alive at a particular time instant $t$ (see [9, pp. 261–262]).

Now, let us provide some more specific details. Given the parameter $\delta \in \mathbf{R}^1_+$ and the real-valued time instant $t \in (-2/\delta, +\infty)$, set $\mathcal{A}_{\delta,t} := 2\delta/(2+\delta t)$. Following [9], we introduce a positive integer-valued parameter $\eta$. We launch our BPS from the Pólya-Aeppli distributed, random number $\mathcal{X}_{\eta,\delta/(2\sqrt{\eta})}$ of the independent particles (possibly sparsely located) at the initial time instant $t = 0$. In view of formula (3.5), the expected number of initial particles $\mathbf{E}\mathcal{X}_{\eta,\delta/(2\sqrt{\eta})} = \eta$.

Note that the initial field here is different from those used in the majority of previous works in this area, but this assumption coincides with the specific initial field that was employed in [9, Sections 4–5]. We elect to assume that our BPS has such Pólya-Aeppli distributed field at time $t = 0$ in order to be consistent with the terminology and results of [9]. In particular, this assumption enables one to trace back this particle system to a Poisson field at time $t = -2/\delta$ (compare to formula (4.15) below and [9, Theorem 2.iv]). Moreover, our Corollary 4.3 below can be regarded as a *local* counterpart of that theorem. We refer to [9, p. 254] for more details on the rationale behind this choice of the initial distribution of the BPS's considered.

Each particle is assumed to perform an independent random spatial motion in $\mathbf{R}^d$ and is assigned the same mass $1/\eta$. This BPS undergoes *critical binary branching*. Namely, at an exponentially distributed time instant with mean $1/\eta$, the particle either dies out with probability $1/2$ or splits into two offspring with the same probability $1/2$. Each newly-born particle is an identical copy of its parent and immediately starts to perform the same spatial motion. The motions, lifetimes and branchings of all particles are assumed to be independent of each other. Following [9], hereinafter we denote this BPS by $\mathcal{L}^{(\eta)}_{\delta,t}$. It constitutes a *measure-valued stochastic process* (see, for example, [6] or [21]).

Subsequently, we introduce the so-called *total mass* process, which is the following real-valued stochastic process:

$$L^{(\eta)}_\delta(t) := \eta^{-1} \cdot \mathcal{L}^{(\eta)}_{\delta,t}(\mathbf{R}^d). \tag{4.1}$$

Here, we can now assume that $t \in (-2/\delta, +\infty)$. This is because in view of [9], the stochastic evolution of this BPS can be traced backward in time up to time instant $-2/\delta$.

Also, it is often more convenient to deal with the *integer-valued* modification of the process (4.1), which is defined as follows:

$$\widetilde{L}_\delta^{(\eta)}(t) := \eta \cdot L_\delta^{(\eta)}(t) = \mathcal{L}_{\delta,t}^{(\eta)}(\mathbf{R}^d). \tag{4.2}$$

Evidently, the integer-valued stochastic process (4.2) represents a random total number of the living particles of the BPS $\mathcal{L}_{\delta,t}^{(\eta)}$ at time $t$.

It then follows from [9, pp. 261–262] that given time instant $t \in (-2/\delta, +\infty)$, the r.v.

$$\widetilde{L}_\delta^{(\eta)}(t) \stackrel{\mathrm{d}}{=} \mathcal{X}_{\eta,\mathcal{A}_{\delta,t}/(2\sqrt{\eta})}. \tag{4.3}$$

The formula (4.3) stipulates that for each such value of $t$, the total number of the living particles $\widetilde{L}_\delta^{(\eta)}(t)$ of this BPS is a member of the Pólya-Aeppli family with parameters $\eta$ and $\mathcal{A}_{\delta,t}/(2\sqrt{\eta})$. A subsequent combination of formulas (3.5) and (4.3) implies that the average number $\eta$ of the living particles of this BPS is *time invariant*, i.e., the branching mechanism is *critical*.

