NEW PROPERTIES AND REPRESENTATIONS FOR MEMBERS OF THE POWER–VARIANCE FAMILY. II

Vladimir Vinogradov*, Richard B. Paris† and Olga Yanushkevichiene‡

Abstract: This is the continuation of Vinogradov, Paris, Yanushkevichiene (2012a) (see [34]). Members of the power-variance family of distributions became popular in stochastic modelling which necessitates a further investigation of their properties. Here, we establish Zolotarev duality of the refined saddlepoint-type approximations for all members of this family thereby providing an interpretation of the Letac–Mora reciprocity of the corresponding NEF's. Several illustrative examples are given. Subtle properties of related special functions are established.

1 Introduction

This is the second part of a series of two articles. The first part of our article quoted as reference [34] is referred to as [VPY1] throughout. The notation is consistent with, and the attached list of references complements that of [VPY1].

Our interest in the PVF is motivated by the elegant mathematical properties of this class (including their connections with particular special functions) as well as by their increasing popularity in stochastic modelling. For instance, the members of the *Poisson-exponential* EDM, which correspond to $p = 3/2$ and are discussed just below formula (3.23) in [VPY1] as well as in Example 4.3, emerge in the studies on the evolution of branching–fluctuating particle systems and their continuous–state limits. We refer to Vinogradov (2007a, 2007c, 2007d), and Hochberg and Vinogradov (2009) for more detail. The latter article discusses such topics as the temporal *forward evolution of the cluster structure* of such particle systems and their limits, and also their *backward evolution towards a Poisson field* (see pp. 256–257 therein). The latter phenomenon is consistent with the fact that specific subclasses of the Poisson-gamma class can sometimes "*degenerate*" into either a Poisson or a gamma distribution (compare Vinogradov (2004a, pp. 1022–1023) or Kaas (2005, p. 9)).

Although the following list of references is incomplete, it still gives some insight into how diverse is the area of applications of the PVF. Thus, *financial* applications were discussed, among others, by Lee and Whitmore (1993), Barndorff-Nielsen and Shephard (2001), and Vinogradov (2002, 2004b, 2008; see also the references therein) who applied Tweedie laws with $p \in (-\infty, 0] \cup (2, +\infty)$ for the description of the random movements of equities. Their use provides an advantage, as compared to the stable laws per se, since in contrast to stable distributions, the majority of the members of the PVF possess *finite* moments, whereas their scaling properties given by [VPY1, form. (3.28) and (3.31)] are quite comparable to those of the class of the stable distributions.

*Department of Mathematics, Ohio University, Athens, OH, 45701 USA, e-mail: vinograd@ohio.edu
†School of Computing, Engineering and Applied Mathematics, University of Abertay Dundee, Dundee, DD1 1HG, UK, e-mail: r.paris@abertay.ac.uk
‡Institute of Mathematics and Informatics, Vilnius University, Akademijos str., 4, LT-08663 Vilnius, Lithuania, e-mail: olga.januskeviciene@mii.vu.lt

In turn, the rapidly growing interest in the development of more adequate stochastic models for numerous sets of *clustered data* necessitates further studies of the delicate properties of the *Poisson-gamma subclass* of the Tweedie family, which corresponds to $p \in (1,2)$ and is characterized by [VPY1, form. (3.23)–(3.25)]. The Poisson-gamma class was originally introduced by Fisher and Cornish (1960, p. 223) to describe "*the total rainfall for a given period*" for many localities. In addition, we refer to Le Cam (1961, p. 167; see also the references therein) for an important special case and a comprehensive review of the early work pertaining to the *stochastic theory of precipitation*. More recently, Jørgensen and de Souza (1997), and Smyth and Jørgensen (2002) employed the Poisson-gamma subclass of the PVF to fit Property & Casualty Insurance data. The Tweedie distributions have also been used to address some problems in *biology* (cf., for example, Kendal (2002, 2007)), *genetics* (cf., for example, Kendal (2004)), and *medicine* (cf., for example, Kendal *et al.* (2000)). Thus, Kendal *et al.* (2000), Kendal (2002, 2004) demonstrated a good fit of members of Tweedie EDM's with $p \in (1,2)$ for numerous biological data as compared to Poisson distributions.

Hougaard (2000) considered some applications of the Tweedie family in survival analysis. The use of the members of the PVF in the time series and longitudinal data analyses for non-normal data is the subject of the papers by Jørgensen and Tsao (1999), and by Jørgensen and Song (1998, 2006, 2007). Kendal and Jørgensen (2011a, 2011b) discussed various relations between the Poisson-gamma distributions, the data clustering, weak convergence to members of the PVF and *Taylor's power law*, whereas Jørgensen *et al.* (2011, Subsec. 2.3) constructed a class of the self–similar–type stochastic processes with *dependent* increments, whose marginals belong to the PVF.

Next, we present a few references that consider applications of the PVF in such areas as *damage accumulation in complex structures* (cf., for example, Ditlevsen (1990, p. 334)), *nature preservation and ecology* (cf., for example, Friis-Hansen and Ditlevsen (2003, form. (17)), and *climatology* (cf., for example, Hasan and Dunn (2011)). It seems appropriate to comment on some of the empirical findings of the latter two articles. Thus, it appears that the empirically confirmed *exponential* law for the oil spills in Øresund and in Great Belt which was established by Friis-Hansen and Ditlevsen (2003, form. (17)) for their compound Poisson model is consistent with the possibility of deriving the *gamma approximation* for the compound Poisson-gamma class described above. At the same time, Hasan and Dunn (2011) demonstrated that for the amount of monthly rainfall in various locations in Australia, one can incur both gamma and the compound Poisson-gamma distributions. In turn, this necessitates the consideration of the *entire* Tweedie family, and not just a part of it. Additional information on possible applications of the Poisson-gamma subclass of the PVF can be found in Withers and Nadarajah (2011, p. 16).

For this Poisson-gamma class, several numerical estimates for the value(s) of the power parameter p have been derived. For instance, Kendal (2004) obtained that $p \approx 1.51$ for a particular set of biological data, whereas for the model considered by Hasan and Dunn (2011), the typical $p \approx 1.6$. Withers and Nadarajah (2011, Sec.'s 4–6) address such aspects of the statistical inference for the class of the Poisson-gamma distributions as the MLE, the construction of the method of moments estimators, and the assessment of their quality.

2 Some auxiliary results on reciprocity and special functions

The property of the reciprocity of a pair of NEF's is applicable in a rather general setting, and has already found numerous interpretations in the fluctuation theory of Lévy processes and random walks (cf., for example, Letac and Mora (1990, Th.'s 5.3 and 5.6), Kokonendji (2001, Th. 1 and comment below that theorem), Kokonendji and Khoudar (2006, Th. 2 and Cor. 3)). Recall that in the context of the Tweedie family, the reciprocity transformation is given by [VPY1, form (4.2)]. Note that a combination of [VPY1, form. (1.1) and (4.2)] stipulates that $\forall p \in (2,3)$, any Tweedie r.v. $Tw_p(\mu, \lambda)$ does not possess a reciprocal (compare Letac and Mora (1990, p. 24)).

Lemma 2.1 *Fix the arbitrary values of* $p \in \Delta \setminus (2,3)$, $\mu \in \Omega_p$ *and* $\lambda \in \mathbf{R}^1_+$. *Then*

$$\text{the collection} \quad \{\zeta_{p,\mu,\lambda}(s), \ -\zeta_{3-p,1/\mu,\lambda}(-s)\} \quad \text{is a pair of inverse functions.} \tag{2.1}$$

Proof of Lemma 2.1. It follows with some effort from Vinogradov (2004a, Prop. 1.1.i-vi). □

Remark 2.2 *The analytical property (2.1) stipulates a correspondence between the c.g.f.'s of the reciprocal pair of the members of the PVF whose values p and p′ of the power parameter satisfy [VPY1, form (4.2)] (compare Tweedie (1984, pp. 584–585), and Letac and Mora (1990, Th. 5.2)). Also, Lemma 2.1 (which refines slightly Letac and Mora (1990, Th. 5.2)) corrects a misprint made by Vinogradov (2004a, Prop. 1.1.vii).*

For $p \leq 0$, the already known probabilistic interpretation of the Letac–Mora reciprocity is expressed in terms of the law of the *first passage times* for the spectrally negative *Hougaard stochastic processes* (see Letac and Mora (1990, p. 25) or Vinogradov (2002, Th. 4.1 and Cor. 4.1)). It is relevant that the same probabilistic property is closely related to *Zolotarev duality*, which is specified in Proposition 2.3 and Corollary 2.4.

In the case of the *extreme* stable laws, Zolotarev's result on the duality of a pair of the p.d.f.'s of the stable distributions with parameters $\{\alpha \in (1,2];\ \beta = -1\}$ and $\{\alpha' := 1/\alpha \in [1/2,1);\ \beta' := 1\}$ can be given in an equivalent form as follows:

Proposition 2.3 *For arbitrary fixed values of* $p \in (-\infty, 0]$ *and* $\lambda \in \mathbf{R}^1_+$, *and* $y \in \mathbf{R}^1_+$,

$$f_{p,0,\lambda}(y) \equiv y^{\rho_p - 1} \cdot f_{3-p,\infty,\lambda}(y^{\rho_p}). \tag{2.2}$$

Proof of Proposition 2.3. The validity of (2.2) is obtained with some effort by combining Zolotarev (1986, form. (2.3.3)), [VPY1, form. (3.9)–(3.11)] with some calculus. □

A subsequent application of the "*exponential tilting transformation*" (which is employed for the derivation of [VPY1, form. (3.14)]) yields the following relationship between the p.d.f.'s of the members of a reciprocal pair of Tweedie r.v.'s $Tw_p(\mu, \lambda)$ and $Tw_{p'}(\mu', \lambda')\}$, whose parameters are related by means of [VPY1, form. (4.2)], in the case where these r.v.'s are obtained by an exponential tilting of the extreme stable laws $Tw_p(0, \lambda)$ with $p \leq 0$, and $Tw_{p'}(+\infty, \lambda')$ with $p' = 3 - p \geq 3$, respectively. (The latter r.v.'s are characterized by [VPY1, form. (3.34)–(3.35)].)

