Euler-type transformations for the generalized hypergeometric function ${}_{r+2}F_{r+1}(x)$

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Abstract

We provide generalizations of two of Euler's classical transformation formulas for the Gauss hypergeometric function extended to the case of the generalized hypergeometric function ${}_{r+2}F_{r+1}(x)$ when there are additional numeratorial and denominatorial parameters differing by unity. The method employed to deduce the latter is also implemented to obtain a Kummer-type transformation formula for ${}_{r+1}F_{r+1}(x)$ that was recently derived in a different way.

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1. Introduction

The well-known Euler transformation formulas for the Gauss hypergeometric function ${}_2F_1(x)$ state that

$${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\,\middle|\,x\right)=(1-x)^{-a}{}_2F_1\left(\begin{matrix}a,c-b\\c\end{matrix}\,\middle|\,\frac{x}{x-1}\right), \tag{1.1}$$

where x lies in the domain $|x|<1$, $\mathrm{Re}\,(x)<\frac{1}{2}$, and

$${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\,\middle|\,x\right)=(1-x)^{c-a-b}\,{}_2F_1\left(\begin{matrix}c-a,c-b\\c\end{matrix}\,\middle|\,x\right) \tag{1.2}$$

valid in $|x|<1$. In [1], Rathie and Paris derived an extension of the first transformation (1.1) that is equivalent to

$${}_3F_2\left(\begin{matrix}a,b, & f+1\\c, & f\end{matrix}\,\middle|\,x\right)=(1-x)^{-a}{}_3F_2\left(\begin{matrix}a,c-b-1, & \xi+1\\c, & \xi\end{matrix}\,\middle|\,\frac{x}{x-1}\right), \tag{1.3a}$$

where

$$\xi = \frac{f(c-b-1)}{f-b} \tag{1.3b}$$

and x lies in the domain $|x| < 1$, $\mathrm{Re}\,(x) < \frac{1}{2}$. Equation (1.3a) was suggested by a less general result due to Exton [2] in the case $f = \frac{1}{2}b$.

In Section 3 we shall derive a generalization of the first Euler transformation (1.1) to include additional numeratorial and denominatorial parameters differing by unity that we record in Theorem 2. We give two methods of proof for this extension: the first method relies on a reduction formula for a certain Kampé de Fériet function, whereas the second method employs a reduction to a finite sum of ${}_2F_1(x)$ functions. Then in Theorem 3 of Section 4 by employing the second method used to obtain Theorem 2, we shall obtain the analogous extension of the second Euler transformation (1.2). A similar approach is also employed to deduce a Kummer-type transformation formula for the generalized hypergeometric function ${}_{r+1}F_{r+1}(x)$; see also [3].

In what follows the Pochhammer symbol $(a)_k$ for integers k is defined by $(a)_k \equiv \Gamma(a+k)/\Gamma(a)$ and the product of r Pochhammer symbols is written as

$$((f_r))_k \equiv (f_1)_k \ldots (f_r)_k,$$

where an empty product ($r = 0$) reduces to unity. A finite sequence (except where noted otherwise) of parameters $f_1, \ldots, f_r$ is denoted simply by (f_r). The symbol $\left\{ {n \atop k} \right\}$ will be used to denote the Stirling numbers of the second kind. These nonnegative integers represent the number of ways to partition a set of n objects into k nonempty subsets and arise for nonnegative integers n in the generating relation

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (-1)^k (-x)_k, \qquad \left\{ {n \atop 0} \right\} = \delta_{0n}, \tag{1.4}$$

where δ_{0n} is the Kronecker symbol; for an introduction to Stirling numbers and their properties, see for example [4].

2. Preliminary results

We shall require two lemmas and a theorem whose proofs are found in [5].

Lemma 1. For nonnegative integers j define

$$S_j \equiv \sum_{k=0}^{\infty} k^j \frac{\lambda_k}{k!}, \qquad S_0 \equiv \sum_{k=0}^{\infty} \frac{\lambda_k}{k!},$$

where the infinite sequence (λ_k) is such that S_j converges for all j. Then

$$S_j = \sum_{\ell=0}^{j} \left\{ {j \atop \ell} \right\} \sum_{k=0}^{\infty} \frac{\lambda_{k+\ell}}{k!}.$$

Lemma 2. Consider the polynomial in n of degree $\mu \geq 1$ given by

$$P_\mu(n) \equiv \alpha_0 n^\mu + \alpha_1 n^{\mu-1} + \ldots + \alpha_{\mu-1} n + \alpha_\mu,$$

where $\alpha_0 \neq 0$ and $\alpha_\mu \neq 0$. Then we may write

$$P_\mu(n) = \alpha_\mu \frac{((\xi_\mu + 1))_n}{((\xi_\mu))_n},$$

where the (ξ_μ) are the nonvanishing zeros of the polynomial $Q_\mu(t)$ defined by

$$Q_\mu(t) \equiv \alpha_0(-t)^\mu + \alpha_1(-t)^{\mu-1} + \ldots + \alpha_{\mu-1}(-t) + \alpha_\mu.$$

The following theorem concerns a specialization of a generalized hypergeometric function in two variables called the Kampé de Fériet function. For a brief introduction to the latter see [5] or [6].

Theorem 1. *Suppose $b \neq f_j$ $(1 \leq j \leq r)$ and $(c-b-r)_r \neq 0$. Then we have the reduction formula for the Kampé de Fériet function*

$$F\,{}^{p:r+1;0}_{q:r+1;0}\left(\begin{array}{l} (a_p) : b, \;\; (f_r+1) \; ; \; \text{—} \\ (b_q) \; : c, \;\;\;\; (f_r) \;\;\;\; ; \; \text{—} \end{array}\middle| -x, x\right)$$

$$= {}_{p+r+1}F_{q+r+1}\left(\begin{array}{ccc} c-b-r, & (a_p), & (\xi_r+1) \\ c, & (b_q), & (\xi_r) \end{array}\middle| x\right),$$

where a solid horizontal line indicates an empty parameter sequence. The (ξ_r) are the nonvanishing zeros of the associated parametric polynomial of degree r given by

$$Q_r(t) = \sum_{j=0}^{r} s_{r-j} \sum_{\ell=0}^{j} \left\{ {j \atop \ell} \right\} (b)_\ell (t)_\ell (c-b-r-t)_{r-\ell}, \tag{2.1}$$

where the s_{r-j} $(0 \leq j \leq r)$ are determined by the generating relation

$$(f_1+x)\ldots(f_r+x) = \sum_{j=0}^{r} s_{r-j} x^j. \tag{2.2}$$

Note that when $f_1 = \ldots = f_r = f$, then $s_{r-j} = \binom{r}{j} f^{r-j}$ for $0 \leq j \leq r$.