Next, recall that we have already defined the Feller diffusion $\mathcal{F}(t)$ with no drift, and which starts from a point source by formulas (1.2)–(1.4). Subsequently, given $\delta \in \mathbf{R}_+^1$, we introduce the *backshifted* Feller diffusion process $\{\mathcal{F}_\delta(t),\ t \in [-2/\delta, +\infty)\}$ as follows:

$$\mathcal{F}_\delta(t) := \mathcal{F}(t + 2/\delta). \tag{4.4}$$

By (1.3),

$$\mathcal{F}_\delta(-2/\delta) = 1. \tag{4.5}$$

It is well known that for an arbitrary *fixed* time instant $s \in (-2/\delta, +\infty)$, the total mass process $\{L_\delta^{(\eta)}(t),\ t \in [s, +\infty)\}$, which is defined by (4.1), converges in the càdlàg space $\mathbf{D}[s, \infty)$ to the backshifted Feller diffusion process $\{\mathcal{F}_\delta(t),\ t \in [s, +\infty)\}$. Namely,

$$L_\delta^{(\eta)}(\cdot) \stackrel{\mathbf{D}[s,\infty)}{\Longrightarrow} \mathcal{F}_\delta(\cdot) \text{ as } \eta \to +\infty \tag{4.6}$$

(see, for example, [6] or [21]).

It is important that all the marginals of the latter limiting diffusion process with continuous trajectories, which emerges on the right-hand side of formula (4.6), belong to the *Poisson-exponential* family. Specifically, given time instant $t \in (-2/\delta, +\infty)$, the r.v.

$$\mathcal{F}_\delta(t) \stackrel{\mathbf{d}}{=} \mathcal{U}_{1,\delta/(2+t\delta)} \tag{4.7}$$

(see, for example, [6] or [21]). Also, a combination of formulas (4.2)–(4.7) implies that for an arbitrary fixed time instant $t \in (-2/\delta, +\infty)$ and as $\eta \to +\infty$,

$$L_\delta^{(\eta)}(t)\ \left(\stackrel{\mathrm{d}}{=} \eta^{-1} \cdot \mathcal{X}_{\eta,\mathcal{A}_{\delta,t}/(2\sqrt{\eta})}\right) \stackrel{\mathbf{d}}{\to} \mathcal{U}_{1,\delta/(2+t\delta)}\ \left(\stackrel{\mathrm{d}}{=} \mathcal{F}_\delta(t)\right) \tag{4.8}$$

(compare [9, Proposition 3.i]). It is easily seen that the result (4.8) on weak convergence can also be derived by establishing the pointwise convergence of the corresponding c.g.f.'s, which are given by formulas (3.4) and (3.6) of Section 3. Also, it is relevant that formula (4.5) is consistent with [9, formula (4.54)]. Indeed, the latter result stipulates that the *backward stochastic evolution* of the limiting backshifted Feller diffusion process $\{\mathcal{F}_\delta(t),\ t \in (-2/\delta, +\infty)\}$ can be traced back up to time $t = -2/\delta$ in the sense that

$$1 = \mathcal{F}_\delta(-2/\delta) \stackrel{\mathbf{P}}{=} \lim_{t \downarrow -2/\delta} \mathcal{F}_\delta(t). \tag{4.9}$$

Next, we proceed with the construction of the asymptotic expansions in the local version of the relationship (4.8) on weak-convergence. To this end, and by analogy to [9, formulas (5.1)–(5.2)], we define $\mathcal{J}_\nu = \mathcal{J}_\nu(\delta,t,u) := I_\nu(2\mathcal{A}_{\delta,t}\sqrt{u})$.