Corollary 2.4 *For arbitrary fixed values of* $p \in (-\infty, 0] \cup [3, +\infty)$, $\mu \in \Omega_p$, $\lambda \in \mathbf{R}^1_+$, *and* $y \in \mathbf{R}^1_+$,

$$f_{p,\mu,\lambda}(y) \equiv y^{\rho_p - 1} \cdot \exp\{-\theta_p \cdot (y-1) + \mathcal{A}_p \cdot (y^{\rho_p} - 1)\} \cdot f_{3-p,1/\mu,\lambda}(y^{\rho_p}). \tag{2.3}$$

Proof of Corollary 2.4. Combine [VPY1, form. (3.7), (3.8), (3.14)] with (2.2). □

In Section 3, we will explain why Zolotarev–type duality per se does not hold in the Poisson-gamma case, for which $p \in (1,2)$ (with the exception of the trivial, *self-reciprocal* case of $p = 3/2$ or $\rho = 1$ considered in Remark 3.2.i). However, Theorem 3.1.i reveals in particular that if one considers the class of the Poisson-gamma distributions then an analogue of Zolotarev–type duality remains valid for the refined saddlepoint–type approximations (with an arbitrary fixed number of refining terms of decreasing magnitude) for the densities of the absolutely continuous components of these distributions. This partly supports the statement made by Letac and Mora (1990, p. 24, lines 19–20).

It is known that the reciprocity of Tweedie EDM's with the values of the reciprocal pair $\{p, p'\}$ comprising the two-point set $\{1; 2\}$ can be interpreted in terms of the well-known result on the *Poisson flow of arrivals in the case of exponential inter-arrival times.* In addition, Theorem 3.1.ii provides the new interpretation of this relationship between two specific c.g.f.'s in terms of an identity relationship which involves the corresponding Poincaré series (see (3.2)).

The remainder of this section contains relevant information on particular special functions.

Definition 2.5 *The general Wright function with* $\mathfrak{p}$ *numeratorial and* $\mathfrak{q}$ *denominatorial gamma functions, where* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are non-negative integers, is given by*

$$ {}_{\mathfrak{p}}\Psi_{\mathfrak{q}}(z) := \sum_{k=0}^{\infty} \frac{\prod_{r=1}^{\mathfrak{p}} \Gamma(a_r \cdot k + c_r)}{\prod_{r=1}^{\mathfrak{q}} \ \Gamma(b_r \cdot k + d_r)} \cdot \frac{z^k}{k!} \ , \tag{2.4} $$

where the parameters a_r*,* b_r *are real and positive,* c_r*,* d_r *are arbitrary real constants and it is supposed that* $a_r \cdot k + c_r \notin \{0, -1, -2, \ldots\}$ $(k \in \{0, 1, 2, \ldots\}; 1 \leq r \leq \mathfrak{p})$.

Numerous asymptotic representations for the general Wright function (2.4) and their heuristic interpretation are isolated into Subsection 5.1. See formulas (5.3) and (5.7)–(5.8) therein.

In its turn, ${}_{\mathfrak{p}}\Psi_{\mathfrak{q}}(z)$ belongs to a wider class of *Fox H*-functions. We refer the reader to Janson (2010) for several additional examples of the probability distributions that are closely related to H-functions, and to Schneider (1986, form. (2.13) and (2.16)).

The specific parts of the proof of [VPY1, Lm. 4.6.i], which is given in Appendix A, are isolated into the separate Subsections 5.1 and 5.2. In addition, the methodology employed in the proof of that lemma is related to the so-called "*Stokes phenomenon*" which is well familiar to specialists in the Theory of Special Functions; see, for example, Paris (2011a, Sec. 1.7). As with any other developed branch of mathematics, the Theory of Special Functions has its own tool-box comprised of *theory-specific* methods and results. According to Paris (2011a, p. 78), "the root cause" of this fundamental phenomenon "is a consequence of *asymptotically approximating a given function, which possesses a certain multi-valued structure, in terms of approximants of a different multi-valued structure.*" It is relevant that such approximants, which the specialists in the Theory of Special Functions frequently call the *algebraic* and the *exponential* (or the *exponentially small*) expansions, respectively, independently emerged in several articles and monographs on analytical Probability Theory without any reference to the Stokes phenomenon. [VPY1, Lm. 4.6.i] delineates this important connection, thereby paving the road for further use of the Stokes phenomenon in Probability Theory and Theoretical Statistics. That lemma plays an essential role in the proof of [VPY1, Th. 4.8], which is given in Appendix B.

3 Extending Zolotarev–Type Duality for Tweedie EDM's

First, observe that an *identity-type* result which would be analogous to the identity (2.3) cannot hold for values of $p \in (1, 2)$ (except for the self-reciprocal case of $p = 3/2$, which is considered in Remark 3.2.i). This is because [VPY1, form. (3.26)] implies that for $1 < p < 2$, the ratio of the functions $f_{p,\mu,\lambda}(y)$ and $y^{\rho_p - 1} \cdot f_{3-p,1/\mu,\lambda}(y^{\rho_p})$ (multiplied by a particular exponential factor) does not converge to 1 as $y \downarrow 0$.

At the same time, the following result, which stipulates an analogue of Zolotarev duality for the members of the PVF for which both p and p' belong to $[1, 2]$, holds for the *Poincaré series* which correspond to the respective reciprocal pairs which are contained in this family. In view of (2.3), the same assertion on a relationship between the *Poincaré series* trivially holds in the case where both p and p' belong to $(-\infty, 0] \cup [3, +\infty)$. Also, in part (ii) of the following assertion we choose to use ρ_2 (which is equal to 0 by [VPY1, form. (2.6)]) instead of just writing 0 in order to stress the similarity between (2.3), (3.1) and (3.2).

Theorem 3.1 *Fix $y \in \mathbf{R}^1_+$. Then*
(i) *For arbitrary fixed values of $p \in \Delta \setminus (\{1\} \cup [2, 3))$, $\mu \in \Omega_p$ and $\lambda \in \mathbf{R}^1_+$, the following Poincaré series are identical:*

$$\mathbf{F}_{p,\mu}(y, \lambda) \equiv y^{\rho_p - 1} \cdot \exp\{-\theta_p \cdot (y - 1) + \mathcal{A}_p \cdot (y^{\rho_p} - 1)\} \cdot \mathbf{F}_{3-p,1/\mu}(y^{\rho_p}, \lambda). \tag{3.1}$$

(ii) *For arbitrary fixed $\mu \in \mathbf{R}^1_+$ and $\lambda \in y^{-1} \cdot \mathbf{N}$, the following Poincaré series are identical:*

$$\mathbf{F}_{2,\mu}(y, \lambda) \equiv y^{\rho_2 - 1} \cdot \exp\{-\theta_2 \cdot (y - 1) + \lambda \cdot \log\ y\} \cdot \mathbf{F}_{1,1/\mu}(y^{\rho_2}, \lambda). \tag{3.2}$$

Proof of Theorem 3.1. **(i)** The validity of (3.1) for $p \in (-\infty, 0] \cup [3, +\infty)$ follows from (2.3). For $p \in (1, 2)$, (3.1) follows from the combination of [VPY1, Prop. 4.1.ii, Th. 4.8.i2 and Cor. 4.9.ii].
(ii) The validity of (3.2) follows from [VPY1, Prop. 4.1.iii and Th. 4.8, parts (i1) and (i3)]. □

Remark 3.2 **(i)** *In the case where $p = 3/2$, a stronger exact result holds, as compared to the identity (3.1) for the corresponding Poincaré series. In this self-reciprocal case, $\rho_{3/2} = 1$. A subsequent combination of [VPY1, form. (3.7)–(3.8), (3.25) and (4.4)] yields the following identity:*

$$f_{3/2,\mu,\lambda}(y) \equiv y^{1-1} \cdot \exp\{-\theta_{3/2} \cdot (y - 1) + \mathcal{A}_{3/2} \cdot (y^1 - 1)\} \cdot f_{3/2,1/\mu,\lambda}(y),$$

where $y \in \mathbf{R}^1_+$ (compare (2.3)).
(ii) *Since $y^{\rho_2} \equiv 1$, the Poincaré series which emerges on the right-hand side of (3.2) does not depend on y. Also, it is straightforward to verify that for arbitrary fixed $\mu \in \mathbf{R}^1_+$ and $\lambda \in \mathbf{R}^1_+$, and for a fixed $y \in \mathbf{R}^1_+$, $\lim_{p \to 2}\{\mathcal{A}_p \cdot (y^{\rho_p} - 1)\} = \lambda \cdot \log\ y$. This limiting result stresses a connection between (3.1) and (3.2).*

4 Special Cases

In this section we concentrate on three special cases in which the function $\phi(\rho, 0, z)$ can be expressed in terms of standard special functions (see (4.1), (4.9) and (4.16)). The first two cases were discussed in a related setting in Zolotarev (1986, form. (2.8.33) and (2.8.31), respectively), whereas the third case was considered, among others, by Vinogradov (2007a, form. (2.5)–(2.6)) and Hochberg and Vinogradov (2009, form. (2.4)–(2.5)). Also, (4.6), (4.14) and (4.22) provide closed-form expressions for the refined saddlepoint-type approximation [VPY1, form. (4.17)] in the cases where p takes on the values 4, 5/2 and 3/2, respectively. Examples 4.1 and 4.2 illustrate part (ii3) of [VPY1, Th. 4.8], whereas Example 4.3 addresses the simplest special case where the conditions of part (ii2) of this theorem are fulfilled.

Example 4.1 *Suppose that $\rho\ (= -\alpha) = -2/3$ or $p = 4$. It can be shown with some effort by combining [VPY1, form. (3.10)] with Zolotarev (1986, form. (2.8.32)) that for complex z,*

$$\phi(-2/3, 0, -z) \equiv \sqrt{\frac{3}{\pi}} \cdot \exp\left(-\frac{2}{27} \cdot z^3\right) \cdot W_{1/2,1/6}\left(\frac{4}{27} \cdot z^3\right). \tag{4.1}$$

Hereinafter, $W_{k,m}(z)$ denotes the Whittaker function; see Abramowitz and Stegun (1965, p. 505). See also Gorenflo et al. (1999, p. 390) for a similar representation.

