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ON THE ASYMPTOTIC EXPANSION OF A BINOMIAL SUM INVOLVING POWERS OF THE SUMMATION INDEX

R. B. PARIS AND P. J. LARCOMBE

Abstract. Work elsewhere [1, 3] has revealed the leading asymptotic behaviour of the binomial sum $S_p(n)$ defined by

$$S_p(n) = \sum_{j=1}^{n} j^p \binom{n+j}{j}$$

in the limit $n \to \infty$ in the case of positive integer p. In this paper, we establish the asymptotic expansion of $S_p(n)$ first for positive integer p and secondly, by means of an integral representation for the sum, for arbitrary values of the index p.

1. Introduction

Consideration of the binomial sum

$$S_p(n) = \sum_{j=1}^{n} j^p \binom{n+j}{j} \tag{1.1}$$

has been motivated by the recent study of a multi-link inverted pendulum enumeration problem [2]. The main properties of $S_p(n)$ are examined for positive integer p in [3], where its explicit evaluation for $1 \leqslant p \leqslant 5$ is given. It is shown among other things that the large-n behaviour is described by

$$S_p(n) \sim 2n^p \binom{2n}{n} \qquad (n \to \infty). \tag{1.2}$$

Two alternative proofs of this result are to be given in a further paper [1]: the first uses an elaborate and lengthy application of the Euler-Maclaurin summation formula, and the second uses a straightforward decomposition of the sum in terms of Stirling numbers of the second kind. These proofs each differ significantly from that in [3].

In this paper we offer two derivations of the asymptotic expansion of $S_p(n)$ as $n \to \infty$. The first approach is valid for positive integer values of p and follows from the above-mentioned decomposition of $S_p(n)$ in terms of the Stirling numbers. The second approach is valid for arbitrary, finite values of p and uses an integral representation for $S_p(n)$ combined with the method of steepest descents.

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2. The expansion for positive integer p

Let $\mathbf{s}(p,j)$ be the Stirling number of the second kind and $[x]_j$ denote (with $[x]_0 = 1$) the usual falling factorial function $[x]_j = x(x-1)\cdots(x-j+1)$. In the second proof in [1] it is established that

$$S_p(n) = \binom{2n}{n}\{2n^p - F(n,p)\}, \tag{2.1}$$

where, with $\alpha_j(p) = (2j+1)\mathbf{s}(p,j)$ and $g(n;j) = [n]_j/(n+j+1)$,

$$F(n,p) = \sum_{j=0}^{p} \alpha_j(p) g(n;j). \tag{2.2}$$

Note that for large n we have $g(n;j) = O(n^{j-1})$.

In order to generate a series expansion for $S_p(n)$ it is necessary to extract terms within $F(n,p)$ of $O(n^{p-r})$ for $r = 1,2,3,\ldots$. Before doing this, we note the following values of the Stirling number

$$\begin{aligned}\mathbf{s}(p,p) &= 1,\\ \mathbf{s}(p,p-1) &= p(p-1)/2,\\ \mathbf{s}(p,p-2) &= p(p-1)(p-2)(3p-5)/24\end{aligned}$$

and that

$$\begin{aligned}g(n;p) &= (n-1)(n-2)\cdots(n-p+1)\left\{1 - \frac{(p+1)}{n} + \frac{(p+1)^2}{n^2} - \frac{(p+1)^3}{n^3} + \cdots\right\}\\ &= A(n,p)B(n,p),\end{aligned} \tag{2.3}$$

say, where $A(n,p) = [n-1]_{p-1}$ is a polynomial of degree $p-1$ and

$$B(n,p) = \sum_{r=0}^{\infty} (-)^r (p+1)^r n^{-r}$$

is a power series (each in n).

