An alternative proof of the extended Saalschütz summation theorem for the r + 3Fr + 2(1) series with applications

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An alternative proof of the extended Saalschütz summation theorem for the ${}_{r+3}F_{r+2}(1)$ series with applications

Y. S. Kim,* Arjun. K. Rathie† and R. B. Paris‡§

Abstract

A simple proof is given of a new summation formula recently added in the literature for a terminating ${}_{r+3}F_{r+2}(1)$ hypergeometric series for the case when r pairs of numeratorial and denominatorial parameters differ by positive integers. This formula represents an extension of the well-known Saalschütz summation formula for a ${}_3F_2(1)$ series. Two applications of this extended summation formula are discussed. The first application extends two identities given by Ramanujan and the second, which also employs a similar extension of the Vandermonde-Chu summation theorem for the ${}_2F_1$ series, extends certain reduction formulas for the Kampé de Fériet function of two variables given by Exton and Cvijović & Miller.

Mathematics Subject Classification: 33C15, 33C20

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1. Introduction

The generalized hypergeometric function ${}_pF_q(z)$ is defined for complex parameters and argument by the series [16, p. 40]

$$ {}_pF_q\left[\begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array}; z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k}{(b_1)_k (b_2)_k \ldots (b_q)_k} \frac{z^k}{k!}. \tag{1.1} $$

When $q \geq p$ this series converges for $|z| < \infty$, but when $q = p - 1$ convergence occurs when $|z| < 1$. However, when only one of the numeratorial parameters a_j is a negative integer or zero, then the series always converges since it is simply a polynomial in z of degree $-a_j$. In (1.1) the Pochhammer symbol, or ascending factorial, $(a)_n$ is given for integer n by

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \left\{ \begin{array}{ll} 1 & (n = 0) \\ a(a+1)\ldots(a+n-1) & (n \geq 1), \end{array} \right. $$

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where Γ is the gamma function. Throughout we shall adopt the convention of writing the finite sequence of parameters $(a_1, \ldots, a_p)$ simply by (a_p) and the product of p Pochhammer symbols by

$$((a_p))_k \equiv (a_1)_k \ldots (a_p)_k,$$

where an empty product $p = 0$ is understood to be unity.

There exist several classical summation theorems for hypergeometric series of specialized argument. These are the theorems of Gauss, Kummer and Bailey for the ${}_2F_1$ series and Saalschütz and Watson for the ${}_3F_2$ series; see, for example, [16, Appendix III]. Various contiguous extensions of these summations theorems have been obtained; see [5, 14] and the references therein. Recent work has been concerned with the extension of the above-mentioned summation theorems to higher-order hypergeometric series with r pairs of numeratorial and denominatorial parameters differing by a set of positive integers (m_r). One of the first results of this type is the generalized Karlsson-Minton summation theorem [11], which extends the first Gauss summation theorem, given by

$$ {}_{r+2}F_{r+1}\left[\begin{matrix} a,\, b,\ (f_r+m_r) \\ 1+b,\quad (f_r) \end{matrix}; 1\right] = \frac{\Gamma(1-a)\Gamma(1+b)}{\Gamma(1+b-a)}\,\frac{(f_1-b)_{m_1}\ldots(f_r-b)_{m_r}}{(f_1)_{m_1}\ldots(f_r)_{m_r}} \tag{1.2}$$

provided $\mathrm{Re}\,(-a) > m-1$, where $m := m_1 + \cdots + m_r$. When $b = -n$, where n is a non-negative integer, the series on the left-hand side terminates and (1.2) reduces to the result originally obtained by Minton [12] when $n \geq m$. A generalization of (1.2) when the series terminates (an extension of the Vandermonde-Chu summation formula for the ${}_2F_1$ series) was derived by Miller [7] in the form

$$ {}_{r+2}F_{r+1}\left[\begin{matrix} -n,\, a,\, (f_r+m_r) \\ c,\qquad (f_r) \end{matrix}; 1\right] = \frac{(c-a-m)_n}{(c)_n}\,\frac{((\xi_m+1))_n}{((\xi))_n} \tag{1.3}$$

for non-negative integer n when it is supposed that $(c-a-m)_m \neq 0$ and $a \neq f_j$ $(1 \leq j \leq r)$, where m is as defined above. The (ξ_m) are the non-vanishing zeros of a certain parametric polynomial $Q_m(t)$ defined in (5.3). The summation theorems of Gauss (second), Kummer, Bailey and Watson have been similarly extended in [15].

The summation theorem which we shall be concerned with in this paper is Saalschütz's summation theorem given by [16, p. 49]

$$ {}_3F_2\left[\begin{matrix} -n,\, a,\, b \\ c,\, -n-\sigma \end{matrix}; 1\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}, \qquad \sigma := c-a-b-1 \tag{1.4}$$

for non-negative integer n. An extension of this theorem to include r pairs of numeratorial and denominatorial parameters differing by positive integers (m_r) has recently been obtained by the authors in [6] in the following form.