Now, set $\mathcal{G}_{\delta,t}(u) := \mathcal{D}_{1,\mathcal{A}_{\delta,t}/2}(u)$. Combine formulas (3.23) and (3.25) of Section 3 to yield that

$$\mathcal{G}_{\delta,t}(u) = \begin{cases} \mathcal{A}_{\delta,t}^2 \cdot (u+1) - 2\mathcal{A}_{\delta,t}\sqrt{u}\mathcal{J}_0/\mathcal{J}_1 & \text{if } u > 0 \\ \mathcal{A}_{\delta,t}^2 - 2\mathcal{A}_{\delta,t} & \text{if } u = 0 \end{cases} \tag{4.10}$$

(compare [9, formula (5.3)]). Also, define $\mathcal{H}_{\delta,t}(u) := \mathcal{E}_{1,\mathcal{A}_{\delta,t}/2}(u)$. It then follows from formulas (3.24) and (3.26) of Section 3 that

$$\begin{aligned} \mathcal{H}_{\delta,t}(u) = \mathcal{A}_{\delta,t}^3\Big( &\frac{3\mathcal{A}_{\delta,t}u^2 + 2(9\mathcal{A}_{\delta,t} - 4)u - 24}{4} \\ + &\frac{3\mathcal{A}_{\delta,t}\mathcal{J}_3}{4\mathcal{J}_1} + \frac{(2 - 3\mathcal{A}_{\delta,t}u)\mathcal{J}_2 + 3u(2 - \mathcal{A}_{\delta,t}u)\mathcal{J}_0)}{\sqrt{u}\mathcal{J}_1}\Big) \text{ if u} > 0; \end{aligned} \tag{4.11}$$

$$\mathcal{H}_{\delta,t}(0) = \mathcal{A}_{\delta,t}^2(\mathcal{A}_{\delta,t}^2 - 6\mathcal{A}_{\delta,t} + 6). \tag{4.12}$$

The parts (i) and (ii) of the following assertion specify Theorems 3.10 and 3.11 for the BPS described above. In addition, part (i) corrects and refines [9, Theorem 3] in an essential way.

**Corollary 4.1.** *Given $\delta \in \mathbf{R}_+^1$, fix $t \in (-2/\delta, \infty)$, and suppose that the integer-valued parameter $\eta$, which constitutes the time-invariant expected number of the living particles, is such that $\eta \to \infty$. Then*
**(i)** *In the case where integer $u \in \mathbf{Z}_+$ is fixed, the marginals of the total mass process $L_\delta^{(\eta)}(t)$ of the BPS admit the following third-order local approximation:*

$$\begin{aligned} \mathbf{P}\{L_\delta^{(\eta)}(t) = u\} &= e^{-\mathcal{A}_{\delta,t}}\chi(\{u=0\}) \\ &+ \frac{f_{1,\mathcal{A}_{\delta,t}/2}(u)}{\eta} \cdot \Big\{1 + \frac{\mathcal{G}_{\delta,t}(u)}{2\eta} + \frac{\mathcal{H}_{\delta,t}(u)}{6\eta^2} + \mathcal{O}(1/\eta^3)\Big\}. \end{aligned} \tag{4.13}$$

*Here, the functions $\mathcal{G}_{\delta,t}(u)$ and $\mathcal{H}_{\delta,t}(u)$ are defined by formulas (4.10)–(4.12).*
**(ii)** *In the case where $u \in \mathbf{Z}_+$ might depend on $\eta$, the following asymptotic representation for the probability function of the marginals of the total mass process $L_\delta^{(\eta)}(t)$, which takes into account large deviations, is valid:*

$$\mathbf{P}\{L_\delta^{(\eta)}(t) = u\} = e^{-\mathcal{A}_{\delta,t}}\chi(\{u=0\}) + \frac{f_{1,\mathcal{A}_{\delta,t}/2}(u)}{\eta} \cdot (1 + o(1)). \tag{4.14}$$

*Proof.* The validity of the parts (i) and (ii) of this corollary follows by setting $\mathcal{C} = \eta$, $\mu = 1$, and $\lambda = \mathcal{A}_{\delta,t}/2 = \delta/(2+\delta t)$ in Theorems 3.10 and 3.11, respectively. It remains to combine these two new representations with formulas (4.2)–(4.3). □