It is immediate[1] that $[n^{p-1}]\{g(n;p)\} = 1$, and so

$$[n^{p-1}]\{F(n,p)\} = \alpha_p(p)[n^{p-1}]\{g(n;p)\} = \alpha_p(p) = 2p+1. \tag{2.4}$$

To next order we have $[n^{p-2}]\{g(n;p-1)\} = 1$ trivially, and construct $[n^{p-2}]\{g(n;p)\}$ as

$$\begin{aligned}[n^{p-2}]\{g(n;p)\} &= [n^{p-1}]\{A(n,p)\}\cdot[n^{-1}]\{B(n,p)\} + [n^{p-2}]\{A(n,p)\}\cdot[n^0]\{B(n,p)\}\\ &= -(p+1) - p(p-1)/2\\ &= -(p^2+p+2)/2,\end{aligned} \tag{2.5}$$

[1] We employ the standard notation $[x^r]\{f(x)\}$ to denote the coefficient of x^r in the expansion of $f(x)$.

whence, with $\alpha_{p-1}(p) = (2p-1)\mathbf{s}(p,p-1)$,

$$\begin{aligned}[n^{p-2}]\{F(n,p)\} &= \alpha_{p-1}(p)[n^{p-2}]\{g(n;p-1)\} + \alpha_p(p)[n^{p-2}]\{g(n;p)\} \\ &= p(p-1)(2p-1)/2 - (2p+1)(p^2+p+2)/2 \\ &= -(3p^2+2p+1). \end{aligned} \tag{2.6}$$

To obtain the $O(n^{p-3})$ term, we first observe that $[n^{p-3}]\{g(n;p-2)\} = 1$ (again trivially). Then, directly from (2.5),

$$[n^{p-3}]\{g(n;p-1)\} = -[(p-1)^2 + (p-1) + 2]/2 = -(p^2 - p + 2)/2, \tag{2.7}$$

and this in turn gives

$$\begin{aligned}[n^{p-3}]\{g(n;p)\} &= [n^{p-1}]\{A(n,p)\} \cdot [n^{-2}]\{B(n,p)\} + [n^{p-2}]\{A(n,p)\} \cdot [n^{-1}]\{B(n,p)\} \\ &\quad + [n^{p-3}]\{A(n,p)\} \cdot [n^0]\{B(n,p)\} \\ &= (p+1)^2 + p(p-1)(p+1)/2 + p(p-1)(p-2)(3p-1)/24 \\ &= (3p^4 + 2p^3 + 33p^2 + 34p + 24)/24. \end{aligned} \tag{2.8}$$

With $\alpha_{p-2}(p) = (2p-3)\mathbf{s}(p,p-2)$, it then follows that

$$\begin{aligned}[n^{p-3}]\{F(n,p)\} &= \alpha_{p-2}(p)[n^{p-3}]\{g(n;p-2)\} + \alpha_{p-1}(p)[n^{p-3}]\{g(n;p-1)\} \\ &\quad + \alpha_p(p)[n^{p-3}]\{g(n;p)\} \\ &= p(p-1)(p-2)(2p-3)(3p-5)/24 \\ &\quad - p(p-1)(2p-1)(p^2-p+2)/4 \\ &\quad + (2p+1)(3p^4+2p^3+33p^2+34p+24)/24 \\ &= (26p^3 + 15p^2 + 25p + 6)/6. \end{aligned} \tag{2.9}$$

Additional terms can be obtained by continuation of this procedure but this becomes increasingly laborious. From (2.4), (2.6) and (2.9) we write

$$\begin{aligned} F(n,p) &= (2p+1)n^{p-1} - (3p^2+2p+1)n^{p-2} \\ &\quad + (26p^3+15p^2+25p+6)n^{p-3}/6 + O(n^{p-4}) + \cdots + O(1) + O(n^{-1}), \end{aligned}$$

so that (2.1) reads

$$\begin{aligned} S_p(n) = 2n^p \binom{2n}{n} \Big\{ 1 &- \frac{(2p+1)}{2n} + \frac{(3p^2+2p+1)}{2n^2} \\ &- \frac{(26p^3+15p^2+25p+6)}{12n^3} + O(n^{-4}) + \cdots + O(n^{-p-1}) \Big\}. \end{aligned} \tag{2.10}$$

Finally, we substitute into (2.10) the known result

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}\left(1-\frac{1}{8n}+\frac{1}{128n^2}+\frac{5}{1024n^3}-\cdots\right) \qquad (n\to\infty)$$

which, after a little algebra, then yields our desired expansion in the form

$$S_p(n) \sim \frac{2^{2n+1}n^p}{\sqrt{\pi n}}\left\{1-\frac{(8p+5)}{8n}+\frac{(192p^2+144p+73)}{128n^2} - \frac{(6656p^3+4416p^2+6808p+1725)}{3072n^3}+\cdots\right\} \tag{2.11}$$

as $n\to\infty$.

