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Exponentially small expansions in the asymptotics of the Wright function

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Abstract

We consider exponentially small expansions present in the asymptotics of the generalised hypergeometric function, or Wright function, ${}_p\Psi_q(z)$ for large $|z|$ that have not been considered in the existing theory. Our interest is principally with those functions of this class that possess either a finite algebraic expansion or no such expansion and with parameter values that produce exponentially small expansions in the neighbourhood of the negative real z axis. Numerical examples are presented to demonstrate the presence of these exponentially small expansions.

Keywords: Asymptotics, exponentially small expansions, Wright function, generalised hypergeometric functions

MSC classification: 33C20, 33C70, 34E05, 41A60

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1. Introduction

We consider the generalised hypergeometric function, or Wright function, defined by

$$ {}_p\Psi_q(z)=\sum_{n=0}^\infty g(n)\,\frac{z^n}{n!},\qquad g(n)=\frac{\prod_{r=1}^p\Gamma(\alpha_rn+a_r)}{\prod_{r=1}^q\Gamma(\beta_rn+b_r)}, \tag{1.1}$$

where p and q are nonnegative integers, the parameters α_r and β_r are real and positive and a_r and b_r are arbitrary complex numbers. We also assume that the α_r and a_r are subject to the restriction

$$\alpha_rn+a_r\neq 0,-1,-2,\ldots\qquad (n=0,1,2,\ldots\ ;\ 1\leq r\leq p) \tag{1.2}$$

so that no gamma function in the numerator in (1.1) is singular. In the special case $\alpha_r=\beta_r=1$, the function ${}_p\Psi_q(z)$ reduces to a multiple of the ordinary hypergeometric function ${}_pF_q((a)_p;(b)_q;z)$ [1, p. 40].

We introduce the parameters associated with $g(n)$ given by

$$\kappa=1+\sum_{r=1}^q\beta_r-\sum_{r=1}^p\alpha_r,\qquad h=\prod_{r=1}^p\alpha_r^{\alpha_r}\prod_{r=1}^q\beta_r^{-\beta_r},$$

$$\vartheta=\sum_{r=1}^p a_r-\sum_{r=1}^q b_r+\tfrac{1}{2}(q-p),\qquad \vartheta'=1-\vartheta. \tag{1.3}$$

If it is supposed that α_r and β_r are such that $\kappa > 0$ then ${}_p\Psi_q(z)$ is uniformly and absolutely convergent for all finite z. If $\kappa = 0$, the sum in (1.1) has a finite radius of convergence equal to h^{-1}, whereas for $\kappa < 0$ the sum is divergent for all nonzero values of z. The parameter κ will be found to play a critical role in the asymptotic theory of ${}_p\Psi_q(z)$ by determining the sectors in the z plane in which its behaviour is either exponentially large, algebraic or exponentially small in character as $|z| \to \infty$.

The determination of the asymptotic expansion of ${}_p\Psi_q(z)$ for $|z| \to \infty$ and finite values of the parameters has a long history; for details, see [2, §2.3]. The earliest asymptotic result concerning (1.1) appears to be due to Stokes [3], who used a discrete analogue of Laplace's method for integrals when $\alpha_r = \beta_r = 1$ and positive values of a_r and b_r, to obtain the leading behaviour of ${}_p\Psi_q(z)$ when $z \to +\infty$. More precise investigations of (1.1) were carried out by Wright [4, 5] and in a long and detailed investigation by Braaksma [6] into the asymptotics of a more general class of integral function than (1.1). An account of the derivation of the asymptotic expansion of ${}_p\Psi_q(z)$ for large $|z|$ based on the Euler-Maclaurin summation formula, together with an application of this theory to the asymptotics of the solutions of a class of high-order ordinary differential equation, is described in [2]. A discussion of the properties of ${}_0\Psi_1(z)$ (the generalised Bessel function) and its application to the solution of fractional diffusion-wave equations has been given in [7].