Theorem 1. *Let (m_r) be a set of positive integers with $m := m_1 + \cdots + m_r$ and let n denote a non-negative integer. Then, with $\sigma := c-a-b-1$, we have*

$$ {}_{r+3}F_{r+2}\left[\begin{matrix} -n,\, a,\, b,\ \ (f_r+m_r) \\ c,\, m-n-\sigma,\ \ (f_r) \end{matrix}; 1\right] = \frac{(c-a-m)_n(c-b-m)_n}{(c)_n(c-a-b-m)_n}\, H_n, \tag{1.5}$$

where

$$H_n := \frac{((\eta_m+1))_n}{((\eta_m))_n}.$$

The (η_m) are the nonvanishing zeros of the associated parametric polynomial $\hat{Q}_m(t)$ of degree m defined in (2.1) and (2.2) below.

The derivation of (1.5) in [6] relied on an Euler-type transformation for the generalized hypergeometric function

$$ {}_{r+2}F_{r+1}\left[\begin{matrix} a,\, b,\, (f_r+m_r) \\ c, \qquad (f_r) \end{matrix};z\right] $$

obtained in [8, 9] and so depended on a special property of such high-order hypergeometric functions. It is the purpose of the present investigation to provide an alternative and more elementary proof of the extended Saalschütz summation theorem (1.5) that does not rely on the above-mentioned Euler-type transformation. We also provide some illustrative examples of Theorem 1. Two applications are discussed involving hypergeometric series when r pairs of numeratorial and denominatorial parameters differ by positive integers (m_r). The first extends two transformations originally obtained by Ramanujan [1] and the second extends two reduction formulas for the Kampé de Fériet function given by Exton [4] and Cvijović & Miller [3].

2. Alternative proof of the extension of Saalschütz's formula (1.5)

Before giving our alternative proof of Theorem 1, we first present below the definition of the associated parametric polynomial $\hat{Q}_m(t)$ appearing therein. For the set of positive integers (m_r) define the integer m by

$$ m := m_1 + \cdots + m_r. $$

Let the quantities (η_m) be the non-vanishing zeros of the polynomial $\hat{Q}_m(t)$ of degree m given by

$$ \hat{Q}_m(t) = \sum_{k=0}^{m} \frac{(-1)^k C_{k,r}(a)_k(b)_k(t)_k}{(c-a-m)_k(c-b-m)_k}\, G_{m,k}(t) \tag{2.1} $$

where

$$ G_{m,k}(t) := {}_3F_2\left[\begin{matrix} -m+k, t+k, c-a-b-m \\ c-a-m+k, c-b-m+k \end{matrix};1\right]. \tag{2.2} $$

The coefficients $C_{k,r}$ $(0 \le k \le m)$ are defined by[1]

$$ C_{k,r} = \frac{1}{\Lambda}\sum_{j=k}^{m} h_j \mathbf{S}_j^{(k)}, \qquad \Lambda = (f_1)_{m_1}\dots(f_r)_{m_r}, \tag{2.3} $$

where $C_{0,r}=1$, $C_{m,r}=1/\Lambda$. Here, $\mathbf{S}_j^{(k)}$ are the Stirling numbers of the second kind and the coefficients h_j $(0 \le j \le m)$ are generated by

$$ (f_1+x)_{m_1}\dots(f_r+x)_{m_r} = \sum_{j=0}^{m} h_j x^j. \tag{2.4} $$

We observe that the polynomial $\hat{Q}_m(t)$ has been normalized so that $\hat{Q}_m(0)=1$ and, for $0 \le k \le m$, the function $G_{m,k}(t)$ is a polynomial in t of degree $m-k$.

In the statement of Theorem 1, the quantity H_n is defined by

$$ H_n = \frac{((\eta_m+1))_n}{((\eta_m))_n} = \frac{(n+\eta_1)\dots(n+\eta_m)}{\eta_1\dots\eta_m}. \tag{2.5} $$

[1] An alternative representation of the coefficients $C_{k,r}$ is given as a terminating ${}_{r+2}F_{r+1}$ hypergeometric series of unit argument in [10].

We observe that to evaluate H_n it is *not necessary to evaluate the zeros* (η_m) of the polynomial $\hat{Q}_m(t)$. Since $\hat{Q}_m(t) = (-1)^m(t-\eta_1)\ldots(t-\eta_m)/(\eta_1\ldots\eta_m)$, it follows immediately from (2.5) that

$$H_n = \hat{Q}_m(-n),$$

where, since $(-n)_m = 0$ for $m > n$,

$$\hat{Q}_m(-n) = \sum_{k=0}^{\min\{m,n\}} \frac{(-1)^k C_{k,r}(a)_k(b)_k(-n)_k}{(c-a-m)_k(c-b-m)_k}\, G_{m,k}(-n). \tag{2.6}$$

Hence, to evaluate H_n it suffices to determine only the associated parametric polynomial $\hat{Q}_m(t)$ and set $t = -n$.