We remark that this formulation has the decomposition (2.1) as its basis and so is necessarily restricted to positive integer values of the index p. The method has relied on the extraction of individual terms within the polynomial $F(n,p)$ and combination with the asymptotic expansion for $\binom{2n}{n}$. Although some aspects of the procedure readily lend themselves to automation by computer algebra, it is required – in accordance with the number of terms in the expansion of $S_p(n)$ sought – to have ever more complicated closed-form expressions for the Stirling numbers $\mathbf{s}(p,p-r)$ with $r\geqslant 3$.

3. The expansion for general values of p

The aim in this section is to derive the asymptotic expansion of $S_p(n)$ for arbitrary (finite) index p. The analysis is presented for real p (positive or negative), but is easily extended to complex values of p. We employ the integral representation

$$\frac{\Gamma(n+j+1)}{\Gamma(j+1)\,n!} = \frac{1}{2\pi i}\int_0^{(1+)} \frac{t^{n+j}}{(t-1)^{n+1}}\,dt,$$

where the integration path is a loop in the positive sense surrounding the point $t=1$. This result is easily established by consideration of the residue of the integrand at the pole $t=1$. Then, from (1.1), we find

$$\begin{aligned} S_p(n) &= \frac{1}{2\pi i}\int_0^{(1+)} \frac{t^n}{(t-1)^{n+1}}\sum_{j=1}^{n} j^p t^j\,dt \\ &= \frac{n^p}{2\pi i}\int_0^{(1+)} \frac{t^{2n}}{(t-1)^{n+1}}\sum_{k=0}^{n-1} t^{-k}\left(1-\frac{k}{n}\right)^p dt. \end{aligned} \tag{3.1}$$

If we introduce the phase function $\psi(t)$ and the amplitude function $F(t)$ by

$$\psi(t) = 2\log t - \log(t-1), \qquad F(t) = \sum_{k=0}^{n-1} t^{-k}\left(1-\frac{k}{n}\right)^p, \tag{3.2}$$

then we can write

$$S_p(n) = \frac{n^p}{2\pi i}\int_0^{(1+)} \frac{e^{n\psi(t)}}{t-1}F(t)\,dt. \tag{3.3}$$

The exponential factor in the integrand in (3.3) has a simple saddle point at $t=2$ (where $\psi'(t)=0$). The steepest descent path through the saddle point is given by $\mathrm{Im}\,\psi(t)=0$, or (with $t=x+iy$)

$$2\arctan\left(\frac{y}{x}\right)=\arctan\left(\frac{y}{x-1}\right).$$

This is readily shown to be the path $(x-1)^2+y^2=1$ and so is a circle of unit radius centred at $t=1$. Let the integration path be as shown in Fig. 1. This path consists of part of the imaginary axis AA' between $\pm i$, the horizontal segments AB and $A'B'$ connecting $\pm i$ to $1\pm i$, respectively, together with the semi-circle C passing through the saddle and the points $1\pm i$ (part of the steepest descent path).

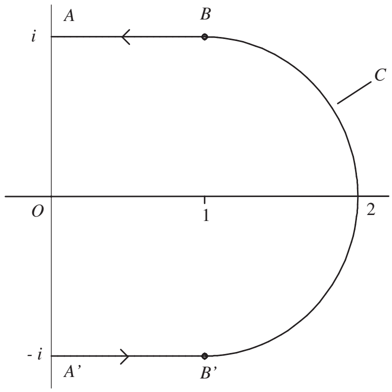

Figure 1: The integration path in the t-plane. The saddle point is at $t=2$.

Let us denote the contributions to the first integral in (3.1) from the different portions of the path by $I_{AA'}$, I_{AB}, $I_{A'B'}$ and I_C. On the path AA', we put $t=iu$, $-1\leqslant u\leqslant 1$ to find

$$|I_{AA'}|=\left|\int_{-i}^{i}\left(\frac{t}{t-1}\right)^n\sum_{j=1}^{n} j^p t^j\,\frac{dt}{t-1}\right|\leqslant 2\int_0^1\left(\frac{u}{\sqrt{1+u^2}}\right)^n\sum_{j=1}^{n} j^p u^j\,\frac{du}{\sqrt{1+u^2}}$$

