An extension of Saalschütz's summation theorem for the series ${}_{r+3}F_{r+2}$

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Abstract

The aim in this research note is to provide an extension of Saalschütz's summation theorem for the series ${}_{r+3}F_{r+2}(1)$ when r pairs of numeratorial and denominatorial parameters differ by positive integers. The result is obtained by exploiting a generalization of an Euler-type transformation recently derived by Miller and Paris [9].

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1. Introduction

The generalized hypergeometric function ${}_pF_q(x)$ may be defined for complex parameters and argument by the series

$$ {}_pF_q\left(\begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array}\middle| x\right) = \sum_{k=0}^\infty \frac{(a_1)_k(a_2)_k \ldots (a_p)_k}{(b_1)_k(b_2)_k \ldots (b_q)_k} \frac{x^k}{k!}. \tag{1.1}$$

When $q = p$ this series converges for $|x| < \infty$, but when $q = p - 1$ convergence occurs when $|x| < 1$. However, when only one of the numeratorial parameters a_j is a negative integer or zero, then the series always converges since it is simply a polynomial in x of degree $-a_j$. In (1.1) the Pochhammer symbol or ascending factorial $(a)_n$ is given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n = 0) \\ a(a+1)\ldots(a+n-1) & (n \in \mathbf{N}), \end{cases}$$

where Γ is the gamma function. The parametric excess s of the above series is defined by

$$s = \sum_{r=1}^q b_r - \sum_{r=1}^p a_r.$$

We shall adopt the convention of writing the finite sequence of parameters $(a_1, \ldots, a_p)$ simply by (a_p) and the product of p Pochhammer symbols by

$$((a_p))_k \equiv (a_1)_k \ldots (a_p)_k,$$

where an empty product $p = 0$ reduces to unity. It is evident that whenever generalized hypergeometric functions of special argument reduce to Gamma functions the results are of considerable importance in applications. Until 1990, only a few classical summation theorems for ${}_2F_1$, ${}_3F_2$ and for higher order series were known. Subsequently, some progress has been made in generalizing these classical summation theorems; see [1, 3, 4, 5, 7, 10, 13, 15].

In our present investigation we shall be concerned with the following summation theorem due to Saalschütz [14, p. 49]

$$ {}_3F_2\left(\begin{matrix} -n,\, a,\, b \\ c, 1+a+b-c-n \end{matrix} \middle| 1\right) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}, \tag{1.2}$$

where n is a nonnegative integer. Such a series is said to be Saalschützian since the parametric excess $s = 1$. As shown in [14, p. 49], this follows from taking the well-known Euler transformation

$$(1-x)^{a+b-c}{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| x\right) = {}_2F_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix} \middle| x\right) \tag{1.3}$$

and equating coefficients of x^n on both sides of the equation. An extension of Saalschütz's theorem has been considered recently by Rakha and Rathie in [13] who showed that

$$ {}_4F_3\left(\begin{matrix} -n,\, a,\, b, & f+1 \\ c,\, 2+a+b-c-n, & f \end{matrix} \middle| 1\right) = \frac{(c-a-1)_n(c-b-1)_n}{(c)_n(c-a-b-1)_n}\,\frac{(\eta+1)_n}{(\eta)_n},$$

where

$$\eta = \frac{(c-a-1)(c-b-1)f}{ab+(c-a-b-1)f}.$$

As an application of this result these authors also established the quadratic transformation

$$(1+x)^{-2a}{}_3F_2\left(\begin{matrix} a,\, b, & f+1 \\ a+b+\frac{3}{2}, & f \end{matrix} \middle| \frac{4x}{(1+x)^2}\right) = {}_4F_3\left(\begin{matrix} 2a,\, a-b-\frac{1}{2}, & \omega_1+1, & \omega_2+1 \\ a+b+\frac{3}{2}, & \omega_1, & \omega_2 \end{matrix} \middle| -x\right), \tag{1.4}$$

where

$$\omega_{1,2} = a \pm \left[a^2 + (a+\tfrac{1}{2})(a+b-\tfrac{1}{2})f/(b-f)\right]^{1/2}.$$