Equation (2.1) may be written in a slightly more compact form by defining

$$\lambda \equiv c-b-r$$

and observing that $\left\{ {n \atop k} \right\} = 0$ when $k > n$. Then, for an arbitrary sequence (B_ℓ), we have

$$\sum_{j=0}^{r} s_{r-j} \sum_{\ell=0}^{j} \left\{ {j \atop \ell} \right\} B_\ell = \sum_{j=0}^{r} s_{r-j} \sum_{\ell=0}^{r} \left\{ {j \atop \ell} \right\} B_\ell$$

$$= \sum_{\ell=0}^{r} B_\ell \left(\sum_{j=0}^{r} \left\{ {j \atop \ell} \right\} s_{r-j} \right) = \sum_{\ell=0}^{r} B_\ell \left(\sum_{j=\ell}^{r} \left\{ {j \atop \ell} \right\} s_{r-j} \right).$$

Thus defining

$$A_\ell \equiv \sum_{j=\ell}^{r} \left\{ {j \atop \ell} \right\} s_{r-j} \tag{2.3}$$

and setting $B_\ell = (b)_\ell (t)_\ell (\lambda - t)_{r-\ell}$, we see that (2.1) may be written as

$$Q_r(t) = \sum_{\ell=0}^{r} A_\ell (b)_\ell (t)_\ell (\lambda - t)_{r-\ell}\,. \tag{2.4}$$

3. The first generalized Euler-type transformation

We are now ready to state and prove Theorem 2 below in two different ways.

Theorem 2. *Suppose* $b \neq f_j \quad (1 \leq j \leq r)$ *and* $(\lambda)_r \neq 0$. *Then*

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r+1) \\ c, & (f_r) \end{matrix} \,\middle|\, x\right) = (1-x)^{-a}\,{}_{r+2}F_{r+1}\left(\begin{matrix} a, \lambda, & (\xi_r+1) \\ c, & (\xi_r) \end{matrix} \,\middle|\, \frac{x}{x-1}\right), \tag{3.1} $$

where $\lambda = c - b - r$ *and* $|x| < 1$, $Re\,(x) < \frac{1}{2}$. *The* (ξ_r) *are the nonvanishing zeros of the associated parametric polynomial* $Q_r(t)$ *given by (2.4). Furthermore, when* $r = 0$, *(3.1) reduces to the first Euler transformation formula for* ${}_2F_1(x)$ *given by (1.1).*

Proof I. Consider

$$ F(y) \equiv (1-y)^{-a}\,{}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r+1) \\ c, & (f_r) \end{matrix} \,\middle|\, \frac{y}{y-1}\right) $$

$$ = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} \frac{((f_r+1))_m}{((f_r))_m} (-y)^m (1-y)^{-a-m}. $$

Since for $|y| < 1$

$$ (1-y)^{-a-m} = \sum_{n=0}^{\infty} \frac{(a+m)_n}{n!} y^n \tag{3.2} $$

we have, upon noting the identity

$$ (\alpha)_{m+n} = (\alpha)_m (\alpha+m)_n = (\alpha)_n (\alpha+n)_m\,, \tag{3.3} $$

$$ \begin{aligned} F(y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (a)_{m+n} \frac{(b)_m}{(c)_m} \frac{((f_r+1))_m}{((f_r))_m} \frac{(-y)^m y^n}{m!\,n!} \\ &= F^{\,1:r+1;0}_{\,0:r+1;0}\left(\begin{matrix} a & : & b, & (f_r+1) & ; & \text{—} \\ \text{—} & : & c, & (f_r) & ; & \text{—} \end{matrix} \,\middle|\, -y, y\right). \end{aligned} $$

Now applying Theorem 1 with $p = 1$, $q = 0$ and $a_1 = a$ we find

$$ F(y) = {}_{r+2}F_{r+1}\left(\begin{matrix} a, c-b-r, & (\xi_r+1) \\ c, & (\xi_r) \end{matrix} \,\middle|\, y\right) $$

so that

$$ (1-y)^{-a}\,{}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r+1) \\ c, & (f_r) \end{matrix} \,\middle|\, \frac{y}{y-1}\right) = {}_{r+2}F_{r+1}\left(\begin{matrix} a, c-b-r, & (\xi_r+1) \\ c, & (\xi_r) \end{matrix} \,\middle|\, y\right). $$

The (ξ_r) are the nonvanishing zeros of the associated parametric polynomial of degree r given by

$$ Q_r(t) = \sum_{\ell=0}^{r} A_\ell (b)_\ell (t)_\ell (c-b-r-t)_{r-\ell}, $$

where the A_ℓ are given by (2.3). Finally, letting $y = x/(x-1)$ we deduce Theorem 2. □

The second proof of Theorem 2 provided next dispenses with the reduction formula for the Kampé de Fériet function given in Theorem 1, but relies instead on (1.1) and Lemmas 1 and 2.

Proof II. Let

$$F(x) \equiv {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r+1) \\ c, & (f_r) \end{matrix} \,\middle|\, x\right). \tag{3.4}$$

Since

$$\frac{((f_r+1))_n}{((f_r))_n} = \frac{(f_1+n)\ldots(f_r+n)}{f_1\ldots f_r},$$

we have

$$F(x) = \frac{1}{f_1\ldots f_r}\sum_{j=0}^{r} s_{r-j}\sum_{n=0}^{\infty} n^j \frac{(a)_n(b)_n}{(c)_n}\,\frac{x^n}{n!},$$

where the coefficients s_{r-j} $(0 \le j \le r)$ are defined by (2.2) and the order of summation has been interchanged. Now applying Lemma 1 to the n-summation we obtain upon use of (3.3)

$$\begin{aligned} F(x) &= \frac{1}{f_1\ldots f_r}\sum_{j=0}^{r} s_{r-j}\sum_{\ell=0}^{j}\left\{\begin{matrix} j \\ \ell \end{matrix}\right\}\sum_{n=0}^{\infty}\frac{(a)_{n+\ell}(b)_{n+\ell}}{(c)_{n+\ell}}\,\frac{x^{n+\ell}}{n!} \\ &= \frac{1}{f_1\ldots f_r}\sum_{j=0}^{r} s_{r-j}\sum_{\ell=0}^{j}\left\{\begin{matrix} j \\ \ell \end{matrix}\right\}\frac{(a)_\ell(b)_\ell}{(c)_\ell}\,x^\ell\sum_{n=0}^{\infty}\frac{(a+\ell)_n(b+\ell)_n}{(c+\ell)_n}\,\frac{x^n}{n!} \\ &= \frac{1}{f_1\ldots f_r}\sum_{j=0}^{r} s_{r-j}\sum_{\ell=0}^{j}\left\{\begin{matrix} j \\ \ell \end{matrix}\right\}\frac{(a)_\ell(b)_\ell}{(c)_\ell}\,x^\ell\,{}_2F_1\left(\begin{matrix} a+\ell, b+\ell \\ c+\ell \end{matrix}\,\middle|\, x\right). \end{aligned} \tag{3.5}$$

