On a new class of summation formulae involving the Laguerre polynomial

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Abstract

By elementary manipulation of series, a general transformation involving the generalized hypergeometric function is established. Kummer's first theorem, the classical Gauss summation theorem and the generalized Kummer summation theorem due to Lavoie, Grondin and Rathie [J. Comput. Appl. Math. **72** (1996) 293–300] are then applied to obtain a new class of summation formulae involving the Laguerre polynomial, which have not previously appeared in the literature. Several related results due to Exton have also been given in corrected form.

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1. Introduction

The generalized hypergeometric function with p numeratorial and q denominatorial parameters is defined by the series [8, p. 41]

$$ {}_pF_q\left[\begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array} ; x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n}{(b_1)_n (b_2)_n \ldots (b_q)_n} \frac{x^n}{n!}. \tag{1.1}$$

When $q = p$ this series converges for $|x| < \infty$, but when $q = p - 1$ convergence occurs when $|x| < 1$ (unless the series terminates). In (1.1) the Pochhammer symbol (or ascending factorial, since $(1)_n = n!$) is defined for any complex number α by

$$(\alpha)_n = \left\{ \begin{array}{ll} \alpha(\alpha+1)\ldots(\alpha+n-1) & (n \in \mathbf{N}) \\ 1 & (n = 0) \end{array} \right.$$

Use of the fundamental relation $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, where Γ is the gamma function, then shows that $(\alpha)_n$ can be written in the form

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \qquad (n \in \mathbf{N} \cup \{0\}).$$

It should be remarked that whenever the hypergeometric function ${}_2F_1$, or generalized hypergeometric functions, reduce to gamma functions the results are very important from the applications point of view. This function has been extensively studied by many authors such as Slater [8] and Exton [3]. Here we shall mention two well-known summation theorems so that the paper is self-contained: the first is Gauss's summation theorem [1, p. 556]

$$ {}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix};1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad (\mathrm{Re}\,(c-a-b)>0) \tag{1.2} $$

and the second is Kummer's summation theorem [1, p. 505]

$$ {}_2F_1\left[\begin{matrix} a,b \\ 1+a-b \end{matrix};-1\right] = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}. \tag{1.3} $$

In 1996, Lavoie, Grondin and Rathie [6] generalized (1.3) in the form

$$ \begin{aligned} {}_2F_1\left[\begin{matrix} a,b \\ 1+a-b+j \end{matrix};-1\right] &= 2^{-a}\pi^{\frac{1}{2}}\frac{\Gamma(1-b)\Gamma(1+a-b+j)}{\Gamma(1-b+\epsilon_j)} \\ &\times\left\{\frac{A_j(a,b)}{\Gamma(\frac{1}{2}a-b+\frac{1}{2}j+1)\Gamma(\frac{1}{2}a+\delta_{j+1})}+\frac{B_j(a,b)}{\Gamma(\frac{1}{2}a-b+\frac{1}{2}j+\frac{1}{2})\Gamma(\frac{1}{2}a+\delta_j)}\right\} \end{aligned} \tag{1.4} $$

for integer values of j. Here, we have defined

$$ \epsilon_j \equiv \tfrac{1}{2}(j+|j|), \qquad \delta_j \equiv \tfrac{1}{2}j-[\tfrac{1}{2}j], \tag{1.5} $$

where $[x]$ denotes the greatest integer less than or equal to x and its modulus is denoted by $|x|$. The coefficients $A_j(a,b)$ and $B_j(a,b)$ for $-5\le j\le 5$ are given in the Table 1. When $j=0$, we remark that (1.4) reduces to Kummer's summation theorem (1.3).

j	$A_j(a,b)$	$B_j(a,b)$
5	$-4(a-b+6)^2+2b(a-b+6)+b^2$	$4(a-b+6)^2+2b(a-b+6)-b^2$
	$+22(a-b+6)-13b-20$	$-34(a-b+6)-b+62$
4	$2(a-b+3)(a-b+1)-(b-1)(b-4)$	$-4(a-b+2)$
3	$-2a+3b-5$	$2a-b+3$
2	$a-b+1$	-2
1	-1	1
0	1	0
-1	1	1
-2	$a-b-1$	2
-3	$2a-3b-4$	$2a-b-2$
-4	$2(a-b-3)(a-b-1)-b(b+3)$	$4(a-b-2)$
-5	$4(a-b-4)^2-2b(a-b-4)-b^2$	$4(a-b-4)^2+2b(a-b-4)-b^2$
	$+8(a-b-4)-7b$	$+16(a-b-4)-b+12$

Table 1: The coefficients A_j and B_j for $-5\le j\le 5$.