The development of exponentially precise asymptotics during the past two decades has shown that retention of exponentially small expansions, which had previously been neglected in asymptotics, are vital for a high-precision description; see the review papers [8, 9] and also [10, §6.3]. An earlier example, which illustrated the advantage of retaining terms that are exponentially small compared with other terms in the asymptotic expansion of a certain integral, was given in [11, p. 76]. Although such terms are negligible in the Poincaré sense, their inclusion can significantly improve the numerical accuracy. In this paper we shall be concerned with exponentially small contributions present in the asymptotic expansion of ${}_p\Psi_q(z)$ for $|z| \to \infty$. Such terms are of particular significance when the parameters in (1.1) are such that there is a sector enclosing the negative real axis in which the dominant asymptotic behaviour of ${}_p\Psi_q(z)$ is either exponentially small or involves a finite algebraic expansion. Numerical examples are given to demonstrate the presence of these exponentially small subdominant contributions.

The paper is structured as follows. In §2 we present a summary of the standard results concerning the asymptotics of ${}_p\Psi_q(z)$ for large $|z|$. In §3 we consider exponentially small expansions present in ${}_0\Psi_q(z)$, which possess no algebraic expansion, and in §4 describe numerical calculations that demonstrate their existence. In §5 we discuss the case when ${}_p\Psi_q(z)$ possesses an algebraic expansion consisting of a finite number of terms. An algorithm for the computation of the coefficients appearing in the exponential expansions of ${}_p\Psi_q(z)$ is given in an appendix.

2. Standard asymptotic theory for $|z| \to \infty$

In this section we state the standard asymptotic expansions of the integral function ${}_p\Psi_q(z)$ as $|z| \to \infty$ with $\kappa > 0$ and finite values of the parameters given in [5, 6]; see also [10, §2.3]. To present these results we first introduce the exponential expansion $E(z)$ and the algebraic expansion $H(z)$ associated with ${}_p\Psi_q(z)$, together with an integral representation that will be used in our discussion.

3.1 Preliminaries. The exponential expansion of $E(z)$ is given by the formal asymptotic sum

$$E(z) = Z^{\vartheta} e^{Z} \sum_{j=0}^{\infty} A_j Z^{-j}, \qquad Z = \kappa(hz)^{1/\kappa}, \tag{2.1}$$

where the coefficients A_j are those appearing in the inverse factorial expansion of $g(s)/s!$ given by

$$\frac{g(s)}{\Gamma(s+1)} = \kappa(h\kappa^\kappa)^s \left\{\sum_{j=0}^{M-1} \frac{A_j}{\Gamma(\kappa s+\vartheta'+j)} + \frac{O(1)}{\Gamma(\kappa s+\vartheta'+M)}\right\} \tag{2.2}$$

for $|s|\to\infty$ uniformly in $|\arg s| \le \pi-\epsilon$, $\epsilon>0$; see (A.1). The leading coefficient A_0 is specified by

$$A_0 = (2\pi)^{\frac{1}{2}(p-q)}\kappa^{-\frac{1}{2}-\vartheta} \prod_{r=1}^p \alpha_r^{a_r-\frac{1}{2}} \prod_{r=1}^q \beta_r^{\frac{1}{2}-b_r}. \tag{2.3}$$

The coefficients A_j are independent of s and depend only on the parameters p, q, α_r, β_r, a_r and b_r. An algorithm for their evaluation in specific cases is described in Appendix A.

The algebraic expansion $H(z)$ follows from the Mellin-Barnes integral representation [10, §2.3]

$${}_p\Psi_q(z) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(s)g(-s)(ze^{\mp\pi i})^{-s}ds, \qquad |\arg(-z)| < \tfrac{1}{2}\pi(2-\kappa) \tag{2.4}$$

where the path of integration is indented near $s=0$ to separate[1] the poles of $\Gamma(s)$ situated at $s=0,-1,-2,\ldots$ from those of $g(-s)$ at