*Remark* 4.2. **(i)** In the case where $u = 0$, the quantity

$$f_{1,\mathcal{A}_{\delta,t}/2}(0)\mathcal{G}_{\delta,t}(0)/2 = \mathcal{A}_{\delta,t}^3 e^{-\mathcal{A}_{\delta,t}}(\mathcal{A}_{\delta,t}/2 - 1),$$

which emerges as a component of the limit as $u \downarrow 0$ of the expression which is present on the right-hand side of (4.13), is consistent with [9, bottom part of formula (5.3)]. Moreover, the continuity of the functions $\mathcal{G}_{\delta,t}(u)$ and $\mathcal{H}_{\delta,t}(u)$ at the origin, which easily follows from formulas (3.25)–(3.26), contradicts the claim on discontinuity at zero made

in [9, p. 265, Remark 9.i]. Hence, Corollary 4.1.i corrects and refines [9, Theorem 3].
**(ii)** The formula (4.14) stipulates that in the case where one is only interested in the marginals of the BPS, the corresponding local approximation by the backshifted Feller diffusion takes into account both the normal and the large deviations.

Next, consider a fixed value of $\eta \in \mathbf{N}$. In view of [9, Theorem 2.iv], the integer-valued Markov process $\{\widetilde{L}_\delta^{(\eta)}(t),\ t \in (-2/\delta, +\infty)\}$, which is defined by formula (4.2) and constitutes the total number of particles alive at time $t$ for the above BPS (with Pólya-Aeppli marginals), must have originated from a Poisson distribution $\mathcal{P}oiss(\eta)$ with mean $\eta$ at time $t = -2/\delta$. Namely,

$$\lim_{t \downarrow -2/\delta} \widetilde{L}_\delta^{(\eta)}(t) =: \widetilde{L}_\delta^{(\eta)}(-2/\delta) \stackrel{\mathrm{d}}{=} \mathcal{P}oiss(\eta), \tag{4.15}$$

where the limit in (4.15) is to be understood in the sense of weak convergence.

It is easily seen that the weak convergence of the marginals, which is stipulated by formula (4.8) in the case where $t > -2/\delta$, is transferred to the left-end point $t = -2/\delta$ in the sense that the expression on the right-hand side of formula (4.15) converges in probability as $\eta \to +\infty$ to a non-random constant limit 1 (compare expressions (4.5) and (4.9)). At the same time, in view of Remark 3.14, the asymptotic behavior of $\widetilde{L}_\delta^{(\eta)}(t)$ in the case where $t \downarrow -2/\delta$ and $\eta \to +\infty$ *simultaneously* might not be consistent with that of $\mathcal{P}oiss(\eta)$ as $\eta \to +\infty$ in the domain of large deviations.

We conclude this section with the *local* counterpart of formula (4.15), which also contains the leading error term.

**Corollary 4.3.** *Given $\delta \in \mathbf{R}_+^1$, $\eta \in \mathbf{N}$, $\ell \in \mathbf{N}$, and as $t \downarrow -2/\delta$ the probability function for the total number of particles $\widetilde{L}_\delta^{(\eta)}(t)$, which are alive at time instant t, admits the following asymptotic representation:*

$$\mathbf{P}\{\widetilde{L}_\delta^{(\eta)}(t) = \ell\} = \pi_\eta(\ell) \cdot \Big(1 + \frac{\ell^2 - (1+2\eta)\ell + \eta^2}{2\delta \cdot \sqrt{\eta}} \cdot (2 + \delta t) + \mathcal{O}((2+\delta t)^2)\Big).$$

*Proof.* It easily follows by setting the parameters $\mu = \eta$, and $\lambda = \mathcal{A}_{\delta,t}/2 = \delta/(2 + \delta t)$ in Theorem 3.13. □