On the other hand, in many branches of pure and applied mathematics, the Laguerre polynomial, which is a terminating form of the confluent hypergeometric function ${}_1F_1$ defined by

$$ L_n^{(\nu)}(x) = \frac{(\nu+1)_n}{n!}\,{}_1F_1\left[\begin{matrix} -n \\ \nu+1 \end{matrix};\ x\right], \tag{1.6} $$

occurs frequently; see, for example, [2, p. 268]. The aim of this note is to obtain a general transformation involving the generalized hypergeometric function by the method of elementary manipulation of series. The well-known Kummer's first theorem, the classical Gauss summation theorem and the generalized Kummer summation theorem in (1.4) are then applied to obtain a new class of summation formulae involving the Laguerre polynomial, which have not previously appeared in the literature. Several related results due to Exton have also been given in corrected form.

2. The main transformation formula

The transformation formula involving the generalized hypergeometric function to be established is

$$\sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{(d)_n}{n!} x^n y^n \, {}_1F_1 \left[\begin{matrix} d+n \\ f \end{matrix} ; \, x \right]$$

$$= \sum_{n=0}^{\infty} \frac{(d)_n}{(f)_n} \frac{x^n}{n!} \, {}_{p+2}F_q \left[\begin{matrix} -n, 1-f-n, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} ; \, y \right]. \tag{2.1}$$

Proof. In order to prove (2.1), we proceed as follows. Denoting the left-hand side of (2.1) by S and expressing ${}_1F_1$ as a series, we have

$$S = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{(d)_n}{n!} \frac{(d+n)_m}{m!(f)_m} x^{n+m} y^n,$$

which, upon use of the identity $(d)_n(d+n)_m = (d)_{n+m}$, becomes

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{(d)_{n+m}}{n!m!(f)_m} x^{n+m} y^n.$$

If we now change m to $m-n$ and make use of a simple formal manipulation for the double series [7, p. 56]

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n-k),$$

we obtain

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{(d)_m x^m y^n}{n!(m-n)!(f)_{m-n}}.$$

By employing the identities

$$(a)_{m-n} = \frac{(-1)^n (a)_m}{(1-a-m)_n}, \qquad (m-n)! = \frac{(-1)^n m!}{(-m)_n},$$

we find, after a little simplification,

$$S = \sum_{m=0}^{\infty} \frac{(d)_m x^m}{(f)_m m!} \sum_{n=0}^{m} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{(1-f-m)_n (-m)_n}{n!} y^n.$$

Expressing the inner series as a ${}_{p+2}F_q(y)$ hypergeometric function, we then easily arrive at the right-hand side of (2.1). This completes the proof of (2.1). □

Remark 2.1. Our main transformation formula is a special case of a general result given by Slater [8, p. 60] and is the corrected form of the result due to Exton [4, Eq. (6)].

3. New class of summation formulae involving the Laguerre polynomial

The summation formulae to be established in this section are given by the following theorem.

Theorem 1. *Let j denote an integer and the quantities ϵ_j and δ_j be as defined in (1.5). Then we have*

$$e^{-x}\sum_{n=0}^\infty \frac{x^n}{(\nu+1+j)_n}\,L_n^{(\nu)}(x)=\pi^{\frac{1}{2}}\frac{\Gamma(\nu+1+j)\Gamma(\nu+1)}{\Gamma(\nu+1+\epsilon_j)}\sum_{n=0}^\infty \frac{(-1)^n(2x)^n}{n!}\,\frac{(\nu+1)_n}{(\nu+1+\epsilon_j)_n}$$
$$\times\left\{\frac{A_j(-n,-\nu-n)}{\Gamma(-\frac{1}{2}n+\delta_{j+1})\Gamma(\frac{1}{2}n+\frac{1}{2}j+\nu+1)}+\frac{B_j(-n,-\nu-n)}{\Gamma(-\frac{1}{2}n+\delta_j)\Gamma(\frac{1}{2}n+\frac{1}{2}j+\nu+\frac{1}{2})}\right\},\qquad (3.1)$$