$$s = (a_r+k)/\alpha_r, \qquad k=0,1,2,\ldots \ (1\le r\le p). \tag{2.5}$$

The upper or lower sign in (2.4) is chosen according as $\arg z>0$ or $\arg z<0$, respectively. In general there will be p such sequences of simple poles though, depending on the values of α_r and a_r, some of these poles could be multiple poles or even ordinary points if any of the $\Gamma(\beta_r s+b_r)$ are singular there. Displacement of the contour to the right over the poles of $g(-s)$ then generates the algebraic expansion of ${}_p\Psi_q(z)$ valid in the sector in (2.4). If it is assumed that the parameters are such that the poles in (2.5) are all simple we obtain the algebraic expansion given by $H(ze^{\mp\pi i})$, where

$$H(z) = \sum_{m=1}^p \alpha_m^{-1} z^{-a_m/\alpha_m} S_{p,q}(z;m) \tag{2.6}$$

and $S_{p,q}(z;m)$ denotes the formal asymptotic sum

$$S_{p,q}(z;m) = \sum_{k=0}^\infty \frac{(-)^k}{k!}\Gamma\left(\frac{k+a_m}{\alpha_m}\right) \frac{\prod_{r=1}^{\prime\, p}\Gamma(a_r-\alpha_r(k+a_m)/\alpha_m)}{\prod_{r=1}^q \Gamma(b_r-\beta_r(k+a_m)/\alpha_m)} z^{-k/\alpha_m}, \tag{2.7}$$

with the prime indicating the omission of the term corresponding to $r=m$ in the product. This expression consists of p expansions each with the leading behaviour z^{-a_m/α_m} $(1\le m\le p)$. When the parameters α_r and a_r are such that some of the poles are of higher order, the expansion (2.7) is invalid and the residues must then be evaluated according to the multiplicity of the poles concerned; this will lead to terms involving $\log z$ in the algebraic expansion.

For future reference, we note that the integral in (2.4) may be analytically continued by bending back the path of integration into a loop with endpoints at infinity in the third and fourth quadrants; see Fig. 1(a). Thus we obtain

$${}_p\Psi_q(z) = \frac{1}{2\pi i}\int_C \Gamma(s)g(-s)(ze^{\mp\pi i})^{-s}ds, \tag{2.8}$$

where C denotes a (possibly indented) loop described in the positive sense that encloses only the poles of $\Gamma(s)$. From Stirling's formula $\Gamma(z)\sim(2\pi)^{\frac{1}{2}}e^{-z}z^{z-\frac{1}{2}}$ for large $|z|$ in $|\arg z|<\pi$,

[1]This is always possible when the condition (1.2) is satisfied.

the dominant behaviour of the modulus of the integrand as $|s| \to \infty$ is controlled by the factor $\exp\{\kappa \mathrm{Re}(s) \log |s|\}$, so that the integral in (2.8) converges *without* restriction on $\arg z$ when $\kappa > 0$ [10, §2.4 and p. 186].

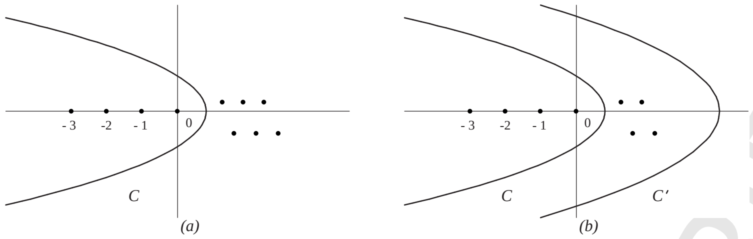

Figure 1: (a) The integration path C and (b) the displaced path C' when there is a finite number of poles on the right of C. The heavy dots denote poles of the integrand.

3.2 Statement of the expansion theorems. The asymptotic expansion of ${}_p\Psi_q(z)$ for large $|z|$ has been given in [4, 5, 6]. We have the following theorems, where throughout we let ϵ denote an arbitrarily small positive quantity.