In [9], a generalization of the Euler transformation (1.3), when r pairs of numeratorial and denominatorial parameters differ by positive integers (m_r), was obtained in the form

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r+m_r) \\ c, & (f_r) \end{matrix} \middle| x\right)$$

$$ = (1-x)^{c-a-b-m}{}_{m+2}F_{m+1}\left(\begin{matrix} c-a-m, c-b-m, & (\eta_m+1) \\ c, & (\eta_m) \end{matrix} \middle| x\right) \tag{1.5}$$

when $|x| < 1$. Here (η_m) are the nonvanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree $m \equiv m_1 + \cdots + m_r$ given by

$$Q_m(t) = \sum_{k=0}^{m} B_k (a)_k (b)_k (t)_k G_{m,k}(t) \tag{1.6}$$

with

$$B_k \equiv (-1)^k A_k (c-a-m+k)_{m-k}(c-b-m+k)_{m-k}$$

and

$$G_{m,k}(t) \equiv {}_3F_2\left(\begin{matrix} -m+k, t+k, c-a-b-m \\ c-a-m+k, c-b-m+k \end{matrix} \middle| 1\right).$$

For $0 \le k \le m$, the function $G_{m,k}(t)$ is a polynomial in t of degree $m-k$. The coefficients A_k are defined by

$$A_k = \sum_{j=k}^{m} \mathbf{S}_j^{(k)} \sigma_{m-j}, \qquad A_0 = (f_1)_{m_1} \ldots (f_r)_{m_r}, \quad A_m = 1, \tag{1.7}$$

where $\mathbf{S}_j^{(k)}$ is the Stirling number of the second kind and the coefficients σ_j $(0 \le j \le m)$ are generated by

$$(f_1+x)_{m_1} \ldots (f_r+x)_{m_r} = \sum_{j=0}^{m} \sigma_{m-j} x^j.$$

The case $m_1 = \cdots = m_r = 1$ in (1.5) has been given earlier in [8], and also in [6] using different methods. When (m_r) is empty we define $m = 0$; in this case (1.5) reduces to the Euler transformation (1.3). In Section 2, we shall employ the same approach described in [14, p. 49] to the transformation formula (1.5) in our proof of the extension of Saalschütz's theorem to ${}_{r+3}F_{r+2}$ series.

2. The extension of Saalschütz's theorem

The extension of Saalschütz's summation theorem is given by the following:

Theorem 1. *Let (m_r) be a set of positive integers and define $m \equiv m_1 + \cdots + m_r$. Let n be a nonnegative integer. Then*

$$ {}_{r+3}F_{r+2}\left(\begin{matrix} -n,\ a,\ b, & (f_r+m_r) \\ c,\ 1+a+b-c+m-n, & (f_r) \end{matrix} \middle| 1\right) = \frac{(c-a-m)_n (c-b-m)_n}{(c)_n (c-a-b-m)_n} \frac{((\eta_m+1))_n}{((\eta_m))_n}, \tag{2.1}$$

where (η_m) are the nonvanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree m defined in (1.6).

Proof: From (1.5) we have

$${}_{m+2}F_{m+1}\left(\begin{matrix} c-a-m, c-b-m, & (\eta_m+1) \\ c, & (\eta_m) \end{matrix} \middle| x\right)$$
$$= (1-x)^{m+a+b-c} {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r+m_r) \\ c, & (f_r) \end{matrix} \middle| x\right).$$

The coefficient of x^n on the left-hand side is

$$\frac{(c-a-m)_n (c-b-m)_n}{(c)_n\, n!} \frac{((\eta_m+1))_n}{((\eta_m))_n} \tag{2.2}$$

and this must equal the coefficient of x^n on the right-hand side. With

$$D_k \equiv (-1)^k \binom{m+a+b-c}{k},$$

this latter coefficient is given by

$$\sum_{k=0}^{n} D_{n-k}\,\frac{(a)_k(b)_k}{(c)_k\,k!}\,\frac{((f_r+m_r))_k}{((f_r))_k}$$

