

Certain transformations and summations for generalized hypergeometric series with integral parameter differences

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Abstract

Certain transformation and summation formulas for generalized hypergeometric series with integral parameter differences are derived.

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1. Introduction

The generalized hypergeometric function ${}_pF_q(x)$ is defined for complex parameters and argument by the series

$${}_pF_q \left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(b_1)_k (b_2)_k \dots (b_q)_k} \frac{x^k}{k!}. \quad (1.1)$$

When $q = p$ this series converges for $|x| < \infty$, but when $q = p - 1$ convergence occurs when $|x| < 1$. However, when only one of the numeratorial parameters a_j is a negative integer or zero, then the series always converges since it is simply a polynomial in x of degree $-a_j$. In (1.1) the Pochhammer symbol, or ascending factorial, $(a)_k$ is defined by $(a)_0 = 1$ and for $k \geq 1$ by $(a)_k = a(a+1) \dots (a+k-1)$. However, for all integers k we write simply

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

In what follows we shall adopt the convention of writing the finite (except where noted otherwise) sequence of parameters (a_1, \dots, a_p) simply by (a_p) and the product of p Pochhammer symbols by

$$((a_p))_k \equiv (a_1)_k \dots (a_p)_k,$$

with an empty product $p = 0$ reducing to unity.

In [1–4] we derived in various ways transformation formulas for the generalized hypergeometric functions ${}_{r+1}F_{r+1}(x)$ and ${}_{r+2}F_{r+1}(x)$, where here and below at least r pairs of numeratorial and

denominatorial parameters differ by arbitrary positive integers. In particular, in [4] we stated without proof that the generalized hypergeometric function ${}_{r+1}F_{r+1}(x)$ in which $r+1$ pairs of numeratorial and denominatorial parameters differ by arbitrary positive integers may be written as a product of e^x and a certain polynomial in x . In Section 3 we shall provide a proof of this result and discuss its implications.

In [1, 4, 5] we showed that essentially the same methods used to obtain the transformation formulas alluded to above may be employed to deduce the Karlsson-Minton and other more general summation formulas for the ${}_{r+2}F_{r+1}(1)$ generalized hypergeometric series with unit argument. In the present investigation we shall derive in Section 4 in a similar manner analogous summation formulas for generalized hypergeometric series of the type ${}_{r+2}F_{r+1}(\frac{1}{2})$ with half unit argument. Finally, in Section 5 we shall consider a certain quadratic transformation for the ${}_3F_2(x)$ hypergeometric function.

2. Preliminary results

We record two lemmas and a theorem that we shall utilize in the sequel. Lemmas 1 and 2 are proved respectively in [1] and [4] and Theorem 1 is proved in [4]. The notation $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ will be employed to denote the Stirling numbers of the second kind. These nonnegative integers represent the number of ways to partition n objects into k nonempty sets and arise for nonnegative integers n in the generating relation [6, p. 262]

$$x^n = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} (-1)^k (-x)_k, \quad \left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\} = \delta_{0n},$$

where δ_{0n} is the Kronecker symbol.

Lemma 1. *For nonnegative integers j define*

$$S_j \equiv \sum_{n=0}^{\infty} n^j \frac{\lambda_n}{n!}, \quad S_0 \equiv \sum_{n=0}^{\infty} \frac{\lambda_n}{n!},$$

where the infinite sequence (λ_n) is such that S_j converges for all j . Then

$$S_j = \sum_{k=0}^j \left\{ \begin{smallmatrix} j \\ k \end{smallmatrix} \right\} \sum_{n=0}^{\infty} \frac{\lambda_{n+k}}{n!}.$$

Lemma 2. *For nonnegative integer s let (a_s) denote a parameter sequence containing s elements, where when $s = 0$ the sequence is empty. Let $(a_s + k)$ denote the sequence when k is added to each element of (a_s) . Let $\mathcal{F}(x)$ denote the generalized hypergeometric function with r numeratorial and denominatorial parameters differing by the positive integers (m_r) , namely*

$$\mathcal{F}(x) \equiv {}_{r+s}F_{r+1} \left(\begin{matrix} (a_s), (f_r + m_r) \\ c, (f_r) \end{matrix} \middle| x \right),$$

where by (1.1) convergence of the series representation for the latter occurs in an appropriate domain depending on the values of s and the elements of the parameter sequence (a_s) . Then