*Remark* 4.4. In contrast to the majority of the results on Poisson convergence, which usually assume the infinite growth of the number of summands in the corresponding triangular array, our Theorem 3.13 and Corollary 4.3, as well as the corresponding assertions of [9], involve a rather uncommon scaling of the parameters. This is because our model stipulates a Poisson number of the geometrically distributed clusters with the Poisson parameter being bounded and approaching a positive constant. In addition, the probability of success in a single trial, which characterizes the geometric distribution involved, approaches 1. In other words, the random size of each cluster collapses into the non-random constant 1, whereas a random number of such clusters is approximated by the Poisson r.v. whose mean tends to constant $\eta$. Hence, this is a rather specific result on the Poisson convergence which clarifies the mechanism of the stochastic backward evolution of the cluster structure of the BPS considered. Recall that this assertion illustrates [12, Theorem 4.5].

It would be interesting to compare our Theorem 3.13 and Corollary 4.3 with other known results on the Poisson convergence which provide the rate of convergence or even the leading error term, such as [13, Corollary 1]. See also Remark 3.14 of Section 3 for a relevant discussion concerning the probabilities of large deviations.

## 5. New Results for Pólya-Aeppli Lévy Processes

First, it is well known that all the members of the classes of Pólya-Aeppli and Poisson-exponential distributions introduced by Definitions 3.1 and 3.2 of Section 3, respectively, are *infinitely divisible* (see, for example, [26, formula (3.4)] and [20, formula (2.10)]). Also, let us point out that the consideration of the following two classes of the corresponding Lévy processes, which are introduced by formulas (5.1)–(5.2) below, is justified by the properties of these two classes of the infinitely divisible, Pólya-Aeppli and Poisson-exponential distributions. They are described in [9, Section 3], [29, Section 3], and [20, Section 3]. See also formulas (5.6)–(5.7) below.

Throughout this section we denote the Pólya-Aeppli and Poisson-exponential Lévy processes, which correspond to specific values of parameters $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$, by $\{\mathfrak{G}_{\mu,\lambda}(t),\ t \geq 0\}$ and $\{\mathfrak{H}_{\mu,\lambda}(t),\ t \geq 0\}$, respectively. It is well known that a generic Lévy process is completely determined by its marginal at time $t = 1$. Hence, it suffices to define these two processes as follows:

$$\mathfrak{G}_{\mu,\lambda}(1) \stackrel{\mathrm{d}}{=} \mathcal{X}_{\mu,\lambda}; \tag{5.1}$$

$$\mathfrak{H}_{\mu,\lambda}(1) \stackrel{\mathrm{d}}{=} \mathcal{U}_{\mu,\lambda}. \tag{5.2}$$

Recall that the specific Pólya-Aeppli and Poisson-exponential r.v.'s $\mathcal{X}_{\mu,\lambda}$ and $\mathcal{U}_{\mu,\lambda}$ which emerge on the right-hand sides of formulas (5.1)–(5.2), are characterized in Definitions 3.1 and 3.2 of Section 3, respectively. Also, let us recall that $\{\Pi_1(t),\ t \geq 0\}$ stands for the Poisson process with unit intensity.

A combination of the compound Poisson-geometric representation (1.1) and formula (3.4) with some algebra yields that given $\rho \in \mathbf{R}^1_+$, $\gamma \in (0,1)$, and for an arbitrary fixed $t \in \mathbf{R}^1_+$, the r.v.

$$\mathcal{R}^{(\rho,\gamma)}(t) \stackrel{\mathrm{d}}{=} \mathcal{X}_{\mu_{\rho,\gamma}(t),\lambda_{\rho,\gamma}(t)}, \tag{5.3}$$

where the parameters $\mu_{\rho,\gamma}(t)$ and $\lambda_{\rho,\gamma}(t)$ are as follows:

