Inequalities, asymptotic expansions and completely monotonic functions related to the gamma function

Chao-Ping Chen, Richard B. Paris

INEQUALITIES, ASYMPTOTIC EXPANSIONS AND COMPLETELY MONOTONIC FUNCTIONS RELATED TO THE GAMMA FUNCTION

CHAO-PING CHEN* AND RICHARD B PARIS

ABSTRACT. In this paper, we present some completely monotonic functions and asymptotic expansions related to the gamma function. Based on the obtained expansions, we provide new bounds for $\Gamma(x+1)/\Gamma(x+\frac{1}{2})$ and $\Gamma\left(x+\frac{1}{2}\right)$.

1. INTRODUCTION

A function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I and satisfies the following inequality:

$$(-1)^n f^{(n)}(x) \geq 0 \qquad (x \in I; \ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad \mathbb{N} := \{1, 2, 3, \ldots\}). \tag{1.1}$$

Dubourdieu [13, p. 98] pointed out that, if a non-constant function f is completely monotonic on $I = (a, \infty)$, then strict inequality holds true in (1.1). See also [16] for a simpler proof of this result. It is known (Bernstein's Theorem) that f is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt} \, d\mu(t),$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$ (see [48, p. 161]). The main properties of completely monotonic functions are given in [48, Chapter IV]. We also refer to [4], where a extensive list of references on completely monotonic functions can be found.

Euler's gamma function:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t, \qquad x > 0$$

is one of the most important functions in mathematical analysis and its applications in various diverse areas. The logarithmic derivative of the gamma function:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is known as the psi (or digamma) function. The derivatives of the psi function $\psi(x)$:

$$\psi^{(n)}(x) := \frac{\mathrm{d}^n}{\mathrm{d}x^n}\{\psi(x)\}, \qquad n \in \mathbb{N}$$

are called the polygamma functions.

In this paper, we present some completely monotonic functions and asymptotic expansions related to the gamma function. Based on the obtained expansions, we provide new bounds for $\Gamma(x+1)/\Gamma(x+\frac{1}{2})$ and $\Gamma\left(x+\frac{1}{2}\right)$.

The numerical values given in this paper have been calculated via the computer program MAPLE 13.

2. LEMMAS

The Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are defined by the generating functions

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad \text{and} \quad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.$$

The rational numbers $B_n = B_n(0)$ and integers $E_n = 2^n E_n(1/2)$ are called Bernoulli and Euler numbers, respectively.

It follows from Problem 154 in Part I, Chapter 4, of [39] that

$$\sum_{j=1}^{2m} \frac{B_{2j}}{(2j)!}t^{2j} < \frac{t}{e^t - 1} - 1 + \frac{t}{2} < \sum_{j=1}^{2m+1} \frac{B_{2j}}{(2j)!}t^{2j} \tag{2.1}$$

for $t > 0$ and $m \in \mathbb{N}_0$. The inequality (2.1) can be also found in [17, 40].

Lemma 1 presents an analogous result to (2.1).

Lemma 1. *For $x > 0$ and $m \in \mathbb{N}$,*

$$\sum_{j=2}^{2m+1} \frac{(1 - 2^{2j})B_{2j}}{j} \frac{x^{2j-1}}{(2j-1)!} < \frac{2}{e^x + 1} - 1 + \frac{x}{2} < \sum_{j=2}^{2m} \frac{(1 - 2^{2j})B_{2j}}{j} \frac{x^{2j-1}}{(2j-1)!}, \tag{2.2}$$

where B_n $(n \in \mathbb{N}_0)$ are the Bernoulli numbers.

Proof. The noted Boole's summation formula (see [45, p. 17]) states for $k \in \mathbb{N}$ that

$$f(1) = \frac{1}{2}\sum_{j=0}^{k-1} \frac{E_j(1)}{j!}\left(f^{(j)}(1) + f^{(j)}(0)\right) + \frac{1}{2(k-1)!}\int_0^1 f^{(k)}(t)E_{k-1}(t)\mathrm{d}t,$$

which can be written for $m \in \mathbb{N}$ as

$$f(1) - f(0) = \sum_{j=1}^m \frac{E_{2j-1}(1)}{(2j-1)!}\left(f^{(2j-1)}(1) + f^{(2j-1)}(0)\right) + \frac{1}{(2m-1)!}\int_0^1 f^{(2m)}(t)E_{2m-1}(t)\mathrm{d}t. \tag{2.3}$$

Applying formula (2.3) to $f(t) = e^{xt}$, we obtain

$$-\frac{2}{e^x + 1} + 1 - \frac{x}{2} = \sum_{j=2}^m \frac{E_{2j-1}(1)}{(2j-1)!}x^{2j-1} + \frac{x}{e^x + 1}\frac{x^{2m-1}}{(2m-1)!}\int_0^1 e^{xt}E_{2m-1}(t)\mathrm{d}t. \tag{2.4}$$

It is well known (see [1, p. 804]) that

$$E_{2m+1}(1-t) = -E_{2m+1}(t) \quad \text{and} \quad E_{2m+1}(\tfrac{1}{2}) = 0.$$

Noting that

$$E_{4m-1}(t) > 0, \quad E_{4m+1}(t) < 0 \quad \text{for} \quad 0 < t < 1/2, \quad m = 1, 2, \ldots,$$

we imply for $x > 0$ that

$$\int_0^1 e^{xt} E_{4m-1}(t)\mathrm{d}t = \int_0^{1/2} \left(e^{xt} - e^{x(1-t)}\right) E_{4m-1}(t)\mathrm{d}t < 0$$

and

$$\int_0^1 e^{xt} E_{4m+1}(t)\mathrm{d}t = \int_0^{1/2} \left(e^{xt} - e^{x(1-t)}\right) E_{4m+1}(t)\mathrm{d}t > 0.$$

Combining these with (2.4), we immediately obtain that for $x > 0$ and $m \in \mathbb{N}$,

$$-\sum_{j=2}^{2m+1} \frac{E_{2j-1}(1)}{(2j-1)!} x^{2j-1} < \frac{2}{e^x + 1} - 1 + \frac{x}{2} < -\sum_{j=2}^{2m} \frac{E_{2j-1}(1)}{(2j-1)!} x^{2j-1}. \tag{2.5}$$

Noting that

$$E_n(1) = \frac{2(2^{n+1} - 1)}{n+1} B_{n+1}, \qquad n \in \mathbb{N},$$

the inequality (2.5) can be written as (2.2). The proof of Theorem 1 is complete.   □

The inequality (2.2) can be written for $x > 0$ and $m \in \mathbb{N}_0$ as

$$\sum_{j=1}^{2m+1} \frac{(1 - 2^{2j}) B_{2j}}{j} \frac{x^{2j-1}}{(2j-1)!} < \frac{2}{e^x + 1} - 1 < \sum_{j=1}^{2m} \frac{(1 - 2^{2j}) B_{2j}}{j} \frac{x^{2j-1}}{(2j-1)!}, \tag{2.6}$$

i.e.,

$$(-1)^{m+1} \left(\frac{2}{e^x + 1} - 1 - \sum_{j=1}^m \frac{(1 - 2^{2j}) B_{2j}}{j} \frac{x^{2j-1}}{(2j-1)!} \right) > 0. \tag{2.7}$$

Lemma 2 ([10]). *Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The gamma function has the following asymptotic expansion:*

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e} \right)^x \left(1 + \sum_{j=1}^\infty \frac{b_j}{x^j} \right)^{x^\ell/r}, \qquad x \to \infty, \tag{2.8}$$

where the coefficients $b_j \equiv b_j(\ell, r) \, (j \in \mathbb{N})$ are given by

$$b_j \equiv b_j(\ell, r) = \sum \frac{r^{k_1 + k_2 + \cdots + k_j}}{k_1! k_2! \cdots k_j!} \left(\frac{B_2}{1 \cdot 2} \right)^{k_1} \left(\frac{B_3}{2 \cdot 3} \right)^{k_2} \cdots \left(\frac{B_{j+1}}{j(j+1)} \right)^{k_j}, \tag{2.9}$$

summed over all nonnegative integers k_j satisfying the equation

$$(1 + \ell)k_1 + (2 + \ell)k_2 + \cdots + (j + \ell)k_j = j.$$

Lemma 3. *Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The gamma function has the following asymptotic expansion:*

$$\Gamma\left(x + \frac{1}{2} \right) \sim \sqrt{2\pi} \left(\frac{x}{e} \right)^x \left(1 + \sum_{j=1}^\infty \frac{c_j}{x^j} \right)^{x^\ell/r}, \qquad x \to \infty, \tag{2.10}$$

where the coefficients $c_j \equiv c_j(\ell, r)\,(j \in \mathbb{N})$ are given by

$$c_j = \sum \frac{(-r)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{(1-2^{-1})B_2}{1\cdot 2}\right)^{k_1}\left(\frac{(1-2^{-3})B_4}{3\cdot 4}\right)^{k_2}\cdots\left(\frac{(1-2^{1-2j})B_{2j}}{(2j-1)(2j)}\right)^{k_j},\tag{2.11}$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1 + (3+\ell)k_2 + \cdots + (2j+\ell-1)k_j = j.$$