Theorem 1 *If* $0 < \kappa < 2$, *then*

$${}_p\Psi_q(z) \sim \begin{cases} E(z) + H(ze^{\mp\pi i}) & \text{in} \quad |\arg z| \leq \frac{1}{2}\pi\kappa \\ H(ze^{\mp\pi i}) & \text{in} \quad |\arg(-z)| \leq \frac{1}{2}\pi(2-\kappa) - \epsilon \end{cases} \tag{2.9}$$

as $|z| \to \infty$. *The upper or lower sign in* $H(ze^{\mp\pi i})$ *is chosen according as* z *lies in the upper or lower half-plane, respectively.*

It is seen that the z plane is divided into two sectors, with a common vertex at $z = 0$, by the rays (the anti-Stokes lines) $\arg z = \pm\frac{1}{2}\pi\kappa$. In the sector $|\arg z| < \frac{1}{2}\pi\kappa$, the asymptotic character of ${}_p\Psi_q(z)$ is exponentially large, whereas in the complementary sector $|\arg(-z)| < \frac{1}{2}\pi(2-\kappa)$, ${}_p\Psi_q(z)$ is algebraic in character. The positive real axis $\arg z = 0$ is a Stokes line, where the algebraic expansion is maximally subdominant.

Theorem 2 *If* $\kappa = 2$ *then*

$${}_p\Psi_q(z) \sim E(z) + E(ze^{\mp 2\pi i}) + H(ze^{\mp\pi i}), \tag{2.10}$$

as $|z| \to \infty$ *in the sector* $|\arg z| \leq \pi$. *The upper or lower signs are chosen according as* $\arg z > 0$ *or* $\arg z < 0$, *respectively.*

The rays $\arg z = \pm\frac{1}{2}\pi\kappa$ now coincide with the negative real axis. It follows that ${}_p\Psi_q(z)$ is exponentially large in character as $|z| \to \infty$ except in the neighbourhood of $\arg z = \pm\pi$, where it is of the mixed type with the algebraic expansion becoming asymptotically significant.

Theorem 3 *When* $\kappa > 2$ *we have*

$${}_p\Psi_q(z) \sim \sum_{r=-P}^{P} E(ze^{2\pi i r}) \qquad (|\arg z| \leq \pi) \tag{2.11}$$

as $|z| \to \infty$ *in the sector* $|\arg z| \leq \pi$. *The integer* P *is chosen such that* $2P + 1$ *is the smallest odd integer satisfying* $2P + 1 > \frac{1}{2}\kappa$.

In this case the asymptotic behaviour of ${}_p\Psi_q(z)$ is exponentially large for all values of arg z. The exponential sums $E(ze^{2\pi ir})$ are exponentially large as $|z|\to\infty$ for values of arg z satisfying $|\arg z+2\pi r|<\frac{1}{2}\pi\kappa$ and $|\arg z|\leq\pi$. The expansion when $\kappa>2$ in [5] was given in terms of the two dominant expansions only, viz. $E(z)+E(ze^{\mp 2\pi i})$, corresponding to $r=0$ and $r=\mp 1$ in (2.11).

In Wright [4, 5] and Braaksma [6], two further theorems were given when $\kappa<2$ covering the subdominant exponential expansion. These are[2]:

Theorem 4 *If* $0<\kappa<2$ *then*

$$ {}_p\Psi_q(z)\sim E(z)+H(ze^{\mp\pi i}) \tag{2.12}$$

as $|z|\to\infty$ *in* $|\arg z|\leq\min\{\pi-\epsilon,\frac{3}{2}\pi\kappa-\epsilon\}$*. The upper or lower sign is chosen according as* $\arg z>0$ *or* $\arg z<0$*, respectively.*

It is clear that when $\frac{2}{3}\leq\kappa<2$ the expansion in a sector that includes the negative real axis must be

$$ {}_p\Psi_q(z)\sim E(z)+E(ze^{\mp 2\pi i})+H(ze^{\mp\pi i})\qquad(|\arg z|\leq\pi). \tag{2.13}$$