We now begin our proof of the summation formula (1.5). We express the ${}_{r+3}F_{r+2}(1)$ in its series form for non-negative integer n as

$$F \equiv {}_{r+3}F_{r+2}\left[\begin{matrix} -n,\ a,\ b, & (f_r+m_r) \\ c,\ m-n-\sigma, & (f_r)\end{matrix}; 1\right] = \sum_{s=0}^{n} \frac{(-n)_s(a)_s(b)_s}{(c)_s(m-n-\sigma)_s s!}\, \frac{((f_r+m_r))_s}{((f_r))_s}.$$

Making use of the fact that for non-negative integer s

$$\frac{(f_r+m_r)_s}{(f_r)_s} = \frac{(f_r+s)_{m_r}}{(f_r)_{m_r}},$$

we find using the definition of the coefficients h_j in (2.4) and $m = m_1 + \cdots + m_r$ that

$$\begin{aligned}
\frac{((f_r+m_r))_s}{((f_r))_s} &= \frac{1}{\Lambda}(s+f_1)_{m_1}\ldots(s+f_r)_{m_r} = \frac{1}{\Lambda}\sum_{j=0}^{m} h_j s^j \\
&= 1 + \frac{1}{\Lambda}\sum_{j=1}^{m} h_j \sum_{k=1}^{j} \mathbf{S}_j^{(k)} s(s-1)\ldots(s-k+1) \\
&= 1 + \sum_{k=1}^{m} s(s-1)\ldots(s-k+1)\frac{1}{\Lambda}\sum_{j=k}^{m} h_j \mathbf{S}_j^{(k)} \\
&= 1 + \sum_{k=1}^{m} C_{k,r}\, s(s-1)\ldots(s-k+1),
\end{aligned}$$

where the coefficients $C_{k,r}$ and the quantity Λ are defined in (2.3).

Then, since $(-n)_m = 0$ for $m > n$ and $C_{0,r} = 1$, we have

$$F = \sum_{s=0}^{n} \frac{(-n)_s(a)_s(b)_s}{(c)_s(m-n-\sigma)_s} \sum_{k=0}^{\min\{m,n\}} \frac{C_{k,r}}{(s-k)!} = \sum_{k=0}^{\min\{m,n\}} C_{k,r} \sum_{s=0}^{n-k} \frac{(-n)_{s+k}(a)_{s+k}(b)_{s+k}}{(c)_{s+k}(m-n-\sigma)_{s+k} s!}$$

upon reversing the order of summation. Employing the result

$$(a)_{s+k} = (a)_k(a+k)_s, \tag{2.7}$$

we obtain

$$F = \sum_{k=0}^{\min\{m,n\}} \frac{C_{k,r}\,(-n)_k(a)_k(b)_k}{(c)_k(m-n-\sigma)_k}\, {}_3F_2\left[\begin{matrix} -n+k,\ a+k,\ b+k \\ c+k,\ m+k-n-\sigma\end{matrix}; 1\right]. \tag{2.8}$$

We now employ the contiguous Saalschütz summation formula, which can be obtained from [13, p. 539, Eq. (85)], in the form

$$
{}_3F_2\left[\begin{matrix}-n,\, a,\, b\\ c,\, p-n-\sigma\end{matrix};1\right]
$$

$$
=\frac{(c-a-p)_n(c-b-p)_n}{(c)_n(c-a-b-p)_n}\,{}_3F_2\left[\begin{matrix}-p,\, -n,\, c-a-b-p\\ c-a-p,\, c-b-p\end{matrix};1\right] \qquad p=0,1,2,\ldots
$$

for non-negative integer n and $\sigma := c-a-b-1$. Then, with $p=m-k$, $n\to n-k$, $a\to a+k$, $b\to b+k$, $c\to c+k$ and $\sigma\to\sigma-k$, we find that the term with index k in (2.8) becomes

$$
\frac{C_{k,r}\,(-n)_k(a)_k(b)_k}{(c)_k(m-n-\sigma)_k}\,\frac{(c-a-m+k)_{n-k}(c-b-m+k)_{n-k}}{(c+k)_{n-k}(c-a-b-m)_{n-k}}
$$

$$
\times{}_3F_2\left[\begin{matrix}-m+k,\, -n+k,\, c-a-b-m\\ c-a-m+k,\, c-b-m+k\end{matrix};1\right]
$$

$$
=\frac{(c-a-m)_n(c-b-m)_n}{(c)_n(c-a-b-m)_n}\,\frac{(-1)^kC_{k,r}\,(-n)_k(a)_k(b)_k}{(c-a-m)_k(c-b-m)_k}\,G_{m,k}(-n),
$$

where we have made use of the identities (2.7) (with $s+k\to n$) and

$$
(m-n-\sigma)_k(c-a-b-m)_{n-k}=(-1)^k(c-a-b-m)_n
$$

and the definition of the polynomial $G_{m,k}(t)$ in (2.2).