where, for $-5\le j\le 5$, the coefficients $A_j(a,b)$ and $B_j(a,b)$ can be obtained from Table 1 by replacing the parameter a by $-n$ and the parameter b by $-\nu-n$. Secondly

$$e^{-x}\sum_{n=0}^\infty \frac{x^n}{(1-\nu+j)_n}\,L_n^{(\nu)}(x)=2^\nu\pi^{\frac{1}{2}}\frac{\Gamma(1-\nu+j)}{\Gamma(1+\epsilon_j)}\sum_{n=0}^\infty \frac{(-1)^n(2x)^n}{(1+\epsilon_j)_n}$$
$$\times\left\{\frac{A_j(-\nu-n,-n)}{\Gamma(-\frac{1}{2}n-\frac{1}{2}\nu+\delta_{j+1})\Gamma(\frac{1}{2}n+\frac{1}{2}j+1-\frac{1}{2}\nu)}+\frac{B_j(-\nu-n,-n)}{\Gamma(-\frac{1}{2}n-\frac{1}{2}\nu+\delta_j)\Gamma(\frac{1}{2}n+\frac{1}{2}j+\frac{1}{2}-\frac{1}{2}\nu)}\right\},\qquad (3.2)$$

where, for $-5\le j\le 5$, the coefficients $A_j(a,b)$ and $B_j(a,b)$ can be obtained from Table 1 by replacing the parameter a by $-\nu-n$ and the parameter b by $-n$. And thirdly

$$e^{-x}\sum_{n=0}^\infty \frac{(-x)^n}{(\mu)_n}\,L_n^{(\nu)}(x)={}_2F_2\left[\begin{matrix}\frac{1}{2}\mu+\frac{1}{2}\nu,\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\\ \mu,\mu+\nu\end{matrix};-4x\right].\qquad (3.3)$$

Any parameter values that lead to results not making sense are tacitly excluded.

Proof. In order to derive the result (3.1) we proceed as follows. If we apply Kummer's first transformation [1, p. 505]

$${}_1F_1\left[\begin{matrix}a\\ c\end{matrix};x\right]=e^x\,{}_1F_1\left[\begin{matrix}c-a\\ c\end{matrix};-x\right],$$

in (2.1) we obtain

$$e^x\sum_{n=0}^\infty \frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{(d)_n}{n!}x^ny^n\,{}_1F_1\left[\begin{matrix}f-d-n\\ f\end{matrix};-x\right]$$
$$=\sum_{n=0}^\infty \frac{(d)_n}{(f)_n}\frac{x^n}{n!}\,{}_{p+2}F_q\left[\begin{matrix}-n,1-f-n,a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};\ y\right].\qquad (3.4)$$

Putting $d=f$ in (3.4), we have

$$e^x\sum_{n=0}^\infty \frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{(f)_n}{n!}x^ny^n\,{}_1F_1\left[\begin{matrix}-n\\ f\end{matrix};-x\right]$$
$$=\sum_{n=0}^\infty \frac{x^n}{n!}\,{}_{p+2}F_q\left[\begin{matrix}-n,1-f-n,a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};\ y\right].$$

If we replace x by $-x$, put $f=\nu+1$ and employ (1.6), there then results

$$e^{-x}\sum_{n=0}^\infty \frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}(-1)^nx^ny^n\,L_n^{(\nu)}(x)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \, {}_{p+2}F_q \left[\begin{matrix} -n, -\nu - n, a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; \; y \right]. \tag{3.5}$$

We remark that (3.5) is the corrected form of the result obtained by Exton [4, Eq. (8)].

