Shift-and-scale model reduction: an alternative stability-preserving approach

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Abstract

A new stability-preserving model order-reduction method is presented for continuous-time systems. It makes use of the relatively new idea of transformed *whole-system* parameter matching for calculating the poles of the reduced-order transfer function. This has the advantage of using more of the system information than traditional methods in the approximation of the poles. The method is seen to be flexible and computationally attractive, relying only on readily available algorithms. It is based on a *shift-and-scale* transformation of the transfer function before applying the order-reduction process. Further, it is shown to be a viable alternative to existing stability-preserving techniques. Some examples illustrate the method.

Keywords: order reduction, stability, Padé least-squares, Markov parameters.

List of Notation

a	real number
a_i	denominator coefficient of system transfer function
α_i	transformed denominator coefficient
b_i	numerator coefficient of system transfer function
β_i	transformed numerator coefficient
$\boldsymbol{c}$	vector of approximate denominator coefficients
c_i	approximate denominator coefficient
d_i	reduced-order numerator coefficient
e_i	reduced-order denominator coefficient
$G(s)$	system transfer function
$G_k(s)$	k'th order transfer function
$\bar{G}(u)$	transformed system transfer function
$\bar{G}_k(u)$	k'th order transformed system transfer function
i	integer value
I_{rel}	relative integral-square-error
j	$\sqrt{-1}$
j	integer value
k	order of reduced model
n	order of system
$\boldsymbol{p}$	vector of Markov parameters
P_i	i'th Markov parameter of $\bar{G}(u)$
$\mathbf{P}$	matrix of Markov parameters P_i
R	radius of circle
s	Laplace transform variable
$T(s)$	transfer function of transient response
$T_k(s)$	k'th order transfer function of transient response
u	transformation variable
$y(t)$	system impulse response
$y_k(t)$	impulse response of k'th order model

1. INTRODUCTION

Model order-reduction techniques are now a well-established part of the control system designer's toolkit. They have become an integral part of overcoming some of the increasingly complex problems associated with modern control requirements in today's industries [**1–3**]. At the forefront of these techniques have been those that deal with the linearised system models in both the time and frequency domains. Each of the more established order-reduction methods has its relative merits and is selected according to the system characteristics being approximated.

The frequency-domain techniques grew naturally from the overwhelming amount of design information gathered by classical methods (e.g. Nyquist, Bode and root-locus plots), leading to the transfer function formation. This approach is still favoured by many designers and consequently the frequency-domain order-reduction methods continue to be of paramount importance.

One of the most desirable properties of any order-reduction technique is that of preserving stability in the approximation. There are certain methods that *guarantee* stability in the reduced-order models from the full system, most of which are built on the pioneering approaches of the Routh [**4**], Schwarz [**5**] and stability equation [**6**] methods. Generally, these methods tend not to give such good overall approximations as the Padé methods [**7-9**], which retain twice the number of whole-system parameters (those that use all of the transfer function's coefficients, e.g. time moments, Markov parameters, etc.) in the reduced-order model that the stability-preserving methods are able to. However, it must be remembered that this is the case only if the Padé reduced-order model happens to be stable, which is not guaranteed.

In an effort to increase the influence of whole-system parameters in a stability-preserving method, Lucas [**10**] recently presented the bilinear method.

This attempts to combine the strengths of the Padé and stability-preserving approaches by using a least-squares Padé matching of the full and reduced-order models' bilinearly-transformed Markov parameters, which guarantees stable poles in the reduced-order transfer function [**11**]. Results from empirical evidence suggest that it indeed rivals the full Padé methods for accuracy of approximation and it has the bonus of guaranteeing stability.

It is now proposed to contribute further to this promising area of research by presenting another stability-preserving method that uses whole-system parameter information in the formation of the reduced model's poles. Like the bilinear method [**10**], use is made of the least-squares Padé algorithm that generates stable models [**11**], but instead of using a bilinear mapping of the transfer function beforehand, a simpler "shift-and-scale" transformation is applied. This is seen to cut down the computational requirement and gives the method extra flexibility by being able to vary the two parameters of the transformation. The merits of the method are discussed and examples are given to illustrate its application.