$$\begin{aligned}
&= \sum_{k=0}^{n} \frac{(-1)^{n-k}\Gamma(1+a+b-c+m)}{(n-k)!\,\Gamma(1+a+b-c+m-n+k)}\,\frac{(a)_k(b)_k}{(c)_k\,k!}\,\frac{((f_r+m_r))_k}{((f_r))_k}\\
&= \frac{(c-a-b-m)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{k!}\,\frac{(a)_k(b)_k}{(c)_k\,(1+a+b-c+m-n)_k}\,\frac{((f_r+m_r))_k}{((f_r))_k}\\
&= \frac{(c-a-b-m)_n}{n!}\;{}_{r+3}F_{r+2}\left(\begin{matrix}-n,\ a,\ b, & (f_r+m_r)\\ c, 1+a+b-c+m-n, & (f_r)\end{matrix}\,\middle|\,1\right),
\end{aligned}$$

where we have used the identities

$$(-n)_k=\frac{(-1)^k n!}{(n-k)!},\qquad \frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha-n)}=(-1)^n(\alpha)_n.$$

Equating this coefficient to that in (2.2) we then obtain the desired summation (2.1) □

3. Examples

In the case $r=1$ and $m_1=m=1$ we have from (2.1)

$${}_4F_3\left(\begin{matrix}-n,\ a,\ b, & f+1\\ c,\ 2+a+b-c-n, & f\end{matrix}\,\middle|\,1\right)=\frac{(c-a-1)_n(c-b-1)_n}{(c)_n(c-a-b-1)_n}\,\frac{(\eta+1)_n}{(\eta)_n}, \tag{3.1}$$

where η is the zero of the first-degree parametric polynomial obtained from (1.6)

$$Q_1(t)=-\{f(c-a-b-1)+ab\}t+(c-a-1)(c-b-1)f,$$

whence

$$\eta=\frac{(c-a-1)(c-b-1)f}{ab+(c-a-b-1)f}.$$

This result was derived in [13] using a different approach.

A special case of (3.1), which may be of some interest, is obtained when $c=1+a-b$, so that $\eta=(a-2b)f/(2f-a)$, to yield

$${}_4F_3\left(\begin{matrix}-n,\ a,\ b, & f+1\\ 1+a-b,\ 1+2b-n, & f\end{matrix}\,\middle|\,1\right)=\frac{(a-2b)_n(-b)_n}{(1+a-b)_n(-2b)_n}\left\{1+\frac{n(2f-a)}{(a-2b)f}\right\},$$

where we have used the fact that $(\eta+1)_n/(\eta)_n=(\eta+n)/\eta$. This result was obtained earlier by Kim, Rathie and Paris by other means who used it to obtain a reduction formula for the Kampé de Fériet function [2]. In the above result, if we let $f=\frac{1}{2}a$ we obtain

$${}_4F_3\left(\begin{matrix}-n,\ a,\ b,\ 1+\frac{1}{2}a\\ c,\ 1+2b-n,\ \frac{1}{2}a\end{matrix}\,\middle|\,1\right)=\frac{(a-2b)_n(-b)_n}{(1+a-b)_n(-2b)_n}\qquad (a\neq 2b),$$

which is a known summation; see [14, Appendix III, Eq. (17)].

Also, in the case $r = 1$ with $m = 2$, we find from (1.7) that $A_0 = f(1+f)$, $A_1 = 2(1+f)$ and $A_2 = 1$. Introducing the abbreviated notation

$$\lambda \equiv c-a-2, \quad \lambda' \equiv c-b-2, \quad \sigma \equiv c-a-b-2,$$

we obtain from (1.6) the quadratic parametric polynomial (with zeros η_1 and η_2) given by

$$Q_2(t) = A_0(\lambda)_2(\lambda')_2\left\{1-\frac{2\sigma t}{\lambda\lambda'}+\frac{(t)_2(\sigma)_2}{(\lambda)_2(\lambda')_2}\right\} - A_1 abt(\lambda+1)(\lambda'+1)\left\{1-\frac{\sigma(1+t)}{(\lambda+1)(\lambda'+1)}\right\}$$

$$+A_2(a)_2(b)_2 t(1+t)$$

$$= A_0(\lambda)_2(\lambda')_2\left\{1-\frac{2Bt}{\lambda\lambda'}+\frac{Ct(1+t)}{(\lambda)_2(\lambda')_2}\right\},$$

where

$$B \equiv \sigma + \frac{ab}{f}, \qquad C \equiv (\sigma)_2 + \frac{2ab\sigma}{f} + \frac{(a)_2(b)_2}{(f)_2}.$$