Hence it follows that

$$
F=\frac{(c-a-m)_n(c-b-m)_n}{(c)_n(c-a-b-m)_n}\sum_{k=0}^{\min\{m,n\}}\frac{(-1)^kC_{k,r}\,(-n)_k(a)_k(b)_k}{(c-a-m)_k(c-b-m)_k}\,G_{m,k}(-n)
$$

$$
=\frac{(c-a-m)_n(c-b-m)_n}{(c)_n(c-a-b-m)_n}\,\hat{Q}_m(-n)
$$

by (2.6). Since, from (2.5), $\hat{Q}_m(-n)=((\eta_m+1))_n/((\eta_m))_n$, this completes the proof of the summation formula in (1.5). $\square$

3. Examples

In the case $r=1$ and $m_1=m=1$, $f_1=f$, the summation theorem (1.5) takes the form

$$
{}_4F_3\left[\begin{matrix}-n,\, a,\, b,\; f+1\\ c,\, 1-n-\sigma,\; f\end{matrix};1\right]=\frac{(c-a-1)_n(c-b-1)_n}{(c)_n(c-a-b-1)_n}\left(1+\frac{n}{\eta}\right) \tag{3.1}
$$

for non-negative integer values of n, where $\sigma=c-a-b-1$ and

$$
\eta=\frac{(c-a-1)(c-b-1)f}{ab+(c-a-b-1)f} \tag{3.2}
$$

is the non-vanishing zero of the first-degree parametric polynomial

$$
\hat{Q}_1(t)=1-\frac{\{(c-a-b-1)f+ab\}t}{(c-a-1)(c-b-1)f}\,.
$$

Remark 1. When $c = 1 + a - b$, we have from (3.2)

$$\eta = \frac{(a-2b)f}{2f-a}. \tag{3.3}$$

If, in addition, $f = \frac{1}{2}a$ $(a \neq 2b)$, we obtain from (3.1)

$${}_4F_3\left(\begin{matrix} -n,\, a,\, b,\, 1+\frac{1}{2}a \\ 1+a-b, 1+2b-n, \frac{1}{2}a \end{matrix}\middle| 1\right) = \frac{(a-2b)_n(-b)_n}{(1+a-b)_n(-2b)_n}$$

which is a known result [16, Appendix III, Eq. (17)].

In the case $r = 1$, $m_1 = 2$, $f_1 = f$, where $C_{0,r} = 1$, $C_{1,r} = 2/f$ and $C_{2,r} = 1/(f)_2$, we have the quadratic parametric polynomial (with zeros η_1 and η_2) given by [9]

$$\hat{Q}_2(t) = 1 - \frac{2Bt}{(c-a-2)(c-b-2)} + \frac{Ct(1+t)}{(c-a-2)_2(c-b-2)_2},$$

where

$$B := \sigma - 1 + \frac{ab}{f}, \qquad C := (\sigma - 1)\left(\sigma + \frac{2ab}{f}\right) + \frac{(a)_2(b)_2}{(f)_2}$$

with σ as above. Hence we obtain

$${}_4F_3\left[\begin{matrix} -n,\, a,\, b,\ \ f+2 \\ c,\, 2-n-\sigma,\ \ f \end{matrix}; 1\right] = \frac{(c-a-2)_n(c-b-2)_n}{(c)_n(c-a-b-2)_n}\left(1+\frac{n}{\eta_1}\right)\left(1+\frac{n}{\eta_2}\right)$$

$$= \frac{(c-a-2)_n(c-b-2)_n}{(c)_n(c-a-b-2)_n}\left\{1+\frac{2Bn}{(c-a-2)(c-b-2)}+\frac{Cn(n-1)}{(c-a-2)_2(c-b-2)_2}\right\}$$

for nonnegative integer values of n.

4. The extension of two transformation formulas of Ramanujan

For our first application of the extension of the Saalschütz summation theorem in (1.5) we obtain two transformation formulas involving ${}_{r+2}F_{r+1}(x^2)$ series, when r pairs of numeratorial and denominatorial parameters differ by positive integers, that generalize results originally given by Ramanujan [1] in the case $r = 0$. Our results are given by the following theorem.

Theorem 2. *Let (m_r) be a set of positive integers with $m := m_1 + \cdots + m_r$. Then, for n arbitrary (not necessarily an integer),*

$$(1-x^2)^{-\frac{1}{2}}{}_{r+2}F_{r+1}\left[\begin{matrix} -n+\frac{1}{2}p,\, n+\frac{1}{2}p,\, (f_r+m_r) \\ p+\frac{1}{2}+m,\, (f_r) \end{matrix}; x^2\right]$$

$$= {}_{m+2}F_{m+1}\left[\begin{matrix} \frac{1}{2}+\frac{1}{2}p-n,\, \frac{1}{2}+\frac{1}{2}p+n,\, (\eta_m+1) \\ p+\frac{1}{2}+m,\, (\eta_m) \end{matrix}; x^2\right] \tag{4.1}$$

when $|x| < 1$, where $p = 0, 1$. The (η_m) are the non-vanishing zeros of the parametric polynomial $\hat{Q}_m(t)$ of degree m defined in (2.1) and (2.2).