2. THE METHOD

Suppose that the full n'th order stable transfer function is given in the usual notation by

$$G(s) = \frac{b_{n-1}s^{n-1} + ... + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0} \qquad (1)$$

It is required to approximate this by the reduced k'th-order transfer function ($k < n$) defined by

$$G_k(s) = \frac{d_{k-1}s^{k-1} + ... + d_1 s + d_0}{s^k + e_{k-1}s^{k-1} + ... + e_1 s + e_0} \qquad (2)$$

In order that the poles of $G_k(s)$ lie in the left half-plane, the procedure begins with mapping the circle

$$|s+a| = R \tag{3}$$

onto the unit circle

$$|u| = 1 \tag{4}$$

by the transformation

$$\left.\begin{aligned} u &= \frac{s+a}{R} \\ \text{or } \; s &= uR - a \end{aligned}\right\} \tag{5}$$

The circle in equation (3), with radius R and centre at $s = -a$ on the real axis, encloses all of the poles of $G(s)$ and is such that

$$0 < R \leq a \tag{6}$$

This last condition ensures that the interior of the circle lies entirely in the left half-plane, as illustrated in Figure 1. Further, equation (3) gives

$$\left|\frac{s+a}{R}\right| = 1$$

and so the mapping of equation (5) means that the region $|s+a| \leq R$ in the s-plane is transformed to the region $|u| \leq 1$ in the u-plane.

Now that the transformed full system transfer function, $\bar{G}(u) \equiv G(uR-a)$, has all of its poles inside the unit circle, the least-squares Padé algorithm [**11**] is applied to $\bar{G}(u)$ as follows.

Suppose that $\bar{G}(u)$ is expanded as a series about $u = \infty$ to give

$$\bar{G}(u) = \frac{\beta_{n-1}u^{n-1} + ... + \beta_1 u + \beta_0}{\alpha_n u^n + \alpha_{n-1}u^{n-1} + ... + \alpha_1 u + \alpha_0} = \sum_{i=0}^{\infty} \frac{P_i}{u^i}$$

where P_i is the *i*'th Markov parameter of the transformed system $\bar{G}(u)$. These parameters are easily calculated by the long division algorithm or by the recurrence relation

$$P_i = \frac{1}{\alpha_n}\left(\beta_{n-i} - \sum_{j=0}^{i-1} P_j \alpha_{n+j-i}\right), \qquad i = 0,1,2,.....$$
$$\beta_j = 0 \quad \text{for } j < 0.$$

The reduced k'th degree denominator polynomial of the approximate transfer function $\bar{G}_k(u)$ is then obtained from solving, in a least-squares sense, the linear set

$$\mathbf{P}\boldsymbol{c} = \boldsymbol{p} \qquad (7)$$

where, $\boldsymbol{c} = [c_{k-1}, c_{k-2}, ..., c_1, c_0]^T$ is the vector of the required denominator coefficients,

$$\mathbf{P} = \begin{bmatrix} P_k & P_{k-1} & . & . & P_2 & P_1 \\ P_{k+1} & P_k & . & . & P_3 & P_2 \\ P_{k+2} & P_{k+1} & . & . & P_4 & P_3 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix} \quad \text{and} \quad \boldsymbol{p} = \begin{bmatrix} -P_{k+1} \\ -P_{k+2} \\ -P_{k+3} \\ . \\ . \\ . \end{bmatrix}.$$

The rows of $\mathbf{P}$ and $\boldsymbol{p}$ may be truncated at a suitable point [**11**] because the poles of $\bar{G}(u)$ are all less than one in magnitude, so $|P_j| \to 0$ as $j \to \infty$.

Equation (7) is equivalent to equating all of the significant Markov parameters of $\bar{G}(u)$ with those of its approximation $\bar{G}_k(u)$. Hence this extensive amount of (transformed) whole-system information is used in its solution given by

$$\boldsymbol{c} = \left(\mathbf{P}^{\mathbf{T}}\mathbf{P}\right)^{-1}\mathbf{P}^{\mathbf{T}}\boldsymbol{p} \qquad (8)$$

The k poles given by this denominator polynomial, $u^k + c_{k-1}u^{k-1} + ... + c_1 u + c_0$, are guaranteed to lie inside the unit circle [**11**]. Consequently, applying the inverse mapping of equation (5) will place these poles back inside the circle $|s + a| = R$ in the s-domain. These poles will be stable because the conditions imposed by equation (6) means that no part of the circle interior lies in the right-hand half of the s-plane.