By [VPY1, form. (3.14)], the p.d.f. $f_{4,\mu,\lambda}(x)$ of Tweedie r.v. $Tw_4(\mu,\lambda)$ is as follows:

$$f_{4,\mu,\lambda}(x) = x^{-1} \cdot \phi(\rho_4, 0, B_{4,\lambda} \cdot x^{\rho_4}) \cdot e^{-\theta_4 \cdot x - \mathcal{A}_4}. \tag{4.2}$$

Here, the argument $x > 0$, and the parameters which emerge in (4.2) are specified below:

$$\rho_4 = -2/3, \quad B_{4,\lambda} = -\lambda^{1/3} \cdot 3^{2/3}/2, \quad \theta_4 = \lambda/(3 \cdot \mu^3), \quad \mathcal{A}_4 = -\lambda/(2 \cdot \mu^2).$$

A combination of (4.1)–(4.2) with [VPY1, form. (4.15)] allows us to construct the Poincaré series for the p.d.f. $f_{4,\mu,\lambda}(x)$ as $x \downarrow 0$ in closed form as well as to specify the values of the Zolotarev polynomials in the case where $\rho = -2/3$ (see (4.4)–(4.6) and (4.7)), respectively). In particular, an application of [VPY1, form. (4.15)] shows that as $z \to +\infty$,

$$\phi(-2/3, 0, -z) \sim \frac{2z^{3/2}}{3 \cdot \sqrt{\pi}} \cdot \exp\left(-\frac{4z^3}{27}\right) \cdot \sum_{k=0}^{\infty} \mathcal{Z}_k\left(-\frac{2}{3}\right) \cdot (8z^3/81)^{-k}. \tag{4.3}$$

In view of the closed-form representations for the Zolotarev polynomials given in [VPY1, Remark 4.3.ii], in the case where $\rho = -2/3$ the first few have the following values:

$$\mathcal{Z}_0\left(-\frac{2}{3}\right) = 1, \quad \mathcal{Z}_1\left(-\frac{2}{3}\right) = \frac{1}{2 \cdot 3^3}\ , \quad \mathcal{Z}_2\left(-\frac{2}{3}\right) = -\frac{5 \cdot 7}{2^3 \cdot 3^6}\ , \quad \mathcal{Z}_3\left(-\frac{2}{3}\right) = \frac{5 \cdot 7 \cdot 11 \cdot 13}{2^4 \cdot 3^{10}}\ , \ldots$$

For simplicity, assume for now that $\mu = \infty$ and $\lambda = 1$. Setting $z := 3^{2/3}/(2x^{2/3})$, we combine (4.2) and (4.3) to obtain that as $x \downarrow 0$,

$$f_{4,\infty,1}(x) \sim \frac{x^{-2}}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{6x^2}\right) \cdot \sum_{k=0}^{\infty} \mathcal{Z}_k\left(-\frac{2}{3}\right) \cdot (9x^2)^k. \tag{4.4}$$

To consider the general case where the values of $\mu \in (0, +\infty]$ and $\lambda \in \mathbf{R}^1_+$ are arbitrary and fixed, we apply a slightly different approach which involves the Whittaker-function representation (4.1). Evidently, its combination with (4.2) yields that for $x \in \mathbf{R}^1_+$,

$$f_{4,\mu,\lambda}(x) = \sqrt{\frac{3}{\pi}} \cdot x^{-1} \cdot \exp\left(-\frac{\lambda}{12 \cdot x^2} + \frac{\lambda}{2 \cdot \mu^2} - \frac{\lambda \cdot x}{3 \cdot \mu^3}\right) \cdot W_{1/2,1/6}\left(\frac{\lambda}{6 \cdot x^2}\right). \tag{4.5}$$

The following Poincaré series for $W_{k,m}(z)$ for large z is given in Gradshteyn and Ryzhik (2007, form. (9.229)):

$$W_{k,m}(z) \sim z^k \cdot e^{-z/2} \cdot \left\{1 + \frac{\{m^2 - (k-\frac{1}{2})^2\}}{1! \cdot z} + \frac{\{m^2 - (k-\frac{1}{2})^2\} \cdot \{m^2 - (k-\frac{3}{2})^2\}}{2! \cdot z^2} + \cdots\right\}.$$

In turn, a combination of the above formula with (4.5) implies that as $x \downarrow 0$,

$$f_{4,\mu,\lambda}(x) \sim \frac{\sqrt{\lambda}}{\sqrt{2\pi \cdot x^4}} \cdot \exp\left(-\frac{\lambda}{6 \cdot x^2} - \frac{\lambda \cdot x}{3 \cdot \mu^3} + \frac{\lambda}{2 \cdot \mu^2}\right) \cdot \sum_{j=0}^{\infty} (-1)^j \cdot \left(\frac{1}{6}\right)_j \cdot \left(-\frac{1}{6}\right)_j \cdot (6x^2/\lambda)^j. \tag{4.6}$$

Here we have utilized the Pochhammer-symbol notation introduced in [VPY1, form. (2.11)]. The representation (4.6) is seen to be consistent with (4.4) (which pertains to the case where $\mu = \infty$ and $\lambda = 1$). Finally, a comparison of the Poincaré series which emerges on the right-hand side of (4.6) with that from (4.4) enables one to evaluate all Zolotarev polynomials in the case when $\rho = -2/3$. Thus, $\forall j \in \mathbf{Z}_+$

$$\mathcal{Z}_j(-2/3) = (-2/3)^j \cdot (1/6)_j \cdot (-1/6)_j. \tag{4.7}$$

It is worth mentioning that a combination of the results presented in Example 4.1 with [VPY1, form. (3.14) and (4.4)] shows that for $x \in \mathbf{R}^1$,

$$f_{-1,\mu,\lambda}(x) \equiv \sqrt{\frac{3}{\pi}} \cdot x^{-1} \cdot \exp\left(-\frac{\lambda \cdot x^3}{12} + \lambda \cdot \mu^2 \cdot x/2 - \lambda \cdot \mu^3/3\right) \cdot W_{1/2,1/6}\left(\frac{\lambda \cdot x^3}{6}\right). \tag{4.8}$$

Here, $\mu \in [0, +\infty)$ and $\lambda \in \mathbf{R}^1_+$ are fixed. Moreover, the Poincaré series for the p.d.f. $f_{-1,\mu,\lambda}(x)$ as $x \to +\infty$ with its terms expressed in the closed form (involving the Pochhammer-symbol notation) can be easily constructed by combining (2.3) with (4.6) and would illustrate [VPY1, Th. 4.8.ii2]. We leave this to the reader, although the consideration of a closely related subclass of the extreme stable laws with index α $(= \alpha_{-1}) = 3/2$ and skewness $\beta = -1$ was overlooked in both Zolotarev (1986, p. 159) and Uchaikin and Zolotarev (1999, Sec. 6.6).

Example 4.2 *Suppose that $\rho = -1/3$ or $p = 5/2$. It can be shown with some effort by combining [VPY1, form. (3.10)] with Zolotarev (1986, form. (2.8.31)) that for complex z,*

$$\phi(-1/3, 0, -z) \equiv \frac{z^{3/2}}{3\pi} \cdot K_{1/3}\left(\frac{2 \cdot z^{3/2}}{3 \cdot \sqrt{3}}\right), \tag{4.9}$$

where $K_\nu(z)$ denotes the modified Bessel function of the second kind (or MacDonald function).

Next, it follows from [VPY1, form. (4.15)] that as $z \to +\infty$,

$$\phi(-1/3, 0, -z) \sim \frac{3^{-1/4} \cdot z^{3/4}}{2\sqrt{\pi}} \cdot \exp\left(-\frac{2 \cdot z^{3/2}}{3 \cdot \sqrt{3}}\right) \cdot \sum_{k=0}^{\infty} \mathcal{Z}_k\left(-\frac{1}{3}\right) \cdot (2z^{3/2}/(9 \cdot \sqrt{3}))^{-k}. \tag{4.10}$$

Observe that [VPY1, Remark 4.3.ii] implies that in the case when $\rho = -2/3$ the first few Zolotarev polynomials are as follows:

$$\mathcal{Z}_0\left(-\frac{1}{3}\right) = 1, \mathcal{Z}_1\left(-\frac{1}{3}\right) = -\frac{5}{2^3 \cdot 3^3}, \mathcal{Z}_2\left(-\frac{1}{3}\right) = \frac{5 \cdot 7 \cdot 11}{2^7 \cdot 3^6}, \mathcal{Z}_3\left(-\frac{1}{3}\right) = -\frac{5 \cdot 7 \cdot 11 \cdot 13 \cdot 17}{2^{10} \cdot 3^{10}}, \ldots$$

By analogy with Example 4.1, we now proceed to construct the Poincaré series for the p.d.f. $f_{5/2,\mu,\lambda}(x)$ as $x \downarrow 0$ in closed form and also to specify the values of the Zolotarev polynomials in the case when $\rho = -1/3$ (see (4.14) and (4.15), respectively).