Proof: We consider the case $p = 0$ and define

$$\begin{aligned} F &\equiv (1-x^2)^{-\frac{1}{2}}\,{}_{r+2}F_{r+1}\left[\begin{matrix} -n,\, n,\, (f_r+m_r) \\ \frac{1}{2}+m,\, (f_r) \end{matrix};x^2\right] \\ &= \sum_{j=0}^{\infty}\frac{(\frac{1}{2})_j}{j!}x^{2j}\sum_{k=0}^{\infty}\frac{(-n)_k(n)_k((f_r+m_r))_k}{(\frac{1}{2}+m)_k((f_r))_k k!}x^{2k} \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_j(-n)_k(n)_k((f_r+m_r))_k}{(\frac{1}{2}+m)_k((f_r))_k\, j!k!}x^{2j+2k}, \qquad (|x|<1) \end{aligned}$$

where we have expressed the Cauchy product as a double sum. Changing the double sum by rows to diagonal summation (see [16, p. 58]) by putting $j \to j-k$ $(0 \le k \le j)$, we find

$$\begin{aligned} F &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(\frac{1}{2})_{j-k}(-n)_k(n)_k((f_r+m_r))_k}{(\frac{1}{2}+m)_k((f_r))_k\,(j-k)!k!}x^{2j} \\ &= \sum_{j=0}^{\infty}\frac{(\frac{1}{2})_j}{j!}x^{2j}\sum_{k=0}^{j}\frac{(-j)_k(-n)_k(n)_k((f_r+m_r))_k}{(\frac{1}{2}+m)_k(\frac{1}{2}+j)_k((f_r))_k k!} \\ &= \sum_{j=0}^{\infty}\frac{(\frac{1}{2})_j}{j!}x^{2j}\,{}_{r+3}F_{r+2}\left[\begin{matrix} -j,\, -n,\, n,\, (f_r+m_r) \\ \frac{1}{2}-j,\, \frac{1}{2}+m,\, (f_r) \end{matrix};1\right], \end{aligned}$$

where we have made use of the identities

$$(\tfrac{1}{2})_{j-k} = \frac{(-1)^k(\frac{1}{2})_k}{(\frac{1}{2}-j)_k}, \qquad \frac{1}{(j-k)!} = \frac{(-1)^k(-j)_k}{j!}. \tag{4.2}$$

If we now identify the parameters a, b and c in (1.5) with n, $-n$ and $\frac{1}{2}+m$ respectively, then we can apply the extension of Saalschütz's summation formula in Theorem 1 to obtain

$$F = \sum_{j=0}^{\infty}\frac{(\frac{1}{2}-n)_j(\frac{1}{2}+n)_j((\eta_m+1))_j}{(\frac{1}{2}+m)_j((\eta_m))_j\, j!}x^{2j} = {}_{m+2}F_{m+1}\left[\begin{matrix} \frac{1}{2}-n,\, \frac{1}{2}+n,\, (\eta_m+1) \\ \frac{1}{2}+m,\, (\eta_m) \end{matrix};x^2\right],$$

thereby establishing the result when $p = 0$. The proof of the case with $p = 1$ is similar and consequently will be omitted. □

In the case $r = 0$ $(m = 0)$, (4.1) reduces to the two identities for n arbitrary

$$(1-x^2)^{-\frac{1}{2}}\,{}_2F_1\left[\begin{matrix} -n,\, n \\ \frac{1}{2} \end{matrix};x^2\right] = {}_2F_1\left[\begin{matrix} \frac{1}{2}-n,\, \frac{1}{2}+n \\ \frac{1}{2} \end{matrix};x^2\right] \tag{4.3}$$

and

$$(1-x^2)^{-\frac{1}{2}}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}-n,\, \frac{1}{2}+n \\ \frac{3}{2} \end{matrix};x^2\right] = {}_2F_1\left[\begin{matrix} 1-n,\, 1+n \\ \frac{3}{2} \end{matrix};x^2\right] \tag{4.4}$$

obtained by Ramanujan [1, p. 99, 35(iii)].

When $r = 1$, $m_1 = m = 1$, $f_1 = f$, we obtain from (4.1) with $p = 0$

$$(1-x^2)^{-\frac{1}{2}}\,{}_3F_2\left[\begin{matrix} -n,\, n,\, f+1 \\ \frac{3}{2},\, f \end{matrix};x^2\right] = {}_3F_2\left[\begin{matrix} \frac{1}{2}-n,\, \frac{1}{2}+n,\, \eta+1 \\ \frac{3}{2},\, \eta \end{matrix};x^2\right], \tag{4.5}$$

where, from (3.2),

$$\eta = \frac{(n^2 - \frac{1}{4})f}{n^2 - \frac{1}{2}f};$$

and when $p = 1$

$$(1 - x^2)^{-\frac{1}{2}}\, {}_3F_2\left[\begin{matrix} \frac{1}{2} - n,\ \frac{1}{2} + n,\ f + 1 \\ \frac{5}{2},\ f \end{matrix}; x^2\right] = {}_3F_2\left[\begin{matrix} 1 - n,\ 1 + n,\ \eta + 1 \\ \frac{5}{2},\ \eta \end{matrix}; x^2\right], \tag{4.6}$$

where, from (3.2),

$$\eta = \frac{(n^2 - 1)f}{n^2 - \frac{1}{2}f - \frac{1}{4}}$$

both for n arbitrary. When $f = \frac{1}{2}$ and $f = \frac{3}{2}$, we remark that (4.5) and (4.6) correctly reduce to (4.3) and (4.4), respectively. The results in (4.5) and (4.6) have been obtained recently by different means in [2].