The result (3.5) has expressed the hypergeometric function in (3.4) as a finite sum of Gauss hypergeometric functions with coefficients involving the Stirling numbers of the second kind and the s_j $(0 \le j \le r)$ which are defined implicitly by the generating relation (2.2).

We can now apply the first Euler transformation (1.1) to the ${}_2F_1(x)$ functions in (3.5) to find

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a+\ell, b+\ell \\ c+\ell \end{matrix}\,\middle|\, x\right) &= (1-x)^{-a-\ell}\,{}_2F_1\left(\begin{matrix} a+\ell,\ c-b \\ c+\ell \end{matrix}\,\middle|\, \frac{x}{x-1}\right) \\ &= (1-x)^{-a-\ell}\sum_{n=0}^{\infty}\frac{(a+\ell)_n}{(c+\ell)_n}\,\frac{(c-b)_n}{n!}\,\left(\frac{x}{x-1}\right)^n \\ &= \frac{(-x)^{-\ell}}{(1-x)^a}\sum_{n=\ell}^{\infty}\frac{(a+\ell)_{n-\ell}(c-b)_{n-\ell}}{(c+\ell)_{n-\ell}(1)_{n-\ell}}\left(\frac{x}{x-1}\right)^n. \end{aligned}$$

Upon noting that

$$\frac{1}{(1)_{n-\ell}} = \frac{(-1)^\ell(-n)_\ell}{n!}, \qquad (a+\ell)_{n-\ell} = \frac{(a)_n}{(a)_\ell} \tag{3.6}$$

and, with $\lambda = c-b-r$ and $p \equiv r-\ell$, that

$$(c-b)_{n-\ell} = \frac{\Gamma(\lambda+n+p)}{\Gamma(\lambda+r)} = \frac{(\lambda)_{n+p}}{(\lambda)_r} = \frac{(\lambda)_n(\lambda+n)_p}{(\lambda)_r}, \tag{3.7}$$

we then obtain

$$\begin{aligned} & x^\ell\,{}_2F_1\left(\begin{matrix} a+\ell, b+\ell \\ c+\ell \end{matrix}\,\middle|\, x\right) \\ &\quad = (1-x)^{-a}\,\frac{(c)_\ell}{(\lambda)_r(a)_\ell}\sum_{n=\ell}^{\infty}\frac{(a)_n(\lambda)_n}{(c)_n n!}\left(\frac{x}{x-1}\right)^n(-n)_\ell(\lambda+n)_{r-\ell}. \end{aligned}$$

Substitution of this last result into (3.5) then leads to

$$F(x)=(1-x)^{-a}\sum_{n=0}^{\infty}\frac{(a)_n(\lambda)_n}{(c)_n n!}\left(\frac{x}{x-1}\right)^n$$

$$\times\frac{1}{f_1\ldots f_r\,(\lambda)_r}\sum_{j=0}^{r}s_{r-j}\sum_{\ell=0}^{j}\left\{{j \atop \ell}\right\}(b)_\ell(-n)_\ell(\lambda+n)_{r-\ell}, \qquad (3.8)$$

where the order of summation has been interchanged and the summation index $n=\ell$ has been replaced by $n=0$ since $(-n)_\ell=0$ when $n<\ell$.

With

$$P_r(n)\equiv\sum_{j=0}^{r}s_{r-j}\sum_{\ell=0}^{j}\left\{{j \atop \ell}\right\}(b)_\ell(-n)_\ell(\lambda+n)_{r-\ell}, \qquad (3.9)$$

it is shown in [5, p. 968] that $P_r(n)$ is a polynomial in n of degree r that takes the form

$$P_r(n)=(f_1-b)\ldots(f_r-b)n^r+\ldots+f_1\ldots f_r\,(\lambda)_r, \qquad (3.10)$$

where, when $r>1$, the intermediate coefficients of powers of n may be computed by using (3.9). Now assuming $b\neq f_j$ $(1\leq j\leq r)$ and $(\lambda)_r\neq 0$, we may invoke Lemma 2 to obtain

$$P_r(n)=f_1\ldots f_r\,(\lambda)_r\frac{((\xi_r+1))_n}{((\xi_r))_n}, \qquad (3.11)$$

where the (ξ_r) are the nonvanishing zeros of

$$Q_r(t)=\sum_{j=0}^{r}s_{r-j}\sum_{\ell=0}^{j}\left\{{j \atop \ell}\right\}(b)_\ell(t)_\ell(\lambda-t)_{r-\ell}. \qquad (3.12)$$

Finally, combining (3.8), (3.9) and (3.11) we have

$${}_{r+2}F_{r+1}\left(\begin{matrix}a,b, & (f_r+1)\\ c, & (f_r)\end{matrix}\,\middle|\,x\right)=(1-x)^{-a}\sum_{n=0}^{\infty}\frac{(a)_n(\lambda)_n}{(c)_n n!}\,\frac{((\xi_r+1))_n}{((\xi_r))_n}\,\left(\frac{x}{x-1}\right)^n$$

and, recalling that (3.12) may be written as (2.4), the proof of Theorem 1 is evident. □

If $r=1$, then $\lambda=c-b-1$. Thus letting $f_1=f$ (so that $s_0=1$, $s_1=f$) and replacing n by $-t$ in (3.10), we see immediately that

$$Q_1(t)=(b-f)t+f(c-b-1)$$

whose nonvanishing zero ξ is given by

$$\xi=\frac{f(c-b-1)}{f-b}\;. \qquad (3.13)$$

Thus, when $r=1$, Theorem 2 reduces to (1.3a) and (1.3b) obtained previously in [1] and subsequently also in [8].