We now put $p = 0$, $q = 1$, $b_1 = \nu + 1 + j$ and $y = -1$ in (3.5) to obtain

$$e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{(\nu + 1 + j)_n} \, L_n^{(\nu)}(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \, {}_2F_1 \left[\begin{matrix} -n, -\nu - n \\ \nu + 1 + j \end{matrix} ; -1 \right] \tag{3.6}$$

for integer values of j and $|x| < \infty$. It is readily seen that the ${}_2F_1$ function on the right-hand side of (3.6) can be evaluated with the help of the generalized Kummer summation theorem in (1.4) to yield

$$e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{(\nu + 1 + j)_n} \, L_n^{(\nu)}(x) = \pi^{\frac{1}{2}} \frac{\Gamma(\nu + 1 + j)\Gamma(\nu + 1)}{\Gamma(\nu + 1 + \epsilon_j)} \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^n}{n!} \, \frac{(\nu + 1)_n}{(\nu + 1 + \epsilon_j)_n}$$

$$\times \left\{ \frac{A_j(-n, -\nu - n)}{\Gamma(-\frac{1}{2}n + \delta_{j+1})\Gamma(\frac{1}{2}n + \frac{1}{2}j + \nu + 1)} + \frac{B_j(-n, -\nu - n)}{\Gamma(-\frac{1}{2}n + \delta_j)\Gamma(\frac{1}{2}n + \frac{1}{2}j + \nu + \frac{1}{2})} \right\},$$

where, for $-5 \leq j \leq 5$, the coefficients are obtained from $A_j(a, b)$ and $B_j(a, b)$ in Table 1 with the parameter a replaced by $-n$ and the parameter b replaced by $-\nu - n$. This completes the proof of (3.1).

To establish (3.2), we put $p = 0$, $q = 1$, $b_1 = 1 - \nu + j$ and $y = -1$ in (3.5) to obtain

$$e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{(1 - \nu + j)_n} \, L_n^{(\nu)}(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \, {}_2F_1 \left[\begin{matrix} -\nu - n, -n \\ 1 - \nu + j \end{matrix} ; -1 \right] \tag{3.7}$$

valid for $|x| < \infty$. The ${}_2F_1$ function on the right-hand side of (3.7) can similarly be evaluated with the help of (1.4) where the coefficients are obtained from $A_j(a, b)$ and $B_j(a, b)$ in Table 1 with a replaced by $-\nu - n$ and b replaced by $-n$. The result (3.2) is obtained in the same manner as that in (3.1) and accordingly we omit the details.

In order to derive (3.3), we put $p = 0$, $q = 1$, $b_1 = \mu$ and $y = 1$ in (3.5) to find

$$e^{-x} \sum_{n=0}^{\infty} \frac{(-x)^n}{(\mu)_n} \, L_n^{(\nu)}(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \, {}_2F_1 \left[\begin{matrix} -n, -\nu - n \\ \mu \end{matrix} ; 1 \right].$$

Use of the Gauss summation theorem (1.2) and the duplication formula for the gamma function then yields

$$e^{-x} \sum_{n=0}^{\infty} \frac{(-x)^n}{(\mu)_n} \, L_n^{(\nu)}(x) = \sum_{n=0}^{\infty} \frac{(-4x)^n}{n!} \, \frac{(\frac{1}{2}\mu + \frac{1}{2}\nu)_n (\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2})_n}{(\mu)_n (\mu + \nu)_n}$$

for $|x| < \infty$. Expressing the series on the right-hand side as a ${}_2F_2$ hypergeometric function, we then arrive at (3.3). This completes the proof of (3.3).

Remark 3.1 The result (3.3) is a corrected form of the result given by Exton in [4, Eq. (12)].

4. Special cases of (3.1) and (3.2)

We present some special cases of (3.1) and (3.2). We first separate these two results into even and odd powers of x by making use of the following identities:

$$(2n)! = 2^{2n} n! (\tfrac{1}{2})_n, \qquad (2n + 1)! = 2^{2n} n! (\tfrac{3}{2})_n, \qquad \Gamma(a - n) = \frac{(-1)^n \Gamma(a)}{(1 - a)_n},$$

$$(a)_{2n} = 2^{2n}(\tfrac{1}{2}a)_n(\tfrac{1}{2}a+\tfrac{1}{2})_n, \qquad (a)_{2n+1} = 2^{2n}a(\tfrac{1}{2}a+\tfrac{1}{2})_n(\tfrac{1}{2}a+1)_n.$$