5. Two reduction formulas for the Kampé de Fériet function

The Kampé de Fériet function is a hypergeometric function of two variables defined by

$$F^{p:r;s}_{q:t;u}\left[\begin{matrix} (\alpha_p) : (a_r); (b_s) \\ (\beta_q) : (c_t); (d_u) \end{matrix} \middle| x,\ y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{((\alpha_p))_{m+n}}{((\beta_q))_{m+n}} \frac{((a_r))_m((b_s))_n}{((c_t))_m((d_u))_n} \frac{x^m y^n}{m!\, n!}, \tag{5.1}$$

where p, q, r, s, t, u are nonnegative integers that correspond to the number of elements in the parameter sets (α_p), (β_q), (a_r), (b_s), (c_t) and (d_u), respectively; for an introduction to this function, see [18, pp. 63–64]. We also have the easily established result [4, Eq. (6)]

$$F^{p:r;s}_{q:t;u}\left[\begin{matrix} (\alpha_p) : (a_r); (b_s) \\ (\beta_q) : (c_t); (d_u) \end{matrix} \middle| x,\ y\right]$$
$$= \sum_{n=0}^{\infty} \frac{((\alpha_p))_n((b_s))_n}{((\beta_q))_n((d_u))_n} \frac{y^n}{n!}\, {}_{r+u+1}F_{s+t}\left[\begin{matrix} -n,\ (a_r),\ (1 - d_u - n) \\ (c_t),\ (1 - b_s - n) \end{matrix}; (-1)^{s-u+1}\frac{x}{y}\right]. \tag{5.2}$$

Reduction formulas represent the Kampé de Fériet function as a generalized hypergeometric function of lower order and of a single variable. The identification of such reductions is of considerable utility in the application of these functions; a compilation can be found in [17, pp. 28–32]. In this section we shall be concerned with reduction formulas for the Kampé de Fériet function when one set of numeratorial and denominatorial parameters differs by positive integers (m_r). One of the first results of this type was obtained by Miller [7] in the form[2]

$$F^{p:r+1;0}_{q:r+1;0}\left[\begin{matrix} (\alpha_p) : a,\ (f_r + m_r);\ - \\ (\beta_q) : \quad b,\ (f_r);\quad - \end{matrix} \middle| -x,\ x\right] = {}_{p+m+1}F_{q+m+1}\left[\begin{matrix} (\alpha_p),\ b - a - m,\ (\xi_m + 1) \\ (\beta_q),\qquad b,\ (\xi_m) \end{matrix}; x\right],$$

where the horizontal line indicates an empty parameter sequence. The (ξ_m) are the nonvanishing zeros of the associated parametric polynomial of degree $m = m_1 + \cdots + m_r$ given by

$$Q_m(t) = \frac{1}{(\lambda)_m}\sum_{k=0}^{m}(b)_k C_{k,r}(t)_k(\lambda - t)_{m-k} \tag{5.3}$$

[2]In [7], Miller gave the case $(m_r) = 1$ but his arguments are easily extended to the case of positive integers (m_r).

which is normalized so that $Q_m(0)=1$, where $\lambda := b-a-m$ and the coefficients $C_{k,r}$ are defined in (2.3). In the case $r=1$, $m_1=m=1$, $f_1=f$, we have

$$Q_1(t)=1+\frac{(b-f)t}{(c-b-1)f},$$

with the nonvanishing zero $\xi_1=\xi$ (provided $c-b-1\neq 0$) given by

$$\xi=\frac{(c-b-1)f}{f-b}. \tag{5.4}$$

Here we shall exploit the result in (1.5) and the extension of the Vandermonde-Chu summation formula in (1.3) to obtain two new reduction formulas.

5.1 First reduction formula

From (5.2), with $y=x$ we obtain

$$F^{p:\,r+2;\,1}_{q:\,r+1;\,0}\left[\begin{array}{l}(\alpha_p):\ a,\ b,\ (f_r+m_r);\ c-a-b-m\\ (\beta_q):\qquad c,\ (f_r);\qquad\quad -\end{array}\middle|\, x,\ x\right]$$

$$\begin{aligned}
&=\sum_{n=0}^\infty \frac{((\alpha_p))_n}{((\beta_q))_n}\,\frac{(c-a-b-m)_n}{n!}x^n\ {}_{r+3}F_{r+2}\left[\begin{array}{c}-n,\ a,\ b,\ \ (f_r+m_r)\\ c,\ m-n-\sigma,\ \ (f_r)\end{array};1\right]\\
&=\sum_{n=0}^\infty \frac{((\alpha_p))_n}{((\beta_q))_n}\,\frac{(c-a-m)_n(c-b-m)_n}{(c)_n}\,\frac{((\eta_m+1))_n}{((\eta_m))_n}\,\frac{x^n}{n!}\\
&={}_{p+m+2}F_{q+m+1}\left[\begin{array}{c}(\alpha_p),\ c-a-m,c-b-m,\ (\eta_m+1)\\ (\beta_q),\qquad c,\qquad (\eta_m)\end{array};x\right]
\end{aligned} \tag{5.5}$$

upon application of (1.5), where we recall that $\sigma := c-a-b-1$ and the (η_m) are the nonvanishing zeros of the parametric polynomial $\hat{Q}_m(t)$ defined in (2.1).