4. The second generalized Euler-type transformation

We now state Theorem 3 below which provides a generalization of the second Euler transformation (1.2). This result will be established by following the second method of proof employed in Section 3.

Theorem 3. *Suppose*[1] $(1+a+b-c)_r \neq 0$ *and* $(\lambda)_r \neq 0$, $(\lambda')_r \neq 0$. *Then*

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, & (f_r+1) \\ c, & (f_r) \end{matrix} \,\middle|\, x\right) = (1-x)^{c-a-b-r} {}_{r+2}F_{r+1}\left(\begin{matrix} \lambda, \lambda', & (\xi_r+1) \\ c, & (\xi_r) \end{matrix} \,\middle|\, x\right) \tag{4.1}$$

valid in $|x|<1$, *where* $\lambda = c-b-r$ *and* $\lambda' = c-a-r$. *The* (ξ_r) *are the nonvanishing zeros of the associated parametric polynomial* $Q_r(t)$ *of degree* r *given by*

$$Q_r(t) = \sum_{\ell=0}^{r} (-1)^\ell A_\ell (a)_\ell (b)_\ell (t)_\ell (\lambda-t)_p (\lambda'-t)_p \, G_{p,\ell}(t), \tag{4.2}$$

where $p \equiv r-\ell$, *the coefficients* A_ℓ *are defined by (2.3) and*

$$G_{p,\ell}(t) \equiv {}_3F_2\left(\begin{matrix} -p,\, \ell+t,\, 1-c+t \\ 1-\lambda-p+t,\, 1-\lambda'-p+t \end{matrix} \,\middle|\, 1\right). \tag{4.3}$$

Furthermore, when $r=0$, *(4.1) reduces to the second Euler transformation given by (1.2).*

Proof. We commence with the expression for $F(x)$ defined by (3.4) as a finite sum of ${}_2F_1(x)$ functions given by (3.5). Application of (1.2) to each of the latter functions yields

$$\begin{aligned} x^\ell {}_2F_1\left(\begin{matrix} a+\ell, b+\ell \\ c+\ell \end{matrix} \,\middle|\, x\right) &= x^\ell (1-x)^{c-a-b-\ell} {}_2F_1\left(\begin{matrix} c-a, c-b \\ c+\ell \end{matrix} \,\middle|\, x\right) \\ &= (1-x)^{c-a-b-r} \sum_{k=0}^{p} \frac{(-p)_k}{k!} \sum_{n=0}^{\infty} \frac{(c-a)_n (c-b)_n}{(c+\ell)_n} \frac{x^{n+\ell+k}}{n!} \end{aligned}$$

upon expansion of the factor $(1-x)^{r-\ell}$ by the binomial theorem, where $p \equiv r-\ell$. If we now change the summation index $n \mapsto n+\ell+k$ and make use of (3.6), (3.7) and the identity

$$(\alpha)_{-k} = (-1)^k/(1-\alpha)_k, \tag{4.4}$$

the right-hand side of the above equation can be written as

$$(1-x)^{c-a-b-r} \sum_{k=0}^{p} \frac{(-p)_k}{k!} \sum_{n=\ell+k}^{\infty} \frac{(c-a)_{n-\ell-k}(c-b)_{n-\ell-k}}{(c+\ell)_{n-\ell-k}(1)_{n-\ell-k}} x^n$$

$$= (1-x)^{c-a-b-r} \frac{(-1)^\ell (c)_\ell}{(\lambda)_r (\lambda')_r}$$

$$\times \sum_{k=0}^{p} \frac{(-p)_k}{k!} \sum_{n=0}^{\infty} \frac{(\lambda)_n (\lambda')_n}{(c)_n} \frac{x^n}{n!} (\lambda+n)_{p-k} (\lambda'+n)_{p-k} (-n)_{\ell+k} (1-c-n)_k,$$

where we have replaced the summation index $n=\ell+k$ by $n=0$ since $(-n)_{\ell+k}=0$ for $n<\ell+k$.

Hence, from (3.5), we obtain

$$F(x) = \frac{(1-x)^{c-a-b-r}}{f_1 \ldots f_r \, (\lambda)_r (\lambda')_r} \sum_{n=0}^{\infty} \frac{(\lambda)_n (\lambda')_n}{(c)_n} \frac{x^n}{n!} P_r(n) \tag{4.5}$$

upon interchanging the order of summation, where now

$$P_r(n) \equiv \sum_{j=0}^{r} s_{r-j} \sum_{\ell=0}^{j} (-1)^\ell \left\{\begin{matrix} j \\ \ell \end{matrix}\right\} (a)_\ell (b)_\ell R_p(n) = \sum_{\ell=0}^{r} (-1)^\ell A_\ell (a)_\ell (b)_\ell R_p(n).$$

[1] The following are necessary conditions for the nonvanishing of the (ξ_r); sufficient conditions are given below.

Here $R_p(n)$ is defined by

$$R_p(n) \equiv \sum_{k=0}^{p} \frac{(-p)_k}{k!} (\lambda+n)_{p-k}(\lambda'+n)_{p-k}(-n)_{\ell+k}(1-c-n)_k$$

and the coefficients A_ℓ are given by (2.3).

We shall show below that $P_r(n)$ is a polynomial in n of degree r having the form

$$P_r(n) = \alpha_0 n^r + \cdots + f_1 \ldots f_r\,(\lambda)_r(\lambda')_r,$$

where α_0 is given by (4.9). Assuming that the coefficient $\alpha_0 \neq 0$ and $(\lambda)_r \neq 0$, $(\lambda')_r \neq 0$, we may then invoke Lemma 2 to obtain

$$P_r(n) = f_1 \ldots f_r\,(\lambda)_r(\lambda')_r\,\frac{((\xi_r+1))_n}{((\xi_r))_n}, \tag{4.6}$$

where the (ξ_r) are the nonvanishing zeros of

$$Q_r(t) = \sum_{\ell=0}^{r}(-1)^\ell A_\ell (a)_\ell (b)_\ell \sum_{k=0}^{p} \frac{(-p)_k}{k!}(\lambda-t)_{p-k}(\lambda'-t)_{p-k}(t)_{\ell+k}(1-c+t)_k \tag{4.7a}$$

$$= \sum_{\ell=0}^{r}(-1)^\ell A_\ell (a)_\ell (b)_\ell (t)_\ell (\lambda-t)_p(\lambda'-t)_p G_{p,\ell}(t). \tag{4.7b}$$

In (4.7b) the identities (3.3) and (4.4) have been employed to express the sum over k in (4.7a) in terms of the ${}_3F_2(1)$ series given by $G_{p,\ell}(t)$ in (4.3). For brevity in the sequel we shall work with $Q_r(t)$ instead of $P_r(n)$.