Since $E(z)$ is exponentially small in $\frac{1}{2}\pi\kappa<|\arg z|\leq\pi$, then in the sense of Poincaré, the expansion $E(z)$ can be neglected and there is no inconsistency between the second expansion in (2.9) and (2.12). Similarly, $E(ze^{-2\pi i})$ is exponentially small compared to $E(z)$ in $0\leq\arg z<\pi$ and there is no inconsistency between (2.12) and (2.13). However, in the neighbourhood of $\arg z=\pi$, these last two expansions are of comparable magnitude and, for real parameters, they combine to generate a real result on $\arg z=\pi$. A similar remark applies to the expansion $E(ze^{2\pi i})$ in $-\pi<\arg z\leq 0$. The expansion (2.13) is discussed in [12].

We observe that, when $\kappa<\frac{2}{3}$, $E(z)$ is exponentially small in the sectors $\frac{1}{2}\pi\kappa<|\arg z|<\frac{3}{2}\pi\kappa$. The behaviour of ${}_p\Psi_q(z)$ in the complementary sector $\frac{3}{2}\pi\kappa<|\arg z|\leq\pi$ is then algebraic and we have

$$ {}_p\Psi_q(z)\sim H(ze^{\mp\pi i})\qquad(\tfrac{3}{2}\pi\kappa+\epsilon\leq|\arg z|\leq\pi;\ 0<\kappa<\tfrac{2}{3}). \tag{2.14}$$

Theorem 5 *If* $p=0$*, so that* $g(s)$ *has no poles and* $\kappa>1$*, then* $H(z)\equiv 0$*. When* $1<\kappa<2$*, we have the expansion*

$$ {}_0\Psi_q(z)\sim E(z)+E(ze^{\mp 2\pi i}) \tag{2.15}$$

as $|z|\to\infty$ *in the sector* $|\arg z|\leq\pi$*. The upper or lower sign is chosen according as* $\arg z>0$ *or* $\arg z<0$*, respectively. The dominant expansion* ${}_0\Psi_q(z)\sim E(z)$ *holds in the reduced sector* $|\arg z|\leq\pi-\epsilon$*.*

It can be seen that (2.15) agrees with (2.13) when $H(z)\equiv 0$. Braaksma [6, p. 331, Eq. (12.18)] gave the expansion (2.15) valid in a sector straddling the negative real axis given by $\pi-\epsilon'\leq\arg z\leq\pi+\epsilon'$, where $0<\epsilon'<\frac{1}{2}\pi(1-\frac{1}{2}\kappa)$.

In [12] an examination of Theorems 4 and 5 has been carried out in some detail. A numerical investigation showed that (2.13) is valid when $\frac{2}{3}\leq\kappa<2$ and that, when $\kappa<\frac{2}{3}$, the exponential expansion $E(z)$ in Theorem 4 switches off (as $|\arg z|$ increases) across the Stokes lines $\arg z=\pm\pi\kappa$, where $E(z)$ is maximally subdominant with respect to $H(ze^{\mp\pi i})$; see [13, §3] for an analytical discussion in a particular case. Similarly in Theorem 5, it was found that the expansions $E(ze^{\mp 2\pi i})$ switch off (as $|\arg z|$ decreases) across the Stokes lines $\arg z=\pm\frac{1}{2}\pi(2-\kappa)$, where they are maximally subdominant with respect to $E(z)$. Thus, although the expansions in (2.12) and (2.15) are valid asymptotic descriptions, more acurate

[2]Wright [5] incorrectly gave the sector in Theorem 4 as $|\arg z|\leq\min\{\pi,\frac{3}{2}\pi\kappa-\epsilon\}$

evaluation will result from taking into account the Stokes phenomenon as certain rays are crossed.