$$\leqslant 2^{1-\frac{1}{2}n}\sum_{j=1}^{n} j^p\int_0^1 u^j\,du\leqslant 2^{1-\frac{1}{2}n}\sum_{j=1}^{n} j^{p-1}\leqslant 2^{1-\frac{1}{2}n}n^{\delta},$$

where $\delta=p$ $(p>1)$, $\delta=1$ $(p\leqslant 1)$. Similarly, on the path AB we put $t=i+u$, $0\leqslant u\leqslant 1$ to find

$$|I_{AB}|=\left|\int_{i}^{1+i}\left(\frac{t}{t-1}\right)^n\sum_{j=1}^{n} j^p t^j\,\frac{dt}{t-1}\right|\leqslant\int_0^1\left(\frac{\sqrt{1+u^2}}{\sqrt{1+(1-u)^2}}\right)^n\sum_{j=1}^{n}\frac{j^p(1+u^2)^{j/2}}{\sqrt{1+(1-u)^2}}\,du$$

$$\leqslant 2^{\frac{1}{2}n}\sum_{j=1}^{n} j^p \int_0^1 (1+u^2)^{j/2}du \leqslant 2^{\frac{1}{2}n}\sum_{j=1}^{n} j^p \int_0^1 (1+u)^{j/2}du$$
$$\leqslant 2^{\frac{1}{2}n}\sum_{j=1}^{n} j^{p-1}2^{\frac{1}{2}j+2} \leqslant 2^{n+2}\sum_{j=1}^{n} j^{p-1} \leqslant 2^{n+2}n^{\delta}.$$

A similar estimate applies for the path $A'B'$. As the saddle point contribution is $O(e^{n\psi(2)}) = O(2^{2n})$ as $n\to\infty$, it is seen that the contributions from the rectilinear parts of the integration path are subdominant in this limit, and hence that the dominant contribution to $S_p(n)$ arises from the path C through the saddle point.

We now consider the contribution

$$I_C = \frac{n^p}{2\pi i}\int_C \frac{e^{n\psi(t)}}{t-1}F(t)\,dt, \tag{3.4}$$

where on the path C we have uniformly $|t|>1$. In order to deal with $F(t)$ we shall require the following lemma.

LEMMA 1. *Let $\Theta\equiv td/dt$ and $\sigma=t/(t-1)$. Then, for $|t|>1$ and non-negative integer r, we have*

$$\sum_{k=0}^{n-1} k^r t^{-k} = (-)^r\Theta^r\sigma + O(n^r t^{-n}) \tag{3.5}$$

as $n\to\infty$.

Proof. We first observe, by differentiation of the geometric series, that $\sum_{k=0}^{\infty} k^r t^{-k} = (-)^r\Theta^r\sigma$ when $|t|>1$. Then, since $|t|>1$,

$$\sum_{k=0}^{n-1} k^r t^{-k} = \left(\sum_{k=0}^{\infty} - \sum_{k=n}^{\infty}\right) k^r t^{-k} = (-)^r\Theta^r\sigma - t^{-n}\sum_{j=0}^{\infty}(n+j)^r t^{-j},$$

where

$$t^{-n}\sum_{j=0}^{\infty}(n+j)^r t^{-j} = t^{-n}\sum_{k=0}^{r}\binom{r}{k} n^{r-k}\sum_{j=0}^{\infty} j^k t^{-j} = t^{-n}\sum_{k=0}^{r}(-)^k\binom{r}{k} n^{r-k}\Theta^k\sigma$$
$$= t^{-n}(n-\Theta)^r\sigma.$$

It then follows that

$$\sum_{k=0}^{n-1} k^r t^{-k} = (-)^r\Theta^r\sigma - t^{-n}(n-\Theta)^r\sigma = (-)^r\Theta^r\sigma + O(n^r t^{-n})$$

as $n\to\infty$, thereby establishing the lemma. □

Let N be a positive integer. Application of the binomial theorem to write $F(t)$ in (3.2) in the form

$$F(t) = \sum_{j=0}^{\infty}\frac{(-)^j}{n^j}\binom{p}{j}\sum_{k=0}^{n-1} k^j t^{-k},$$

shows that, provided $|t|>1$,

$$F(t)=\sum_{j=0}^{N-1}\frac{(-)^j}{n^j}\binom{p}{j}\sum_{k=0}^{n-1}k^jt^{-k}+O(n^{-N})\,\Theta^N\sigma=\sum_{j=0}^{N-1}\binom{p}{j}\frac{\Theta^j\sigma}{n^j}+O(t^{-n})+O(n^{-N})\,\Theta^N\sigma$$