The numerator of $G_k(s)$ can be found either by using the complete Padé least-squares algorithm [**11**], which effectively matches exactly the first k Markov parameters between $\bar{G}(u)$ and $\bar{G}_k(u)$ before the inverse transformation, or by using an *optimal* algorithm [**12**]. It is interesting to note that if the former method is used then the first k Markov parameters of $G(s)$ are also preserved in $G_k(s)$. This is because the shift-and-scale transformation gives these as a linear combination of those of $\bar{G}(u)$. However, as in the bilinear method [**10**], it is proposed to use the readily available algorithm [**12**] for calculating the numerator polynomial that minimises the integral-square-error (ISE) value

$$\int_0^\infty \{y(t) - y_k(t)\}^2 \, dt$$

where $y(t)$ and $y_k(t)$ are the responses of the full and reduced-order models respectively to an impulse input. This gives a fair basis on which to compare various stability-preserving methods if their transfer function numerators are also calculated by this algorithm.

In summary, the procedure followed by the proposed shift-and-scale method to obtain the reduced-order models is as follows:

- use the transformation $s = uR - a$ on the full transfer function $G(s)$ to obtain $\bar{G}(u)$;
- use the least-squares Padé algorithm [**11**] with the Markov parameters of $\bar{G}(u)$ to calculate the reduced-degree denominator coefficients of $\bar{G}_k(u)$;
- apply the inverse mapping $u = \dfrac{s+a}{R}$ to the denominator of $\bar{G}_k(u)$ to obtain the denominator of $G_k(s)$;

- use the ISE minimisation algorithm [**12**] to calculate the numerator polynomial of $G_k(s)$.

3. CHOOSING THE TRANSFORMATION PARAMETER VALUES

Central to the proposed method are the values of the parameters a and R in the transformation from the s-plane to the u-plane, given by equation (5). The circle $|s+a| = R$ must enclose the poles of $G(s)$ and so, if no knowledge of the pole locations is at hand, then either this has to be calculated or a trial-and-error approach can be employed.

Finding the pole locations can be a relatively straightforward process with today's computing power, but some locations might still be troublesome and time-consuming. However, much computation might be avoided if the exact locations are not calculated and, instead, a test is used to indicate whether the poles lie inside the circle or not. Such a Routh-type test, using the denominator coefficients of $\bar{G}(u)$, is available in the u-plane [**13**] to determine the pole positions relative to the unit circle. This efficient test has been used in stability studies for discrete-time systems over the years and is a preferable alternative to the older Jury test [**14**] that uses determinants. A brief account of the algorithm is given in the Appendix for ease of reference.

Armed with this procedure, it is dependent on the designer to try various values of a and R to come up with a suitable model. Although this has the advantage of bringing a degree of flexibility into the order-reduction process, it also has the disadvantage of perhaps being too dependent on the skill of the designer in selecting these parameter values. To help choose initial values for these parameters, it is suggested that

$$a = R = a_{n-1} \tag{9}$$

where, from equation (1), $-a_{n-1}$ is the sum of the poles of $G(s)$. Reasons for this choice include the following.

(i) Provided that the poles of $G(s)$ do not contain too many with large imaginary parts (high oscillations) then the circle with centre at $s = -a_{n-1}$ and radius a_{n-1} will almost certainly contain all of the poles.

(ii) Poles with real parts ≤ -2 are usually associated with the fast-acting modes of the system, whose influence decays quickly in the time-response. Very often, the slower acting poles have real parts between 0 and −1. The restriction that $a \geq R$ means that the difference between these values, $a - R$, is often quite small when compared to the value of a if slow and fast acting poles are present. Hence letting $a = R$ seems reasonable as an initial step.

(iii) Experience with many transfer functions has shown that "better" models tend to be derived if the system poles are not too close to the circle boundary. This means that larger rather than smaller values of a are the norm (see the examples). A possible explanation for this might be that, as the reduced-order model's poles are constrained to lie always within the defined circular region, then having system poles very close to the boundary makes it likely that their approximations move closer to the circle centre. In some approximations it may be necessary to have approximate poles that are further from the circle centre in order to compensate adequately for the reduction in system modes. This is seen to happen in the two given examples.

In the above suggestions, it must be remembered that experimentation with the values of a and R will be inevitable for some systems. The relative separation of the system poles will obviously influence the final choice of these parameters. The examples that follow give some indication of the effects of varying them.