Then, from (2.1),

$$ {}_4F_3\left(\begin{matrix} -n,\ a,\ b, & f+2 \\ c,\ 3+a+b-c-n, & f\end{matrix}\,\middle|\,1\right) = \frac{(c-a-2)_n(c-b-2)_n}{(c)_n(c-a-b-2)_n}\,\frac{(\eta_1+1)_n}{(\eta_1)_n}\,\frac{(\eta_2+1)_n}{(\eta_2)_n}. \tag{3.2}$$

For example, if $a=\frac{3}{2}$, $b=2$, $c=\frac{5}{4}$ and $f=1$ we find

$$Q_2(t) = \tfrac{3465}{256}\{1+\tfrac{40}{99}t-\tfrac{16}{495}t(1+t)\},$$

which has the zeros $\eta_1 = -\frac{9}{4}$ and $\eta_2 = \frac{55}{4}$.

Finally, we consider the case $r=2$ with $m_1 = m_2 = 1$ (so that $m=2$). If we take $a=\frac{1}{3}$, $b=\frac{1}{2}$, $c=1$, $f_1=\frac{1}{4}$ and $f_2=2$, we find the parametric polynomial

$$Q_2(t) = \tfrac{1}{72}\{12+\tfrac{31}{2}t+\tfrac{15}{2}t(1+t)\}$$

which has the zeros $\eta_1 = -\frac{2}{3}$ and $\eta_2 = -\frac{12}{5}$. Then from (2.1) we obtain

$$ {}_5F_4\left(\begin{matrix} -n,\ a,\ b,\ \frac{5}{4},\ 3 \\ c,\ 3+a+b-c-n,\ \frac{1}{4},\ 2\end{matrix}\,\middle|\,1\right) = \frac{(c-a-2)_n(c-b-2)_n}{(c)_n(c-a-b-2)_n}\,\frac{(\frac{1}{3})_n\,(-\frac{7}{5})_n}{(-\frac{2}{3})_n\,(-\frac{12}{5})_n}.$$

Similarly, other results can also be obtained.

Remark. Two other quadratic transformations, different from that in (1.4), have been presented by Miller and Paris [9, Section 6] in the form

$$(1+x)^{-2a}{}_3F_2\left(\begin{matrix} a,\ a+\frac{1}{2},\ f+1 \\ c,\ f\end{matrix}\,\middle|\,\frac{x^2}{(1+x)^2}\right) = {}_4F_3\left(\begin{matrix} 2a,\ c-\frac{3}{2},\ \xi_1+1,\ \xi_2+1 \\ 2c-1,\ \xi_1,\ \xi_2\end{matrix}\,\middle|\,-2x\right) \tag{3.3}$$

provided $c \neq \frac{3}{2}$, where

$$\xi_{1,2} = 2f-\tfrac{1}{2} \pm [(2f-\tfrac{1}{2})^2 - 4(c-\tfrac{3}{2})f]^{1/2},$$

and

$$(1+x)^{-2a}{}_3F_2\left(\begin{matrix} a,\ a+\frac{1}{2},\ f+1 \\ c,\ f\end{matrix}\,\middle|\,\frac{4x}{(1+x)^2}\right) = {}_4F_3\left(\begin{matrix} 2a,\ 2a-c+1,\ \eta_1+1,\ \eta_2+1 \\ c,\ \eta_1,\ \eta_2\end{matrix}\,\middle|\,x\right) \tag{3.4}$$

provided $c \neq 2a+1$, where

$$\eta_{1,2} = a \pm [a^2 - (2a-c+1)f]^{1/2}.$$

In the same paper, the authors also derived the extension of both these quadratic transformations in the more general case of r pairs of numeratorial and denominatorial parameters differing by positive integers. It should be pointed out that the results in (3.3) and (3.4) were also obtained in [11] and [12], repectively, following different methods.

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