In the case $r=1$, $m_1=m=1$, $f_1=f$ we obtain the reduction formula

$$F^{p:\,3;\,1}_{q:\,2;\,0}\left[\begin{array}{l}(\alpha_p):\ a,\ b,\ f+1;\ c-a-b-1\\ (\beta_q):\qquad c,\ f;\qquad -\end{array}\middle|\, x,\ x\right]$$

$$={}_{p+3}F_{q+2}\left[\begin{array}{c}(\alpha_p),\ c-a-1,c-b-1,\ \eta+1\\ (\beta_q),\qquad c,\ \eta\end{array};x\right], \tag{5.6}$$

where η is defined in (3.2).

Remark 2. If we take $c=1+a-b$ in (5.6), then η is given by (3.3). The special case $f=\frac{1}{2}a$ then corresponds to $\eta=\infty$ $(a\neq 2b)$ and we obtain Exton's reduction formula [4, Eq. (11)]

$$F^{p:\,3;\,1}_{q:\,2;\,0}\left[\begin{array}{l}(\alpha_p):\ \ a,\ b,\ 1+\tfrac{1}{2}a;\ \ -2b\\ (\beta_q):\ 1+a-b,\ \tfrac{1}{2}a;\ \ -\end{array}\middle|\, x,\ x\right]={}_{p+2}F_{q+1}\left[\begin{array}{l}(\alpha_p),\ a-2b,\ -b\\ (\beta_q),\ \ 1+a-b\end{array};x\right].$$

5.2 Second reduction formula

We consider another reduction formula when $p=q=1$ for the function

$$F(x,x)\equiv F^{1:\,r+2;\,1}_{1:\,r+1;\,0}\left[\begin{array}{l}\alpha:\ a,\ \beta-d,\ (f_r+m_r);\ d\\ \beta:\qquad c,\ (f_r);\qquad -\end{array}\middle|\, x,\ x\right]$$

thereby generalizing a result obtained by Cvijović and Miller in [3] in the case $r = 0$. Our derivation follows closely that given by these authors.

Making use of the identity (2.7) we find when $|x| < 1$

$$\begin{aligned} F(x,x) &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{n+k}}{(\beta)_{n+k}} \frac{(a)_k(\beta-d)_k(d)_n}{(c)_k} \frac{((f_r+m_r))_k}{((f_r))_k} \frac{x^{n+k}}{n!\,k!} \\ &= \sum_{k=0}^{\infty} \frac{(\alpha)_k(a)_k(\beta-d)_k}{(\beta)_k(c)_k} \frac{((f_r+m_r))_k}{((f_r))_k} \frac{x^k}{k!}\, {}_2F_1\left[\begin{matrix}\alpha+k,\, d\\ \beta+k\end{matrix};x\right]. \end{aligned}$$

Now applying Euler's first transformation [16, p. 31]

$${}_2F_1\left[\begin{matrix}a,\, b\\ c\end{matrix};x\right] = (1-x)^{-a}\, {}_2F_1\left[\begin{matrix}a,\, c-b\\ c\end{matrix};\frac{x}{x-1}\right],$$

we obtain

$$\begin{aligned} &F(x,x)\\ &= (1-x)^{-\alpha}\sum_{k=0}^{\infty} \frac{(\alpha)_k(a)_k(\beta-d)_k}{(\beta)_k(c)_k k!} \frac{((f_r+m_r))_k}{((f_r))_k} \left(\frac{x}{1-x}\right)^k {}_2F_1\left[\begin{matrix}\alpha+k,\, \beta-d+k\\ \beta+k\end{matrix};\frac{x}{x-1}\right] \\ &= (1-x)^{-\alpha}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^k(\alpha)_k(a)_k(\beta-d)_k}{(\beta)_k(c)_k k!} \frac{((f_r+m_r))_k}{((f_r))_k} \left(\frac{x}{x-1}\right)^{n+k} \frac{(\alpha+k)_n(\beta-d+k)_n}{(\beta+k)_n\, n!} \\ &= (1-x)^{-\alpha}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{n+k}(\beta-d)_{n+k}}{(\beta)_{n+k}} \frac{(-1)^k(a)_k((f_r+m_r))_k}{(c)_k((f_r))_k\, n!\, k!} \left(\frac{x}{x-1}\right)^{n+k} \end{aligned}$$

valid when $|x| < 1$ and $|x/(x-1)| < 1$; that is in the domain $|x| < 1$, $\Re(x) < \frac{1}{2}$.