Now, we apply [VPY1, form. (3.14)] to conclude that the p.d.f. $f_{5/2,\mu,\lambda}(x)$ of the r.v. $Tw_{5/2}(\mu, \lambda)$ takes on the following form:

$$f_{5/2,\mu,\lambda}(x) = x^{-1} \cdot \phi(\rho_{5/2}, 0, B_{5/2,\lambda} \cdot x^{\rho_{5/2}}) \cdot e^{-\theta_{5/2} \cdot x - \mathcal{A}_{5/2}}. \tag{4.11}$$

Here, $x > 0$, and the parameters which emerge on the right-hand side of (4.11) are as follows:

$$\rho_{5/2} = -1/3, \quad B_{5/2,\lambda} = -12^{1/3} \cdot \lambda^{2/3}, \quad \theta_{5/2} = 2 \cdot \lambda/(3 \cdot \mu^{3/2}), \quad \mathcal{A}_{5/2} = -2 \cdot \lambda/\sqrt{\mu}.$$

For simplicity, for now we will concentrate on the case characterized by $\mu = \infty$ and $\lambda = 1$. Let us use (4.10) setting the argument z of the Wright function $\phi(-1/3, 0, -z)$, which emerges in that formula, to be equal to $2 \cdot (3/2)^{1/3} \cdot x^{-1/3}$ (compare to (4.11) and the above expression where the parameters emerging in (4.11) are specified). After some algebra, we find that as $x \downarrow 0$,

$$\begin{aligned} f_{5/2,\infty,1}(x) &\sim \frac{\exp[-\frac{4}{3} \cdot x^{-1/2}]}{\sqrt{2\pi \cdot x^{5/2}}} \cdot \sum_{j=0}^{\infty} \mathcal{Z}_j(-1/3) \cdot (3/4)^j \cdot x^{j/2} = \frac{\exp[-\frac{4}{3} \cdot x^{-1/2}]}{\sqrt{2\pi \cdot x^{5/2}}} \\ &\times \left(1 - \frac{5}{3 \cdot 2^5} x^{1/2} + \frac{5 \cdot 7 \cdot 11}{3^2 \cdot 2^{11}} x - \frac{5 \cdot 7 \cdot 11 \cdot 13 \cdot 17}{3^4 \cdot 2^{16}} x^{3/2} + \frac{5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23}{3^5 \cdot 2^{23}} x^2 - \cdots\right). \end{aligned} \tag{4.12}$$

In the general case where $\mu \in (0, +\infty]$ and $\lambda \in \mathbf{R}^1_+$ are arbitrary and fixed, a combination of (4.9) and (4.11) implies that for $x \in \mathbf{R}^1_+$,

$$f_{5/2,\mu,\lambda}(x) = \frac{2 \cdot \lambda}{\sqrt{3} \cdot \pi \cdot x^{3/2}} \cdot \exp\Big[- \frac{2 \cdot \lambda \cdot x}{3 \cdot \mu^{3/2}} + \frac{2 \cdot \lambda}{\mu^{1/2}}\Big] \cdot K_{1/3}\Big(\frac{4 \cdot \lambda}{3 \cdot \sqrt{x}}\Big). \tag{4.13}$$

The implementation of an approach parallel to that used in Example 4.1 now requires the consideration of the following Poincaré series for $K_{1/3}(z)$ for large values of z, which can be found in Abramowitz and Stegun (1965, form. (9.7.2)):

$$K_{1/3}(z) \sim \sqrt{\pi/(2z)} \cdot e^{-z}$$
$$\times \Big\{ 1 + \frac{4/9 - 1}{2^3 \cdot z} + \frac{(4/9-1) \cdot (4/9-9)}{2! \cdot (2^3 \cdot z)^2} + \frac{(4/9-1) \cdot (4/9-9) \cdot (4/9-25)}{3! \cdot (2^3 \cdot z)^3} + \dots \Big\}.$$

It then follows from (4.13) that as $x \downarrow 0$,

$$\begin{aligned} f_{5/2,\mu,\lambda}(x) \sim \frac{\sqrt{\lambda}}{\sqrt{2\pi \cdot x^{5/2}}} \cdot \exp\Big[- \frac{4 \cdot \lambda}{3} x^{-1/2} - \frac{2 \cdot \lambda \cdot x}{3 \cdot \mu^{3/2}} + \frac{2 \cdot \lambda}{\sqrt{\mu}}\Big] \\ \times \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \cdot \Big(\frac{3}{2^3 \cdot \lambda}\Big)^j \cdot (1/6)_j \cdot (5/6)_j \cdot x^{j/2}. \end{aligned} \tag{4.14}$$

It is evident that (4.14) is consistent with (4.12). Similar to Example 4.1, we now derive the following closed-form expression (in terms of the Pochhammer symbols) for the Zolotarev polynomials in the case when $\rho = -1/3$. Thus, it follows by comparison of (4.12) with (4.14) that $\forall j \in \mathbf{Z}_+$,

$$\mathcal{Z}_j(-1/3) = \frac{(-1/2)^j}{j!} \cdot (1/6)_j \cdot (5/6)_j. \tag{4.15}$$

Example 4.3 *Suppose that $\rho = 1$ or $p = 3/2$. The validity of the following identity*

$$\phi(1, 0, z) \equiv z^{-1/2} \cdot I_1(2 \cdot z^{1/2}), \tag{4.16}$$

$\forall z \in \mathbb{C} \setminus \{0\}$ can be established with a little effort; it is then trivially extended by continuity to be valid at the origin. Hereinafter, $I_1(\cdot)$ denotes the modified Bessel function of the first kind; see Abramowitz and Stegun (1965, form. (9.6.6) and (9.6.10)).

By Fisher and Cornish (1960, p. 222), the law of the r.v. $Tw_{3/2}(\mu, \lambda)$ has an absolutely continuous component in $\mathbf{R}^1_+$ whose density $f_{3/2,\mu,\lambda}(x)$ admits the following representation:

$$f_{3/2,\mu,\lambda}(x) = \frac{2 \cdot \lambda}{\sqrt{x}} \cdot \exp\big\{ -\theta_{3/2} \cdot (x + \mu) \big\} \cdot I_1(4 \cdot \lambda \cdot \sqrt{x}) \tag{4.17}$$

(compare Vinogradov (2007a, form. (2.6))). In addition, it follows from [VPY1, form. (3.25)] that

$$f_{3/2,\mu,\lambda}(x) = x^{-1} \cdot \phi(\rho_{3/2}, 0, B_{3/2,\lambda} \cdot x^{\rho_{3/2}}) \cdot e^{-\theta_{3/2} \cdot x - \mathcal{A}_{3/2}}. \tag{4.18}$$

Here, $x > 0$, and the parameters which emerge on the right-hand side of (4.18) are as follows: $\rho_{3/2} = 1,\ B_{3/2,\lambda} = 4 \cdot \lambda^2,\ \theta_{3/2} = 2 \cdot \lambda / \sqrt{\mu},\ \mathcal{A}_{3/2} = 2 \cdot \Phi_{3/2} = 2 \cdot \lambda \cdot \sqrt{\mu}$. It is straightforward to demonstrate that formulas (4.16), (4.17) and (4.18) are consistent.

Next, a combination of (4.16) with the Poincaré series for $I_1(z)$ given in Abramowitz and Stegun (1965, form. (9.7.1)) yields that as $x \to +\infty$,

$$\begin{aligned}\phi(1,0,4\lambda^2\cdot x) &\sim \sqrt{\frac{\lambda}{2\pi}}\cdot x^{1/4}\cdot\exp[4\lambda\cdot\sqrt{x}]\cdot\sum_{k=0}^{\infty}\frac{(-\frac{1}{2})_k\cdot(\frac{3}{2})_k}{k!\cdot 2^{3k}\cdot\lambda^k}\cdot x^{-k/2}\\ &= \sqrt{\frac{\lambda}{2\pi}}\cdot x^{1/4}\cdot\exp[4\lambda\cdot\sqrt{x}]\cdot\left\{1-\frac{3}{2^5\cdot\lambda\cdot\sqrt{x}}-\frac{3\cdot 5}{2^{11}\cdot\lambda^2\cdot x}+\cdots\right\}.\end{aligned} \tag{4.19}$$

At the same time, it easily follows from [VPY1, form. (3.14)] (with $z := 4\lambda^2\cdot x$) that as $x \to +\infty$,

$$\phi(1,0,4\lambda^2\cdot x) \sim \left(\frac{\lambda}{2\pi}\right)^{1/2}\cdot x^{1/4}\cdot\exp[4\lambda\cdot\sqrt{x}]\cdot\sum_{k=0}^{\infty}\mathcal{Z}_k(1)\cdot(4\lambda\cdot\sqrt{x})^{-k}. \tag{4.20}$$

Similar to the previous examples, [VPY1, Remark 4.3.ii] yields that for $\rho = 1$, the first few Zolotarev polynomials are as follows: $\mathcal{Z}_0(1) = 1$, $\mathcal{Z}_1(1) = -3/2^3$, $\mathcal{Z}_2(1) = -3\cdot 5/2^7$, $\mathcal{Z}_3(1) = -3\cdot 5\cdot 7/2^{10}$.

It is evident that (4.19) and (4.20) are consistent. Their comparison implies that $\forall j \in \mathbf{Z}_+$,

$$\mathcal{Z}_j(1) = \frac{(-\frac{1}{2})_j\cdot(\frac{3}{2})_j}{2^j\cdot j!}. \tag{4.21}$$

Finally, a combination of (4.18), (4.20) and (4.21) yields that as $x \to +\infty$,

$$f_{3/2,\mu,\lambda}(x) \sim \sqrt{\frac{\lambda}{2\pi}}\cdot x^{-3/4}\cdot\exp\{-\theta_{3/2}\cdot(\sqrt{x}-\sqrt{\mu})^2\}\cdot\sum_{k=0}^{\infty}\frac{(-\frac{1}{2})_k\cdot(\frac{3}{2})_k}{k!\cdot 2^{3k}\cdot\lambda^k}\cdot x^{-k/2}. \tag{4.22}$$

Note that the main term of the Poincaré series which emerges on the right-hand side of (4.22) was previously given by Vinogradov (2007a, form. (2.7)).

5 Appendix A. Proof of [VPY1, Lemma 4.6.i]

We will demonstrate that the coefficients of the Poincaré series for the function $\phi(\rho,0,z)$ when $\rho > 0$ and argument $z \to +\infty$ have the same form as the respective coefficients — the Zolotarev polynomials $\mathcal{Z}_k(\rho)$ — of the Poincaré series for $\phi(\rho,0,z)$ when $\rho \in (-1,0)$ and argument $z \to -\infty$. Recall that these series are given in [VPY1, form. (4.14) and (4.15)], respectively.