Proof. The following asymptotic expansion can be found [27, p. 32]

$$\ln\Gamma\left(x+\frac{1}{2}\right) \sim x\ln x - x + \ln\sqrt{2\pi} + \sum_{j=1}^{\infty}\frac{B_{2j}(\frac{1}{2})}{2j(2j-1)}x^{1-2j}, \qquad x \to \infty.\tag{2.12}$$

It is well-known (see [1, p. 805]) that

$$B_n(\tfrac{1}{2}) = -(1-2^{1-n})B_n, \qquad n \in \mathbb{N}_0,$$

and then the expansion (2.12) can be rewritten as

$$\frac{\Gamma(x+\frac{1}{2})}{\sqrt{2\pi}\,(x/e)^x} = \exp\left(\sum_{k=1}^{m}\frac{-(1-2^{1-2k})B_{2k}}{2k(2k-1)x^{2k-1}} + \mathcal{R}_m(x)\right), \qquad x \to \infty,\tag{2.13}$$

where $\mathcal{R}_m(x) = O(1/x^{2m+1})$. Further, we have

$$\left(\frac{\Gamma(x+\frac{1}{2})}{\sqrt{2\pi}\,(x/e)^x}\right)^{r/x^\ell} = e^{r\mathcal{R}_m(x)/x^\ell}\exp\left(\sum_{k=1}^{m}\frac{-r(1-2^{1-2k})B_{2k}}{2k(2k-1)x^{2k+\ell-1}}\right)$$

$$= e^{r\mathcal{R}_m(x)/x^\ell}\prod_{k=1}^{m}\left[1 + \left(\frac{-r(1-2^{1-2k})B_{2k}}{2k(2k-1)x^{2k+\ell-1}}\right) + \frac{1}{2!}\left(\frac{-r(1-2^{1-2k})B_{2k}}{2k(2k-1)x^{2k+\ell-1}}\right)^2 + \cdots\right]$$

$$= e^{r\mathcal{R}_m(x)/x^\ell}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\frac{1}{k_1!k_2!\cdots k_m!}$$

$$\times \left(\frac{-r(1-2^{-1})B_2}{1\cdot 2}\right)^{k_1}\left(\frac{-r(1-2^{-3})B_4}{3\cdot 4}\right)^{k_2}\cdots\left(\frac{-r(1-2^{1-2m})B_{2m}}{(2m-1)(2m)}\right)^{k_m}$$

$$\times \frac{1}{x^{(1+\ell)k_1+(3+\ell)k_2+\cdots+(2m+\ell-1)k_m}}.\tag{2.14}$$

On the other hand, from (2.13) it follows that for any positive integer m,

$$\left(\frac{\Gamma(x+\frac{1}{2})}{\sqrt{2\pi}\,(x/e)^x}\right)^{r/x^\ell} = 1 + \sum_{j=1}^{m}\frac{c_j}{x^j} + O(1/x^{m+1})\tag{2.15}$$

for some real numbers $c_1, \ldots, c_m$.

Equating the coefficients by the equal powers of x in (2.14) and (2.15), we see that

$$c_j = \sum \frac{(-r)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{(1-2^{-1})B_2}{1\cdot 2}\right)^{k_1}\left(\frac{(1-2^{-3})B_4}{3\cdot 4}\right)^{k_2}\cdots\left(\frac{(1-2^{1-2j})B_{2j}}{(2j-1)(2j)}\right)^{k_j},$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1 + (3+\ell)k_2 + \cdots + (2j+\ell-1)k_j = j.$$

This completes the proof of Lemma 3. $\square$

Lemma 4. *Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The following asymptotic expansion holds:*

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \sim \sqrt{x} \left(1 + \sum_{j=1}^{\infty} \frac{p_j}{x^j}\right)^{x^\ell/r}, \qquad x \to \infty, \tag{2.16}$$

where the coefficients $p_j \equiv p_j(\ell, r)$ $(j \in \mathbb{N})$ are given by

$$p_j = \sum \frac{r^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!} \left(\frac{(2^2-1)B_2}{1\cdot 1\cdot 2^2}\right)^{k_1} \left(\frac{(2^4-1)B_4}{2\cdot 3\cdot 2^4}\right)^{k_2} \cdots \left(\frac{(2^{2j}-1)B_{2j}}{j(2j-1)2^{2j}}\right)^{k_j}, \tag{2.17}$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1 + (3+\ell)k_2 + \cdots + (2j+\ell-1)k_j = j.$$

Proof. From (3.7) we obtain the following asymptotic expansion:

$$\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \sim \sqrt{x} \exp\left(\sum_{j=1}^{\infty} \left(1 - \frac{1}{2^{2j}}\right) \frac{B_{2j}}{j(2j-1)x^{2j-1}}\right), \qquad x \to \infty. \tag{2.18}$$

Write (2.18) as

$$\frac{\Gamma(x+1)}{\sqrt{x}\Gamma(x+\frac{1}{2})} \sim \exp\left(\sum_{k=1}^{m} \frac{(2^{2k}-1)B_{2k}}{k(2k-1)2^{2k}x^{2k-1}} + R_m(x)\right), \qquad x \to \infty, \tag{2.19}$$

where $R_m(x) = O(1/x^{2m+1})$. Further, we have

$$\left(\frac{\Gamma(x+1)}{\sqrt{x}\Gamma(x+\frac{1}{2})}\right)^{r/x^\ell} = e^{rR_m(x)/x^\ell} \exp\left(\sum_{k=1}^{m} \frac{r(2^{2k}-1)B_{2k}}{k(2k-1)2^{2k}x^{2k+\ell-1}}\right)$$

$$= e^{rR_m(x)/x^\ell} \prod_{k=1}^{m} \left[1 + \left(\frac{r(2^{2k}-1)B_{2k}}{k(2k-1)2^{2k}x^{2k+\ell-1}}\right) + \frac{1}{2!}\left(\frac{r(2^{2k}-1)B_{2k}}{k(2k-1)2^{2k}x^{2k+\ell-1}}\right)^2 + \cdots\right]$$

$$= e^{rR_m(x)/x^\ell} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \frac{1}{k_1!k_2!\cdots k_m!}$$

$$\times \left(\frac{r(2^2-1)B_2}{1\cdot 1\cdot 2^2}\right)^{k_1} \left(\frac{r(2^4-1)B_4}{2\cdot 3\cdot 2^4}\right)^{k_2} \cdots \left(\frac{r(2^{2m}-1)B_{2m}}{m(2m-1)2^{2m}}\right)^{k_m}$$

$$\times \frac{1}{x^{(1+\ell)k_1+(3+\ell)k_2+\cdots+(2m+\ell-1)k_m}}. \tag{2.20}$$

On the other hand, from (2.19) it follows that for any positive integer m,

$$\left(\frac{\Gamma(x+1)}{\sqrt{x}\Gamma(x+\frac{1}{2})}\right)^{r/x^\ell} = 1 + \sum_{j=1}^{m} \frac{p_j}{x^j} + O(1/x^{m+1}) \tag{2.21}$$

for some real numbers $p_1, \ldots, p_m$.