To determine the degree of the polynomial $Q_r(t)$ we employ Sheppard's transformation of the ${}_3F_2(1)$ series given in [7, p. 141], namely

$${}_3F_2\left(\begin{matrix}-n, a, b\\ d, e\end{matrix}\,\middle|\,1\right) = \frac{(d-a)_n(e-a)_n}{(d)_n(e)_n}\,{}_3F_2\left(\begin{matrix}-n,\ a,\ 1-s\\ a-n-d+1,\ a-n-e+1\end{matrix}\,\middle|\,1\right),$$

where n is a nonnegative integer and $s = d+e-a-b+n$ is the parametric excess. Application of this identity to $G_{p,\ell}(t)$ then leads to

$$G_{p,\ell}(t) = \frac{(1-\lambda-p-\ell)_p(1-\lambda'-p-\ell)_p}{(1-\lambda-p+t)_p(1-\lambda'-p+t)_p}\,{}_3F_2\left(\begin{matrix}-p,\ t+\ell,\ 1-s\\ \lambda+\ell,\ \lambda'+\ell\end{matrix}\,\middle|\,1\right),$$

where $s = 1+a+b-c+r$. Finally, employing the identity (4.4) we obtain from (4.7b)

$$Q_r(t) = \sum_{\ell=0}^{r}(-1)^\ell A_\ell (a)_\ell (b)_\ell (t)_\ell (\lambda+\ell)_p(\lambda'+\ell)_p\,{}_3F_2\left(\begin{matrix}-p,\ t+\ell,\ 1-s\\ \lambda+\ell,\ \lambda'+\ell\end{matrix}\,\middle|\,1\right). \tag{4.8}$$

It is now seen that t only appears in a single numeratorial parameter of the ${}_3F_2(1)$ series on the right-hand side of (4.8). Consequently ${}_3F_2(1)$ is a polynomial in t of degree $p = r-\ell$ provided that $s \neq 1, 2, \ldots, p$, which is equivalent to the condition $(1+a+b-c)_r \neq 0$. Since $(t)_\ell$ is a polynomial in t of degree ℓ, it follows that $Q_r(t)$ is a polynomial in t of degree $\ell + p = r$ and hence must have the form

$$Q_r(t) = \alpha_0(-t)^r + \cdots + \alpha_{r-1}(-t) + \alpha_r.$$

In the latter the coefficient of t^r is $(-1)^r\alpha_0$ which can be determined as follows. The highest power of t in the ${}_3F_2(1)$ series in (4.8) arises from the last term when it is expressed as a k-summation; that is when $k = p$

$$\frac{(-1)^p(t+\ell)_p(1-s)_p}{(\lambda+\ell)_p(\lambda'+\ell)_p} = \frac{(-1)^p(1-s)_p}{(\lambda+\ell)_p(\lambda'+\ell)_p}t^p + \cdots.$$

Then, recalling that $p = r - \ell$, we find

$$(-1)^r \alpha_0 = \sum_{\ell=0}^{r} (-1)^{p+\ell} A_\ell (a)_\ell (b)_\ell (1-s)_p$$

thus giving

$$\alpha_0 = (-1)^r (1+a+b-c)_r \sum_{\ell=0}^{r} (-1)^\ell A_\ell \, \frac{(a)_\ell (b)_\ell}{(1+a+b-c)_\ell}. \tag{4.9}$$

Since $Q_r(0) = \alpha_r$, the coefficient α_r is obtained as follows. By noting that when $t = 0$ the only contribution to the double sum in (4.7a) arises when $k = \ell = 0$ we obtain

$$\alpha_r = A_0 (\lambda)_r (\lambda')_r = s_r (\lambda)_r (\lambda')_r = f_1 \ldots f_r \, (\lambda)_r (\lambda')_r, \tag{4.10}$$

where (2.2) and (2.3) have been utilized.

Then, provided $\alpha_0 \neq 0$, $\alpha_r \neq 0$ by Lemma 2, the zeros (ξ_r) of the associated parametric polynomial $Q_r(t)$ are nonvanishing. This requires that $(\lambda)_r \neq 0$ and $(\lambda')_r \neq 0$ for the coefficient $\alpha_r \neq 0$; a necessary condition for $\alpha_0 \neq 0$ is $(1+a+b-c)_r \neq 0$, since if this is satisfied then $(1+a+b-c)_\ell \neq 0$ for $\ell < r$, so that the ℓ-summation in (4.9) exists as a finite value. A sufficient condition for $\alpha_0 \neq 0$ is that the finite sum in (4.9) does not vanish. With these restrictions, it then follows from (4.5) and (4.6) that

$$F(x) = (1-x)^{c-a-b-r} \sum_{n=0}^{\infty} \frac{(\lambda)_n (\lambda')_n}{(c)_n} \, \frac{((\xi_r + 1))_n}{((\xi_r))_n} \, \frac{x^n}{n!},$$

thereby establishing Theorem 3. $\square$

If $r = 1$, then $\lambda = c - b - 1$ and $\lambda' = c - a - 1$. Then letting $f_1 = f$ we find from (4.7a) (or immediately from (4.9) and (4.10)) that

$$Q_1(t) = ((c - \lambda - \lambda' - 1)f - ab)t + \lambda\lambda' f.$$

This yields the transformation formula

$${}_3F_2 \left(\left. \begin{matrix} a, b, & f+1 \\ c, & f \end{matrix} \right| x \right) = (1-x)^{c-a-b-1} {}_3F_2 \left(\left. \begin{matrix} c-a-1, \; c-b-1, & \xi+1 \\ c, & \xi \end{matrix} \right| x \right), \tag{4.11a}$$

where ξ is the nonvanishing zero of $Q_1(t)$ given by

$$\xi = \frac{f(c-a-1)(c-b-1)}{ab + f(c-a-b-1)}. \tag{4.11b}$$

This result has also been obtained by Maier [8] who employed different methods. We remark that, for the second Euler-type transformation, the zero ξ given by (4.11b) depends on the four parameters a, b, c and f, whereas for the first Euler-type transformation, the zero ξ given by (3.13) is *independent* of the parameter a.