From Wright (1935a, pp. 257–258), the function $\phi(\rho,0,z)$ admits a Poincaré series of the form [VPY1, form. (4.14)] when $\rho > 0$ and $z \to +\infty$, with the Zolotarev polynomials $\mathcal{Z}_k(\rho)$ replaced by certain (potentially different) functions of ρ. For now, we shall denote these coefficients by $\widetilde{\mathcal{Z}}_k(\rho)$. Wright (1935a, p. 258) stipulated that $\widetilde{\mathcal{Z}}_1(\rho) \equiv \mathcal{Z}_1(\rho) \equiv -(\rho+2)\cdot(2\rho+1)/24$ and described the following algorithm for the derivation of these coefficients. Fix an arbitrary $\rho \in \mathbf{R}^1_+$ and consider the following auxiliary function of argument $v \in (0,+\infty)$, which is indexed by ρ:

$$\mathcal{G}_\rho(v) := \left\{1+\frac{\rho+2}{3}\cdot v+\frac{(\rho+2)\cdot(\rho+3)}{3\cdot 4}\cdot v^2+\ldots\right\}^{1/2}.$$

For non-negative integer m, denote the coefficient of v^{2m} in the Taylor series expansion of the function $\mathcal{G}_\rho(v)^{-2m-1}$ in ascending powers of v about zero by $\mathcal{K}_m(\rho)$. Then

$$\widetilde{\mathcal{Z}}_k(\rho) = \mathcal{K}_m(\rho)\cdot 2\pi\cdot((\rho+1)/2)^{m+1/2}/\Gamma(m+1/2). \tag{5.1}$$

However, we shall employ a different algorithm for the determination of these coefficients which is explained in Subsection 5.1. Our use of an alternative method is quite justifiable, since it employs the machinery of the Theory of Special Functions, which was already developed for a wider class introduced by Definition 2.5.

5.1 Asymptotics of the General Wright Function ${}_{\mathfrak{p}}\Psi_{\mathfrak{q}}(z)$

First, we introduce some notation. Consider the following parameters associated with (2.4):

$$\kappa := 1 + \sum_{r=1}^{\mathfrak{q}} b_r - \sum_{r=1}^{\mathfrak{p}} a_r, \quad h := \prod_{r=1}^{\mathfrak{p}} a_r^{a_r} \cdot \prod_{r=1}^{\mathfrak{q}} b_r^{-b_r}, \quad \vartheta := \sum_{r=1}^{\mathfrak{p}} c_r - \sum_{r=1}^{\mathfrak{q}} d_r + \tfrac{1}{2} \cdot (\mathfrak{q} - \mathfrak{p}),$$

$$\vartheta' := 1 - \vartheta, \quad A := (2\pi)^{(\mathfrak{p}-\mathfrak{q})/2} \cdot \kappa^{-\vartheta - 1/2} \cdot \prod_{r=1}^{\mathfrak{p}} a_r^{c_r - 1/2} \cdot \prod_{r=1}^{\mathfrak{q}} b_r^{-d_r + 1/2}. \tag{5.2}$$

If a_r and b_r are such that $\kappa > 0$ then it follows by the ratio test that the function ${}_{\mathfrak{p}}\Psi_{\mathfrak{q}}(z)$ is uniformly and absolutely convergent for all finite (complex) z.

The asymptotic expansion of ${}_{\mathfrak{p}}\Psi_{\mathfrak{q}}(z)$ for large complex z is discussed in Wright (1935b) and Braaksma (1963); see also Paris and Kaminski (2001, pp. 55–58). In particular, these results stipulate that for $0 < \kappa < 2$,

$${}_{\mathfrak{p}}\Psi_{\mathfrak{q}}(z) \sim \begin{cases} E_{\mathfrak{p},\mathfrak{q}}(z) + H_{\mathfrak{p},\mathfrak{q}}(z \cdot e^{\mp\pi i}) & \text{in} \quad |\arg z| \le \tfrac{1}{2} \cdot \pi\kappa \\ H_{\mathfrak{p},\mathfrak{q}}(z \cdot e^{\mp\pi i}) & \text{in} \quad |\arg(-z)| < \tfrac{1}{2} \cdot \pi(2-\kappa) \end{cases} \tag{5.3}$$

as $|z| \to \infty$, where the upper or lower signs are chosen according as z lies in the upper or lower half-plane of $\mathbb{C}$, respectively. The quantity $E_{\mathfrak{p},\mathfrak{q}}(z)$ denotes the *exponential expansion* defined by

$$E_{\mathfrak{p},\mathfrak{q}}(z) := A \cdot X^{\vartheta} \cdot e^X \cdot \sum_{j=0}^{\infty} C_j(\rho) \cdot X^{-j}, \qquad X := \kappa \cdot (h \cdot z)^{1/\kappa}, \tag{5.4}$$

where the coefficients $C_j(\rho)$ (with $C_0(\rho) \equiv 1$) are those appearing in the following *inverse factorial expansion* for positive integer M

$$\frac{1}{\Gamma(s+1)} \cdot \frac{\prod_{r=1}^{\mathfrak{p}} \Gamma(a_r \cdot s + c_r)}{\prod_{r=1}^{\mathfrak{q}} \Gamma(b_r \cdot s + d_r)} = \kappa \cdot A \cdot (h \cdot \kappa^{\kappa})^s \cdot \left\{ \sum_{j=0}^{M-1} \frac{C_j(\rho)}{\Gamma(\kappa s + \vartheta' + j)} + \frac{\mathcal{O}(1)}{\Gamma(\kappa s + \vartheta' + M)} \right\} \tag{5.5}$$

valid for $|s| \to \infty$ in the sector $|\arg s| < \pi$. Throughout the remainder of this section, M denotes an arbitrary fixed positive integer.

The quantity $H_{\mathfrak{p},\mathfrak{q}}(z)$ denotes the *algebraic expansion* whose precise form depends on the parameters of the numeratorial gamma functions in (2.4) and results from the evaluation of the residues at the poles of a *Mellin-Barnes integral representation* for ${}_{\mathfrak{p}}\Psi_{\mathfrak{q}}(z)$. (We refer to Paris and Kaminski (2001) for a thorough consideration of such representations.) In the most straightforward case, where all the poles are simple, we have

$$H_{\mathfrak{p},\mathfrak{q}}(z) = \sum_{r=1}^{\mathfrak{p}} a_m^{-1} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \cdot \frac{\prod_{r=1}^{\mathfrak{p}'} \Gamma(c_r - a_r \cdot k_m)}{\prod_{r=1}^{\mathfrak{q}} \Gamma(d_r - b_r \cdot k_m)} \cdot \Gamma(k_m) \cdot z^{-k_m}, \quad k_m = \frac{k + c_m}{a_m}, \tag{5.6}$$

which holds provided that the sequences k_m for $k = 0, 1, 2, \ldots$ $(1 \le m \le \mathfrak{p})$ are all distinct. Here, the prime denotes the omission of the term corresponding to $r = m$ in the product. When the parameters a_r and c_r are such that some of the poles are of higher order, the residues must be evaluated according to the multiplicity of the poles concerned and will result in terms involving $\log z$ in the algebraic expansion.

When $0 < \kappa < 2$, we have from (5.3) the following single dominant exponential expansion:

$$ {}_{\mathfrak{p}}\Psi_{\mathfrak{q}}(z) \sim E_{\mathfrak{p},\mathfrak{q}}(z) \tag{5.7} $$

as $z \to +\infty$. When $\kappa \geq 2$, the expansion of ${}_{\mathfrak{p}}\Psi_{\mathfrak{q}}(z)$ as $z \to +\infty$ consists of additional exponential expansions of the type $E_{\mathfrak{p},\mathfrak{q}}(z)$, but with the argument of z rotated by multiples of 2π. However, as these additional expansions are *subdominant* compared to $E_{\mathfrak{p},\mathfrak{q}}(z)$ on the positive real z-axis, we may still say that the *dominant* expansion of ${}_{\mathfrak{p}}\Psi_{\mathfrak{q}}(z)$ as $z \to +\infty$ is given by (5.7) when $\kappa > 0$.

It is important that for the purposes of this work we had to establish a new, more subtle result on the behavior of the general Wright function ${}_{\mathfrak{p}}\Psi_{\mathfrak{q}}(z)$ as compared to (5.3). Thus, in Subsection 5.2 we shall require the following asymptotic expansion for the function ${}_{\mathfrak{p}}\Psi_{\mathfrak{q}}(z)$ in the case where $\mathfrak{p} = 0$, $\mathfrak{q} = 1$ and $0 < \kappa < 1$ on certain critical rays in $\mathbb{C}$, which are known as *Stokes lines*.

A more precise form of (5.3) when $0 < \kappa < 1$, which specifies the behavior on these rays, is given by

$$ {}_{\mathfrak{p}}\Psi_{\mathfrak{q}}(z) \sim \begin{cases} E_{\mathfrak{p},\mathfrak{q}}(z) + H_{\mathfrak{p},\mathfrak{q}}(z \cdot e^{\mp\pi i}) & \text{in} \quad |\arg z| < \pi \cdot \kappa \\ \frac{1}{2} \cdot \hat{E}_{\mathfrak{p},\mathfrak{q}}(z) + H^{o}_{\mathfrak{p},\mathfrak{q}}(z \cdot e^{\mp\pi i}) & \text{on} \quad \arg z = \pm\pi \cdot \kappa \\ H_{\mathfrak{p},\mathfrak{q}}(z \cdot e^{\mp\pi i}) & \text{in} \quad |\arg(-z)| < \pi \cdot (1-\kappa), \end{cases} \tag{5.8} $$

where in the middle expression the upper and lower signs are chosen together. The superscript "o" signifies that the algebraic expansions $H_{\mathfrak{p},\mathfrak{q}}(z \cdot e^{\mp\pi i})$ are optimally truncated at, or near, the terms of least magnitude and $\hat{E}_{p,q}(z)$ denotes the expansion $E_{p,q}(z)$ augmented by the addition of a series of the form $\sum \hat{B}_j \cdot |X|^{-j-1/2}$ on the rays $\arg z = \pm\pi \cdot \kappa$. Paris (2011b) was able to prove (5.8) only in the following two special cases:
(i) $\mathfrak{p} = 1, \mathfrak{q} \geq 0$ and **(ii)** $\mathfrak{p} \geq 1, \mathfrak{q} = 0$ with all the a_r equal.
We will apply the middle equation in (5.8) to the case of ${}_{\mathfrak{p}}\Psi_{\mathfrak{q}}(\cdot)$ with $\mathfrak{p} = 1$, $\mathfrak{q} = 0$ in (5.13).