4. ILLUSTRATIVE EXAMPLES

Example 1

Consider the 5th order transfer function

$$G(s) = \frac{5s^4 + 20s^3 + 30s^2 + 20s - 75}{s^5 + 20s^4 + 137 \cdot 5s^3 + 362 \cdot 5s^2 + 414s + 170}$$

which has poles at –1, –1·5 ± 0·5j, –8 ± 2j and has zeros at 1, –3, –1 ± 2j. Reducing this to 2nd and 3rd order models by the proposed method, using equation (9), gives $a = R = 20$ and the transfer functions

$$G_2(s) = \frac{4 \cdot 9038s - 1 \cdot 4429}{s^2 + 15 \cdot 4138s + 57 \cdot 1336} \quad \text{with } I_{rel} = 6 \cdot 11\%$$

(poles at –6·20 and –9·21)

$$G_3(s) = \frac{5 \cdot 007s^3 + 0 \cdot 914s - 1 \cdot 156}{s^3 + 16 \cdot 194s^2 + 70 \cdot 404s + 1 \cdot 351} \quad \text{with } I_{rel} = 5 \cdot 35\%$$

(poles at –0·019 and –8·087 ± 2·165j)

where I_{rel} is the *relative* ISE measured by $I_{rel} = \text{ISE} \div \int_0^\infty y^2(t)\, dt$.

It is seen that both of the reduced-order models give good relative ISE values and the transient impulse response of $G_2(s)$ is compared with that of $G(s)$ in Figure 2, confirming the accuracy of the approximation. Further, it is of interest to vary values of a and R to demonstrate the built-in flexibility of the method and to compare the respective ISE values of the approximations. Table 1 shows these values and it is seen that, for 3rd order models, sometimes better results are obtained by perturbing the value of R from that of a.

For the two cases in Table 1 where a and R are different values, it is noticeable that, while the 3rd order model's ISE is improved, the corresponding 2nd order model's ISE deteriorates. This is not uncommon in order-reduction methods that depend on choosing parameter values for the reduction process [**10**].

However, inspection of the pole distributions of the 2nd and 3rd order models provide a possible clue as to why the two approximations give these different characteristics.

Table 1

R	a	I_{rel}%	
		k = 2	k = 3
5	5	6·17	2·58
7	7	6·14	3·53
12	12	6·11	4·80
20·5	21	8·47	1·88
25	25	6·12	5·47
25	25·5	8·55	1·88
30	30	6·13	5·60

For R = 20·5 and a = 21, the 2nd order model has poles at –2·29 and –11·36, while those of the corresponding 3rd order model are at –0·507 and –8·065 ± 2·121j. Comparing these values with those of $G_2(s)$ ($R = a = 20$) shows that the 2nd order model has a wider range of real poles than $G_2(s)$. The fast acting pole at –11·36 is further away from the apparently significant system pair at –8 ± 2j than that of $G_2(s)$ at –9·21, so it probably contributes less response energy at important parts of the time spectrum due to the faster decay. However, the 3rd order approximation not only approximates the significant system pole-pair very well, but also has its real pole at –0·507 closer to the remaining system poles than the one given by $G_3(s)$ at –0·019. Similar characteristics are exhibited by the model for R = 25 and a = 25·5.

It is also instructive to compare these results with those obtained from the bilinear method [**10**] and the more traditional Routh [**4**] and Schwarz [**5**] methods. Table 2 shows the I_{rel} values for various values of the sampling parameter T in the

bilinear method. It is seen that the better models have approximately the same I_{rel} values as those given by the shift-and-scale method above.

Table 2

T	I_{rel}%	
	k = 2	k = 3
2	51·1	13·8
1	5·99	4·49
0·5	10·6	1·65
0·25	6·4	2·07
0·1	5·9	3·7

However, the Routh and Schwarz methods, used with the optimal numerator algorithm [**12**], give disappointing results with respective 3rd order models' I_{rel} values of 73·8% and 80·6%. This is most probably due to the fact that these two methods do not use whole-system parameter information in formulating the approximate poles.

Example 2

Consider the 6th order transfer function

$$G(s) = \frac{4s^5 + 28s^4 + 64s^3 + 32s^2 - 68s - 60}{s^6 + 34s^5 + 455s^4 + 2990s^3 + 9754s^2 + 13656s + 9360}$$

which has poles at $-1 \pm j$, $-6 \pm 3j$, $-10 \pm 2j$ and zeros at ± 1, -3, $-2 \pm j$. Reducing this to orders 2 and 3 by the shift-and-scale method, with $R = a = 34$, give the transfer functions

$$G_2(s) = \frac{3 \cdot 9380s - 9 \cdot 8385}{s^2 + 24 \cdot 2607s + 180 \cdot 8530} \quad \text{with } I_{rel} = 1 \cdot 37\%$$