$$\mathcal{F}(x) = \frac{1}{A_0} \sum_{k=0}^m x^k A_k \frac{((a_s))_k}{(c)_k} {}_sF_1 \left(\begin{matrix} (a_s + k) \\ c + k \end{matrix} \middle| x \right),$$

where $m = m_1 + \cdots + m_r$, the coefficients A_k are defined by

$$A_k \equiv \sum_{j=k}^m \left\{ \begin{smallmatrix} j \\ k \end{smallmatrix} \right\} \sigma_{m-j}, \quad A_0 = (f_1)_{m_1} \cdots (f_r)_{m_r}, \quad A_m = 1 \quad (2.1)$$

and the σ_j ($0 \leq j \leq m$) are generated by the relation

$$(f_1 + x)_{m_1} \cdots (f_r + x)_{m_r} = \sum_{j=0}^m \sigma_{m-j} x^j. \quad (2.2)$$

Theorem 1. Let (m_r) be a nonempty sequence of positive integers and define $m \equiv m_1 + \cdots + m_r$. Then if $b \neq f_j$ ($1 \leq j \leq r$), $(\lambda)_m \neq 0$, where $\lambda \equiv c - b - m$, we have the transformation formula

$${}_{r+1}F_{r+1} \left(\begin{matrix} b, (f_r + m_r) \\ c, (f_r) \end{matrix} \middle| x \right) = e^x {}_{m+1}F_{m+1} \left(\begin{matrix} \lambda, (\xi_m + 1) \\ c, (\xi_m) \end{matrix} \middle| -x \right), \quad (2.3)$$

where $|x| < \infty$. The (ξ_m) are the nonvanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree m given by

$$Q_m(t) = \sum_{j=0}^m \sigma_{m-j} \sum_{k=0}^j \left\{ \begin{matrix} j \\ k \end{matrix} \right\} (b)_k (t)_k (\lambda - t)_{m-k},$$

where the σ_j ($0 \leq j \leq m$) are determined by the generating relation (2.2).

In the following Section 3 we shall consider the generalized hypergeometric function $w(x)$ defined for $|x| < \infty$ by

$$w(x) \equiv {}_{r+1}F_{r+1} \left(\begin{matrix} (f_{r+1} + m_{r+1}) \\ (f_{r+1}) \end{matrix} \middle| x \right),$$

where (m_{r+1}) is a sequence of positive integers. It is evident that $w(x)$ is an entire function.

3. Properties of $w(x)$

If in Theorem 1 we set $b = f_{r+1} + m_{r+1}$, $c = f_{r+1}$ and define $m \equiv m_1 + \cdots + m_r$, $M \equiv m + m_{r+1}$, then $\lambda = -M$, $(-M)_m \neq 0$ and we find from (2.3) that

$$w(x) = e^x {}_{m+1}F_{m+1} \left(\begin{matrix} -M, (\xi_m + 1) \\ f_{r+1}, (\xi_m) \end{matrix} \middle| -x \right).$$

The (ξ_m) are the nonvanishing zeros of the associated parametric polynomial of degree m given by

$$Q_m(t) = \sum_{j=0}^m \sigma_{m-j} \sum_{k=0}^j \left\{ \begin{matrix} j \\ k \end{matrix} \right\} (f_{r+1} + m_{r+1})_k (t)_k (-M - t)_{m-k},$$

where the σ_j ($0 \leq j \leq m$) are generated by (2.2).

Thus it is evident that $w(x)$ is proportional to a polynomial in x of degree at most M which we define as

$$\mathcal{P}_M(x) \equiv {}_{m+1}F_{m+1} \left(\begin{matrix} -M, (\xi_m + 1) \\ f_{r+1}, (\xi_m) \end{matrix} \middle| -x \right).$$

Moreover, since e^x can never vanish it follows that the entire function $w(x)$ has at most M zeros in the complex plane. However, we shall obtain an explicit representation for $\mathcal{P}_M(x)$ which shows that its degree is exactly M .

