A note on a hypergeometric transformation formula due to Slater with an application

Y. S. Kim

Department of Mathematics Education, Wonkwang University,
Iksan, Korea

A. K. Rathie

School of Mathematical and Physical Sciences, Central University of Kerala,
Periye P.O. Dist. Kasaragad 671123, Kerala State, India

R. B. Paris

Division of Computing and Mathematics,
University of Abertay Dundee, Dundee DD1 1HG, UK

Abstract

In this note we state (with minor corrections) and give an alternative proof of a very general hypergeometric transformation formula due to Slater. As an application, we obtain a new hypergeometric transformation formula for a $_5F_4(-1)$ series with one pair of parameters differing by unity expressed as a linear combination of two $_3F_2(1)$ series.

1   Introduction

The generalized hypergeometric function with p numeratorial and q denominatorial parameters is defined by the series [7, p. 41]

$$_pF_q \begin{bmatrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{bmatrix} ; z \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n}{(b_1)_n (b_2)_n \ldots (b_q)_n} \frac{z^n}{n!}, \tag{1.1}$$

where for nonnegative integer n the Pochhammer symbol (or ascending factorial) is defined by $(a)_0 = 1$ and for $n \geq 1$ by $(a)_n = a(a+1)\ldots(a+n-1)$. When $p \leq q$ the above series on the right-hand side of (1.1) converges for $|z| < \infty$, but when $p = q+1$ convergence occurs when $|z| < 1$ (unless the series terminates).

By employing Bailey's transform of double series, Slater [7, p. 60, Eq. (2.4.10)] derived the following very general hypergeometric transformation formula (written here in corrected form)

$$\sum_{n=0}^{\infty} \frac{((a))_n((d))_n((v))_{2n}}{((h))_n((g))_n((f))_{2n}\, n!} x^n y^n z^{2n} \times_{U+D+V} F_{E+F+G} \left[\begin{array}{c} (u),(d)+n,(v)+2n \\ (e),(g)+n,(f)+2n \end{array} ; xwz \right]$$

$$= \sum_{n=0}^{\infty} \frac{((d))_n((u))_n((v))_n}{((e))_n((f))_n((g))_n\, n!} x^n w^n z^n$$

$$\times_{A+E+V+1} F_{U+H+F} \left[\begin{array}{c} -n,(a),1-n-(e),(v)+n \\ (h),1-n-(u),(f)+n \end{array} ; (-1)^{1+E-U} w^{-1} yz \right], \quad (1.2)$$

where we have adopted the convention of writing the finite sequence of parameters $(a_1,\ldots,a_A)$ simply by (a) and the product of Pochhammer symbols by

$$((a))_n \equiv (a_1)_n \ldots (a_A)_n,$$

where an empty product is understood to be unity. The general result (1.2) contains as special cases very many relationships between generalized hypergeometric functions.

Several interesting special cases of (1.2) have been obtained by Exton [1, 2]. In particular, he gave the transformation [2]

$$\sum_{n=0}^{\infty} \frac{((g))_n(c)_n(d)_n}{((h))_n(f)_{2n}} \frac{x^n y^n}{n!} \times {}_2F_1 \left[\begin{array}{c} c+n, d+n \\ f+2n \end{array} ; x \right]$$

$$= \sum_{n=0}^{\infty} \frac{(c)_n(d)_n}{(f)_n} \frac{x^n}{n!} \times {}_{G+1}F_{H+1} \left[\begin{array}{cc} -n, & (g) \\ f+n, & (h) \end{array} ; -y \right].$$
$$(1.3)$$

This result follows from (1.2) by setting the parameters $(e), (g), (u), (v) = 0$, $(d) = (c,d)$, replacing (a) by (g) and letting $w = z = 1$.

Hypergeometric identities and transformation formulas have wide applications in numerous areas of mathematics including in series systems of symbolic computer algebra manipulation. A list of such useful identities and transformation formulas can be found in Slater's book [7, Appendix III]. We have the following known hypergeometric identities:

$$_4F_3 \left[\begin{array}{c} a, 1+\frac{1}{2}a, b, c \\ \frac{1}{2}a, 1+a-b, 1+a-c \end{array} ; 1 \right]$$

$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}a-b-c+\frac{1}{2})}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(\frac{1}{2}a-b+\frac{1}{2})\Gamma(\frac{1}{2}a-c+\frac{1}{2})} \quad (1.4)$$

provided $\Re(a-2b-2c) > -1$, and

$$_5F_4\left[\begin{array}{c} a,1+\frac{1}{2}a,b,c,d \\ \frac{1}{2}a,1+a-b,1+a-c,1+a-d \end{array};1\right]$$