(poles at −12·130 ± 5·806j)

$$G_3(s) = \frac{3 \cdot 9964s^2 + 1 \cdot 8343s - 4 \cdot 2972}{s^3 + 27 \cdot 4374s^2 + 260 \cdot 8782s + 885 \cdot 7780} \quad \text{with } I_{rel} = 0 \cdot 008\%$$

(poles at −11·233 and −8·102 ± 3·635j)

Both of these approximations are seen to be excellent and the transient impulse response of $G_2(s)$ is compared with that of $G(s)$ in Figure 3. Again, it is interesting to vary the values of the parameters a and R to compare the I_{rel} values; these are listed in Table 3. Also listed in Table 4 are the I_{rel} values for reduced-order models obtained by the bilinear method for different values of the sampling parameter T.

Table 3

R	a	I_{rel} %	
		$k = 2$	$k = 3$
6	6	7·84	0·18
6	6·5	7·32	0·14
10	10	3·25	0·028
10	10·5	3·20	0·025
15	15	2·12	0·013
15	15·5	2·11	0·012
30	30	1·42	0·006
29·5	30	1·43	0·006
40	40	1·31	0·009
50	50	1·24	0·007

Table 4

T	I_{rel} %	
	$k = 2$	$k = 3$
1	58·57	13·57
0·5	34·07	0·35
0·25	7·59	0·002
0·1	0·64	0·001

It is seen that the shift-and-scale method produces consistently good results for 2nd and 3rd order models. For this system, perturbing the value of R from that of a does not seem to be of any significant benefit because the reduced-order models' poles do not change much for a given order. However, the models appear to get slightly better for values of R and a that are larger than those given by equation (9). The bilinear method also produces excellent 3rd order models for the sampling time parameter T between 0·1 and 0·5, and an excellent 2nd order model for $T = 0·1$.

Again, the Routh and Schwarz methods used with the optimal numerator procedure produce disappointing results, giving 3rd order model I_{rel} values of 96·3% and 97·8% respectively.

It should be noted that step and ramp responses can also be approximated by the proposed method. This is done by considering the transfer function of the *transient* part of the response curve [**10,12**]. In the case of a step input, the transfer function $T(s) = \dfrac{G(s) - G(0)}{s}$ is approximated by $T_k(s)$ and the final k'th order transfer function would be given as $G_k(s) = G(0) + sT_k(s)$, thus ensuring the correct steady-state response value.

5. CONCLUDING REMARKS

A new stability-preserving model order-reduction method has been given that takes into account the whole-system parameters when calculating the approximate transfer function's denominator polynomial. A simple linear shift transformation followed by a scaling in the s-plane ensures that all of the system's poles are mapped into the unit circle. The standard least-squares Padé algorithm [**11**] is then applied to obtain the stable approximate poles. After applying the inverse transformation to the reduced-degree denominator polynomial, the optimal numerator is calculated by a readily available Padé-type algorithm [**12**].

The computational requirement of the model is quite modest and is in fact less than that for the bilinear method [**10**]. The transformation parameters a and R have to be estimated reliably before applying the technique and a suggestion has been given in equation (9) as to how this can be achieved. Estimating these parameters is an area for future work to refine the method and perhaps to cut down on the experimentation with these values. However, being able to vary the parameter values does give the method a range of flexibility that most other stability-preserving methods do not have.

Illustrative examples were given and showed that the proposed shift-and-scale method is capable of producing good approximations to rival those of the

bilinear method. Consequently, it should prove to be a useful contribution to the system designer's toolkit for model order reduction.

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APPENDIX

To determine the location of the roots of the polynomial

$$f(u) \equiv a_0 + a_1 u + a_2 u^2 + \ldots + a_n u^n$$

relative to the region enclosed by the unit circle $|u| = 1$, the coefficients may be arranged in the following table [**13**]:

a_0	a_1	a_2	.	.	.	a_n
a_n	a_{n-1}	a_{n-2}	.	.	.	a_0
b_0	b_1	b_2	.	.	b_{n-1}	
b_{n-1}	b_{n-2}	b_{n-3}	.	.	b_0	
c_0	c_1	.	.	c_{n-2}		
c_{n-2}	c_{n-3}	.	.	c_0		
.	.	.				
.	.	.				