Theorem 2. Let (m_{r+1}) be a sequence of positive integers such that $M \equiv m_1 + \cdots + m_{r+1}$ and let (f_{r+1}) be a sequence of complex numbers such that $(f_1)_{m_1} \cdots (f_{r+1})_{m_{r+1}}$ is nonvanishing. Then

$$w(x) = e^x \mathcal{P}_M(x),$$

where $\mathcal{P}_M(x)$ is a polynomial of degree M given by

$$\mathcal{P}_M(x) = \frac{1}{B_0} \sum_{k=0}^M B_k x^k, \quad B_k \equiv \sum_{j=k}^M \left\{ \begin{matrix} j \\ k \end{matrix} \right\} \rho_{M-j}.$$

Here

$$B_0 = (f_1)_{m_1} \dots (f_{r+1})_{m_{r+1}}, \quad B_M = 1,$$

and the ρ_j ($0 \leq j \leq M$) are generated by the relation

$$(f_1 + x)_{m_1} \dots (f_{r+1} + x)_{m_{r+1}} = \sum_{j=0}^M \rho_{M-j} x^j.$$

Proof: Note that

$$\frac{((f_{r+1} + m_{r+1}))_n}{((f_{r+1}))_n} = \frac{(f_1 + n)_{m_1}}{(f_1)_{m_1}} \dots \frac{(f_{r+1} + n)_{m_{r+1}}}{(f_{r+1})_{m_{r+1}}},$$

where the numeratorial expression on the right-hand side of the latter may be written as

$$(f_1 + n)_{m_1} \dots (f_{r+1} + n)_{m_{r+1}} = \sum_{j=0}^M \rho_{M-j} n^j. \quad (3.1)$$

Thus by (1.1)

$$\begin{aligned} w(x) &\equiv {}_{r+1}F_{r+1} \left(\begin{matrix} (f_{r+1} + m_{r+1}) \\ (f_{r+1}) \end{matrix} \middle| x \right) = \sum_{n=0}^{\infty} \frac{((f_{r+1} + m_{r+1}))_n}{((f_{r+1}))_n} \frac{x^n}{n!} \\ &= \frac{1}{(f_1)_{m_1} \dots (f_{r+1})_{m_{r+1}}} \sum_{j=0}^M \rho_{M-j} \sum_{n=0}^{\infty} n^j \frac{x^n}{n!}, \end{aligned}$$

where the order of the summations has been interchanged.

Now employing Lemma 1 we have

$$\sum_{n=0}^{\infty} n^j \frac{x^n}{n!} = \sum_{k=0}^j \left\{ \begin{matrix} j \\ k \end{matrix} \right\} \sum_{n=0}^{\infty} \frac{x^{n+k}}{n!} = e^x \sum_{k=0}^j \left\{ \begin{matrix} j \\ k \end{matrix} \right\} x^k$$

so that

$$w(x) = \frac{e^x}{(f_1)_{m_1} \dots (f_{r+1})_{m_{r+1}}} \sum_{j=0}^M \rho_{M-j} \sum_{k=0}^j \left\{ \begin{matrix} j \\ k \end{matrix} \right\} x^k,$$

where

$$\sum_{j=0}^M \rho_{M-j} \sum_{k=0}^j \left\{ \begin{matrix} j \\ k \end{matrix} \right\} x^k = \sum_{k=0}^M \left(\sum_{j=k}^M \left\{ \begin{matrix} j \\ k \end{matrix} \right\} \rho_{M-j} \right) x^k.$$

Defining

$$B_k \equiv \sum_{j=k}^M \left\{ \begin{matrix} j \\ k \end{matrix} \right\} \rho_{M-j} \quad (0 \leq k \leq M),$$

we note that when $k = 0$ the only contribution to the j -summation comes from $j = 0$, so that by using (3.1) we find

$$B_0 = \rho_M = (f_1)_{m_1} \dots (f_{r+1})_{m_{r+1}}.$$

In addition when $k = M$, the only contribution to the latter summation comes from $j = M$, so that by again using (3.1) we see that

$$B_M = \rho_0 = 1.$$

Thus

$$w(x) = \frac{e^x}{B_0} \sum_{k=0}^M B_k x^k \quad (3.2)$$

which evidently completes the proof. \square

Thus we also have the following.

Corollary 1. *The entire function $w(x)$ has exactly M zeros in the complex plane.*

We remark that Ki and Kim [7] only show the existence of at most M zeros for $w(x)$, whereas from (3.2) we can in principle obtain all of the M zeros. For example,

$${}_2F_2 \left(\begin{matrix} f+1, g+1 \\ f, g \end{matrix} \middle| x \right) = e^x \left(\frac{1}{fg} x^2 + \frac{f+g+1}{fg} x + 1 \right),$$

and here the zeros of $\mathcal{P}_2(x)$ are

$$x_{1,2} = -\frac{1}{2}(f+g+1 \pm [(f-g)^2 + 2f + 2g + 1]^{1/2}).$$

4. Summation formulas for ${}_{r+2}F_{r+1}(\frac{1}{2})$

In [4] we employed Lemma 2 to readily obtain a generalization of the Karlsson-Minton summation formula which is given in the following.