by (3.5). Insertion of the above representation into (3.4) then yields

$$I_C=n^p\left\{\sum_{r=0}^{N-1}\binom{p}{r}\frac{J_r}{n^r}+R_N\right\},$$

where

$$J_r=\frac{1}{2\pi i}\int_C\frac{e^{n\psi(t)}}{t-1}\,\Theta^r\sigma\,dt$$

and

$$R_N=\frac{1}{2\pi i}\int_C\frac{e^{n\psi(t)}}{t-1}\{O(t^{-n})+\Theta^N\sigma\,O(n^{-N})\}dt=J_N\,O(n^{-N})$$

upon absorbing the exponentially subdominant contribution resulting from $O(t^{-n})$ into the $O(n^{-N})$ term.

The asymptotic expansion of the integrals J_r is discussed in the Appendix. From (A.5), we have for positive integers M_r

$$J_r=\frac{2^{2n+1}(-)^r}{\sqrt{\pi n}}\left\{\sum_{j=0}^{M_r-1}\frac{(-)^jA_j^{(r)}}{n^j}+O(n^{-M_r})\right\}$$

for large n, where the coefficients $A_j^{(r)}$ for $r+j\leqslant 5$ are given in Table 2. If we set the index $M_r=N-r$ $(0\leqslant r\leqslant N-1)$, we obtain

$$\begin{aligned}I_C&=\frac{2^{2n+1}n^p}{\sqrt{\pi n}}\left\{\sum_{r=0}^{N-1}\frac{(-)^r}{n^r}\binom{p}{r}\left\{\sum_{j=0}^{N-r}\frac{(-)^jA_j^{(r)}}{n^j}+O(n^{-N+r})\right\}+O(n^{-N})\right\}\\&=\frac{2^{2n+1}n^p}{\sqrt{\pi n}}\left\{\sum_{k=0}^{N-1}\frac{(-)^k}{n^k}\sum_{r=0}^{k}\binom{p}{r}A_{k-r}^{(r)}+O(n^{-N})\right\}\end{aligned}\tag{3.6}$$

upon setting $k=r+j$ and summing the double sum diagonally.

Then, from (3.4) and (3.6), the dominant contribution to $S_p(n)$ takes the form

$$S_p(n)\sim\frac{2^{2n+1}n^p}{\sqrt{\pi n}}\sum_{k=0}^{\infty}\frac{(-)^kc_k}{n^k}\qquad(n\to\infty)\tag{3.7}$$

valid for arbitrary, finite values of p, where the coefficients c_k are defined by

$$c_k=\sum_{r=0}^{k}\binom{p}{r}A_{k-r}^{(r)}.$$

Use of the values of the coefficients $A_j^{(r)}$ in Table 2 shows that the c_k for $k \leqslant 5$ have the values

$$c_0=1, \quad c_1=\frac{1}{8}(8p+5), \quad c_2=\frac{1}{128}(192p^2+144p+73),$$
$$c_3=\frac{1}{3072}(6656p^3+4416p^2+6808p+1725),$$
$$c_4=\frac{1}{32768}(102400p^4+30720p^3+214400p^2+52320p+18459),$$
$$c_5=\frac{1}{3932160}(17727488p^5-6082560p^4+70517760p^3-5950080p^2 +21964072p+2222325).$$

It is seen that the coefficients up to and including c_3 agree with those obtained in Section 2 valid for positive integer p. We note the presence of negative terms in the coefficient c_5.

We present the results of numerical calculations. In Table 1 we show the absolute value of the relative error in the computation of the sum $S_p(n)$ by means of the asymptotic expansion (3.7) for different values of n, p and truncation index j. In each case the value of n has been chosen so that the optimal truncation point of the asymptotic series in (3.7) corresponds to $j>5$. Consequently, the relative error progressively decreases with each increment in the truncation index, thereby confirming the validity of the asymptotic series (3.7).