Equating the coefficients by the equal powers of x in (2.20) and (2.21), we see that

$$p_j = \sum \frac{r^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!} \left(\frac{(2^2-1)B_2}{1\cdot 1\cdot 2^2}\right)^{k_1} \left(\frac{(2^4-1)B_4}{2\cdot 3\cdot 2^4}\right)^{k_2} \cdots \left(\frac{(2^{2j}-1)B_{2j}}{j(2j-1)2^{2j}}\right)^{k_j},$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1 + (3+\ell)k_2 + \cdots + (2j+\ell-1)k_j = j.$$

This completes the proof of Lemma 4.   □

Lemma 5. *For $t \neq 0$,*

$$\frac{1}{2t^2} + \frac{1}{12} - \frac{7}{240}t^2 + \frac{31}{6048}t^4 - \frac{127}{172800}t^6 < \frac{\cosh t}{\cosh(2t) - 1} < \frac{1}{2t^2} + \frac{1}{12} - \frac{7}{240}t^2 + \frac{31}{6048}t^4.$$
(2.22)

Proof. We only prove the second inequality in (2.22). The proof of the first inequality in (2.22) is analogous. By using the power series expansion of $\cosh t$, we find that

$$\left(\frac{1}{2t^2} + \frac{1}{12} - \frac{7}{240}t^2 + \frac{31}{6048}t^4\right)\left(\cosh(2t) - 1\right) - \cosh t = \sum_{n=4}^{\infty} \frac{\alpha_n}{(2n+2)!}t^{2n}$$

with

$$\alpha_n = 2^{2n-4}\left(\frac{104}{3} + \frac{10513}{1260}n + \frac{11117}{1890}n^2 - \frac{1577}{1260}n^3 - \frac{4303}{1890}n^4 - \frac{31}{63}n^5 + \frac{62}{189}n^6\right)$$
$$\quad - 2(n+1)(2n+1)$$
$$= 2^{2n-4}\left(339 + \frac{1005649}{1260}(n-4) + \frac{1355243}{1890}(n-4)^2 + \frac{382391}{1260}(n-4)^3\right.$$
$$\quad \left. + \frac{125897}{1890}(n-4)^4 + \frac{155}{21}(n-4)^5 + \frac{62}{189}(n-4)^6\right) - 2(n+1)(2n+1)$$
$$> 2^{2n-4} \cdot 339 - 2(n+1)(2n+1)$$
$$> 0 \quad \text{for} \quad n \geq 4.$$

Hence, the second inequality in (2.22) holds. This completes the proof of Lemma 5.   □

3. COMPLETELY MONOTONIC FUNCTIONS

It is known in [45, p. 64] that

$$\frac{t}{e^t - 1} - 1 + \frac{t}{2} = \sum_{j=1}^{n} \frac{B_{2j}}{(2j)!}t^{2j} + (-1)^n t^{2n+2}\nu_n(t), \qquad n \geq 0, \tag{3.1}$$

where

$$\nu_n(t) = \sum_{k=1}^{\infty} \frac{2}{(t^2 + 4\pi^2 k^2)(2\pi k)^{2n}} > 0.$$

It is easy to see that (3.1) implies (2.1).

The noted Binet's first formula [44, p. 16] states that

$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \int_0^{\infty}\left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,dt, \qquad x > 0. \tag{3.2}$$

Combining (3.1) with (3.2), Xu and Han [46] educed in 2009 that for every $m \in \mathbb{N}_0$, the function

$$R_m(x) = (-1)^m\left[\ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x + x - \ln\sqrt{2\pi} - \sum_{j=1}^{m}\frac{B_{2j}}{2j(2j-1)x^{2j-1}}\right] \tag{3.3}$$

is completely monotonic on $(0, \infty)$.

For $m = 0$, complete monotonicity property of $R_m(x)$ was proved by Muldoon [38]. Alzer [2] first proved in 1997 that $R_m(x)$ is completely monotonic on $(0, \infty)$. In 2006, Koumandos [17]

proved double inequality (2.1), and then used (2.1) and (3.2) to give the proof of complete mono-tonicity property of $R_m(x)$. In 2009, Koumandos and Pedersen [18, Theorem 2.1] strengthened this result.

Based on the inequality (2.2), in this section we prove that for every $m \in \mathbb{N}_0$, the function

$$F_m(x) = (-1)^m \left[\ln \left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \right) - \frac{1}{2} \ln x - \sum_{j=1}^{m} \left(1 - \frac{1}{2^{2j}} \right) \frac{B_{2j}}{j(2j-1)x^{2j-1}} \right] \qquad (3.4)$$

is completely monotonic on $(0, \infty)$. This result is similar to complete monotonicity property of $R_m(x)$ in (3.3).

Theorem 1. *For every $m \in \mathbb{N}_0$, the function $F_m(x)$, defined by (3.4), is completely monotonic on $(0, \infty)$.*

Proof. The logarithm of the gamma function has the following integral representation (see [1, p. 258]):

$$\ln \Gamma(x) = \int_0^{\infty} \left[(x-1)e^{-t} - \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} \right] \frac{dt}{t}, \qquad x > 0. \qquad (3.5)$$

By using (3.5) and the following representations:

$$\ln x = \int_0^{\infty} \frac{e^{-t} - e^{-xt}}{t} dt, \qquad x > 0$$

in [1, p. 230, 5.1.32] and

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^{\infty} t^{r-1} e^{-xt} dt, \qquad x > 0 \quad \text{and} \quad r > 0$$

in [1, p. 255, 6.1.1], we find that

$$F_m(x) = (-1)^m \left[\int_0^{\infty} \left(-\frac{1}{e^{t/2} + 1} + \frac{1}{2} \right) \frac{e^{-xt}}{t} dt \right.$$

$$\left. - \sum_{j=1}^{m} \left(1 - \frac{1}{2^{2j}} \right) \frac{B_{2j}}{j(2j-1)!} \int_0^{\infty} t^{2j-2} e^{-xt} dt \right]$$

$$= \frac{1}{2} \int_0^{\infty} (-1)^{m+1} \lambda_m(t) \frac{e^{-xt}}{t} dt, \qquad (3.6)$$

where

$$\lambda_m(t) = \frac{2}{e^{t/2} + 1} - 1 - \sum_{j=1}^{m} \frac{(1 - 2^{2j}) B_{2j}}{j(2j-1)!} \left(\frac{t}{2} \right)^{2j-1}.$$

By (2.7), we have $(-1)^{m+1} \lambda_m(t) > 0$ for $t > 0$ and $m \in \mathbb{N}_0$. From (3.6) we obtain that for every $m \in \mathbb{N}_0$,

$$(-1)^n F_m^{(n)}(x) = \frac{1}{2} \int_0^{\infty} (-1)^{m+1} \lambda_m(t) t^{n-1} e^{-xt} dt > 0$$

for $x > 0$ and $n \in \mathbb{N}_0$. The proof of Theorem 1 is complete. $\qquad \square$

Under the inequality $(-1)^n F_m^{(n)}(x) > 0$ for $x > 0$ and $m, n \in \mathbb{N}_0$, we obtain the following

Corollary 1. (i) *Let* $m \in \mathbb{N}_0$. *Then for* $x > 0$,

$$\sqrt{x} \exp\left(\sum_{j=1}^{2m}\left(1 - \frac{1}{2^{2j}}\right)\frac{B_{2j}}{j(2j-1)x^{2j-1}}\right) < \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}$$

$$< \sqrt{x}\exp\left(\sum_{j=1}^{2m+1}\left(1 - \frac{1}{2^{2j}}\right)\frac{B_{2j}}{j(2j-1)x^{2j-1}}\right). \tag{3.7}$$

(ii) *Let* $m, n \in \mathbb{N}$. *Then for* $x > 0$,

$$\sum_{j=1}^{2m}\left(1 - \frac{1}{2^{2j}}\right)\frac{2B_{2j}}{(2j)!}\frac{(2j+n-2)!}{x^{2j+n-1}}$$

$$< (-1)^n\left(\psi^{(n-1)}(x+1) - \psi^{(n-1)}\left(x+\frac{1}{2}\right)\right) + \frac{(n-1)!}{2x^n}$$

$$< \sum_{j=1}^{2m-1}\left(1 - \frac{1}{2^{2j}}\right)\frac{2B_{2j}}{(2j)!}\frac{(2j+n-2)!}{x^{2j+n-1}}. \tag{3.8}$$

By using the obtained results above, we here present inequalities and integral representations for the constant π.