Theorem 3. *Suppose (m_r) is a sequence of positive integers such that $m = m_1 + \dots + m_r$. Then provided that $\operatorname{Re}(c - a - b) > m$ we have*

$${}_{r+2}F_{r+1} \left(\begin{matrix} a, b, (f_r + m_r) \\ c, (f_r) \end{matrix} \middle| 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^m \frac{A_k}{A_0} \frac{(-1)^k (a)_k (b)_k}{(1+a+b-c)_k},$$

where the A_k ($0 \leq k \leq m$) are given by (2.1).

However we may also utilize Lemma 2 to obtain summation formulas for ${}_{r+2}F_{r+1}(\frac{1}{2})$. To this end we note the following (see, for example, [8, Section 7.3.7 (3)–(6)]) summations for ${}_2F_1(\frac{1}{2})$ given by

$${}_2F_1 \left(\begin{matrix} a, b \\ \frac{1}{2}a + \frac{1}{2}b - \frac{1}{2} \end{matrix} \middle| \frac{1}{2} \right) = \sqrt{\pi} \left\{ \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b + \frac{1}{2})} + \frac{2\Gamma(\frac{1}{2}a + \frac{1}{2}b - \frac{1}{2})}{\Gamma(a)\Gamma(b)} \right\}, \quad (4.1)$$

$${}_2F_1 \left(\begin{matrix} a, b \\ \frac{1}{2}a + \frac{1}{2}b \end{matrix} \middle| \frac{1}{2} \right) = \sqrt{\pi} \left\{ \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b)}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}a + \frac{1}{2})} + \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b)}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b + \frac{1}{2})} \right\}, \quad (4.2)$$

$${}_2F_1 \left(\begin{matrix} a, b \\ \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2} \end{matrix} \middle| \frac{1}{2} \right) = \sqrt{\pi} \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b + \frac{1}{2})}, \quad (4.3)$$

$${}_2F_1 \left(\begin{matrix} a, b \\ \frac{1}{2}a + \frac{1}{2}b + 1 \end{matrix} \middle| \frac{1}{2} \right) = \frac{2\sqrt{\pi}}{a-b} \left\{ \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b + 1)}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b + \frac{1}{2})} - \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b + 1)}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}a + \frac{1}{2})} \right\}. \quad (4.4)$$

Setting $x = \frac{1}{2}$, $s = 2$, $(a_s) = (a, b)$, $c = \frac{1}{2}(a + b + n)$, where $n = -1, 0, 1, 2$, in the first two equations of Lemma 2, we see that

$$\begin{aligned} {}_{r+2}F_{r+1} \left(\begin{matrix} a, b, & (f_r + m_r) \\ \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}n, & (f_r) \end{matrix} \middle| \frac{1}{2} \right) \\ = \sum_{k=0}^m \left(\frac{1}{2} \right)^k \frac{A_k}{A_0} \frac{(a)_k (b)_k}{(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}n)_k} {}_2F_1 \left(\begin{matrix} a + k, b + k \\ \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}n + k \end{matrix} \middle| \frac{1}{2} \right), \end{aligned} \quad (4.5)$$

where when $r = 0$, (f_r) and $(f_r + m_r)$ are empty and we define $m \equiv 0$. Thus the Gauss series ${}_2F_1(\frac{1}{2})$ in (4.5) with respectively $n = -1, 0, 1, 2$ correspond to the left-hand sides of (4.1)–(4.4) with $a \mapsto a + k$ and $b \mapsto b + k$. Combining these with (4.5) we therefore deduce the following.