$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)} \quad (1.5)$$

provided $\Re(a-b-c-d) > -1$. If we put $c = -n$ in (1.4) and $d = -n$ in (1.5), where n is a non-negative integer, we have the terminating forms

$$_4F_3\left[\begin{array}{c} -n,a,1+\frac{1}{2}a,b \\ \frac{1}{2}a,1+a-b,1+a-c \end{array};1\right] = \frac{(1+a)_n(\frac{1}{2}+\frac{1}{2}a-b)_n}{(\frac{1}{2}+\frac{1}{2}a)_n(1+a-b)_n} \quad (1.6)$$

and

$$_5F_4\left[\begin{array}{c} -n,a,1+\frac{1}{2}a,b,c \\ \frac{1}{2}a,1+a-b,1+a-c,1+a-d \end{array};1\right] = \frac{(1+a)_n(1+a-b-c)_n}{(1+a-b)_n(1+a-c)_n}. \quad (1.7)$$

Furthermore, by taking $x = 1$ in (1.3) and making use of Gauss' summation theorem, Exton [2] obtained the following transformation formula

$$_{G+2}F_{H+2}\left[\begin{array}{cc} (g), & c,d \\ (h), & f-c,f-d \end{array};y\right]$$

$$= \frac{\Gamma(f-c)\Gamma(f-d)}{\Gamma(f)\Gamma(f-c-d)}\sum_{n=0}^{\infty}\frac{(c)_n(d)_n}{(f)_n n!}\, _{G+1}F_{H+1}\left[\begin{array}{cc} -n, & (g) \\ f+n, & (h) \end{array};-y\right] \quad (1.8)$$

When $G = H + 1$, this result holds for $|y| \leq 1$ when the parameters are such that the series on the left is defined and is convergent. Using the results (1.6) and (1.7) in (1.8), he then deduced the following known hypergeometric identity:

$$_5F_4\left[\begin{array}{c} a,1+\frac{1}{2}a,b,c,d \\ \frac{1}{2}a,1+a-b,1+a-c,1+a-d \end{array};-1\right]$$

$$= \frac{\Gamma(1+a-c)\Gamma(1+a-d)}{\Gamma(1+a)\Gamma(1+a-c-d)}\, _3F_2\left[\begin{array}{c} c,d,\frac{1}{2}+\frac{1}{2}a-b \\ \frac{1}{2}+\frac{1}{2}a,1+a-b \end{array};1\right]. \quad (1.9)$$

Again, it is tacitly assumed that the hypergeometric series in (1.9) is convergent. This requires that the parametric excess s, which is defined as the difference between the sum of denominator and numerator parameters, should satisfy $\Re(s) > -1$ when $y = -1$ and $\Re(s) > 0$ when $y = 1$.

Recently, some progress has been achieved in generalizing various hypergeometric identities; for this we refer to the papers cited in [3, 5]. In 2010, Kim *et al.* [3] generalized several well-known identities and in particular obtained the following result involving one pair of numeratorial and denominatorial parameters differing by unity:

$$
{}_4F_3\left[\begin{array}{c} a, b, c, d+1 \\ 1+a-b, 1+a-c, d \end{array} ; 1\right] = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}
$$

$$
\times \left\{ \frac{a}{2d} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}a-b-c)}{\Gamma(\frac{1}{2}+\frac{1}{2}a-b)\Gamma(\frac{1}{2}+\frac{1}{2}a-c)} + \left(1 - \frac{a}{2d}\right) \frac{\Gamma(1+\frac{1}{2}a)\,\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)} \right\}
$$

$$(1.10)$$

provided $\Re(a-2b-2c) > -1$ and $d \neq 0, -1, -2, \ldots$. The result (1.10) may be regarded as a generalization of (1.5).

The aim in this note is to provide first another method of derivation of Slater's general transformation in (1.2) and to point out two significant misprints in [7, p. 60, Eq. (2.4.10)]. As an application, we give a generalization of the ${}_5F_4(-1)$ summation in (1.9) to the case when a pair of numeratorial and denominatorial parameters differ by unity. Our result will be established with the help of Exton's transformation formula (1.8) and the summation in (1.10).