The problem of finding new and sharp inequalities for the gamma function Γ and in particular about the Wallis ratio

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} = \frac{1}{\sqrt{\pi}} \cdot \frac{(2n)!!}{(2n-1)!!}, \qquad n \in \mathbb{N} \tag{3.9}$$

has attracted the attention of many researchers (see [11, 19, 20, 33] and references therein). Some inequalities for π can be found (see, for example, [15, 21, 34, 35]). Here, we employ the special double factorial notation as follows:

$$(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!,$$
$$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1) = \pi^{-1/2}2^n\Gamma(n+\tfrac{1}{2}),$$
$$0!! = 1, \qquad (-1)!! = 1$$

(see [1, p. 258]). Very recently, Lin [21, Theorem 2.4] proved that for all $n \in \mathbb{N}$,

$$\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{n}\exp\left(-\frac{1}{4n} + \frac{1}{96n^3} - \frac{1}{320n^5} + \frac{17}{7168n^7} - \frac{31}{9216n^9}\right)$$

$$< \pi < \left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{n}\exp\left(-\frac{1}{4n} + \frac{1}{96n^3} - \frac{1}{320n^5} + \frac{17}{7168n^7}\right). \tag{3.10}$$

Setting $x = n$ in (3.7), we obtain estimate for the constant π:

$$\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{n}\exp\left(-\sum_{j=1}^{2m+1}\left(1 - \frac{1}{2^{2j}}\right)\frac{2B_{2j}}{j(2j-1)n^{2j-1}}\right)$$

$$< \pi < \left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{n}\exp\left(-\sum_{j=1}^{2m}\left(1 - \frac{1}{2^{2j}}\right)\frac{2B_{2j}}{j(2j-1)n^{2j-1}}\right). \tag{3.11}$$

Obviously, (3.11) is a generalization of (3.10).

Formula (3.6) gives the following integral representation:

$$\ln\left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right) - \frac{1}{2}\ln x - \sum_{j=1}^{m}\left(1 - \frac{1}{2^{2j}}\right)\frac{B_{2j}}{j(2j-1)x^{2j-1}}$$

$$= -\frac{1}{2}\int_{0}^{\infty}\left(\frac{2}{e^{t/2}+1} - 1 - \sum_{j=1}^{m}\frac{(1-2^{2j})B_{2j}}{j(2j-1)!}\left(\frac{t}{2}\right)^{2j-1}\right)\frac{e^{-xt}}{t}\,\mathrm{d}t, \tag{3.12}$$

which implies

$$\psi(x+1) - \psi\left(x+\frac{1}{2}\right) - \frac{1}{2x} + \sum_{j=1}^{m}\left(1 - \frac{1}{2^{2j}}\right)\frac{B_{2j}}{jx^{2j}}$$

$$= \frac{1}{2}\int_{0}^{\infty}\left(\frac{2}{e^{t/2}+1} - 1 - \sum_{j=1}^{m}\frac{(1-2^{2j})B_{2j}}{j(2j-1)!}\left(\frac{t}{2}\right)^{2j-1}\right)e^{-xt}\,\mathrm{d}t \tag{3.13}$$

for $x > 0$ and $m \in \mathbb{N}_0$. Formulas (3.12) and (3.13) can provide integral representations for the constant π. For example, setting $(x, m) = (1/2, 0)$ in (3.12) yields

$$\int_{0}^{\infty}\left(1 - \frac{2}{e^{u}+1}\right)\frac{e^{-u}}{u}\,\mathrm{d}u = \ln\left(\frac{\pi}{2}\right). \tag{3.14}$$

Setting $(x, m) = (1/4, 1)$ in (3.13) yields

$$\int_{0}^{\infty}\left(\frac{2}{e^{u}+1} - 1 + \frac{u}{2}\right)e^{-u/2}\,\mathrm{d}u = 4 - \pi. \tag{3.15}$$

Many formulas exist for the representation of π, and a collection of these formulas is listed [42, 43]. For more history of π see [6, 7, 14].

Very recently, Mortici et al. [36] proved some completely monotonic functions and inequalities associated with the ratio of gamma functions.

4. ASYMPTOTIC EXPANSIONS

Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}, \qquad n \in \mathbb{N} := \{1, 2, \ldots\} \tag{4.1}$$

has many applications in statistical physics, probability theory and number theory. Actually, it was first discovered in 1733 by the French mathematician Abraham de Moivre (1667–1754) in the form

$$n! \sim \text{constant} \cdot \sqrt{n}(n/e)^{n}$$

when he was studying the Gaussian distribution and the central limit theorem. Afterwards, the Scottish mathematician James Stirling (1692–1770) found the missing constant $\sqrt{2\pi}$ when he was trying to give the normal approximation of the binomial distribution.

Stirling's series for the gamma function is given (see [1, p. 257, Eq. (6.1.40)]) by

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^{x}\exp\left(\sum_{m=1}^{\infty}\frac{B_{2m}}{2m(2m-1)x^{2m-1}}\right) \tag{4.2}$$

$$= \sqrt{2\pi x}\left(\frac{x}{e}\right)^{x}\exp\left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \cdots\right)$$

as $x \to \infty$, where B_n ($n \in \mathbb{N}_0$) are the Bernoulli numbers. The following asymptotic formula is due to Laplace:

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + \cdots\right) \qquad (4.3)$$

as $x \to \infty$ (see [1, p. 257, Eq. (6.1.37)]). The expression (4.3) is sometimes incorrectly called Stirling's series (see [12, pp. 2-3]). Stirling's formula is in fact the first approximation to the asymptotic formula (4.3). Stirling's formula has attracted much interest of many mathematicians and have motivated a large number of research papers concerning various generalizations and improvements (see, for example, [8, 9, 22, 23, 24, 25, 26, 29, 30, 31, 32, 37] and the references cited therein). See also an overview at [28].

Windschitl (see [5, p. 128] and [49]) had noted that for $x > 8$, the approximation

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x} + \frac{1}{810x^6}\right)^{x/2} \qquad (4.4)$$

gives at least eight decimal places of the gamma function. The formula (4.4) derives

$$\Gamma(x+1) = \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x}\right)^{x/2} \left(1 + O\left(\frac{1}{x^5}\right)\right), \qquad x \to \infty. \qquad (4.5)$$

Inspired by (4.4) and (4.5), Alzer [3] proved in 2009 that for all $x > 0$,

$$\sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x}\right)^{x/2} \left(1 + \frac{\alpha}{x^5}\right) < \Gamma(x+1) < \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x}\right)^{x/2} \left(1 + \frac{\beta}{x^5}\right)$$
$$(4.6)$$

with the best possible constants $\alpha = 0$ and $\beta = 1/1620$.

In 2014, Lu *et al.* [24] extended Windschitl's formula as follows:

$$\Gamma(n+1) \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(n \sinh\left(\frac{1}{n} + \frac{a_7}{n^7} + \frac{a_9}{n^9} + \frac{a_{11}}{n^{11}} + \cdots\right)\right)^{n/2}, \qquad (4.7)$$

where

$$a_7 = \frac{1}{810}, \quad a_9 = -\frac{67}{42\,525}, \quad a_{11} = \frac{19}{8505}, \ldots. \qquad (4.8)$$

However, the authors did not give the general formula for the coefficients a_j ($j \geq 7$) in (4.7). Subsequently, Chen [9] gave a recurrence relation formula for determining the coefficient of n^{-j} ($j \in \mathbb{N}$) in (4.7). Also in [9], Chen developed Windschitl's approximation formula to a new asymptotic expansion:

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x}\right)^{x/2 + \sum_{j=0}^{\infty} r_j x^{-j}}, \qquad x \to \infty, \qquad (4.9)$$

and provided a recurrence relation for determining the coefficients r_j in (4.9).

Smith [41, Eq. (43)] presented the following analogous result to (4.5):

$$\Gamma\left(x + \frac{1}{2}\right) = \sqrt{2\pi} \left(\frac{x}{e}\right)^x \left(2x \tanh\frac{1}{2x}\right)^{x/2} \left(1 + O\left(\frac{1}{x^5}\right)\right), \qquad x \to \infty. \qquad (4.10)$$

Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. We here determine the coefficients $a_j \equiv a_j(\ell, r)$ and $d_j \equiv d_j(\ell, r)$ (for $j \in \mathbb{N}$) such that

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x} + \sum_{j=1}^{\infty} \frac{a_j}{x^j}\right)^{x^\ell/r}$$

and

$$\Gamma\left(x + \frac{1}{2}\right) \sim \sqrt{2\pi} \left(\frac{x}{e}\right)^x \left(2x \tanh \frac{1}{2x} + \sum_{j=1}^{\infty} \frac{d_j}{x^j}\right)^{x^\ell/r}$$

as $x \to \infty$.