After some straightforward algebra we arrive at the sum (3.1) in the form

$$\begin{aligned} e^{-x}\sum_{n=0}^{\infty}&\frac{x^n}{(\nu+1+j)_n}\,L_n^{(\nu)}(x)\\ &=\pi^{\frac{1}{2}}\frac{\Gamma(\nu+1)\Gamma(\nu+1+j)}{\Gamma(\nu+1+\epsilon_j)}\left\{\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}\,\frac{(\frac{1}{2}\nu+\frac{1}{2})_n(\frac{1}{2}\nu+1)_n\,\Lambda_1(n,j)}{(\frac{1}{2})_n(\frac{1}{2}\nu+\frac{1}{2}+\frac{1}{2}\epsilon_j)_n(\frac{1}{2}\nu+1+\frac{1}{2}\epsilon_j)_n}\right.\\ &\qquad\left.-\frac{2x(\nu+1)}{\nu+1+\epsilon_j}\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}\,\frac{(\frac{1}{2}\nu+1)_n(\frac{1}{2}\nu+\frac{3}{2})_n\,\Lambda_2(n,j)}{(\frac{3}{2})_n(\frac{1}{2}\nu+1+\frac{1}{2}\epsilon_j)_n(\frac{1}{2}\nu+\frac{3}{2}+\frac{1}{2}\epsilon_j)_n}\right\}, \end{aligned} \tag{4.1}$$

where

$$\Lambda_1(n,j)=\frac{A_j(-2n,-\nu-2n)}{\Gamma(\nu+1+\frac{1}{2}j)\Gamma(\delta_{j+1})}\,\frac{(1-\delta_{j+1})_n}{(\nu+1+\frac{1}{2}j)_n}+\frac{B_j(-2n,-\nu-2n)}{\Gamma(\nu+\frac{1}{2}+\frac{1}{2}j)\Gamma(\delta_j)}\,\frac{(1-\delta_j)_n}{(\nu+\frac{1}{2}+\frac{1}{2}j)_n},$$

$$\Lambda_2(n,j)=\frac{A_j(-2n-1,-\nu-2n-1)}{\Gamma(\nu+\frac{3}{2}+\frac{1}{2}j)\Gamma(\delta_{j+1}-\frac{1}{2})}\,\frac{(\frac{3}{2}-\delta_{j+1})_n}{(\nu+\frac{3}{2}+\frac{1}{2}j)_n}+\frac{B_j(-2n-1,-\nu-2n-1)}{\Gamma(\nu+1+\frac{1}{2}j)\Gamma(\delta_j-\frac{1}{2})}\,\frac{(\frac{3}{2}-\delta_j)_n}{(\nu+1+\frac{1}{2}j)_n},$$

and, for $-5\le j\le 5$, the coefficients in $\Lambda_1(n,j)$ are obtained from the $A_j(a,b)$ and $B_j(a,b)$ in Table 1 by replacing a by $-2n$ and b by $-\nu-2n$, and the coefficients in $\Lambda_2(n,j)$ by replacing a by $-2n-1$ and b by $-\nu-2n-1$. Similarly, (3.2) becomes

$$\begin{aligned} e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{(1-\nu+j)_n}\,L_n^{(\nu)}(x)&=\frac{2^{\nu}\pi^{\frac{1}{2}}\Gamma(1-\nu+j)}{\Gamma(1+\epsilon_j)}\left\{\sum_{n=9}^{\infty}\frac{(-x^2)^n}{n!}\,\frac{(1)_n\,\Lambda_3(n,j)}{(\frac{1}{2}+\frac{1}{2}\epsilon_j)_n(1+\frac{1}{2}\epsilon_j)_n}\right.\\ &\qquad\left.-\frac{2x}{1+\epsilon_j}\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}\,\frac{(1)_n\,\Lambda_4(n,j)}{(1+\frac{1}{2}\epsilon_j)_n(\frac{3}{2}+\frac{1}{2}\epsilon_j)_n}\right\}, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} \Lambda_3(n,j)&=\frac{A_j(-\nu-2n,-2n)}{\Gamma(1-\frac{1}{2}\nu+\frac{1}{2}j)\Gamma(-\frac{1}{2}\nu+\delta_{j+1})}\,\frac{(\frac{1}{2}\nu+1-\delta_{j+1})_n}{(1-\frac{1}{2}\nu+\frac{1}{2}j)_n}\\ &\qquad+\frac{B_j(-\nu-2n,-2n)}{\Gamma(\frac{1}{2}-\frac{1}{2}\nu+\frac{1}{2}j)\Gamma(-\frac{1}{2}\nu+\delta_j)}\,\frac{(\frac{1}{2}\nu+1-\delta_j)_n}{(\frac{1}{2}-\frac{1}{2}\nu+\frac{1}{2}j)_n},\\ \Lambda_4(n,j)&=\frac{A_j(-\nu-2n-1,-2n-1)}{\Gamma(\frac{3}{2}-\frac{1}{2}\nu+\frac{1}{2}j)\Gamma(-\frac{1}{2}\nu-\frac{1}{2}+\delta_{j+1})}\,\frac{(\frac{1}{2}\nu+\frac{3}{2}-\delta_{j+1})_n}{(\frac{3}{2}-\frac{1}{2}\nu+\frac{1}{2}j)_n}\\ &\qquad+\frac{B_j(-\nu-2n-1,-2n-1)}{\Gamma(1-\frac{1}{2}\nu+\frac{1}{2}j)\Gamma(-\frac{1}{2}\nu-\frac{1}{2}+\delta_j)}\,\frac{(\frac{1}{2}\nu+\frac{3}{2}-\delta_j)_n}{(1-\frac{1}{2}\nu+\frac{1}{2}j)_n}, \end{aligned}$$