3. Functions with no algebraic expansion

We first consider the case of (1.1) with $p = 0$ so that, from (2.6), the algebraic expansion $H(z) \equiv 0$. We write $\xi = ze^{\mp\pi i}$, where the upper or lower sign is chosen according as $0 \le \arg z \le \pi$ or $-\pi \le \arg z \le 0$, respectively. Then we have

$$ {}_0\Psi_q(z) = \sum_{n=0}^{\infty} \frac{(-\xi)^n}{n!} \left\{ \prod_{r=1}^{q} \Gamma(\beta_r n + b_r) \right\}^{-1}, $$

where, from (1.3),

$$ \kappa = 1 + \sum_{r=1}^{q} \beta_r > 1, \qquad h = \prod_{r=1}^{q} \beta_r^{-\beta_r}, \qquad \vartheta = \tfrac{1}{2}q - \sum_{r=1}^{q} b_r. $$

From (2.8), use of the reflection formula for the gamma function produces

$$ \begin{aligned} {}_0\Psi_q(z) &= \frac{1}{2\pi i} \int_C \Gamma(s) \left\{ \prod_{r=1}^{q} \Gamma(b_r - \beta_r s) \right\}^{-1} \xi^{-s} ds \\ &= \frac{(2\pi)^{-q}}{2\pi i} \int_C \Gamma(s) \prod_{r=1}^{q} \Gamma(1 - b_r + \beta_r s)\, \Xi(s)\, \xi^{-s} ds, \end{aligned} \tag{3.1} $$

where

$$ \Xi(s) = 2^q \prod_{r=1}^{q} \sin \pi(b_r - \beta_r s). \tag{3.2} $$

Since there are no poles of the integrand in (3.1) to the right of the contour of integration we are free to expand C as far to the right as we please (but with endpoints at infinity still in $\mathrm{Re}(s) < 0$), so that $|s|$ is everywhere large on the expanded loop. We shall continue to denote this expanded loop by C. On the expanded loop C we can employ the inverse factorial expansion obtained from (A.2)

$$ \Gamma(s) \prod_{r=1}^{q} \Gamma(1 - b_r + \beta_r s) = \frac{\kappa(h\kappa^\kappa)^{-s}}{(2\pi)^{-q}} \left\{ \sum_{j=0}^{M-1} (-)^j A_j \Gamma(\kappa s + \vartheta - j) + \rho_M(s) \Gamma(\kappa s + \vartheta - M) \right\} \tag{3.3} $$

valid for $|s| \to \infty$ in $|\arg s| \le \pi - \epsilon$, where M is a positive integer. The remainder function $\rho_M(s)$ is analytic in s except at the points $s = (b_r - 1 - k)/\beta_r$, $k = 0, 1, 2, \ldots$ $(1 \le r \le q)$ and $s = -k$, where the left-hand side of (3.3) has poles, and is such that $\rho_M(s) = O(1)$ as $|s| \to \infty$ in $|\arg s| \le \pi - \epsilon$. The coefficients A_j are the *same as those appearing in the expansion* (2.2) with $p = 0$, where

$$ A_0 = (2\pi)^{-\frac{1}{2}q} \kappa^{-\frac{1}{2} - \vartheta} \prod_{r=1}^{q} \beta_r^{\frac{1}{2} - b_r} \tag{3.4} $$

from (2.3). An algorithm for the computation of the A_j is described in Appendix A.

The function $\Xi(s)$ introduced in (3.2) has the expansion

$$ \Xi(s) = \sum_{k=1}^{N} \hat{B}_k e^{-\pi i \omega_k s} \qquad (N = 2^q), \tag{3.5} $$

where

$$\omega_1 = \sum_{r=1}^{q} \beta_r = \kappa - 1, \qquad \omega_k \geq \omega_{k+1}, \qquad \omega_N = -\omega_1,$$

and

$$\hat{B}_1 = i^{-q} \exp(\pi i \sum_{r=1}^{q} b_r) = e^{-\pi i \vartheta}, \qquad \hat{B}_N = e^{\pi i \vartheta}.$$