Theorem 2. *Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The gamma function has the following asymptotic expansion:*

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x} + \sum_{j=1}^{\infty} \frac{a_j}{x^j}\right)^{x^\ell/r}, \qquad x \to \infty, \tag{4.11}$$

with the coefficients $a_j \equiv a_j(\ell, r)\, (j \in \mathbb{N})$ given by

$$a_{2j-1} = b_{2j-1}(\ell, r), \quad a_{2j} = b_{2j}(\ell, r) - \frac{1}{(2j+1)!}, \tag{4.12}$$

where $b_j(\ell, r)\, (j \in \mathbb{N})$ can be calculated using (2.9).

Proof. The Maclaurin expansion of $\sinh t$ with $t = 1/x$ gives

$$x \sinh \frac{1}{x} = 1 + \sum_{j=1}^{\infty} \frac{1}{(2j+1)! x^{2j}}, \qquad x \neq 0. \tag{4.13}$$

In view of (2.8) and (4.13), we can let

$$\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right)^{r/x^\ell} - x \sinh \frac{1}{x} \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j}, \qquad x \to \infty, \tag{4.14}$$

where $a_j\, (j \in \mathbb{N})$ are real numbers to be determined. It follows that

$$\sum_{j=1}^{\infty} \frac{b_j}{x^j} - \sum_{j=1}^{\infty} \frac{1}{(2j+1)! x^{2j}} \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j}, \qquad x \to \infty, \tag{4.15}$$

where $b_j \equiv b_j(\ell, r)\, (j \in \mathbb{N})$ are given in (2.9). Equating coefficients of equal powers of x in (4.15) yields (4.12). The proof of Theorem 2 is complete. $\qquad\square$

Remark 1. *Setting $(r, \ell) = (2, 1)$ in (4.11) yields full asymptotic expansion of Windschitl's formula (4.4):*

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x} + \frac{1}{810 x^6} - \frac{163}{170100 x^8} + \frac{1019}{680400 x^{10}} - \cdots\right)^{x/2} \tag{4.16}$$

as $x \to \infty$.

Theorem 3. *Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The gamma function has the following asymptotic expansion:*

$$\Gamma\left(x + \frac{1}{2}\right) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x \tanh\frac{1}{2x} + \sum_{j=1}^{\infty} \frac{d_j}{x^j}\right)^{x^\ell/r}, \qquad x \to \infty, \tag{4.17}$$

with the coefficients $d_j \equiv d_j(\ell, r)\,(j \in \mathbb{N})$ given by

$$d_{2j-1} = c_{2j-1}(\ell, r), \quad d_{2j} = c_{2j}(\ell, r) - \frac{4(2^{2j+2} - 1)B_{2j+2}}{(2j+2)!}, \tag{4.18}$$

where $c_j(\ell, r)\,(j \in \mathbb{N})$ can be calculated using (2.11).

Proof. The power series expansion of $\tanh t$ with $t = 1/(2x)$ gives

$$2x \tanh\frac{1}{2x} = 1 + \sum_{j=2}^{\infty} \frac{4(2^{2j} - 1)B_{2j}}{(2j)!} \frac{1}{x^{2j-2}}, \qquad |x| > \frac{1}{\pi}. \tag{4.19}$$

In view of (2.10) and (4.19), we can let

$$\left(\frac{\Gamma(x + \frac{1}{2})}{\sqrt{2\pi}(x/e)^x}\right)^{r/x^\ell} - 2x \tanh\frac{1}{2x} \sim \sum_{j=1}^{\infty} \frac{d_j}{x^j}, \qquad x \to \infty, \tag{4.20}$$

where $d_j\,(j \in \mathbb{N})$ are real numbers to be determined. It follows that

$$\sum_{j=1}^{\infty} \frac{c_j}{x^j} - \sum_{j=1}^{\infty} \frac{4(2^{2j+2} - 1)B_{2j+2}}{(2j+2)!} \frac{1}{x^{2j}} \sim \sum_{j=1}^{\infty} \frac{d_j}{x^j}, \qquad x \to \infty, \tag{4.21}$$

where $c_j \equiv c_j(\ell, r)\,(j \in \mathbb{N})$ are given in (2.11). Equating coefficients of equal powers of x in (4.21) yields (4.18). The proof of Theorem 3 is complete. □

Remark 2. *Setting $(r, \ell) = (2, 1)$ in (4.17) yields the following full asymptotic expansion:*

$$\Gamma\left(x + \frac{1}{2}\right) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x \tanh\frac{1}{2x} - \frac{31}{25920x^6} + \frac{6829}{5443200x^8} - \cdots\right)^{x/2} \tag{4.22}$$

as $x \to \infty$.

From (4.5) and (4.10), we derive

$$\frac{\Gamma(x + 1)}{\Gamma\left(x + \frac{1}{2}\right)} \sim \sqrt{x}\left(\cosh\frac{1}{2x}\right)^x \left(1 + O\left(\frac{1}{x^5}\right)\right), \qquad x \to \infty.$$

This fact motivated us to observe the following

Theorem 4. *Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The following asymptotic expansion holds:*

$$\frac{\Gamma(x + 1)}{\Gamma\left(x + \frac{1}{2}\right)} \sim \sqrt{x}\left(\cosh\frac{1}{2x} + \sum_{j=1}^{\infty} \frac{q_j}{x^j}\right)^{x^\ell/r}, \qquad x \to \infty, \tag{4.23}$$

with the coefficients $q_j \equiv q_j(\ell, r)\,(j \in \mathbb{N})$ given by

$$q_{2j-1} = p_{2j-1}(\ell, r), \quad q_{2j} = p_{2j}(\ell, r) - \frac{1}{2^{2j}(2j)!}, \tag{4.24}$$

where $p_j(\ell, r)\,(j \in \mathbb{N})$ can be calculated using (2.17).

Proof. The Maclaurin expansion of $\cosh t$ with $t = 1/(2x)$ gives

$$\cosh\frac{1}{2x} = 1 + \sum_{j=1}^{\infty}\frac{1}{2^{2j}(2j)!}\frac{1}{x^{2j}}, \qquad |x| \neq 0. \tag{4.25}$$

In view of (2.16) and (4.25), we can let

$$\left(\frac{\Gamma(x+1)}{\sqrt{x}\Gamma(x+\frac{1}{2})}\right)^{r/x^{\ell}} - \cosh\frac{1}{2x} \sim \sum_{j=1}^{\infty}\frac{q_j}{x^j}, \qquad x \to \infty, \tag{4.26}$$

where $q_j\ (j \in \mathbb{N})$ are real numbers to be determined. It follows that

$$\sum_{j=1}^{\infty}\frac{p_j}{x^j} - \sum_{j=1}^{\infty}\frac{1}{2^{2j}(2j)!}\frac{1}{x^{2j}} \sim \sum_{j=1}^{\infty}\frac{q_j}{x^j}, \qquad x \to \infty, \tag{4.27}$$

where $p_j \equiv p_j(\ell, r)\ (j \in \mathbb{N})$ are given in (2.17). Equating coefficients of equal powers of x in (4.27) yields (4.24). The proof of Theorem 3 is complete. $\qquad\square$

Remark 3. *Setting* $(r, \ell) = (1, 1)$ *in* (4.23) *yields the following full asymptotic expansion:*

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \sim \sqrt{x}\left(\cosh\frac{1}{2x} + \frac{7}{5760x^6} - \frac{65}{64512x^8} + \cdots\right)^x, \qquad x \to \infty. \tag{4.28}$$

5. Inequalities

Theorem 5. *For* $x > 0$,

$$\sqrt{x}\left(\cosh\frac{1}{2x}\right)^x\left(1 + \frac{\theta_1}{x^5}\right) < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{x}\left(\cosh\frac{1}{2x}\right)^x\left(1 + \frac{\theta_2}{x^5}\right) \tag{5.1}$$

with the best possible constants

$$\theta_1 = 0 \quad and \quad \theta_2 = \frac{7}{5760}. \tag{5.2}$$

Proof. We first prove the inequality (5.1) with $\theta_1 = 0$ and $\theta_2 = \frac{7}{5760}$. That is

$$\sqrt{x}\left(\cosh\frac{1}{2x}\right)^x < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{x}\left(\cosh\frac{1}{2x}\right)^x\left(1 + \frac{7}{5760x^5}\right), \qquad x > 0. \tag{5.3}$$

Lambert's continued fraction [47, p. 349]

$$\tanh(z) = \cfrac{z}{1 + \cfrac{z^2}{3 + \cfrac{z^2}{5 + \cfrac{z^2}{7 + \cdots}}}}$$

is valid for all values of z. Hence for $x > 0$,

$$\frac{6x}{12x^2+1} = \frac{1/(2x)}{1 + \frac{(1/(2x))^2}{3}} < \tanh\frac{1}{2x} < \frac{1/(2x)}{1 + \frac{(1/(2x))^2}{3 + \frac{(1/(2x))^2}{5}}} = \frac{60x^2+1}{12x(10x^2+1)}. \tag{5.4}$$

The proof of the inequality (5.3) make use of the inequality (5.4).