2 Derivation of (1.2)

In order to derive (1.2) we proceed as follows. Denoting the left-hand side of (1.2) by S and expressing the hypergeometric functions ${}_{U+D+V}F_{E+F+G}$ in its series form, we find after some simplification

$$
S = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{((a))_n((u))_m}{((h))_n((e))_m} \frac{((d))_{n+m}((v))_{2n+m}}{((g))_{n+m}((f))_{2n+m}} \frac{x^{n+m}y^n w^m z^{2n+m}}{n!\, m!},
$$

where we have made use of the elementary identities

$$
((\alpha))_n((\alpha)+n)_m = ((\alpha))_{n+m}, \qquad ((\alpha))_{2n}((\alpha)+2n)_m = ((\alpha))_{2n+m}.
$$

We now replace m by $m-n$ and apply the result [4, p. 56, Lemma 10(1)]

$$
\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k)
$$

for convergent double series, to obtain

$$
S = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{((a))_n((u))_{m-n}}{((h))_n((e))_{m-n}} \frac{((d))_m((v))_{n+m}}{((g))_m((f))_{n+m}} \frac{x^m y^n w^{m-n} z^{n+m}}{n!\,(m-n)!}.
$$

Employing the identity $((\alpha))_{n+m} = ((\alpha))_m ((\alpha) + m)_n$, together with

$$(\alpha)_{m-n} = \frac{(-1)^n (\alpha)_m}{(1 - \alpha - m)_n}, \qquad (m-n)! = \frac{(-1)^n m!}{(-m)_n},$$

we then find

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{((d))_m ((v))_m ((v) + m)_n ((a))_n ((u))_{m-n}}{((g))_m ((f))_m ((f) + m)_n ((h))_n ((e))_{m-n}} \frac{x^m y^n w^{m-n} z^{m+n}}{n! \, (m-n)!}$$

$$= \sum_{m=0}^{\infty} \frac{((d))_m ((v))_m ((u))_m}{((g))_m ((f))_m ((e))_m} \frac{x^m w^m z^m}{m!}$$

$$\times \sum_{n=0}^{m} \frac{(-m)_n (1 - (e) - m)_n ((a))_n ((v) + m)_n}{(1 - (u))_n ((h))_n ((f) + m)_n} (-1)^{(1+E-U)n} \frac{w^{-n} y^n z^n}{n!}.$$

Finally, expressing the inner series as a hypergeometric function, we easily arrive at the right-hand side of (1.2). This completes the proof of the transformation formula (1.2).

3   A new transformation formula

The transformation formula for the $_5F_4(1)$ series with a pair of numeratorial and denominatorial parameters differing by unity to be established is given by the following theorem.

Theorem 1. *For $e \neq 0, -1, -2, \ldots,$ the following summation holds true*

$$_5F_4 \left[\begin{array}{c} a, \ b, \ c, \ d, \ 1+e \\ 1 + a - b, 1 + a - c, 1 + a - d, e \end{array} ; -1 \right]$$

$$= \frac{\Gamma(1 + a - c)\Gamma(1 + a - d)}{\Gamma(1 + a)\Gamma(1 + a - c - d)} \left\{ \frac{a}{2e} \, _3F_2 \left[\begin{array}{c} c, d, \frac{1}{2} + \frac{1}{2}a - b \\ \frac{1}{2} + \frac{1}{2}a, 1 + a - b \end{array} ; 1 \right] \right.$$

$$\left. + \left(1 - \frac{a}{2e} \right) \, _3F_2 \left[\begin{array}{c} c, d, 1 + \frac{1}{2}a - b \\ 1 + \frac{1}{2}a, 1 + a - b \end{array} ; 1 \right] \right\} \qquad (3.1)$$

provided $\Re(a - c - d) > \max\{-1, \Re(b) - \frac{3}{2}\}$.

Proof. Our derivation follows in a straightforward manner from Exton's transformation formula (1.8). If we take $y = -1$, $G = 3$, $H = 2$, $g_1 = a$, $g_2 = 1 + e$, $g_3 = b$, $h_1 = e$, $h_2 = 1 + a - b$ and $f = 1 + a$ in (1.8), we obtain after some simplification

$$F \equiv \, _5F_4 \left[\begin{array}{c} a, \ b, \ c, \ d, \ 1+e \\ 1 + a - b, 1 + a - c, 1 + a - d, e \end{array} ; -1 \right]$$

$$= \frac{\Gamma(1+a-c)\Gamma(1+a-d)}{\Gamma(1+a)\Gamma(1+a-c-d)} \sum_{n=0}^{\infty} \frac{(c)_n(d)_n}{(1+a)_n n!} \, {}_4F_3 \left[\begin{array}{c} -n, a, b, 1+e \\ 1+a-b, 1+a+n, e \end{array} ; 1 \right].$$
(3.2)