and, for $-5\le j\le 5$, the coefficients in $\Lambda_3(n,j)$ are obtained from the $A_j(a,b)$ and $B_j(a,b)$ in Table 1 by replacing a by $-\nu-2n$ and b by $-2n$, and the coefficients in $\Lambda_4(n,j)$ by replacing a by $-\nu-2n-1$ and b by $-2n-1$.

4.1 Special cases of (4.1)

If we take $j=0$ in (4.1), we obtain, after a little simplification,

$$e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{(\nu+1)_n}\,L_n^{(\nu)}(x)={}_0F_1\left[\begin{matrix}-\\ \nu+1\end{matrix};-x^2\right]=x^{-\nu}\Gamma(\nu+1)J_\nu(2x) \tag{4.3}$$

for $\nu \neq -1, -2, \ldots$, where $J_\nu(z)$ is the Bessel function of the first kind. This is the corrected form of the result obtained by Exton in [4, Eq. (16)] and can be found in [5, p. 1038].

If we take $j = 1$ and $j = -1$ in (4.1), we obtain after some routine algebra the following results

$$e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{(\nu+2)_n} L_n^{(\nu)}(x) = {}_1F_2 \left[\begin{matrix} \frac{1}{2}\nu + \frac{1}{2} \\ \frac{1}{2}\nu + \frac{3}{2}, \nu + 1 \end{matrix} ; -x^2 \right] - \frac{x}{\nu+2} \, {}_1F_2 \left[\begin{matrix} \frac{1}{2}\nu + 1 \\ \frac{1}{2}\nu + 2, \nu + 2 \end{matrix} ; -x^2 \right] \tag{4.4}$$

for $\nu \neq -1, -2, \ldots$, and

$$\begin{aligned} e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{(\nu)_n} L_n^{(\nu)}(x) &= {}_0F_1 \left[\begin{matrix} - \\ \nu \end{matrix} ; -x^2 \right] + \frac{x}{\nu} \, {}_0F_1 \left[\begin{matrix} - \\ \nu + 1 \end{matrix} : -x^2 \right] \\ &= x^{1-\nu} \Gamma(\nu) \{ J_{\nu-1}(2x) + J_\nu(2x) \}, \end{aligned} \tag{4.5}$$

for $\nu \neq 0, -1, -2, \ldots$.

If we put $\nu = \pm\frac{1}{2}$ in (4.3) and (4.5) we obtain, upon evaluation of the resulting Bessel functions of half-integer orders in terms of trigonometric functions, the special summations

$$\sum_{n=0}^{\infty} \frac{x^n}{(\frac{1}{2})_n} L_n^{(-1/2)}(x) = e^x \cos(2x), \qquad \sum_{n=0}^{\infty} \frac{x^n}{(\frac{3}{2})_n} L_n^{(1/2)}(x) = e^x \frac{\sin(2x)}{2x},$$

$$\sum_{n=0}^{\infty} \frac{x^n}{(\frac{1}{2})_n} L_n^{(1/2)}(x) = e^x \{ \cos(2x) + \sin(2x) \},$$

and

$$\sum_{n=0}^{\infty} \frac{x^n}{(-\frac{1}{2})_n} L_n^{(-1/2)}(x) = e^x \{ (1-2x)\cos(2x) + 2x \sin(2x) \}.$$