Theorem 4. *Let (m_r) be an arbitrary sequence of positive integers such that $m \equiv m_1 + \dots + m_r$ and (f_r) a sequence of complex numbers such that $(f_1)_{m_1} \dots (f_r)_{m_r} \neq 0$. Then*

$$\begin{aligned} {}_{r+2}F_{r+1} \left(\begin{matrix} a, b, & (f_r + m_r) \\ \frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}, & (f_r) \end{matrix} \middle| \frac{1}{2} \right) \\ = \frac{2\sqrt{\pi}}{A_0} \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b - \frac{1}{2})}{\Gamma(a)\Gamma(b)} \sum_{k=0}^m \left(\frac{1}{2} \right)^k A_k \left\{ 1 + \frac{\Gamma(a+k)\Gamma(b+k)(a+b-1+2k)}{4\Gamma(\frac{1}{2}a + \frac{1}{2} + \frac{1}{2}k)\Gamma(\frac{1}{2}b + \frac{1}{2} + \frac{1}{2}k)} \right\}, \end{aligned} \quad (4.6)$$

$$\begin{aligned} {}_{r+2}F_{r+1} \left(\begin{matrix} a, b, & (f_r + m_r) \\ \frac{1}{2}a + \frac{1}{2}b, & (f_r) \end{matrix} \middle| \frac{1}{2} \right) \\ = \frac{2^{a+b-2}}{A_0\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^m 2^k A_k \left\{ \Gamma(\frac{1}{2}a + \frac{1}{2}k)\Gamma(\frac{1}{2}b + \frac{1}{2} + \frac{1}{2}k) - \Gamma(\frac{1}{2}b + \frac{1}{2}k)\Gamma(\frac{1}{2}a + \frac{1}{2} + \frac{1}{2}k) \right\}, \end{aligned} \quad (4.7)$$

$$\begin{aligned} {}_{r+2}F_{r+1} \left(\begin{matrix} a, b, & (f_r + m_r) \\ \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}, & (f_r) \end{matrix} \middle| \frac{1}{2} \right) \\ = \frac{2^{a+b-2}}{A_0\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(a)\Gamma(b)} \sum_{k=0}^m 2^k A_k \Gamma(\frac{1}{2}a + \frac{1}{2}k)\Gamma(\frac{1}{2}b + \frac{1}{2}k), \end{aligned} \quad (4.8)$$

$$\begin{aligned} {}_{r+2}F_{r+1} \left(\begin{matrix} a, b, & (f_r + m_r) \\ \frac{1}{2}a + \frac{1}{2}b + 1, & (f_r) \end{matrix} \middle| \frac{1}{2} \right) \\ = \frac{2^{a+b-2}}{A_0\sqrt{\pi}} \frac{a+b}{a-b} \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^m 2^k A_k \left\{ \Gamma(\frac{1}{2}b + \frac{1}{2}k)\Gamma(\frac{1}{2}a + \frac{1}{2}k + \frac{1}{2}) - \Gamma(\frac{1}{2}a + \frac{1}{2}k)\Gamma(\frac{1}{2}b + \frac{1}{2}k + \frac{1}{2}) \right\}, \end{aligned} \quad (4.9)$$

where the A_k ($0 \leq k \leq m$) are given by (2.1).

We have used the duplication formula

$$\sqrt{\pi} \Gamma(2z) = 2^{2z-1} \Gamma(z) \Gamma(z + \frac{1}{2}) \quad (4.10)$$

to obtain (4.7)–(4.9). Thus, when $r = 0$, the sequences (f_r) and $(f_r + m_r)$ are empty so that $m = 0$ and (4.6)–(4.9) reduce respectively to (4.1)–(4.4). Since (4.9) is not valid when $a = b$ we may use l'Hôpital's rule and (4.10) to obtain the limiting case of (4.9), namely

$${}_{r+2}F_{r+1} \left(\begin{matrix} a, a, & (f_r + m_r) \\ a + 1, & (f_r) \end{matrix} \middle| \frac{1}{2} \right) = \frac{2^{a-1}a}{A_0} \sum_{k=0}^m A_k(a)_k \left\{ \psi(\frac{1}{2}a + \frac{1}{2}k + \frac{1}{2}) - \psi(\frac{1}{2}a + \frac{1}{2}k) \right\},$$

where ψ is the digamma or psi function. When $r = 0$ we retrieve

$${}_2F_1 \left(\begin{matrix} a, a \\ a+1 \end{matrix} \middle| \frac{1}{2} \right) = 2^{a-1} a \left\{ \psi\left(\frac{1}{2}a + \frac{1}{2}\right) - \psi\left(\frac{1}{2}a\right) \right\}$$

which is found in [8, Section 7.3.7 (16)] in an equivalent form.