5. Generalized Kummer-type transformation

If we set $p = q = 0$ in the results given in Theorem 1, we immediately obtain, upon recalling (2.3), the following [5]

Theorem 4. *Suppose $b \neq f_j$ $(1 \leq j \leq r)$ and $(c-b-r)_r \neq 0$. Then*

$$ {}_{r+1}F_{r+1}\left(\begin{array}{cc} b, & (f_r+1) \\ c, & (f_r) \end{array} \middle| x \right) = e^x \, {}_{r+1}F_{r+1}\left(\begin{array}{cc} c-b-r, & (\xi_r+1) \\ c, & (\xi_r) \end{array} \middle| -x \right), \tag{5.1} $$

where the (ξ_r) are the nonvanishing zeros of the associated parametric polynomial of degee r given by (2.4).

Note that when $r=0$, (5.1) reduces to Kummer's classical transformation formula for the confluent hypergeometric function, namely

$$ {}_1F_1\left(\begin{array}{c} b \\ c \end{array} \middle| x \right) = e^x \, {}_1F_1\left(\begin{array}{c} c-b \\ c \end{array} \middle| -x \right). \tag{5.2} $$

We indicate below another proof of Theorem 4 that does not rely on the reduction formula for the Kampé de Fériet function given in Theorem 1. We shall omit the obvious details since this proof is very similar to Proof II of Theorem 2. Theorem 4 has also been discussed in [3], where a slightly different derivation that utilizes the generating relation (1.4) has been given.

Proof. Calling the left-hand side of (5.1) $F(x)$, we follow the same procedure leading to (3.5) to find

$$ \begin{aligned} F(x) &= \frac{1}{f_1 \ldots f_r} \sum_{j=0}^{r} s_{r-j} \sum_{\ell=0}^{j} \left\{ \begin{array}{c} j \\ \ell \end{array} \right\} \frac{(b)_\ell}{(c)_\ell} x^\ell \, {}_1F_1\left(\begin{array}{c} b+\ell \\ c+\ell \end{array} \middle| x \right) \\ &= \frac{e^x}{f_1 \ldots f_r} \sum_{j=0}^{r} s_{r-j} \sum_{\ell=0}^{j} \left\{ \begin{array}{c} j \\ \ell \end{array} \right\} \frac{(b)_\ell}{(c)_\ell} x^\ell \, {}_1F_1\left(\begin{array}{c} c-b \\ c+\ell \end{array} \middle| -x \right), \end{aligned} \tag{5.3} $$

where we have utilized the Kummer transformation (5.2). Now

$$ x^\ell \, {}_1F_1\left(\begin{array}{c} c-b \\ c+\ell \end{array} \middle| -x \right) = x^\ell \sum_{n=0}^{\infty} \frac{(c-b)_n}{(c+\ell)_n} \frac{(-x)^n}{n!} = (-1)^\ell \sum_{n=\ell}^{\infty} \frac{(c-b)_{n-\ell}}{(c+\ell)_{n-\ell}} \frac{(-x)^n}{(1)_{n-\ell}}, $$

so that using the identities in (3.6) and (3.7) we may write the last expression above as

$$ \frac{(c)_\ell}{(\lambda)_r} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(c)_n} \frac{(-x)^n}{n!} (-n)_\ell (\lambda+n)_{r-\ell}, $$

where $\lambda = c-b-r$ and we have replaced the summation index $n=\ell$ by $n=0$ since $(-n)_\ell = 0$ for $n < \ell$.

Substitution of this result into (5.3), followed by an interchange in the order of summations, then leads to

$$ F(x) = \frac{e^x}{f_1 \ldots f_r (\lambda)_r} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(c)_n} \frac{(-x)^n}{n!} \sum_{j=0}^{r} s_{r-j} \sum_{\ell=0}^{j} \left\{ \begin{array}{c} j \\ \ell \end{array} \right\} (b)_\ell (-n)_\ell (\lambda+n)_{r-\ell}. $$

Now recalling (3.9) we therefore deduce

$$ {}_{r+1}F_{r+1}\left(\begin{array}{cc} b, & (f_r+1) \\ c, & (f_r) \end{array} \middle| x \right) = \frac{e^x}{f_1 \ldots f_r (\lambda)_r} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(c)_n} \frac{(-x)^n}{n!} P_r(n). $$

Finally, utilizing (3.11) we have for the right-hand side of the above equation

$$ e^x \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(c)_n} \frac{((\xi_r+1))_n}{((\xi_r))_n} \frac{(-x)^n}{n!}, $$

where the (ξ_r) are the nonvanishing zeros of $Q_r(t)$ given by (3.12), or equivalently by (2.4). The proof of Theorem 4 is evidently complete. $\square$

It is noteworthy that the generalized first Euler-type and Kummer-type transformations have the same associated parametric polynomial. See [3, 5] for additional results pertaining to the latter polynomial.

We note that when $r = 1$, Theorem 4 yields

$$ {}_2F_2\left(\begin{matrix} b, & f+1 \\ c, & f \end{matrix} \middle| x\right) = e^x\, {}_2F_2\left(\begin{matrix} c-b-1, & \xi+1 \\ c, & \xi \end{matrix} \middle| -x\right), \tag{5.4} $$

where ξ is given by (3.13). This Kummer-type transformation was first deduced by Paris [9] and generalized a transformation for the above ${}_2F_2(x)$ with $f = \frac{1}{2}b$ given by Exton [2] and rederived using other methods by Miller [10]. Other derivations of (5.4) have been recorded in [1, 11, 12].

We remark that the methods employed herein have been utilized in [13] to obtain a generalization of the Karlsson-Minton summation formula. They can also be brought to bear on quadratic transformations of certain specializations of the generalized hypergeometric function in (3.4); these quadratic transformations will be given elsewhere [14]. As an example, we have