It can be shown that the exponential expansion $E_{\mathfrak{p},\mathfrak{q}}(z)$, which emerges on the right-hand side of (5.8), is dominant as $|z| \to \infty$ in the sector $|\arg z| < \frac{1}{2} \cdot \pi \cdot \kappa$ and is exponentially small in the sectors $\frac{1}{2} \cdot \pi \cdot \kappa < |\arg z| < \pi \cdot \kappa$. The rays $\arg z = \pm\pi \cdot \kappa$, where $E_{\mathfrak{p},\mathfrak{q}}(z)$ is *maximally subdominant* with respect to the algebraic expansion, are the Stokes lines. In the neighborhood of these rays, the asymptotic structure of ${}_{\mathfrak{p}}\Psi_{\mathfrak{q}}(z)$ is associated with a *Stokes phenomenon*, where the coefficient (the *Stokes multiplier*) of the subdominant exponential term undergoes a smooth, but rapid, transition (at fixed large $|z|$) from the value 1, when $|\arg z|$ is somewhat less than $\pi \cdot \kappa$, to the value 0, when $|\arg z|$ is somewhat greater than $\pi \cdot \kappa$. This transition is approximated by an error function whose argument is a measure of the angular separation from the Stokes lines; see Paris and Kaminski (2001, Ch. 6) for details and the references therein. On the Stokes lines, the multiplier has the value 1/2 to leading order.

5.2 The Asymptotic Expansions of $\phi(\rho, 0, \pm z)$

In the case of the function $\phi(\rho, 0, z)$ with $\rho > 0$, which is defined by [VPY1, form. (2.2)], we have $\mathfrak{p} = 0$, $\mathfrak{q} = 1$ (with $b_1 = \rho$ and $d_1 = 0$) and, from (5.2), the associated parameters

$$ \kappa = 1 + \rho > 0, \qquad h = \rho^{-\rho}, \qquad \vartheta = \vartheta' = 1/2, \qquad A = (2\pi)^{-1/2} \cdot \rho^{1/2}/\kappa. \tag{5.9} $$

A combination of (5.9) with the statement made below (5.2) implies that the function $\phi(\rho, 0, z) = {}_0\Psi_1(z)$ is uniformly and absolutely convergent for all finite (complex) z.

It then follows from (5.6) and (5.7) that $H_{0,1}(z) \equiv 0$ and hence, for each fixed $\rho \in \mathbf{R}^1_+$,

$$\phi(\rho, 0, z) \sim \left(\frac{\rho}{2\pi \cdot \kappa}\right)^{1/2} \cdot (h \cdot z)^{1/(2\kappa)} \cdot \exp[\kappa \cdot (hz)^{1/\kappa}] \cdot \sum_{j=0}^{\infty} C_j(\rho) \cdot (\kappa \cdot (hz)^{1/\kappa})^{-j} \tag{5.10}$$

as $z \to +\infty$. Substitution of the above parameter values into (5.10) then leads to the expansion

$$\phi(\rho, 0, z) \sim \frac{(\rho \cdot z)^{1/2 \cdot (1+\rho)}}{\sqrt{2\pi \cdot (1+\rho)}} \cdot \exp[(1+\rho) \cdot (\rho^{-\rho} \cdot z)^{1/(1+\rho)}] \cdot \sum_{j=0}^{\infty} \rho^j \cdot C_j(\rho) \cdot \left((1+\rho) \cdot (\rho z)^{1/(1+\rho)}\right)^{-j}$$

as $z \to +\infty$, which may be compared with [VPY1, form. (4.14)]. It now remains to determine the coefficients $C_j(\rho)$. In addition, we intoduce the related coefficients $\hat{\mathcal{Z}}_j(\rho)$, which we define by

$$C_j(\rho) \equiv \rho^{-j} \cdot \hat{\mathcal{Z}}_j(\rho). \tag{5.11}$$

In fact, the quantities $\mathcal{Z}_j(\rho)$, $\widetilde{\mathcal{Z}}_j(\rho)$, and $\hat{\mathcal{Z}}_j(\rho)$, which are defined by [VPY1, form. (4.12)], (5.1) and (5.11), respectively, turn out to be identical. However, this result was not given *a priori*, and its validity will follow from our proof. Hence, for now we should assume that these coefficients may be different.

In our discussion on the asymptotics of the function $\phi(\rho, 0, -z)$ as $z \to +\infty$ with $\rho \in (-1, 0)$, we shall denote the associated parameters κ, h, ϑ and A with a hat to distinguish them from those in (5.9). With $\mathfrak{r} := -\rho$ $(0 < \mathfrak{r} < 1)$, we replace the gamma function on the right-hand side of [VPY1, form. (4.14)] by the equivalent expression obtained from the following reflection formula for the gamma function:

$$-z \cdot \Gamma(-z) \cdot \Gamma(z) = \pi / \sin(\pi z). \tag{5.12}$$

Hence,

$$\begin{aligned} \phi(\rho, 0, -z) &= \frac{1}{2\pi i} \cdot \sum_{k=1}^{\infty} \frac{\Gamma(1+\mathfrak{r}k)}{k!} \cdot \left\{(z \cdot e^{\pi i \hat{\kappa}})^k - (z \cdot e^{-\pi i \hat{\kappa}})^k\right\} \\ &= \frac{1}{2\pi i} \cdot \left\{{}_1\Psi_0(z \cdot e^{\pi i \hat{\kappa}}) - {}_1\Psi_0(z \cdot e^{-\pi i \hat{\kappa}})\right\}, \end{aligned} \tag{5.13}$$

where, from (2.4) and (5.2) with $\mathfrak{p} = 1$, $\mathfrak{q} = 0$ and $a_1 = \mathfrak{r}$, $c_1 = 1$, we have the associated parameters

$$\hat{\kappa} = 1 - \mathfrak{r}, \qquad \hat{h} = \mathfrak{r}^{\mathfrak{r}}, \qquad \hat{\vartheta} = \hat{\vartheta}' = 1/2, \qquad \hat{A} = (2\pi\mathfrak{r})^{1/2}/\hat{\kappa}.$$

For $z \to +\infty$, the arguments $z \cdot e^{\pm \pi i \hat{\kappa}}$ in (5.13) are situated on the Stokes lines $\arg z = \pm\pi \cdot \hat{\kappa}$, respectively. Consequently, from (5.13) and the second relation in (5.8) we obtain

$$\phi(\rho, 0, -z) \sim \frac{1}{2\pi i} \cdot \left\{\tfrac{1}{2} \cdot \hat{E}_{1,0}(z \cdot e^{\pi i \hat{\kappa}}) - \tfrac{1}{2} \cdot \hat{E}_{1,0}(z \cdot e^{-\pi i \hat{\kappa}}) + H^o_{1,0}(z \cdot e^{\pi i \hat{\kappa} - \pi i}) - H^o_{1,0}(z \cdot e^{-\pi i \hat{\kappa} + \pi i})\right\}$$

as $z \to +\infty$, where

$$\hat{E}_{1,0}(z) := E_{1,0}(z) \pm i \cdot (|X| \cdot e^{\mp \pi \cdot i})^{\vartheta} \cdot e^{-|X|} \cdot \sum_{j=0}^{\infty} \hat{B}_j \cdot |X|^{-j-1/2} \qquad (\arg z = \pm\pi \cdot \kappa)$$

and from (5.6) the algebraic expansion is given by

$$H_{1,0}(z) = \frac{1}{\mathfrak{r}} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \cdot \Gamma\left(\frac{k+1}{\mathfrak{r}}\right) \cdot z^{-(k+1)/\mathfrak{r}}.$$

The coefficients $\hat{B}_j$, which we do not specify here, depend on $C_j(\rho)$ and are real. It is readily seen that $H_{1,0}(z \cdot e^{\pi i\hat{\kappa}-\pi i}) - H_{1,0}(z \cdot e^{-\pi i\hat{\kappa}+\pi i}) \equiv 0$, so that the algebraic expansions present in the combination (5.13) cancel. From (5.4), together with the fact that since $\vartheta = 1/2$ the series involving the coefficients $\hat{B}_j$ present in $\hat{E}_{1,0}(z)$ cancel in the combination (5.13), we therefore deduce that

$$\begin{aligned}\phi(\rho, 0, -z) &\sim \frac{1}{4\pi i} \cdot \left\{E_{1,0}(ze^{\pi i\hat{\kappa}}) - E_{1,0}(ze^{-\pi i\hat{\kappa}})\right\} \\ &= \left(\frac{\mathfrak{r}}{2\pi\hat{\kappa}}\right)^{1/2} \cdot (\hat{h} \cdot z)^{1/(2\hat{\kappa})} \cdot \exp[-\hat{\kappa} \cdot (\hat{h} \cdot z)^{1/\hat{\kappa}}] \cdot \sum_{j=0}^{\infty}(-1)^j \hat{C}_j(\mathfrak{r}) \left(\hat{\kappa} \cdot (\hat{h} \cdot z)^{1/\hat{\kappa}}\right)^{-j} \\ &= \frac{(-\rho \cdot z)^{1/(2(1+\rho))}}{\sqrt{2\pi(1+\rho)}} \cdot \exp[-(1+\rho) \cdot ((-\rho)^{-\rho} \cdot z)^{1/(1+\rho)}] \\ &\qquad \times \sum_{j=0}^{\infty}(-\mathfrak{r})^j \cdot \hat{C}_j(\mathfrak{r}) \cdot \left((1+\rho) \cdot (-\rho z)^{1/(1+\rho)}\right)^{-j}\end{aligned} \tag{5.14}$$

as $z \to +\infty$, upon substitution of the above values for $\hat{\kappa}$ and $\hat{h}$ and replacement of $\mathfrak{r}$ by $-\rho$. Comparison with [VPY1, form. (4.14)] shows that the coefficients $\hat{C}_j(\mathfrak{r})$ in (5.14) are specified by $(-\mathfrak{r})^j \cdot \hat{C}_j(\mathfrak{r}) = \mathcal{Z}_j(\rho)$, whence

$$\hat{C}_j(-\rho) = \rho^{-j} \cdot \mathcal{Z}_j(\rho) \qquad (j = 0, 1, 2, \ldots). \tag{5.15}$$

5.3 Calculation of the Coefficients $C_j(\rho)$

From (5.5), the coefficients $C_j(\rho)$ in the Poincaré series in (5.10) are defined by means of the inverse factorial expansion

$$\frac{\Gamma(\kappa \cdot s + \frac{1}{2})}{\Gamma(s+1) \cdot \Gamma(\rho \cdot s)} = \kappa \cdot A(h \cdot \kappa^{\kappa})^s \cdot \left\{\sum_{j=0}^{M-1} \frac{C_j(\rho)}{(\kappa \cdot s + \frac{1}{2})_j} + \frac{\mathcal{O}(1)}{(\kappa \cdot s + \frac{1}{2})_M}\right\} \tag{5.16}$$

as $|s| \to \infty$ in $|\arg s| < \pi$, where the parameters κ, h and A are specified in (5.9). The algorithm for the determination of the coefficients $C_j(\rho)$ that we describe here was presented in Paris and Kaminski (2001, p. 46); see also Paris (2010, Appendix).