In addition, if we let $r = 1$, $m_1 = 1$ and $f_1 = f$, so that $A_0 = f$, $A_1 = 1$, then we obtain from (4.9) and (4.10) after some straightforward algebra the result (when $a \neq b$)

$${}_3F_2 \left(\begin{matrix} a, b, f+1 \\ \frac{1}{2}a + \frac{1}{2}b + 1, f \end{matrix} \middle| \frac{1}{2} \right) = \sqrt{\pi} \left(\frac{a+b}{a-b} \right) \Gamma\left(\frac{1}{2}a + \frac{1}{2}b\right) \left\{ \frac{(a-f)/f}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}a + \frac{1}{2})} - \frac{(b-f)/f}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b + \frac{1}{2})} \right\},$$

which has been obtained by Rathie and Pogány in [9].

5. A quadratic transformation for ${}_3F_2(x)$

In [4] by utilizing Lemma 2 we derived two quadratic transformations for

$${}_{r+2}F_{r+1} \left(\begin{matrix} a, a + \frac{1}{2}, (f_r + m_r) \\ c, (f_r) \end{matrix} \middle| X \right),$$

where $X \equiv x^2/(1 \pm x)^2$ and $X \equiv 4x/(1+x)^2$. As discussed below certain parametric conditions guaranteeing the existence of a quadratic transformation for the Gauss function ${}_2F_1(x)$ restrict the existence of more general quadratic transformations for ${}_{r+2}F_{r+1}(x)$ when the decomposition Lemma 2 is employed. We illustrate this by deducing a quadratic transformation that necessarily is restricted to $r = 1$ thus giving the quadratic transformation for ${}_3F_2(x)$ in the following theorem.

Theorem 5. Suppose $a, b \neq \frac{1}{2}$ and $f \neq a + b + \frac{1}{2}$. Then

$$\begin{aligned} {}_3F_2 \left(\begin{matrix} a, b, f+1 \\ a+b+\frac{1}{2}, f \end{matrix} \middle| 4x(1-x) \right) \\ = (1-2x)^{-1} {}_4F_3 \left(\begin{matrix} 2a-1, 2b-1, \xi_1+1, \xi_2+1 \\ a+b+\frac{1}{2}, \xi_1, \xi_2 \end{matrix} \middle| x \right), \end{aligned} \quad (5.1)$$

where ξ_1, ξ_2 are given by

$$\xi_{1,2} = \alpha + \frac{1}{2} \pm [(\alpha + \frac{1}{2})^2 - 2\alpha f]^{1/2}, \quad \alpha = \frac{(2a-1)(2b-1)}{2(a+b-f)-1}. \quad (5.2)$$

The transformation (5.1) holds in a neighborhood of $x = 0$.

Proof: From Lemma 2 we have the expansion of ${}_{r+2}F_{r+1}(z)$, with r pairs of parameters differing by the positive integers (m_r) , as a finite sum of ${}_2F_1(z)$ functions in the form

$${}_{r+2}F_{r+1} \left(\begin{matrix} a, b, (f_r + m_r) \\ c, (f_r) \end{matrix} \middle| z \right) = \frac{1}{A_0} \sum_{k=0}^m z^k A_k \frac{(a)_k (b)_k}{(c)_k} {}_2F_1 \left(\begin{matrix} a+k, b+k \\ c+k \end{matrix} \middle| z \right), \quad (5.3)$$

where the coefficients A_k are defined by (2.1). If we set $r = 1$, $m_1 = 1$ and $f_1 = f$ in the above expansion we obtain the particular case

$${}_3F_2 \left(\begin{matrix} a, b, f+1 \\ c, f \end{matrix} \middle| z \right) = {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| z \right) + \frac{abz}{cf} {}_2F_1 \left(\begin{matrix} a+1, b+1 \\ c+1 \end{matrix} \middle| z \right). \quad (5.4)$$

Now let $z = X \equiv 4x(1 - x)$ and define

$$F(x) \equiv {}_3F_2 \left(\begin{matrix} a, b, f+1 \\ a+b+\frac{1}{2}, f \end{matrix} \middle| X \right).$$

Then, from (5.4) with $c = a + b + \frac{1}{2}$, we find

$$F(x) = {}_2F_1 \left(\begin{matrix} a, b \\ a+b+\frac{1}{2} \end{matrix} \middle| X \right) + \frac{ab}{cf} X {}_2F_1 \left(\begin{matrix} a+1, b+1 \\ a+b+\frac{3}{2} \end{matrix} \middle| X \right). \quad (5.5)$$