$$\mu_{\rho,\gamma}(t) = (\rho/\gamma)t; \tag{5.4}$$

$$\lambda_{\rho,\gamma}(t) = \sqrt{\rho\gamma t}/(2(1-\gamma)). \tag{5.5}$$

Evidently, representation (5.3) stipulates that all the marginals of the compound Poisson-geometric process $\{\mathcal{R}^{(\rho,\gamma)}(t),\ t \geq 0\}$ are Pólya-Aeppli distributed with the values of the parameters $\mu_{\rho,\gamma}(t)$ and $\lambda_{\rho,\gamma}(t)$ as those specified by formulas (5.4) and (5.5), respectively. In addition, since a Lévy process can be regarded as a continuous-time analogue of the sequence of partial sums of the related i.i.d.r.v.'s, one ascertains that given $\mu \in \mathbf{R}^1_+$, $\lambda \in \mathbf{R}^1_+$, and for an arbitrary fixed $t \in \mathbf{R}^1_+$, the r.v.

$$\mathfrak{G}_{\mu,\lambda}(t) \stackrel{\mathrm{d}}{=} \mathcal{X}_{\mu t,\sqrt{t}\lambda} \tag{5.6}$$

(compare formulas (5.6) and (3.22)).

Similar to representation (5.6), [20, formula (3.2)] implies that given $\mu \in \mathbf{R}^1_+$, $\lambda \in \mathbf{R}^1_+$, and for an arbitrary fixed $t \in \mathbf{R}^1_+$, the r.v.

$$\mathfrak{H}_{\mu,\lambda}(t) \stackrel{\mathrm{d}}{=} \mathcal{U}_{\mu t,\sqrt{t}\lambda}. \tag{5.7}$$

Since a generic compound Poisson-exponential Lévy process is a *subordinator*, it can be employed in lieu of the "time" argument to make a random time change. Therefore, one may consider a *mixed Poisson process* with time $t$ replaced by a particular compound Poisson-exponential process. We refer to [8] for more details on the mixed Poisson processes.

The following representation result of the Theory of Stochastic Processes characterizes a generic Pólya-Aeppli Lévy process as the mixed Poisson process of unit intensity with the random time change being a specific Poisson-exponential Lévy process. It is a stochastic-processes-theory analogue of Lemma 3.3 of Section 3, which concerns a closely related Poisson-mixture representation for a Pólya-Aeppli distribution.

**Proposition 5.1.** *Given $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$, the Pólya-Aeppli Lévy process defined by formula (5.1), considered as an element of the càdlàg space $\mathbf{D}[0,\infty)$, admits the following representation as a mixed Poisson process:*

$$\mathfrak{G}_{\mu,\lambda}(\cdot) \stackrel{\mathbf{D}[0,\infty)}{=\!=\!=} \Pi_1(\mathfrak{H}_{\mu,\lambda}(\cdot)). \tag{5.8}$$

*Proof.* The equality in law between the Lévy processes which emerge on both sides of formula (5.8), considered as the elements of $\mathbf{D}[0,\infty)$, easily follows from a combination of Lemma 3.3 of Section 3 with [14, Propositions 3.2 and 4.1] and the well-known above-quoted fact that a generic Lévy process is completely determined by its marginal at time $t = 1$. Some relevant comments and observations can also be found in [3, p. 19 and Example 1, pp. 27–28]. □

*Remark* 5.2. The derivation of the validity of representation (5.8) starting from the fact that all the marginals of two Lévy processes coincide is straightforward, since a composition of Lévy processes is also a Lévy process. It would be interesting to determine whether a mixed Poisson process with unit intensity and the random time change being (a version of) the Feller diffusion (1.2)–(1.4) can be characterized as a BPS similar to that described in Section 4. In view of Lemma 3.3 of Section 3, all the marginals of these two stochastic processes coincide. We conjecture that the laws of these two stochastic processes also coincide in the càdlàg space $\mathbf{D}[0,\infty)$. However, the consideration of this hypothesis is beyond the scope of this paper.

The following assertion provides the tail asymptotics for the short-term behavior of a generic Pólya-Aeppli Lévy process.