The lower bound in (5.3) is obtained by considering the function $f(x)$ defined for $x > 0$ by

$$f(x) = \ln\Gamma(x+1) - \ln\Gamma\left(x + \frac{1}{2}\right) - \frac{1}{2}\ln x - x\ln\left(\cosh\frac{1}{2x}\right).$$

Differentiation yields

$$f'(x) = \psi(x+1) - \psi\left(x + \frac{1}{2}\right) - \frac{1}{2x} - \ln\left(\cosh\frac{1}{2x}\right) + \frac{1}{2x}\tanh\frac{1}{2x} \tag{5.5}$$

and

$$f''(x) = \psi'(x+1) - \psi'\left(x + \frac{1}{2}\right) + \frac{1}{2x^2} - \frac{1}{4x^3} + \frac{1}{4x^3}\left(\tanh\frac{1}{2x}\right)^2$$

$$> \psi'(x+1) - \psi'\left(x + \frac{1}{2}\right) + \frac{1}{2x^2} - \frac{1}{4x^3} + \frac{1}{4x^3}\left(\frac{6x}{12x^2+1}\right)^2 =: g(x),$$

by applying the left-hand inequality of (5.4). By using the recurrence formula

$$\psi'(x+1) = \psi'(x) - \frac{1}{x^2}, \tag{5.6}$$

we find that

$$g(x) - g(x+1) = \frac{h(x)}{4x^3(12x^2+1)^2(x+1)^3(2x+1)^2(12x^2+24x+13)^2}$$

with

$$h(x) = 1540909 + 8983929(x-1) + 22585735(x-1)^2 + 32006952(x-1)^3$$
$$+ 27981444(x-1)^4 + 15459984(x-1)^5 + 5274000(x-1)^6$$
$$+ 1016064(x-1)^7 + 84672(x-1)^8.$$

Hence, for $x \geq 1$,

$$g(x) > g(x+1) \quad \text{and} \quad g(x) > g(x+n).$$

Therefore, for $x \geq 1$,

$$g(x) > \lim_{n \to \infty} g(x+n) = 0 \quad \text{and} \quad f''(x) > 0.$$

We then obtain that

$$f'(x) < \lim_{t \to \infty} f'(t) = 0 \quad \text{for} \quad x \geq 1. \tag{5.7}$$

We now show that (5.7) is also valid for $0 < x \leq 1$. It follows from (5.5) that

$$-f'(x) = y_1(x) + y_2(x)$$

with

$$y_1(x) = \ln\left(\cosh\frac{1}{2x}\right) - \frac{1}{2x}\tanh\frac{1}{2x}$$

and

$$y_2(x) = -\psi(x+1) + \psi\left(x + \frac{1}{2}\right) + \frac{1}{2x}.$$

Differentiation yields

$$y_1'(x) = \frac{1}{4x^3\left(\cosh\frac{1}{2x}\right)^2} > 0.$$

Using the following representations:

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}}\right) dt$$

in [1, p. 259, 6.3.21] and

$$\frac{1}{x} = \int_0^\infty e^{-xt} dt,$$

we conclude that

$$y_2(x) = \int_0^\infty \left(\frac{1}{2} - \frac{1}{e^{t/2}+1} \right) e^{-xt} dt$$

and

$$y_2'(x) = -\int_0^\infty \left(\frac{1}{2} - \frac{1}{e^{t/2}+1} \right) t e^{-xt} dt < 0.$$

Let $0 \le r \le x \le s \le 1$. Since $y_1(x)$ is increasing and $y_2(x)$ is decreasing, we obtain

$$-f'(x) \ge y_1(r) + y_2(s) =: \sigma_1(r, s).$$

We divide the interval $[0, 1]$ into 100 subintervals:

$$[0, 1] = \bigcup_{k=0}^{99} \left[\frac{k}{100}, \frac{k+1}{100} \right] > 0 \quad \text{for} \quad k = 0, 1, 2, \ldots, 99.$$

By direct computation we get

$$\sigma_1 \left(\frac{k}{100}, \frac{k+1}{100} \right) > 0 \quad \text{for} \quad k = 0, 1, 2, \ldots, 99.$$

Hence,

$$-f'(x) > 0 \quad \text{for} \quad x \in \left[\frac{k}{100}, \frac{k+1}{100} \right] \quad \text{and} \quad k = 0, 1, 2, \ldots, 99.$$

This implies that $f'(x)$ is negative on $(0, 1]$.

We then obtain that for all $x > 0$,

$$f(x) > \lim_{t \to \infty} f(t) = 0.$$

This means that the first inequality in (5.3) holds for $x > 0$.

The upper bound in (5.3) is obtained by considering the function $u(x)$ defined for $x > 0$ by

$$u(x) = \ln \Gamma(x+1) - \ln \Gamma \left(x + \frac{1}{2} \right) - \frac{1}{2} \ln x - x \ln \left(\cosh \frac{1}{2x} \right) - \ln \left(1 + \frac{7}{5760x^5} \right).$$

Differentiation yields

$$u'(x) = \psi(x+1) - \psi \left(x + \frac{1}{2} \right) - \frac{1}{2x} - \ln \left(\cosh \frac{1}{2x} \right) + \frac{1}{2x} \tanh \frac{1}{2x} + \frac{35}{x(5760x^5+7)} \tag{5.8}$$

and

$$u''(x) = \psi'(x+1) - \psi' \left(x + \frac{1}{2} \right) + \frac{1}{2x^2} - \frac{1}{4x^3} + \frac{1}{4x^3} \left(\tanh \frac{1}{2x} \right)^2 - \frac{35(34560x^5+7)}{x^2(5760x^5+7)^2}$$

$$< \psi'(x+1) - \psi' \left(x + \frac{1}{2} \right) + \frac{1}{2x^2} - \frac{1}{4x^3}$$

$$+ \frac{1}{4x^3} \left(\frac{60x^2+1}{12x(10x^2+1)} \right)^2 - \frac{35(34560x^5+7)}{x^2(5760x^5+7)^2} =: v(x),$$

by applying the right-hand inequality of (5.4). By using the recurrence formula (5.6), we find that

$$v(x) - v(x+1) = -\frac{w_1(x)}{576w_2(x)}$$

with

$$w_1(x) = 868270269104794924464036897 + 18765499992969382192107571641(x-1)$$
$$+ \cdots + 14820540300656640000000(x-1)^{30}$$

has all coefficients positive, and

$$w_2(x) = x^5(10x^2+1)^2(5760x^5+7)^2(x+1)^5(2x+1)^2(10x^2+20x+11)^2$$
$$\times (5760x^5 + 28800x^4 + 57600x^3 + 57600x^2 + 28800x + 5767)^2.$$

Hence, for $x \geq 1$,

$$v(x) < v(x+1) \quad \text{and} \quad v(x) < v(x+n).$$

Therefore, for $x \geq 1$,

$$v(x) < \lim_{n \to \infty} v(x+n) = 0 \quad \text{and} \quad u''(x) < 0.$$

We then obtain that

$$u'(x) > \lim_{t \to \infty} u'(t) = 0 \quad \text{for} \quad x \geq 1. \tag{5.9}$$

We now show that (5.9) is also valid for $0 < x \leq 1$. It follows from (5.8) that

$$u'(x) = y_3(x) + y_4(x),$$

where $y_3(x) = -y_2(x)$ and

$$y_4(x) = -\ln\left(\cosh\frac{1}{2x}\right) + \frac{1}{2x}\tanh\frac{1}{2x} + \frac{35}{x(5760x^5+7)}.$$

Differentiation yields

$$y_4'(x) = -\frac{1}{4x^3\left(\cosh\frac{1}{2x}\right)^2} - \frac{35(34560x^5+7)}{x^2(5760x^5+7)^2} < 0.$$

Let $0 \leq r \leq x \leq s \leq 1$. Since $y_3(x)$ is increasing and $y_4(x)$ is decreasing, we obtain

$$u'(x) \geq y_3(r) + y_4(s) =: \sigma_2(r,s).$$

The same as above, we divide the interval $[0,1]$ into 100 subintervals. By direct computation we get

$$\sigma_2\left(\frac{k}{100}, \frac{k+1}{100}\right) > 0 \quad \text{for} \quad k = 0,1,2,\ldots,99.$$

Hence,

$$u'(x) > 0 \quad \text{for} \quad x \in \left[\frac{k}{100}, \frac{k+1}{100}\right] \quad \text{and} \quad k = 0,1,2,\ldots,99.$$

This implies that $u'(x)$ is positive on $(0,1]$.