We now make use of two quadratic transformation formulas for the Gauss hypergeometric function [10, Section 15.3, (22) and (24)], namely

$${}_2F_1 \left(\begin{matrix} a, b \\ a+b-j+\frac{1}{2} \end{matrix} \middle| X \right) = (1-2x)^{-j} {}_2F_1 \left(\begin{matrix} 2a-j, 2b-j \\ a+b-j+\frac{1}{2} \end{matrix} \middle| x \right), \quad (5.6)$$

where $j = 0, 1$. Application of these transformations with $j = 0$ and $j = 1$ respectively to the first and second hypergeometric functions in (5.5) then leads to

$$F(x) = (1-2x)^{-1} \left\{ (1-2x) {}_2F_1 \left(\begin{matrix} 2a, 2b \\ a+b+\frac{1}{2} \end{matrix} \middle| x \right) + \frac{4ab}{cf} x(1-x) {}_2F_1 \left(\begin{matrix} 2a+1, 2b+1 \\ a+b+\frac{3}{2} \end{matrix} \middle| x \right) \right\}. \quad (5.7)$$

From (1.1) it can be seen that for arbitrary values of c

$$\begin{aligned} {}_2F_1 \left(\begin{matrix} 2a, 2b \\ c \end{matrix} \middle| x \right) &= \sum_{n=0}^{\infty} \frac{(2a)_n (2b)_n}{(c)_n} \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(2a-1)_n (2b-1)_n}{(c)_n} \frac{x^n}{n!} \frac{(2a-1+n)(2b-1+n)}{(2a-1)(2b-1)}, \end{aligned}$$

$$\begin{aligned} x {}_2F_1 \left(\begin{matrix} 2a, 2b \\ c \end{matrix} \middle| x \right) &= \sum_{n=1}^{\infty} \frac{(2a)_{n-1} (2b)_{n-1}}{(c)_{n-1}} \frac{x^n}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{(2a-1)_n (2b-1)_n}{(c)_n} \frac{x^n}{n!} \frac{n(c-1+n)}{(2a-1)(2b-1)}, \end{aligned}$$

$$\begin{aligned} x {}_2F_1 \left(\begin{matrix} 2a+1, 2b+1 \\ c+1 \end{matrix} \middle| x \right) &= \sum_{n=1}^{\infty} \frac{(2a+1)_{n-1} (2b+1)_{n-1}}{(c+1)_{n-1}} \frac{x^n}{(n-1)!} \\ &= \frac{c}{4ab} \sum_{n=0}^{\infty} \frac{(2a-1)_n (2b-1)_n}{(c)_n} \frac{x^n}{n!} \frac{n(2a-1+n)(2b-1+n)}{(2a-1)(2b-1)}, \end{aligned}$$

and

$$\begin{aligned} x^2 {}_2F_1 \left(\begin{matrix} 2a+1, 2b+1 \\ c+1 \end{matrix} \middle| x \right) &= \sum_{n=2}^{\infty} \frac{(2a+1)_{n-2} (2b+1)_{n-2}}{(c+1)_{n-2}} \frac{x^n}{(n-2)!} \\ &= \frac{c}{4ab} \sum_{n=0}^{\infty} \frac{(2a-1)_n (2b-1)_n}{(c)_n} \frac{x^n}{n!} \frac{n(n-1)(c-1+n)}{(2a-1)(2b-1)}, \end{aligned}$$

where obvious adjustments to the summation index n have been made. Then, upon setting $c = a + b + \frac{1}{2}$ in the above sums, we obtain from (5.7)

$$F(x) = \frac{(1-2x)^{-1}}{(2a-1)(2b-1)} \sum_{n=0}^{\infty} \frac{(2a-1)_n(2b-1)_n}{(a+b+\frac{1}{2})_n} \frac{x^n}{n!} P_2(n),$$

where, with $\gamma \equiv f^{-1}(a+b-\frac{1}{2})-1$,

$$\begin{aligned} P_2(n) &= (2a-1+n)(2b-1+n) \left(1 + \frac{n}{f}\right) - n \left(2 + \frac{n-1}{f}\right) (a+b-\frac{1}{2}+n) \\ &= \gamma n^2 + \left(\gamma + \frac{1}{f}(2a-1)(2b-1)\right) n + (2a-1)(2b-1). \end{aligned} \quad (5.8)$$