Next, we introduce the *scaled gamma function* $\Gamma^*(z)$ defined by

$$\Gamma^*(z) := (2\pi)^{-1/2} \cdot e^z \cdot z^{1/2-z} \cdot \Gamma(z) \tag{5.17}$$

which, in view of (5.12), satisfies

$$\Gamma^*(-z) \cdot \Gamma^*(z) = 1/(1 - e^{\pm 2\pi i z}), \tag{5.18}$$

where the upper or lower sign is chosen according as $0 < \arg z < \pi$ or $-\pi < \arg z < 0$, respectively. Then for $a \in \mathbf{R}^1_+$ and arbitrary b, this enables us to write

$$\Gamma(a \cdot s + b) = (2\pi)^{1/2} \cdot e^{-as} \cdot (as)^{as+b-1/2} \cdot \mathfrak{e}(a \cdot s; b) \cdot \Gamma^*(a \cdot s + b),$$

where

$$\mathfrak{e}(u; b) := e^{-b} \cdot (1 + b/u)^{u+b-1/2} = \exp\{(u + b - 1/2) \cdot \log(1 + b/u) - b\}.$$

From [VPY1, form. (4.8)], we obtain the large-s expansions

$$\Gamma^*(a \cdot s + b) = 1 + \frac{1}{12a \cdot s} + \frac{1 - 24b}{288 \cdot (as)^2} + \mathcal{O}(s^{-3});$$

$$\mathfrak{e}(a\cdot s;b)=1+\frac{b(b-1)}{2\cdot as}+\frac{b^2}{24\cdot(as)^2}\cdot(3b^2-10b+9)+\mathcal{O}(s^{-3}).$$

Then, (5.16) can be written in terms of the scaled gamma function in the form

$$R(s)\cdot\Upsilon(s)=\sum_{j=0}^{M-1}\frac{C_j(\rho)}{(\kappa s+\frac{1}{2})_j}+\frac{\mathcal{O}(1)}{(\kappa\cdot s+\frac{1}{2})_M}\ , \tag{5.19}$$

where

$$\Upsilon(s):=\frac{\Gamma^*(\kappa\cdot s+\frac{1}{2})}{\Gamma^*(s+1)\cdot\Gamma^*(\rho\cdot s)}\ ,\qquad R(s):=\frac{\mathfrak{e}(\kappa\cdot s;\frac{1}{2})}{\mathfrak{e}(s;1)\ \cdot\mathfrak{e}(\rho\cdot s;0)}\ .$$

Setting $\chi:=(\kappa\cdot s)^{-1}$, we obtain from (5.19) after some routine algebra that

$$\begin{aligned}R(s)\cdot\Upsilon(s)\ &=\ 1-\frac{(2\rho+1)\cdot(\rho+1)}{24\cdot\rho}\cdot\chi+\frac{(2\rho+1)^2\cdot(\rho+1)^2}{1152\cdot\rho^2}\cdot\chi^2+\mathcal{O}(\chi^3)\\ &=\ C_0(\rho)+C_1(\rho)\cdot\chi+(C_2(\rho)-\frac{1}{2}\cdot C_1(\rho))\cdot\chi^2+\mathcal{O}(\chi^3).\end{aligned}$$

Equating coefficients of powers of χ, we therefore find

$$C_0(\rho)=1,\quad C_1(\rho)=-\frac{(2\rho+1)\cdot(\rho+2)}{24\cdot\rho}\ ,\quad C_2(\rho)=\frac{(2\rho+1)\cdot(\rho+2)}{1152\cdot\rho^2}\cdot\ (2\rho^2-19\rho+2).$$

Higher coefficients are obtained by continuation of this expansion process with the help of *Mathematica*. In this manner we have determined the coefficients $C_j(\rho)$ up to $j=30$. By comparison with the Zolotarev polynomials $\mathcal{Z}_k(\rho)$ in [VPY1, Remark 4.3.ii], we verified numerically that

$$C_j(\rho)=\rho^{-j}\cdot\mathcal{Z}_j(\rho)\qquad(j\le 30). \tag{5.20}$$

5.4 Analytical Proof of Relation (5.20) for Any Integer $j\ge 0$

From (5.5), the coefficients $\hat{C}_j(\mathfrak{r})$ which appear in (5.14)–(5.15) are specified by the inverse factorial expansion

$$\frac{\Gamma(\hat{\kappa}s+\frac{1}{2})\cdot\Gamma(1+\mathfrak{r}s)}{\Gamma(s+1)}=\hat{\kappa}\cdot\hat{A}\cdot(\hat{h}\cdot\hat{\kappa}^{\hat{\kappa}})^s\cdot\left\{\sum_{j=0}^{M-1}\frac{\hat{C}_j(r)}{(\hat{\kappa}s+\frac{1}{2})_j}+\frac{\mathcal{O}(1)}{(\hat{\kappa}s+\frac{1}{2})_M}\right\} \tag{5.21}$$

valid as $|s|\to\infty$ in the sector $|\arg s|<\pi$, where we recall that M is an arbitrary positive integer. Next, we employ (5.17) to rewrite $\Gamma(1+\mathfrak{r}s)$ and $\Gamma(\rho\cdot s)$ which emerge in (5.21) and (5.16), respectively. A combination of these expressions with (5.16) and (5.21), and the formula $\Gamma(1+z)=z\cdot\Gamma(z)$ stipulates that under the same assumptions on the complex values of s,

$$\frac{\Gamma(\hat{\kappa}s+\frac{1}{2})\cdot\Gamma^*(\mathfrak{r}s)}{\Gamma(s+1)}=s^{-\mathfrak{r}s-1/2}\cdot e^{\mathfrak{r}s}\cdot\hat{\kappa}^{\hat{\kappa}s}\cdot\left\{\sum_{j=0}^{M-1}\frac{\hat{C}_j(\mathfrak{r})}{(\hat{\kappa}s+\frac{1}{2})_j}+\frac{\mathcal{O}(1)}{(\hat{\kappa}s+\frac{1}{2})_M}\right\},$$

$$\frac{\Gamma(\kappa s+\frac{1}{2})}{\Gamma(s+1)\cdot\Gamma^*(\rho s)}=s^{\rho s-1/2}\cdot e^{-\rho s}\cdot\kappa^{\kappa s}\cdot\left\{\sum_{j=0}^{M-1}\frac{C_j(\rho)}{(\kappa s+\frac{1}{2})_j}+\frac{\mathcal{O}(1)}{(\kappa s+\frac{1}{2})_M}\right\}.$$

In turn, a combination of the above two representations with some algebra implies that

$$\sum_{j=0}^{M-1} \frac{C_j(\rho)}{(\kappa s+\frac{1}{2})_j} + \frac{\mathcal{O}(1)}{(\kappa s+\frac{1}{2})_M} = \frac{\Lambda_{\mathfrak{r},\rho}(s)}{\Gamma^*(\mathfrak{r}s)\cdot\Gamma^*(\rho s)} \cdot \left\{ \sum_{j=0}^{M-1} \frac{\hat{C}_j(\mathfrak{r})}{(\hat{\kappa}s+\frac{1}{2})_j} + \frac{\mathcal{O}(1)}{(\hat{\kappa}s+\frac{1}{2})_M} \right\}$$

as $|s| \to \infty$ in the sector $|\arg s| < \pi$, where

$$\Lambda_{\mathfrak{r},\rho}(s) := \frac{\Gamma(\kappa s+1/2)}{\Gamma(\hat{\kappa}s+1/2)} \cdot \frac{\hat{\kappa}^{\hat{\kappa}s}}{\kappa^{\kappa s}} \cdot (s/e)^{-(\mathfrak{r}+\rho)\cdot s}.$$

Now, recall that in our case, $\mathfrak{r} = -\rho$. Therefore, $\hat{\kappa} = \kappa$ and $\Lambda_{-\rho,\rho}(s) \equiv 1$. From (5.18),

$$\Gamma^*(-\rho s)\cdot\Gamma^*(\rho s) = 1/(1-e^{2\pi i\rho s}) \sim 1$$

for $|s| \to \infty$ in $0 < \arg s < \pi$ and $\rho > 0$. This yields the result that for an arbitrary $M \in \mathbf{N}$,

$$\sum_{j=0}^{M-1} \frac{C_j(\rho)}{(\kappa s+\frac{1}{2})_j} = \sum_{j=0}^{M-1} \frac{\hat{C}_j(-\rho)}{(\kappa s+\frac{1}{2})_j} + \frac{\mathcal{O}(1)}{(\kappa s+\frac{1}{2})_M}$$

as $|s| \to \infty$ in $0 < \arg s < \pi$, where the exponentially small terms present in $\Gamma^*(-\rho s)\cdot\Gamma^*(\rho s)$ have been absorbed into the order term. By comparison of corresponding coefficients and a simple induction argument combined with (5.15), it then follows that

$$C_j(\rho) = \hat{C}_j(-\rho) = \rho^{-j}\cdot\mathcal{Z}_j(\rho) \qquad (j=0,1,2,\ldots). \tag{5.22}$$

It remains to combine (5.11) and (5.22) to get that $\hat{\mathcal{Z}}_j(\rho) = \mathcal{Z}_j(\rho)$. □

6 Appendix B. Proof of [VPY1, Theorem 4.8]

(ii) The validity of **(ii1)** is established by combining [VPY1, form. (2.2), (2.9), (4.1), and (4.9)] with some routine algebra. In particular, one should set $z := y\cdot\lambda+1$ in [VPY1, form. (4.9)] in order to derive the Poincaré series for $1/(y\cdot\lambda)!$.
(ii3) The proof is obtained by combining [VPY1, form. (2.6), (2.9), (3.7)–(3.9), (3.14), (4.1) and (4.15)].
(ii2) In the case where $p \in (-\infty, 0]$, we first combine the result of part (ii3) with [VPY1, Prop. 4.1.ii and form. (2.6), (2.9), (3.7)–(3.9), (3.14), (4.1)–(4.3)] and (2.3). This implies that $f_{p,\mu,\lambda}(y)$ admits the following Poincaré series as $y \to +\infty$:

$$f_{p,\mu,\lambda}(y) \sim \mathcal{F}_{p,\mu,\lambda}(y)\cdot\sum_{k=0}^{\infty} \frac{\mathcal{Z}_k(\rho_{3-p})\cdot(2-p)^{2k}}{(y^{p-1}\cdot\lambda)^k}\ . \tag{6.1}$$

The rest follows from a combination of [VPY1, form. (4.2)–(4.3) and (4.13)] with (6.1).