We introduce the variables

$$X = \kappa(h\xi)^{1/\kappa}, \qquad X_k = X e^{\pi i \omega_k/\kappa} \tag{3.6}$$

and make use of the standard result [10, §3.3]

$$\frac{\kappa}{2\pi i} \int_C \Gamma(\kappa s + \vartheta - j) z^{-\kappa s} ds = z^{\vartheta - j} e^{-z}$$

valid for all arg z when C is a loop in the positive sense enclosing all the poles of the integrand with endpoints at infinity in $\mathrm{Re}(s) < 0$; compare Fig. 1(a). Substitution of (3.3) and (3.5) into (3.1) (with the expanded contour C) then leads to

$$\begin{aligned} {}_0\Psi_q(z) &= \sum_{k=1}^{N} \hat{B}_k \left\{ \frac{\kappa}{2\pi i} \int_C X_k^{-\kappa s} \sum_{j=0}^{M-1} (-)^j A_j \Gamma(\kappa s + \vartheta - j)\, ds + R_{M,k} \right\} \\ &= \sum_{k=1}^{N} \hat{B}_k \left\{ X_k^{\vartheta} e^{-X_k} \sum_{j=0}^{M-1} (-)^j A_j X_k^{-j} + R_{M,k} \right\}, \end{aligned}$$

where the remainder $R_{M,k}$ is given by

$$R_{M,k} = \frac{\kappa}{2\pi i} \int_C \rho_M(s) \Gamma(\kappa s + \vartheta - M)\, X_k^{-\kappa s} ds.$$

It is shown in [6, §10.1; 10, p. 72, Lemma 2.8] that an order estimate for the above remainder integral is

$$|R_{M,k}| = O(X_k^{\vartheta - M} e^{-X_k}) \qquad (|X_k| \to \infty)$$

valid in the sector $|\arg X_k| < \pi$; that is, in the sectors $|\arg \xi + \pi\omega_k| < \pi\kappa$ $(1 \leq k \leq N)$. Since we can write $\omega_k = \omega_1 - \lambda_k$, with $0 \leq \lambda_k \leq 2\omega_1$, it follows that the sectors of validity for the order estimate of $R_{M,k}$ correspond to

$$-\pi(2\omega_1 - \lambda_k) - \pi < \arg \xi < \pi(1 + \lambda_k) \qquad (1 \leq k \leq N).$$

The common sector of validity of the above order estimate for $R_{M,k}$ with $1 \leq k \leq N$ is consequently $|\arg \xi| \leq \pi - \epsilon$. Hence we obtain the expansion valid for $|z| \to \infty$ in $|\arg(-z)| \leq \pi - \epsilon$

$${}_0\Psi_q(z) = \sum_{k=1}^{N} \hat{B}_k X_k^{\vartheta} e^{-X_k} \left\{ \sum_{j=0}^{M-1} (-)^j A_j X_k^{-j} + O(X_k^{-M}) \right\}.$$

An equivalent expansion valid in a narrower sector has been given in [14, p. 370].

We now define the formal exponential expansion $E_*(\xi)$ by

$$E_*(\xi) = X^{\vartheta} e^{-X} \sum_{j=0}^{\infty} (-)^j A_j X^{-j}, \tag{3.7}$$

where X is defined in (3.6). Since, from (2.1) and (3.6), $X = Ze^{\mp\pi i/\kappa}$ then some straightforward algebra using the values of ω_j, $\hat{B}_j$ $(j = 1, N)$ defined above shows that

$$\hat{B}_1 E_*(\xi e^{\pi i\omega_1}) + \hat{B}_N E_*(\xi e^{\pi i\omega_N}) = E(z) + E(ze^{\mp 2\pi i}), \tag{3.8}$$

where $E(z)$ is defined in (2.9). We then finally obtain the expansion

$${}_0\Psi_q(z) \sim E(z) + E(ze^{\mp 2\pi i}) + \sum_{k=2}^{N-1} \hat{B}_k E_*(\xi e^{\pi i\omega_k}), \qquad \xi = ze^{\mp\pi i} \tag{3.9}$$

valid as $|z| \to \infty$ in $|\arg(-z)| \le \pi - \epsilon$.