We then obtain that for all $x > 0$,

$$u(x) < \lim_{t \to \infty} u(t) = 0.$$

This means that the second inequality in (5.3) holds for $x > 0$.

The inequality (5.1) can be written as

$$\theta_1 < x^5 \left[\frac{\Gamma(x+1)}{\sqrt{x}\,\Gamma\left(x+\frac{1}{2}\right)\left(\cosh\frac{1}{2x}\right)^x} - 1 \right] < \theta_2, \qquad x > 0.$$

We find that

$$\theta_1 \leq \lim_{x\to 0^+} x^5 \left[\frac{\Gamma(x+1)}{\sqrt{x}\,\Gamma\left(x+\frac{1}{2}\right)\left(\cosh\frac{1}{2x}\right)^x} - 1 \right] = 0$$

and

$$\lim_{x\to\infty} x^5 \left[\frac{\Gamma(x+1)}{\sqrt{x}\,\Gamma\left(x+\frac{1}{2}\right)\left(\cosh\frac{1}{2x}\right)^x} - 1 \right] = \frac{7}{5760} \leq \theta_2.$$

Hence, inequality (5.1) holds with the best possible constants given in equation (5.2). The proof of Theorem 5 is complete. $\square$

Remark 4. *From (5.3), we derive new inequality for the constant π:*

$$\left(\frac{1}{\sqrt{n}\left(\cosh\frac{1}{2n}\right)^n \left(1+\frac{7}{5760n^5}\right)} \frac{(2n)!!}{(2n-1)!!} \right)^2 < \pi < \left(\frac{1}{\sqrt{n}\left(\cosh\frac{1}{2n}\right)^n} \frac{(2n)!!}{(2n-1)!!} \right)^2 \quad (5.10)$$

for $n \in \mathbb{N}$.

Theorem 6. *For $x \geq 1$,*

$$\sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x\tanh\frac{1}{2x}\right)^{x/2}\left(1-\frac{31}{51840x^5}\right) < \Gamma\left(x+\frac{1}{2}\right) < \sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x\tanh\frac{1}{2x}\right)^{x/2}.$$
$$(5.11)$$

Proof. From the well-known continued fraction for ψ' (see [47, p. 373])

$$\psi'\left(x+\frac{1}{2}\right) = \cfrac{1}{x+\cfrac{a_1}{x+\cfrac{a_2}{x+\cdots}}}, \qquad x > 0,$$

where

$$a_p = \frac{p^4}{4(2p-1)(2p+1)}, \qquad p = 1, 2, \ldots,$$

we find that $x > 0$,

$$\frac{20x(84x^2+71)}{3(560x^4+520x^2+27)} = \cfrac{1}{x+\cfrac{\frac{1}{12}}{x+\cfrac{\frac{4}{15}}{x+\cfrac{\frac{81}{140}}{x}}}} < \psi'\left(x+\frac{1}{2}\right) < \cfrac{1}{x+\cfrac{\frac{1}{12}}{x+\frac{\frac{4}{15}}{x}}} = \frac{4(15x^2+4)}{3x(20x^2+7)}.$$
$$(5.12)$$

The proof of the inequality (5.11) make use of the inequalities (2.22) and (5.12).

The upper bound in (5.11) is obtained by considering the function $U(x)$ defined for $x \geq 1$ by

$$U(x) = \ln\Gamma\left(x+\frac{1}{2}\right) - \ln(\sqrt{2\pi}) - x\ln x + x - \frac{x}{2}\ln(2x) - \frac{x}{2}\ln\left(\tanh\frac{1}{2x}\right).$$

Differentiation yields

$$U'(x) = \psi\left(x+\frac{1}{2}\right) - \frac{3}{2}\ln x - \frac{1}{2} - \frac{1}{2}\ln 2 - \frac{1}{2}\ln\left(\tanh\frac{1}{2x}\right) + \frac{1}{2x\sinh\frac{1}{x}}.$$

Differentiating $U'(x)$ and applying the inequalities (2.22) and (5.12) yield

$$U''(x) = \psi'\left(x + \frac{1}{2}\right) + \frac{\cosh\frac{1}{x}}{x^3(\cosh\frac{2}{x} - 1)} - \frac{3}{2x}$$

$$< \frac{4(15x^2 + 4)}{3x(20x^2 + 7)} + \frac{1}{2x} + \frac{1}{12x^3} - \frac{7}{240x^5} + \frac{31}{6048x^7} - \frac{3}{2x}$$

$$= -\frac{3074x^2 - 1085}{30240x^7(20x^2 + 7)} < 0 \quad \text{for} \quad x \geq 1.$$

We then obtain that for $x \geq 1$,

$$U'(x) > \lim_{t \to \infty} U'(t) = 0 \Longrightarrow U(x) < \lim_{t \to \infty} U(t) = 0.$$

This means that the second inequality in (5.11) holds for $x \geq 1$.

The lower bound in (5.11) is obtained by considering the function $F(x)$ defined for $x \geq 1$ by

$$F(x) = \ln \Gamma\left(x + \frac{1}{2}\right) - \ln(\sqrt{2\pi}) - x \ln x + x - \frac{x}{2}\ln(2x)$$

$$- \frac{x}{2}\ln\left(\tanh\frac{1}{2x}\right) - \ln\left(1 - \frac{31}{51840x^5}\right).$$

Differentiation yields

$$F'(x) = \psi\left(x + \frac{1}{2}\right) - \frac{3}{2}\ln x - \frac{1}{2} - \frac{1}{2}\ln 2 - \frac{1}{2}\ln\left(\tanh\frac{1}{2x}\right)$$

$$+ \frac{1}{2x\sinh\frac{1}{x}} - \frac{155}{x(51840x^5 - 31)}.$$

Differentiating $F'(x)$ and applying the inequalities (2.22) and (5.12) yield

$$F''(x) = \psi'\left(x + \frac{1}{2}\right) + \frac{\cosh\frac{1}{x}}{x^3(\cosh\frac{2}{x} - 1)}$$

$$- \frac{8062156800x^{11} - 96422400x^5 - 9642240x^6 + 2883x + 9610}{2x^2(51840x^5 - 31)^2}$$

$$> \frac{20x(84x^2 + 71)}{3(560x^4 + 520x^2 + 27)} + \frac{1}{2x} + \frac{1}{12x^3} - \frac{7}{240x^5} + \frac{31}{6048x^7} - \frac{127}{172800x^9}$$

$$- \frac{8062156800x^{11} - 96422400x^5 - 9642240x^6 + 2883x + 9610}{2x^2(51840x^5 - 31)^2}$$

$$= \frac{G(x)}{1209600x^9(560x^4 + 520x^2 + 27)(51840x^5 - 31)^2},$$

where

$$
\begin{aligned}
G(x) =\ & 35836321468874877 + 500332411677183040(x-1) \\
& + 3243650443203650720(x-1)^2 + 12942574029209436800(x-1)^3 \\
& + 35510225676993688800(x-1)^4 + 70870406789351495040(x-1)^5 \\
& + 106102603246810676800(x-1)^6 + 121054817642726534400(x-1)^7 \\
& + 105767748316645632000(x-1)^8 + 70425167657151744000(x-1)^9 \\
& + 35177843816429875200(x-1)^{10} + 12782474258840064000(x-1)^{11} \\
& + 3194045448118272000(x-1)^{12} + 491271286947840000(x-1)^{13} \\
& + 35090806210560000(x-1)^{14}.
\end{aligned}
$$

Hence, $F''(x) > 0$ for $x \geq 1$. We then obtain that for $x \geq 1$,

$$
F'(x) < \lim_{t \to \infty} F'(t) = 0 \Longrightarrow F(x) > \lim_{t \to \infty} F(t) = 0.
$$

This means that the first inequality in (5.11) holds for $x \geq 1$. The proof of Theorem 6 is complete. $\qquad\square$

Remark 5. *Some computer experiments indicate that for $x > 0$,*

$$
\sqrt{2\pi} \left(\frac{x}{e} \right)^x \left(2x \tanh \frac{1}{2x} \right)^{x/2} \left(1 - \frac{\vartheta_1}{x^5} \right)
$$
$$
< \Gamma \left(x + \frac{1}{2} \right) < \sqrt{2\pi} \left(\frac{x}{e} \right)^x \left(2x \tanh \frac{1}{2x} \right)^{x/2} \left(1 - \frac{\vartheta_2}{x^5} \right) \tag{5.13}
$$

with the best possible constants $\vartheta_1 = \frac{31}{51840}$ and $\vartheta_2 = 0$.