For $p \in (1,2)$, the proof involves a combination of [VPY1, form. (2.6), (2.9), (3.7)–(3.9), (3.25), (4.1), (4.14)].
(i) The proof of **(i1)** is identical to that of (ii1), since one obtains the same Poincaré series for $1/(y\cdot\lambda)!$ when a factor of the product $y\cdot\lambda$ approaches $+\infty$, whereas the other one is kept fixed.
(i2) The proof relies on a combination of the rightmost representation in [VPY1, form. (3.33)] with

parts (ii2) and (ii3) in the cases where $p \in (-\infty, 0] \cup (1, 2)$ and $p \in (2, +\infty)$, respectively. Thus, we get the following Poincaré series as $y \in \mathbf{R}^1_+$ is fixed and $\lambda \to +\infty$:

$$f_{p,\mu,\lambda}(y) \sim \lambda^{1/(2-p)} \cdot \mathcal{F}_{p,\Phi_p^{1/(2-p)},1}(y \cdot \lambda^{1/(2-p)}) \cdot \sum_{k=0}^{\infty} \frac{\mathcal{Z}_k(\rho_p) \cdot (p-1)^{2k}}{((y \cdot \lambda^{1/(2-p)})^{2-p} \cdot 1)^k} \,. \tag{6.2}$$

Recall that Φ_p is given by [VPY1, form. (3.6)]. Next, the validity of the identity

$$\lambda^{1/(2-p)} \cdot \mathcal{F}_{p,\Phi_p^{1/(2-p)},1}(y \cdot \lambda^{1/(2-p)}) \equiv \mathcal{F}_{p,\mu,\lambda}(y) \tag{6.3}$$

follows by combining [VPY1, form. (2.9) and (4.1)]. The rest is obtained from a combination of (6.2)–(6.3).
(i3) It is derived by combining [VPY1, form. (2.9), (3.4), (4.1) and (4.9)] with some algebra. □

Acknowledgement

V.V. is grateful to the Fields Institute, Institute of Mathematics and Informatics of Vilnius University, University of Toronto and York University for their hospitality. O.Ya. appreciates financial support of the Robert and Rene Glidden Visiting Professorship and is indebted to the Department of Mathematics at Ohio University for its hospitality. We thank A. Feuerverger, A. Kuznetsov, G. Letac, L. Lorch, Yu. Luchko, N. Madras, A. Meir, S. Nadarajah, M. Reynolds, V.M. Tihomirov, C.S. Withers for help, as well as the managing editor and the anonymous referee for their feedback.

References

[1] Abramowitz, M., Stegun, I. A. (1965). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* New York: Dover.

[2] Barndorff-Nielsen, O.E., Shephard, N. (2001). Normal modified stable processes. *Teor. Ǐmovirnost. ta Matem. Statyst.* 65:1–19.

[3] Ditlevsen, O. (1990). Asymptotic first–passage time distributions in compound Poisson processes. *Struct. Saf.* 8:327–336.

[4] Fisher, R.A., Cornish, E.A. (1960). The percentile points of distributions having known cumulants. *Technometrics* 2:209–225.

[5] Friis-Hansen, P., Ditlevsen, O. (2003). Nature preservation acceptance model applied to tanker oil spill simulations. *Struct. Saf.* 25:1–34.

[6] Gradshteyn, I.S., Ryzhik, I.M. (2007). *Table of Integrals, Series and Products.* 7th edn. Oxford: Academic.

[7] Hasan, M.M., Dunn, P.K. (2011). Two Tweedie distributions that are near–optimal for modelling monthly rainfall in Australia. *Int. J. Climatol.* 31:1389-1397.

[8] Janson, S. (2010). Moments of gamma type and the Brownian supremum process area. *Probability Surveys* 7:1–52 (addendum pp. 207-208).

[9] Jørgensen, B., de Souza, M.C.P. (1994). Fitting Tweedie's compound Poisson model to insurance claims data. *Scand. Actuarial J.* 69–93.

[10] Jørgensen, B., Martínez, J.R., Demétrio, C.G.B. (2011). Self–similarity and Lamperti convergence for families of stochastic processes. *Lith. Math. J.* 51:342-361.

[11] Jørgensen, B., Song, P.X.-K. (1998). Stationary time series models with exponential dispersion model margins. *J. Appl. Probab.* 35:78–92.

[12] Jørgensen, B., Song, P.X.-K. (2006). Diagnosis of stationarity in state space models for longitudinal data. *Far East J. Theor. Stat.* 19:43-59.

[13] Jørgensen, B., Song, P.X.-K. (2007). Stationary state space models for longitudinal data. *Can. J. Stat.* 34:1-23.

[14] Jørgensen, B., Tsao, M. (1999). Dispersion models and longitudinal data analysis. *Statist. Med.* 18:2257–2270.

[15] Kaas, R. (2005). Compound Poisson distribution and GLM's Tweedie's distribution. In: Proceedings of the Contact Forum *3rd Actuarial and Financial Mathematics Day*, Brussels: KVAB, pp. 3-12. http://lstat.kuleuven.be/research/seminars_events/files/3afmd/Kaas.PDF

[16] Kendal, W.S. (2002). Spatial aggregation of the Colorado potato beetle described by an exponential dispersion model. *Ecol. Model.* 151:261–269.

[17] Kendal, W.S. (2004). A scale invariant clustering of genes on human chromosome 7. *BMC Evol. Biol.* 4:(3). http://www.biomedcentral.com/1471-2148/4/3

[18] Kendal, W.S. (2007). Scale invariant correlations between genes and SNPs on human chromosome 1 reveal potential evolutionary mechanisms. *J. Theor. Biol.* 245:329–340.

[19] Kendal, W.S., Jørgensen, B. (2011a). Taylor's power law and fluctuation scaling explained by a central–limit–like convergence. *Phys. Rev. E* 83, 066115 (7 pp.).

[20] Kendal, W.S., Jørgensen, B. (2011b). Tweedie convergence: a mathematical basis for Taylor's power law, $1/f$ noise, and multifractality. *Phys. Rev. E* 83, 066120 (10 pp.).

[21] Kendal, W.S., Lagerwaard, F.J., Agboola, O. (2000). Characterization of the frequency distribution for human hematogenous metastases: Evidence for clustering and a power variance function. *Clin. Exp. Metastas.* 18:219–229.

[22] Kokonendji, C.C. (2001). First passage times on zero and one and natural exponential families. *Statist. Probab. Lett.* 51:293–298.

[23] Kokonendji, C.C., Khoudar, M. (2006). On Lévy measures for infinitely divisible natural exponential families. *Statist. Probab. Lett.* 76:1364–1368.

[24] Le Cam, L. (1961). A stochastic description of precipitation. In: Neyman, J., ed. Proceedings of *4th Berkeley Symp. Math. Statist. Probab.*, Berkeley: University of California Press, vol. 3, pp. 165-186.

[25] Lee, M.-L.T., Whitmore, G.A. (1993). Stochastic processes directed by randomized time. *J. Appl. Probab.* 30:302–314.

[26] Letac, G., Mora, M. (1990). Natural real exponential families with cubic variance functions. *Ann. Statist.* 18:1–37.

[27] Schneider, W.R. (1986). Stable distributions: Fox function representation and generalization. In: Albeverio, S. et al., eds. Stochastic Processes in Classical and Quantum Systems. *Lecture Notes in Physics*. Berlin: Springer, vol. 262, pp. 497–511.

[28] Smyth, G.K., Jørgensen, B. (2002). Fitting Tweedie's compound Poisson model to insurance claims data: dispersion modelling. *ASTIN Bull.* 32:143–157.

[29] Vinogradov, V. (2002). On a class of Lévy processes used to model stock price movements with possible downward jumps. *C.R. Math. Rep. Acad. Sci. Canada* 24:152–159.

[30] Vinogradov, V. (2004b). On a model for stock price movements and the power-variance family. *C.R. Math. Rep. Acad. Sci. Canada* 26:102–109.

[31] Vinogradov, V. (2007c). On structural and asymptotic properties of some classes of distributions. *Acta Appl. Math.* 97:335–351.

[32] Vinogradov, V. (2007d). Local approximations for branching particle systems. *Commun. Stoch. Anal.* 1:293–309.

[33] Vinogradov, V. (2008). Properties of certain Lévy and geometric Lévy processes. *Commun. Stoch. Anal.* 2:193–208.

[34] Vinogradov, V., Paris, R.B., Yanushkevichiene, O.L. (2012a). New properties and representations for members of the power-variance family. I. *Lith. Math. J.* (in press).

[35] Wright, E. M. (1935b). The asymptotic expansion of the generalized hypergeometric function. *J. London Math. Soc.* 10:286–293.