Theorem 5. *Suppose* $(c-r-\frac{1}{2})_r \neq 0$. *Then*

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} a,\ a+\frac{1}{2}, & (f_r+1) \\ c, & (f_r) \end{matrix} \middle| \frac{x^2}{(1\mp x)^2}\right) $$

$$ = (1\mp x)^{2a}\ {}_{2r+2}F_{2r+1}\left(\begin{matrix} 2a,\ c-r-\frac{1}{2}, & (\xi_{2r}+1) \\ 2c-1, & (\xi_{2r}) \end{matrix} \middle| \pm 2x\right) \tag{5.5} $$

valid in $|x| < \frac{1}{2}$. *The* (ξ_{2r}) *are the nonvanishing zeros of the associated parametric polynomial of degree* $2r$ *given by*

$$ Q_{2r}(t) = \sum_{\ell=0}^{r} 2^{-2\ell} A_\ell\, (t)_{2\ell}\, (c-r-\tfrac{1}{2}-t)_{r-\ell}, $$

where the coefficients A_ℓ *are defined by (2.3).*

When $r = 0$, we note that (5.5) reduces to the Gaussian quadratic transformation

$$ {}_2F_1\left(\begin{matrix} a,\ a+\frac{1}{2} \\ c, \end{matrix} \middle| \frac{x^2}{(1\mp x)^2}\right) = (1\mp x)^{2a}\ {}_2F_1\left(\begin{matrix} 2a,\ c-\frac{1}{2} \\ 2c-1, \end{matrix} \middle| \pm 2x\right) $$

given in [15, Eq. (15.3.20)]. When $r = 1$, we see with $f_1 = f$ that the associated parametric polynomial is

$$ Q_2(t) = \tfrac{1}{4}t^2 + (\tfrac{1}{4} - f)t + f(c - \tfrac{3}{2}), $$

which possesses the zeros

$$ \xi_{1,2} = 2f - \tfrac{1}{2} \pm [(2f - \tfrac{1}{2})^2 - 4f(c - \tfrac{3}{2})]^{1/2}. $$

Thus we obtain the quadratic transformation

$$ {}_3F_2\left(\begin{matrix} a,\ a+\frac{1}{2}, & f+1 \\ c, & f \end{matrix} \middle| \frac{x^2}{(1\mp x)^2}\right) = (1\mp x)^{2a}\ {}_4F_3\left(\begin{matrix} 2a,\ c-\frac{3}{2}, & \xi_1+1, & \xi_2+1 \\ 2c-1, & \xi_1, & \xi_2 \end{matrix} \middle| \pm 2x\right) $$

as found in an equivalent form by Rakha *et al.* [16].

6. Examples

We give below some examples of the transformation formulas developed in this paper. The case corresponding to $r = 1$ of these transformations is covered by (1.3a), (1.3b), (4.11a), (4.11b) and (5.4). When $r = 2$, the associated parametric polynomial for the generalized first Euler-type and Kummer-type transformations is given by [5]

$$Q_2(t) = \alpha t^2 - ((\alpha+\beta)\lambda + \beta)t + f_1 f_2 \lambda(\lambda+1), \tag{6.1}$$

where $\lambda = c - b - 2$ and

$$\alpha = (f_1 - b)(f_2 - b), \qquad \beta = f_1 f_2 - b(b+1).$$

If we take $b = \frac{3}{2}$, $c = 1$ and $f_1 = \frac{1}{2}$, $f_2 = \frac{3}{4}$ then

$$Q_2(t) = \tfrac{3}{4}(t^2 - \tfrac{17}{4}t + \tfrac{15}{8}),$$

which possesses the zeros $\xi_1 = \frac{1}{2}$ and $\xi_2 = \frac{15}{4}$. From (3.1) and (5.1), we therefore have the first Euler and Kummer-type transformation formulas

$${}_4F_3\left(\begin{array}{c} a,\ \frac{3}{2},\ \frac{3}{2},\ \frac{7}{4} \\ 1,\ \frac{1}{2},\ \frac{3}{4} \end{array}\middle| x\right) = (1-x)^{-a}\,{}_4F_3\left(\begin{array}{c} a,\ -\frac{5}{2},\ \frac{3}{2},\ \frac{19}{4} \\ 1,\ \frac{1}{2},\ \frac{15}{4} \end{array}\middle| \frac{x}{x-1}\right),$$

where a is a free parameter, and

$${}_3F_3\left(\begin{array}{c} \frac{3}{2},\ \frac{3}{2},\ \frac{7}{4} \\ 1,\ \frac{1}{2},\ \frac{3}{4} \end{array}\middle| x\right) = e^x\,{}_3F_3\left(\begin{array}{c} -\frac{5}{2},\ \frac{3}{2},\ \frac{19}{4} \\ 1,\ \frac{1}{2},\ \frac{15}{4} \end{array}\middle| -x\right).$$

The associated parametric polynomial (4.7a) for the second Euler-type transformation when $r = 2$, $a = b = 1$, $c = \frac{1}{2}$ and $f_1 = \frac{1}{3}$, $f_2 = \frac{1}{6}$ is

$$Q_2(t) = -\tfrac{5}{72}(11t^2 + 47t - \tfrac{45}{4}),$$

so that $\xi_1 = -\frac{9}{2}$ and $\xi_2 = \frac{5}{22}$. Hence, from (4.1) with $\lambda = \lambda' = -\frac{5}{2}$, we have the second Euler-type transformation

$${}_4F_3\left(\begin{array}{c} 1,\ 1,\ \frac{4}{3},\ \frac{7}{6} \\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{6} \end{array}\middle| x\right) = (1-x)^{-7/2}\,{}_4F_3\left(\begin{array}{c} -\frac{5}{2},\ -\frac{5}{2},\ -\frac{7}{2},\ \frac{27}{22} \\ \frac{1}{2},\ -\frac{9}{2},\ \frac{5}{22} \end{array}\middle| x\right).$$

The zeros of the associated parametric polynomial (6.1) in this case are $\xi_{1,2} = (11 \pm \sqrt{97})/8$ (independently of a), so from (3.1) we have the first Euler-type transformation

$${}_4F_3\left(\begin{array}{c} a,\ 1,\ \frac{4}{3},\ \frac{7}{6} \\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{6} \end{array}\middle| x\right) = (1-x)^{-a}\,{}_4F_3\left(\begin{array}{c} a,\ -\frac{5}{2},\ \frac{1}{8}(19+\sqrt{97}),\ \frac{1}{8}(19-\sqrt{97}) \\ \frac{1}{2},\ \frac{1}{8}(11+\sqrt{97}),\ \frac{1}{8}(11-\sqrt{97}) \end{array}\middle| \frac{x}{x-1}\right).$$

We remark that when the associated parametric polynomials $Q_r(t)$ given by (3.12) and (4.7a) are of degree $r \geq 2$ and the parameters are all real, the possibility of complex zeros (ξ_r) arises; see also [3]. For example, if $a = b = 1$, $c = \frac{3}{2}$ and $f_1 = \frac{1}{3}$, $f_2 = \frac{1}{4}$ the associated parametric polynomial given by (4.7a) for the second Euler-type transformation becomes

$$Q_2(t) = \tfrac{17}{48}(t^2 + \tfrac{3}{17}t + \tfrac{9}{68})$$

and the zeros are $\xi_{1,2} = \frac{3}{34}(-1 \pm 4i)$.