j	$n=50$, $p=-\frac{1}{2}$	$n=50$, $p=\frac{3}{2}$	$n=100$, $p=\frac{1}{2}$	$n=100$, $p=\frac{3}{4}$
0	2.348×10^{-3}	4.205×10^{-2}	1.123×10^{-2}	1.371×10^{-2}
1	1.574×10^{-4}	2.233×10^{-3}	1.502×10^{-4}	2.249×10^{-4}
2	3.928×10^{-6}	1.145×10^{-4}	2.293×10^{-6}	3.932×10^{-6}
3	2.550×10^{-7}	5.819×10^{-6}	3.294×10^{-8}	6.799×10^{-8}
4	1.786×10^{-8}	2.936×10^{-7}	5.256×10^{-10}	1.201×10^{-9}
5	1.544×10^{-9}	1.477×10^{-8}	6.841×10^{-12}	2.061×10^{-11}

Table 1: The relative error in the computation of $S_p(n)$ by (3.7) for different truncation index j.

4. Concluding remarks

We have presented two methods to generate an asymptotic expansion for the sum $S_p(n)$ in (1.1) for large n, completing an examination of properties of the sum discussed elsewhere. The first method is valid for positive integer p and relies on a decomposition of $S_p(n)$ in terms of the Stirling numbers of the second kind. The second method is valid for arbitrary p and is based on an integral representation combined with an application of the method of steepest descents.

When $p=0$, we have the well-known result [4, p. 619]

$$S_0(n)+1=\sum_{j=0}^{n}\binom{n+j}{j}=\binom{2n+1}{n}=\frac{2^{2n+1}}{\sqrt{\pi}}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+2)}.$$

By means of the expansion of the ratio of two gamma functions [5, p. 119], [6, p. 50] this produces the large-n expansion

$$S_0(n)\sim\frac{2^{2n+1}}{\sqrt{\pi n}}\left\{1-\frac{5}{8n}+\frac{73}{128n^2}-\frac{575}{1024n^3}+\frac{18459}{32768n^4}-\frac{148155}{262144n^5}+\cdots\right\}.$$

This is seen to be in agreement with (3.7) when the coefficients c_k are evaluated at $p=0$.

Appendix A: The expansion of the integrals J_r

We consider the asymptotic expansion of the integrals

$$J_r=\frac{1}{2\pi i}\int_C\frac{e^{n\psi(t)}}{t-1}\Theta^r\sigma\,dt\qquad(r=0,1,2,\ldots)\tag{A.1}$$

for $n\to\infty$, where $\psi(t)$ is defined in (3.2), C is the semi-circular path described in Section 3 that passes through the saddle point $t=2$, $\Theta\equiv td/dt$ and $\sigma=t/(t-1)$. Routine calculations show that

$$\Theta^r\sigma=\frac{(-)^r tq_r(t)}{(t-1)^{r+1}},$$

where

$$q_0(t)=q_1(t)=1,\ \ q_2(t)=t+1,\ \ q_3(t)=t^2+4t+1,$$
$$q_4(t)=t^3+11t^2+11t+1,\ \ q_5(t)=t^4+26t^3+66t^2+26t+1,\ldots .$$

We now make the change of variable $t\mapsto\tau$ in (A.1) given by

$$-\tau^2=\psi(t)-2\log 2=\frac{(t-2)^2}{4}-\frac{(t-2)^3}{4}+\frac{7(t-2)^4}{32}+\ldots,$$

so that the point $t=2$ corresponds to $\tau=0$. Straightforward differentiation gives

$$\frac{dt}{d\tau}=-\frac{2\tau t(t-1)}{t-2}$$

and inversion of the above series yields

$$t-2=2i\tau-2\tau^2-\frac{3}{2}i\tau^3+\tau^4+\frac{29i\tau^5}{48}-\frac{\tau^6}{3}-\frac{11i\tau^7}{64}+\ldots .\tag{A.2}$$

Then, since the endpoints of C at $t=1\pm i$ correspond to $\tau=\tau_0=\pm(\log 2)^{1/2}$, we find

$$J_r=\frac{2^{2n}}{2\pi i}\int_{-\tau_0}^{\tau_0}\frac{e^{-n\tau^2}}{t-1}\Theta^r\sigma\frac{dt}{d\tau}d\tau=\frac{2^{2n+1}}{\pi}\int_{-\tau_0}^{\tau_0}e^{-n\tau^2}Q_r(\tau)\,d\tau,\tag{A.3}$$