Making the change of summation index $n \to n-k$, reversing the order of summation and using the second identity in (4.2), we then find

$$\begin{aligned} F(x,x) &= (1-x)^{-\alpha}\sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta-d)_n}{(\beta)_n n!} \left(\frac{x}{x-1}\right)^n \sum_{k=0}^{n} \frac{(-n)_k(a)_k((f_r+m_r))_k}{(c)_k((f_r))_k\, k!} \\ &= (1-x)^{-\alpha}\sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta-d)_n}{(\beta)_n n!} \left(\frac{x}{x-1}\right)^n {}_{r+2}F_{r+1}\left[\begin{matrix}-n,\, a,\, (f_r+m_r)\\ c,\quad (f_r)\end{matrix};1\right] \\ &= (1-x)^{-\alpha}\, {}_{m+3}F_{m+2}\left[\begin{matrix}\alpha,\, \beta-d,\, c-a-m,\, (\xi_m+1)\\ \beta,\, c,\quad (\xi_m)\end{matrix};\frac{x}{x-1}\right] \end{aligned} \tag{5.7}$$

upon use of the extension of the Vandermonde-Chu summation theorem in (1.3), where the (ξ_m) are the nonvanishing zeros of the associated parametric polynomial $Q_m(t)$ defined in (5.3).

When $r = 0$ ($m = 0$) we recover the Cvijović-Miller result [3] given by

$$F^{1:2;1}_{1:1;0}\left[\begin{matrix}\alpha:\ a,\ \beta-d;\ d\\ \beta:\quad c;\quad -\end{matrix}\middle|\, x,\ x\right] = (1-x)^{-\alpha}{}_3F_2\left[\begin{matrix}\alpha,\, \beta-d,\, c-a\\ \beta,\, c\end{matrix};\frac{x}{x-1}\right].$$

When $r = 1$, $m_1 = m = 1$, $f_1 = f$, we have the reduction formula

$$F^{1:3;1}_{1:2;0}\left[\begin{matrix}\alpha:\ a,\ \beta-d,\ f+1;\ d\\ \beta:\quad c,\ f;\quad -\end{matrix}\middle|\, x,\ x\right] = (1-x)^{-\alpha}{}_4F_3\left[\begin{matrix}\alpha,\, \beta-d,\, c-a-1,\, \xi+1\\ \beta,\, c,\quad \xi\end{matrix};\frac{x}{x-1}\right], \tag{5.8}$$

where, from (5.4) with b replaced by a,

$$\xi = \frac{(c-a-1)f}{f-a}.$$

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References

[1] B. C. Berndt, *Ramanujan's Notebooks*, vol. 2, Springer-Verlag, 1989.

[2] J. Choi, A. K. Rathie and P. Chopra, A new proof of the extended Saalschütz summation theorem for the series ${}_4F_3$ and its applications, Honam Math. J. **35** (2013) 407–415.

[3] D. Cvijović and A. R. Miller, A reduction formula for the Kampé de Fériet function, Appl. Math. Letters **23** (2010) 769–771.

[4] H. Exton, On the reducibility of the Kampé de Fériet function, J. Comput. Appl. Math. **83** (1997) 119–121.

[5] Y. S. Kim, M. A. Rakha and A. K. Rathie, Extensions of certain classical summation theorems for the series ${}_2F_1$, ${}_3F_2$ and ${}_4F_3$ with applications in Ramanujan summations, Int. J. Math. Math. Sci. 309503,

[6] Y. S. Kim, A. K. Rathie and R. B. Paris, An extension of Saalschütz's summation theorem for the series ${}_{r+3}F_{r+2}$, Integral Transforms and Special Functions **24** (2013) 916–921.

[7] A. R. Miller, Certain summation and transformation formulas for generalized hypergeometric series, J. Comp. Appl. Math. **231** (2009) 964–972.

[8] A. R. Miller and R. B. Paris, Euler-type transformations for the generalized hypergeometric function ${}_{r+2}F_{r+1}(x)$, Zeit. angew. Math. Phys. **62** (2011) 31–45.

[9] A. R. Miller and R. B. Paris, Transformation formulas for the generalized hypergeometric function with integral parameter differences, Rocky Mountain J. Math **43** (2013) 291–327.

[10] A. R. Miller and R. B. Paris, On a result related to transformations and summations of generalized hypergeometric series, Math. Communications **17** (2012) 205–210.

[11] A. R. Miller and H. M. Srivastava, Karlsson–Minton summation theorems for the generalized hypergeometric series of unit argument, Integral Transforms and Special Functions **21** (2010) 603–612.

[12] B. M. Minton, Generalized hypergeometric functions with integral parameter differences, J. Math. Phys. **11** (1970) 1375–1376.

[13] A. P. Prudnikov, Y. A. Brychkov and O. I. Marichev, *Integrals and Series: More Special Functions*, vol. 3, Gordon and Breach Science, New York, 1988.

[14] Rakha, M. A. and Rathie, A. K., Generalizations of classical summation theorems for the series ${}_2F_1$ and ${}_3F_2$ with applications, Integral Transforms and Special Functions **22** (2011) 823–840.

[15] A. K. Rathie and R. B. Paris, Extension of some classical summation theorems for the generalized hypergeometric series with integral parameter differences, J. Classical Anal. **3** (2013) 109–127.

[16] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.

[17] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood, Chichester, 1985.

[18] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood, Chichester, 1984.