**Theorem 5.3.** *Given $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$, consider the Pólya-Aeppli Lévy process*

$$\{\mathfrak{G}_{\mu,\lambda}(t),\ t \geq 0\}.$$

*Suppose that the integer-valued $y \to \infty$, and the time argument $t \leq Const/y$. Then*

$$\mathbf{P}\{\mathfrak{G}_{\mu,\lambda}(t) = y\} \sim \frac{1}{\theta+1} \exp\left\{y\left(\frac{\theta}{\theta+1} - \log(\theta+1)\right)\right\} \cdot f_{\mathfrak{H}_{\mu,\lambda}(t)}(y/(\theta+1)). \tag{5.9}$$

*Proof.* It easily follows from Theorem 3.7.ii of Section 3. In addition, note that the closed-form expressions for the probability function of the r.v. $\mathfrak{G}_{\mu,\lambda}(t)$ and for the density component $f_{\mathfrak{H}_{\mu,\lambda}(t)}(y/(\theta+1))$ which emerge in formula (5.9) are easily derived by combining Definition 3.1 with formulas (3.7), (5.6), and (5.7). □

We conclude this paper with a new theorem which takes into account large deviations for the class of Pólya-Aeppli Lévy processes, and its corollary. This corollary is related to the long-time behavior of the so-called *average process* constructed starting from a generic Pólya-Aeppli Lévy process.

**Theorem 5.4.** *Given $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$, consider the Pólya-Aeppli Lévy process*

$$\{\mathfrak{G}_{\mu,\lambda}(t),\ t \geq 0\}.$$

*Suppose that the integer-valued argument $y$ and the time instant $t$ vary in such way that $y \to \infty$ and $y \cdot t \to \infty$. Then*

$$\begin{aligned}\mathbf{P}\{\mathfrak{G}_{\mu,\lambda}(t) = y\} &\sim \frac{1}{\sqrt{2\pi y}(1 + 4\mathcal{Z}^{-1}y/t)^{1/4}} \cdot (\theta + 1)^{-y} \cdot \exp\{-t\mathcal{M}\} \\ &\times \left( \frac{\sqrt{1 + 4\mathcal{Z}^{-1}y/t} + 1}{\sqrt{1 + 4\mathcal{Z}^{-1}y/t} - 1} \right)^{y} \exp\Big\{ \frac{\mathcal{Z}}{2} t\big(1 + \sqrt{1 + 4\mathcal{Z}^{-1}y/t}\big) \Big\}.\end{aligned} \tag{5.10}$$

*Proof.* It easily follows from a combination of Theorem 3.7.i of Section 3 with the above formulas (5.6)–(5.7). ☐

**Corollary 5.5.** *Given $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$, consider the average process*

$$\mathfrak{N}_{\mu,\lambda}(t) := \mathfrak{G}_{\mu,\lambda}(t)/t,$$

*which is constructed starting from the corresponding Pólya-Aeppli Lévy process*

$$\{\mathfrak{G}_{\mu,\lambda}(t),\ t \geq 0\}.$$

*Suppose that $t \in \mathbf{R}^1_+$ and $x \in \mathbf{R}^1_+$ vary in such way that the product $t \cdot x \in \mathbf{Z}_+$, and that both $t \cdot x \to \infty$, $t^2 \cdot x \to \infty$. Then*

$$\begin{aligned}\mathbf{P}\{\mathfrak{N}_{\mu,\lambda}(t) = x\} &\sim \frac{1}{\sqrt{2\pi tx}(1 + 4\mathcal{Z}^{-1}x)^{1/4}} \cdot (\theta + 1)^{-tx} \cdot \exp\{-t\mathcal{M}\} \\ &\times \left( \frac{\sqrt{1 + 4\mathcal{Z}^{-1}x} + 1}{\sqrt{1 + 4\mathcal{Z}^{-1}x} - 1} \right)^{tx} \exp\Big\{ \frac{\mathcal{Z}}{2} t\big(1 + \sqrt{1 + 4\mathcal{Z}^{-1}x}\big) \Big\}.\end{aligned}$$

*Proof.* It easily follows from representation (5.10) and Corollary 2.4. ☐

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