The quadratic $P_2(n)$ can be factored to yield

$$P_2(n) = \gamma(n + \xi_1)(n + \xi_2),$$

where, provided $a, b \neq \frac{1}{2}$ and $f \neq a+b-\frac{1}{2}$, ξ_1 and ξ_2 are given by (5.2). Since $\gamma\xi_1\xi_2 = (2a-1)(2b-1)$, we therefore see that

$$\begin{aligned} F(x) &= (1-2x)^{-1} \sum_{n=0}^{\infty} \frac{(2a-1)_n(2b-1)_n}{(a+b+\frac{1}{2})_n} \frac{x^n}{n!} \frac{(n+\xi_1)(n+\xi_2)}{\xi_1\xi_2} \\ &= (1-2x)^{-1} \sum_{n=0}^{\infty} \frac{(2a-1)_n(2b-1)_n}{(a+b+\frac{1}{2})_n} \frac{(\xi_1+1)_n(\xi_2+1)_n}{(\xi_1)_n(\xi_2)_n} \frac{x^n}{n!}, \end{aligned}$$

where we have made use of the fact that $(\lambda+1)_n/(\lambda)_n = (n+\lambda)/\lambda$. This yields (5.1) and so completes the proof. \square

We remark that when $f = a+b-\frac{1}{2}$, the coefficient $\gamma = 0$ and the polynomial $P_2(n)$ becomes linear with $\xi_1 = f$. The formula (5.1) then reduces to the quadratic transformation (5.6) with $j = 1$.

If we try to apply the same reasoning to the more general function with r pairs of parameters differing by the positive integers (m_r)

$${}_{r+2}F_{r+1} \left(\begin{matrix} a, b, & (f_r + m_r) \\ a + b + \frac{1}{2}, & (f_r) \end{matrix} \middle| X \right), \quad (5.9)$$

we obtain from (5.3) a series of $m+1$ terms involving the Gauss functions

$$F_k \equiv {}_2F_1 \left(\begin{matrix} a+k, b+k \\ a+b+k+\frac{1}{2} \end{matrix} \middle| X \right) \quad (0 \leq k \leq m).$$

A quadratic transformation for ${}_2F_1(\alpha, \beta; \gamma | x)$ exists if and only if any of the quantities

$$\pm(1-\gamma), \quad \pm(\alpha-\beta), \quad \pm(\alpha+\beta-\gamma)$$

are such that either one of them equals $\frac{1}{2}$ or two of them are equal [10, p. 560]. In this case, the third condition above for the functions F_k , with $0 \leq k \leq m$, has the form $\alpha + \beta - \gamma = k - \frac{1}{2}$; that is, a quadratic transformation exists only when $k = 0$ and $k = 1$. Consequently, we are compelled to take $r = 1$, $m = 1$ in (5.9).

References

- [1] A. R. Miller, Certain summation and transformation formulas for generalized hypergeometric series, J. Comput. Appl. Math. **231** (2009) 964–972.
- [2] A. R. Miller and R. B. Paris, A generalized Kummer-type transformation for the ${}_pF_p(x)$ hypergeometric function, Canadian Math. Bulletin (to appear 2011).
- [3] A. R. Miller and R. B. Paris, Euler-type transformations for the generalized hypergeometric function ${}_{r+2}F_{r+1}(x)$ (submitted for publication 2010).
- [4] A. R. Miller and R. B. Paris, Transformation formulas for the generalized hypergeometric function with integral parameter differences (submitted for publication 2010).
- [5] A. R. Miller and H. M. Srivastava, Karlsson-Minton summation theorems for the generalized hypergeometric series of unit argument, Integral Transforms and Special Functions (in press 2010).
- [6] R. L. Graham, D. E. Knuth and O. Patashnik, Concrete Mathematics, second ed., Addison-Wesley, Upper Saddle River, 1994.
- [7] H. Ki and Y.-O. Kim, On the zeros of some generalized hypergeometric functions, J. Math. Anal. Appl. **243** (2000) 249–260.
- [8] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, Integrals and Series, vol. 3, Gordon and Breach, New York, 1990.
- [9] A. K. Rathie and T. K. Pogány, New summation formula for ${}_3F_2(\frac{1}{2})$ and a Kummer-type II transformation of ${}_2F_2(x)$, Mathematical Communications **13** (2008) 63–66.
- [10] M. Abramowitz and I. A. Stegun (Eds.), Handbook of Mathematical Functions, Dover, New York, 1965.