Although the double inequality (5.11) is given only for $x \geq 1$, its main utility is in the evaluation of $\Gamma \left(x + \frac{1}{2} \right)$ for large values of the argument.

REFERENCES

[1] M. Abramowitz, I. A. Stegun (Editors), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Applied Mathematics Series **55**, Ninth printing, National Bureau of Standards, Washington, D.C., 1972.

[2] H. Alzer, On some inequalities for the gamma and psi functions, Math. Comp. **66** (1997) 373–389.

[3] H. Alzer, Sharp upper and lower bounds for the gamma function, Proc. Royal Soc. Edinburgh **139A** (2009) 709–718.

[4] H. Alzer, C. Berg, Some classes of completely monotonic functions, Ann. Acad. Sci. Fenn. Math. **27** (2002) 445–460.

[5] G.E. Andrews, B.C. Berndt, Ramanujan's Lost Notebook, Part IV, Springer Science Business Media New York 2013.

[6] P. Beckmann, A History of Pi, St. Martin's Press, New York, New York, 1971.

[7] L. Berggren, J. Borwein, P. Borwein, eds., Pi: A Source Book, 2nd ed., Springer, New York, 2000.

[8] C.-P. Chen, Unified treatment of several asymptotic formulas for the gamma function, Numer. Algor. **64** (2013) 311–319.

[9] C.-P. Chen, Asymptotic expansions of the gamma function related to Windschitl's formula, Appl. Math. Comput. **245** (2014) 174–180.

[10] C.-P. Chen, L. Lin, Remarks on asymptotic expansions for the gamma function, Appl. Math. Lett. **25** (2012) 2322–2326.

[11] C.-P. Chen, F. Qi, The best bounds in Wallis' inequality, Proc. Amer. Math. Soc. **133** (2005) 397–401.

[12] E.T. Copson. Asymptotic Expansions, Cambridge University Press, 1965.

[13] J. Dubourdieu, Sur un théorème de M. S. Bernstein relatif à la transformation de Laplace-Stieltjes, Compositio Math. **7** (1939), 96–111 (in French).

[14] W. Dunham, *Journey Through Genious, The Great Theorems of Mathematics*, Penguin, 1990.

[15] J. Gurland, On Wallis' formula, Amer. Math. Monthly **63** (1956) 643–645.

[16] H. van Haeringen, Completely monotonic and related functions, J. Math. Anal. Appl. **204** (1996) 389–408.

[17] S. Koumandos, Remarks on some completely monotonic functions, J. Math. Anal. Appl. **324** (2006) 1458–1461.

[18] S. Koumandos, H.L. Pedersen, Completely monotonic functions of positive order and asymptotic expansions of the logarithm of Barnes double gamma function and Euler's gamma function, J. Math. Anal. Appl. **355** (2009) 33–40.

[19] V. Lampret, Wallis sequence estimated through the Euler-Maclaurin formula: even from the Wallis product π could be computed fairly accurately, Austral. Math. Soc. Gaz. **31** (2004) 328–339.

[20] V. Lampret, An asymptotic approximation of Wallis' sequence, Cent. Eur. J. of Math. **10** (2012) 775–787.

[21] L. Lin, Further refinements of Gurland's formula for π, J. Ineq. Appl. **2013** (2013), Article 48, doi:10.1186/1029-242X-2013-48

[22] D. Lu, A generated approximation related to Burnside's formula, J. Number Theory **136** (2014) 414–422.

[23] D. Lu, J. Feng, C. Ma, A general asymptotic formula of the gamma function based on the Burnsides formula, J. Number Theory **145** (2014) 317–328.

[24] D. Lu, L. Song, C. Ma, A generated approximation of the gamma function related to Windschitl's formula, J. Number Theory **140** (2014) 215–225.

[25] D. Lu, X. Wang, A generated approximation related to Gosper's formula and Ramanujan's formula, J. Math. Anal. Appl. **406** (2013) 287–292.

[26] D. Lu, X. Wang, A new asymptotic expansion and some inequalities for the gamma function. J. Number Theory **140** (2014) 314–323.

[27] Y.L. Luke, The Special Functions and their Approximations, vol. I, Academic Press, New York, 1969.

[28] P. Luschny, Approximation formulas for the factorial function $n!$, `http://www.luschny.de/math/factorial/approx/SimpleCases.html`.

[29] C. Mortici, A continued fraction approximation of the gamma function, J. Math. Anal. Appl. **402** (2013) 405–410.

[30] C. Mortici, Ramanujan formula for the generalized Stirling approximation, Ramanujan J. **217** (2010) 2579–2585.

[31] C. Mortici, Sharp inequalities related to Gosper's formula, C. R. Math. Acad. Sci. Paris **348** (2010) 137–140.

[32] C. Mortici, New improvements of the Stirling formula, Appl. Math. Comput. **217** (2010) 699–704.

[33] C. Mortic, New approximation formulas for evaluating the ratio of gamma functions, Math. Comput. Modelling **52** (2010) 425–433.

[34] C. Mortici, Refinements of Gurland's formula for pi, Comput. Math. Appl. **62** (2011) 2616–2620.

[35] C. Mortici, Estimating π from the Wallis sequence, Math. Commun. **17** (2012) 489–495.

[36] C. Mortici, V.G. Cristea, D. Lu, Completely monotonic functions and inequalities associated to some ratio of gamma function. Appl. Math. Comput. **240** (2014) 168–174.

[37] G. Nemes, New asymptotic expansion for the Gamma function, Arch. Math. **95** (2010) 161–169.

[38] M. E. Muldoon, Some monotonicity properties and characterizations of the gamma function, Aequationes Math. **18** (1978) 54–63.

[39] G. Pólya, G. Szegö, Problems and Theorems in Analysis, Vol. I and II, Springer-Verlag, Berlin, Heidelberg, 1972.

[40] Z. Sasvári, Inequalities for binomial coefficients. J. Math. Anal. Appl. **236** (1999) 223–226.

[41] W.D. Smith, The gamma function revisited, (2006), `http://schule.bayernport.com/gamma/gamma05.pdf`.

[42] A. Sofo, Some representations of π, Austral. Math. Soc. Gaz. **31** (2004) 184–189.

[43] A. Sofo, π and some other constants, J. Inequal. Pure Appl. Math. **6** (2005), no. 5, Article 138. (electronic).

[44] H.M. Srivastava, J. Choi, Series associated with the zeta and related functions, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

[45] N.M. Temme, Special Functions: An introduction to the classical functions of mathematical physics, Wiley, 1996.

[46] Y. Xu, X. Han, Complete monotonicity properties for the gamma function and Barnes G-function, Sci. Magna **5** (2009) 47–51.

[47] H.S. Wall, Analytic Theory of Continued Fractions, AMS Chelsea Publishing, 2000.

[48] D.V. Widder, The Laplace Transform, Princeton University Press, Princeton, 1941.

[49] `http://www.rskey.org/gamma.htm`.

C.-P. Chen: School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City 454000, Henan Province, China

E-mail address: chenchaoping@sohu.com

R.B. Paris: School of Engineering, Computing and Applied Mathematics, University of Abertay, Dundee DD1 1HG, UK

E-mail address: R.Paris@abertay.ac.uk