Notice that the *even-numbered* rows consist of the elements of the row immediately above them in reverse order. The first row contains the coefficients of $f(u)$ arranged in ascending order of powers of *u*. The rest of the *odd-numbered* rows are calculated by the Routh recurrence relation [**4**],

$$b_i = a_{i+1} - a_{n-i-1}\frac{a_0}{a_n} \qquad i = 0,1,2,\ldots.,n-1$$

$$c_i = b_{i+1} - b_{n-i-2}\frac{b_0}{b_{n-1}} \qquad i = 0,1,2,\ldots.,n-2$$

etc.

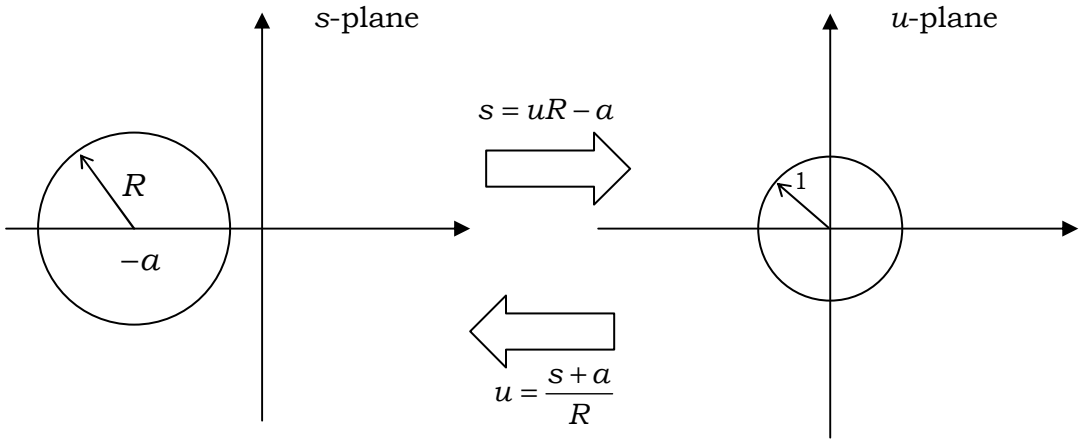
The table is exhausted after $2n + 2$ rows.

The condition to be satisfied for all of the roots of $f(u)$ to lie inside the unit circle is that the even numbered first-column elements in the table, i.e.

$$a_n,\ b_{n-1},\ c_{n-2},\ \ldots..$$

are all of the same sign [**13**].

Figure 1



Legends for Graphs

Figure 2:

——————— Impulse response of $G(s)$

- - - - - - - - - - - - - Impulse response of $G_2(s)$

Figure 3:

——————— Impulse response of $G(s)$

- - - - - - - - - - - - - Impulse response of $G_2(s)$

Figure 2

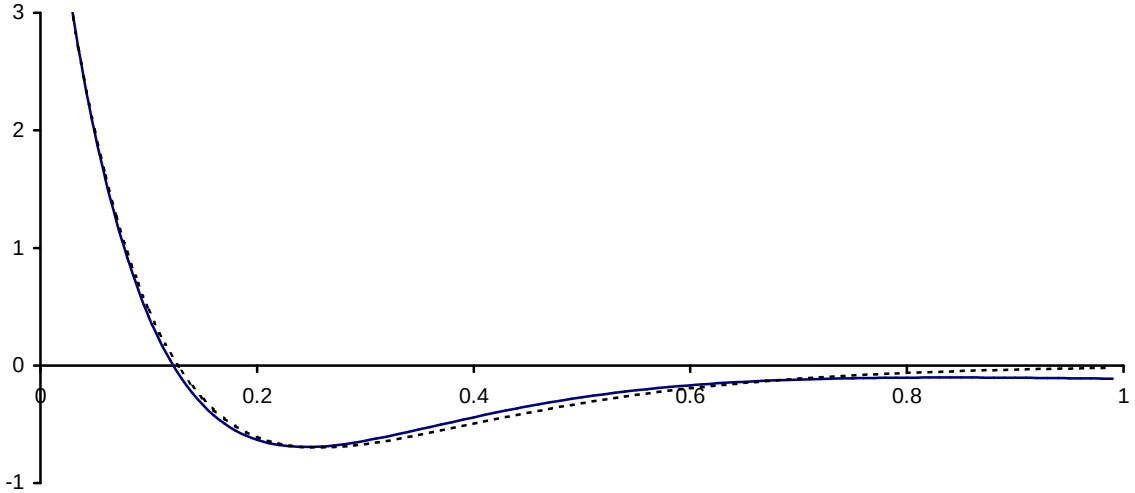


Figure 3