We remark in passing that, since the Laguerre polynomials with degree $\nu = \pm\frac{1}{2}$ are related to the Hermite polynomials $H_n(x)$ by [1, p. 779]

$$L_n^{(-1/2)}(x^2) = \frac{(-1)^n}{2^{2n} n!} H_{2n}(x), \qquad L_n^{(1/2)}(x^2) = \frac{(-1)^n}{2^{2n} n! x} H_{2n+1}(x),$$

we obtain the summations

$$\sum_{n=0}^{\infty} \frac{(-x^2)^n}{(2n)!} H_{2n}(x) = e^{x^2} \cos(2x^2), \qquad \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} H_{2n+1}(x) = e^{x^2} \sin(2x^2),$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-x^2)^n}{(2n-1)!} H_{2n}(x) &= 2x^2 e^{x^2} \{ \cos(2x^2) - \sin(2x^2) \}, \\ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!} H_{2n+1}(x) &= 2x^2 e^{x^2} \{ \cos(2x^2) + \sin(2x^2) \}. \end{aligned}$$

4.2 Special cases of (4.2)

If we take $j = 0$ in (4.2), we obtain, after a little simplification,

$$e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{(1-\nu)_n} L_n^{(\nu)}(x) = {}_1F_2 \left[\begin{matrix} \frac{1}{2}\nu + \frac{1}{2} \\ \frac{1}{2}, 1 - \frac{1}{2}\nu \end{matrix} ; -x^2 \right] + \frac{2x\nu}{1-\nu} \, {}_1F_2 \left[\begin{matrix} \frac{1}{2}\nu + 1 \\ \frac{3}{2}, \frac{3}{2} - \frac{1}{2}\nu \end{matrix} ; -x^2 \right] \tag{4.6}$$

for $\nu \neq 1, 2, \ldots$. The result in (4.6) is the corrected form of that obtained by Exton in [4, Eqs. (14)].

If we take $j = 1$ and $j = -1$ in (4.2), we obtain after some routine algebra the following results

$$e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{(2-\nu)_n}\,L_n^{(\nu)}(x) = (1-\nu)\,{}_1F_2\left[\begin{matrix}\frac{1}{2}\nu+\frac{1}{2}\\ \frac{3}{2},1-\frac{1}{2}\nu\end{matrix};-x^2\right] + \nu\,{}_1F_2\left[\begin{matrix}\frac{1}{2}\nu+1\\ \frac{3}{2},\frac{3}{2}-\frac{1}{2}\nu\end{matrix};-x^2\right]$$
$$-\frac{x(1-\nu^2)}{2-\nu}\,{}_2F_3\left[\begin{matrix}1,\frac{1}{2}\nu+\frac{3}{2}\\ \frac{3}{2},2,2-\frac{1}{2}\nu\end{matrix};-x^2\right] + x\nu\,{}_2F_3\left[\begin{matrix}1,\frac{1}{2}\nu+1\\ \frac{3}{2},2,\frac{3}{2}-\frac{1}{2}\nu\end{matrix};-x^2\right] \tag{4.7}$$

for $\nu \neq 2, 3, \ldots$, and

$$e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{(-\nu)_n}\,L_n^{(\nu)}(x) = \frac{1}{2}{}_1F_2\left[\begin{matrix}\frac{1}{2}\nu+1\\ \frac{1}{2},\frac{1}{2}-\frac{1}{2}\nu\end{matrix};-x^2\right] + \frac{1}{2}{}_1F_2\left[\begin{matrix}\frac{1}{2}\nu+\frac{1}{2}\\ \frac{1}{2},-\frac{1}{2}\nu\end{matrix};-x^2\right]$$
$$-\frac{x(\nu+1)}{\nu}\,{}_1F_2\left[\begin{matrix}\frac{1}{2}\nu+\frac{3}{2}\\ \frac{3}{2},1-\frac{1}{2}\nu\end{matrix};-x^2\right] - x\,{}_1F_2\left[\begin{matrix}\frac{1}{2}\nu+1\\ \frac{3}{2},\frac{1}{2}-\frac{1}{2}\nu\end{matrix};-x^2\right] \tag{4.8}$$

for $\nu \neq 0, 1, 2, \ldots$. Similarly, other results can be obtained.

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