where

$$Q_r(\tau)=\frac{(-)^r i\tau t^2 q_r(t)}{2(t-2)(t-1)^{r+1}}. \tag{A.4}$$

We present only the calculation of the expansion of J_0, the details for J_r with $r=1,2,\ldots$ being similar. With the help of *Mathematica* and (A.2), we obtain the expansion of $Q_0(\tau)$ in powers of τ given by[2]

$$Q_0(\tau)=\frac{i\tau t^2}{2(t-1)(t-2)}\stackrel{\mathrm{e}}{=}\sum_{k=0}^{\infty}(-)^k a_k\tau^{2k},$$

where

$$a_0=1,\quad a_1=\frac{5}{4},\quad a_2=\frac{73}{96},\quad a_3=\frac{115}{384},\quad a_4=\frac{879}{10240},\quad a_5=\frac{1411}{73728},\ldots .$$

The limits of integration in (A.3) may be extended to $\pm\infty$ (thereby describing the full steepest descent path in the t-plane) with the introduction of an exponentially small error of $O(2^{-n})$. Neglecting exponentially small terms, we then obtain

$$J_0\simeq\frac{2^{2n+1}}{\pi}\int_{-\infty}^{\infty}e^{-n\tau^2}Q_0(\tau)\,d\tau\sim\frac{2^{2n+1}}{\sqrt{\pi n}}\sum_{k=0}^{\infty}\frac{(-)^k a_k\Gamma(k+\frac{1}{2})}{n^k\Gamma(\frac{1}{2})}\qquad(n\to\infty)$$

upon straightforward evaluation of the integrals in terms of the gamma function.

By means of similar calculations the expansions of $Q_r(\tau)$ for $1\leqslant r\leqslant 5$ are found to be

$$\begin{aligned}
Q_1(\tau)&\stackrel{\mathrm{e}}{=}1-\frac{21}{4}\tau^2+\frac{745}{96}\tau^4-\frac{833}{128}\tau^6+\frac{38959}{10240}\tau^8-\ldots,\\
Q_2(\tau)&\stackrel{\mathrm{e}}{=}3-\frac{127}{4}\tau^2+\frac{2665}{32}\tau^4-\frac{44681}{384}\tau^6+\ldots,\\
Q_3(\tau)&\stackrel{\mathrm{e}}{=}13-\frac{945}{4}\tau^2+\frac{93205}{96}\tau^4-\ldots,\\
Q_4(\tau)&\stackrel{\mathrm{e}}{=}75-\frac{8359}{4}\tau^2+\ldots,\\
Q_5(\tau)&\stackrel{\mathrm{e}}{=}541+O(\tau^2).
\end{aligned}$$

An analogous procedure then yields the desired expansion of J_r in the form

$$J_r\sim\frac{2^{2n+1}(-)^r}{\sqrt{\pi n}}\sum_{j=0}^{\infty}(-)^j A_j^{(r)}n^{-j} \tag{A.5}$$

as $n\to\infty$, where the coefficients $A_j^{(r)}$ for $r+j\leqslant 5$ are listed in Table 2. From (A.2) and (A.4), it is seen that $A_0^{(r)}=Q_r(0)=q_r(2)$.

[2]We use the symbol $\stackrel{\mathrm{e}}{=}$ to indicate that only *even* powers of τ are included. The terms involving odd powers of τ do not enter into our calculations and so are not shown.

$r \backslash j$	0	1	2	3	4	5
0	1	$\frac{5}{8}$	$\frac{73}{128}$	$\frac{575}{1024}$	$\frac{18459}{32768}$	$\frac{148155}{262144}$
1	1	$\frac{21}{8}$	$\frac{745}{128}$	$\frac{12495}{1024}$	$\frac{818139}{32768}$	
2	3	$\frac{127}{8}$	$\frac{7995}{128}$	$\frac{223405}{1024}$		
3	13	$\frac{945}{8}$	$\frac{93205}{128}$			
4	75	$\frac{8359}{8}$				
5	541					

Table 2: The coefficients $A_j^{(r)}$ for $r+j \leqslant 5$.

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R. B. Paris
School of Computing, University of Abertay Dundee
Dundee DD1 1HG, UK
e-mail: r.paris@abertay.ac.uk

P. J. Larcombe
School of Computing and Mathematics, University of Derby
Kedleston Road, Derby DE22 1GB, UK
e-mail: p.j.larcombe@derby.ac.uk