Finally, we consider a third example by setting $f_1 = \ldots = f_r = f$ and $a = b = c$ in (3.1) and (4.1). Thus, for $|x| < 1$, defining

$$F(x) \equiv {}_{r+1}F_r \left(\begin{matrix} a, & f+1, & \ldots, & f+1 \\ & f, & \ldots, & f \end{matrix} \middle| x \right)$$

since $\lambda = \lambda' = -r$ we have

$$\begin{aligned} F(x) &= (1-x)^{-a} {}_{r+1}F_r \left(\begin{matrix} -r, & (\xi_r + 1) \\ & (\xi_r) \end{matrix} \middle| \frac{x}{x-1} \right) \\ &= (1-x)^{-a-r} {}_{r+2}F_{r+1} \left(\begin{matrix} -r, -r, & (\eta_r + 1) \\ a, & (\eta_r) \end{matrix} \middle| x \right), \end{aligned}$$

where the (ξ_r) and (η_r) are the nonvanishing zeros[2] of respective parametric polynomials of degree r. The above equation shows that $F(x)$ is proportional to some polynomial in x of degree r. We shall show that this polynomial may be written explicitly.

For since

$$\left(\frac{(f+1)_n}{(f)_n} \right)^r = \left(1 + \frac{n}{f} \right)^r = f^{-r} \sum_{k=0}^{r} \binom{r}{k} n^k f^{r-k},$$

for positive integer r, we have

$$F(x) = \sum_{k=0}^{r} \binom{r}{k} f^{-k} \sum_{n=0}^{\infty} n^k \frac{(a)_n x^n}{n!},$$

where we have interchanged the order of summation. Now employing Lemma 1, we see that

$$\sum_{n=0}^{\infty} n^k \frac{(a)_n x^n}{n!} = \sum_{j=0}^{k} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} \sum_{n=0}^{\infty} (a)_{n+j} \frac{x^{n+j}}{n!} = (1-x)^{-a} \sum_{j=0}^{k} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} (a)_j \left(\frac{x}{1-x} \right)^j,$$

where we have made use of (3.2) and (3.3). Thus

$$F(x) = (1-x)^{-a} \sum_{k=0}^{r} \binom{r}{k} f^{-k} \sum_{j=0}^{k} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} (a)_j \left(\frac{x}{1-x} \right)^j. \tag{6.2}$$

Next introducing the coefficients

$$\gamma_j(f) \equiv \sum_{k=j}^{r} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} \binom{r}{k} f^{-k} \tag{6.3}$$

in (6.2) after an interchange of the order of summation, we deduce that

$$F(x) = (1-x)^{-a} \sum_{j=0}^{r} (a)_j \gamma_j(f) \left(\frac{x}{1-x} \right)^j, \tag{6.4}$$

where we note that the right-hand side of this equation provides the analytic continuation of ${}_{r+1}F_r(x)$ for $x \neq 1$. Now defining the polynomial in x of degree r

$$\Xi_r(a, f|x) \equiv \sum_{j=0}^{r} (a)_j \gamma_j(f)\, x^j,$$

[2]For the (η_r) to be nonvanishing when $a = b = c$ we must have $(1+a)_r \neq 0$ and α_0 given by (4.9) must also be nonzero.

we have

$$ {}_{r+1}F_r\left(\begin{matrix} a, & f+1, & \ldots, & f+1 \\ & f, & \ldots, & f \end{matrix}\,\middle|\, x\right) = (1-x)^{-a}\,\Xi_r\left(a, f \,\middle|\, \frac{x}{1-x}\right). \tag{6.5} $$

This leads to the results

$$ {}_{r+1}F_r\left(\begin{matrix} -r, & (\xi_r+1) \\ & (\xi_r) \end{matrix}\,\middle|\, x\right) = \Xi_r(a, f|x) $$

and

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} -r, -r, & (\eta_r+1) \\ a, & (\eta_r) \end{matrix}\,\middle|\, x\right) = (1-x)^{r}\,\Xi_r\left(a, f \,\middle|\, \frac{x}{1-x}\right). $$

Moreover, since [4, (6.15), p. 265]

$$ \sum_{k=0}^{r} \left\{\begin{matrix} k \\ j \end{matrix}\right\} \binom{r}{k} = \left\{\begin{matrix} r+1 \\ j+1 \end{matrix}\right\}, $$

we see from (6.3) and (6.4) when $f = 1$ that

$$ {}_{r+1}F_r\left(\begin{matrix} a, & 2, \ldots, 2 \\ & 1, \ldots, 1 \end{matrix}\,\middle|\, x\right) = (1-x)^{-a}\sum_{j=0}^{r}(a)_j \left\{\begin{matrix} r+1 \\ j+1 \end{matrix}\right\} \left(\frac{x}{1-x}\right)^{j}. \tag{6.6} $$

The results given by (6.5) and (6.6) do not appear to be recorded in the literature. However, (6.6) is given by Prudnikov *et al.* [17, Section 7.10.1, p. 572] in an equivalent form only for $a = 2$.

7. Concluding remarks

In this paper we have derived Euler and Kummer-type transformation formulas for the generalized hypergeometric functions ${}_{r+2}F_{r+1}(x)$ and ${}_{r+1}F_{r+1}(x)$ in which r numeratorial and corresponding denominatorial parameters differ by unity. However, the methods presented herein may be further developed to extend these results to where the corresponding parameters differ by arbitrary positive integers [14]. The question naturally arises as to whether such transformations may be extended even further to cases where there are no restrictions whatsoever on the parameters. In answering this we note that at least for the particular case of ${}_2F_2(a, b; c, d|x)$, a Kummer-type transformation has been found [9, Eq. (3)] for the latter function in terms of an infinite series of ${}_2F_2(-x)$ functions. Thus we can only hope that the developments presented in this work as in [3, 5, 14] will stimulate further interest and research in this important area of classical special functions.

Just as the mathematical properties of the Gauss hypergeometric function ${}_2F_1(x)$ and its confluent form ${}_1F_1(x)$ are already of immense and significant utility in mathematical physics and numerous other areas of pure and applied mathematics, the elucidation and discovery of properties of the generalized hypergeometric functions considered herein should certainly eventually prove useful to further developments in the broad areas alluded to above.

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