The terminating ${}_4F_3(1)$ series appearing on the right-hand side of (3.2) can be evaluated with the help of the summation in (1.10). If we take $c = -n$ in this latter summation formula, where n is a non-negative integer, we have

$$\, {}_4F_3 \left[\begin{array}{c} -n, a, b, d+1 \\ 1+a-b, 1+a+n, d \end{array} ; 1 \right]$$

$$= \frac{(1+a)_n}{(1+a-b)_n} \left\{ \frac{a}{2d} \frac{(\frac{1}{2}+\frac{1}{2}a-b)_n}{(\frac{1}{2}+\frac{1}{2}a)_n} + \left(1 - \frac{a}{2d} \right) \frac{(1+\frac{1}{2}a-b)_n}{(1+\frac{1}{2}a)_n} \right\}. \quad (3.3)$$

It is of interest to mention parenthetically that, when $d = \frac{1}{2}a$, (1.10) and (3.3) reduce to (1.4) and (1.6) respectively. Then we find that

$$F = \frac{\Gamma(1+a-c)\Gamma(1+a-d)}{\Gamma(1+a)\Gamma(1+a-c-d)} \left\{ \frac{a}{2e} \sum_{n=0}^{\infty} \frac{(c)_n(d)_n(\frac{1}{2}+\frac{1}{2}a-b)_n}{(\frac{1}{2}+\frac{1}{2}a)_n(1+a-b)_n n!} \right.$$

$$\left. + \left(1 - \frac{a}{2e} \right) \sum_{n=0}^{\infty} \frac{(c)_n(d)_n(1+\frac{1}{2}a-b)_n}{(1+\frac{1}{2}a)_n(1+a-b)_n n!} \right\}.$$

Identification of the resulting series on the right-hand side as ${}_3F_2(1)$ series then leads to the result stated in (3.1). The convergence of the hypergeometric series appearing on the left and right-hand sides of (3.1) requires $\Re(a-b-c-d) > -\frac{3}{2}$ and $\Re(a-c-d) > -1$, respectively; combination of these two conditions yields the stated condition following (3.1). This completes the proof of the theorem.

If we put $e = \frac{1}{2}a$ in (3.1), we recover Exton's result in (1.9). Thus (3.1) may be regarded as a generalization of (1.9), which we hope may prove to be of interest.

Further, if $d = 1+a-b$ in (3.1), we find

$$\, {}_3F_2 \left[\begin{array}{c} a, \, c, \, 1+e \\ 1+a-c, \, e \end{array} ; -1 \right]$$

$$= \frac{\Gamma(1+a-c)}{\Gamma(1+a)} \left\{ \frac{a}{2e} \, {}_2F_1 \left[\begin{array}{c} c, \frac{1}{2}+\frac{1}{2}a-b \\ \frac{1}{2}+\frac{1}{2}a \end{array} ; 1 \right] + \left(1 - \frac{a}{2e} \right) {}_2F_1 \left[\begin{array}{c} c, 1+\frac{1}{2}a-b \\ 1+\frac{1}{2}a \end{array} ; 1 \right] \right\}$$

$$= \frac{\Gamma(1+a-c)}{\Gamma(1+a)} \left\{ \frac{a}{2e} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}a)}{\Gamma(\frac{1}{2}+\frac{1}{2}a-c)} + \left(1 - \frac{a}{2e} \right) \frac{\Gamma(1+\frac{1}{2}a)}{\Gamma(1+\frac{1}{2}a-c)} \right\} \quad (3.4)$$

provided $\Re(c) < \frac{1}{2}$. [We note that in the evaluation of the ${}_2F_1(1)$ series by Gauss' theorem we need the dummy condition $\Re(b - c) > 0$.] This formula

has been obtained by different means in [3, Eq. (5.10)] and [6, Eq. (4.7)]. If $e = \frac{1}{2}a$, (3.4) reduces to the summation formula given in [7, p. 245, III.21]; if, in addition, $c = -n$, where n is a non-negative integer, (3.4) reduces to (III.25) in [7, p. 245].

References

[1] H. Exton, New hypergeometric identities arising from Gauss's second summation theorem, J. Comput. Appl. Math. **88** (1997) 269–274.

[2] H. Exton, New hypergeometric transformations, J. Comp. Appl. Math. **92**(1998) 135–137.

[3] Y. S. Kim, M. A. Rakha and A. K. Rathie, Extensions of certain classical summation theorems for the series $_2F_1$, $_3F_2$ and $_4F_3$ with applications in Ramanujan summations, Int. J. Math. Math. Sci. 309503, 26 pages (2010).

[4] E. D. Rainville, *Special Functions*, Macmillan, New York, 1960.

[5] M. A. Rakha and A. K. Rathie, Generalizations of classical summation theorems for the series $_2F_1$ and $_3F_2$ with applications, Integral Transforms and Special Functions **22** (2011) 823–840.

[6] A. K. Rathie and R. B. Paris, Extension of some classical summation theorems for the generalized hypergeometric series with integral parameter differences, J. Classical Anal. **3** (2013) 109–127.

[7] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.