Remark: In certain cases the function

$${}_1\Psi_1(z) = \sum_{n=0}^{\infty} \frac{z^n\,\Gamma(\alpha n + a)}{n!\,\Gamma(\beta n + b)},$$

with $\beta = M\alpha$, where M is a positive integer, can be expressed in terms of a ${}_0\Psi_{M-1}(z)$ function. To see this we apply the multiplication formula for the gamma function [15, p. 256]

$$\Gamma(nz) = (2\pi)^{\frac{1}{2}(1-n)} n^{nz-\frac{1}{2}} \prod_{j=0}^{n-1} \Gamma(z + \frac{j}{n}) \qquad (n \in \mathrm{N}) \tag{3.10}$$

to obtain

$${}_1\Psi_1(z) = (2\pi)^{\frac{1}{2}(M-1)} M^{\frac{1}{2}-b} \sum_{n=0}^{\infty} \frac{(M^{-\beta}z)^n}{n!} \frac{\Gamma(\alpha n + a)}{\prod_{r=0}^{M-1} \Gamma(\alpha n + M^{-1}(b + r))}.$$

If a takes on any of the values $b_r' = (b + r)/M$ $(r = 0, 1, 2, \ldots, M - 1)$ then the gamma function in the numerator will cancel with one of the gamma functions in the denominator and the associated algebraic expansion $H(z) \equiv 0$. We then have

$${}_1\Psi_1(z) = (2\pi)^{\frac{1}{2}(M-1)} M^{\frac{1}{2}-b}\, {}_0\Psi_{M-1}(M^{-\beta}z) \qquad (\beta = M\alpha) \tag{3.11}$$

with the parameters b_r appearing in ${}_0\Psi_{M-1}$ given by $\{b_1', b_2', \ldots, *, \ldots, b_M'\}$, where the asterisk denotes the omission of one of the values. A similar reduction applies when there are one or more additional gamma functions in the numerator with the same α and appropriate values of the a_r.

4. Numerical examples

We consider cases of ${}_0\Psi_q(z)$ with $q \ge 2$, since the expansion (3.9) for the so-called generalised Bessel function ${}_0\Psi_1(z)$ (also called the Wright function in [7]) with $q = 1$ has $N = 2$ and so merely reproduces Theorem 5 when $1 < \kappa < 2$; when $\kappa \ge 2$, the expansion is covered by Theorems 2 and 3. A hyperasymptotic expansion of this function for a more extended range of parameters is given in [16].

For our first example we take $q = 2$ and consider the function

$${}_0\Psi_2(z) \equiv F(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\beta_1 n + b_1)\Gamma(\beta_2 n + b_2)}, \tag{4.1}$$

which is associated with the parameters $\kappa = 1 + \beta_1 + \beta_2$, $\vartheta = 1 - b_1 - b_2$ and $h = \beta_1^{-\beta_1}\beta_2^{-\beta_2}$. In the expansion (3.5) with $N = 4$ it is easily found that

$$\hat{B}_2 = e^{\pi i(b_1 - b_2)}, \quad \omega_2 = \beta_1 - \beta_2, \qquad \hat{B}_3 = e^{-\pi i(b_1 - b_2)}, \quad \omega_3 = \beta_2 - \beta_1.$$

From (3.9), we then obtain the asymptotic expansion

$$F(z) \sim E(z) + E(ze^{\mp 2\pi i}) \\ + e^{\pi i(b_1-b_2)} E_*(\xi e^{\pi i(\beta_1-\beta_2)}) + e^{-\pi i(b_1-b_2)} E_*(\xi e^{-\pi i(\beta_1-\beta_2)}) \tag{4.2}$$

for $|z| \to \infty$ in $|\arg(-z)| \leq \pi - \epsilon$. Comparison with Theorems 5 and 2 shows that when $\kappa \leq 2$ there are two additional exponential expansions present. These additional expansions are both exponentially small on $\arg z = \pm\pi$ when $\kappa \leq 2$, since $\cos(\pi|\beta_1 - \beta_2|/\kappa) > 0$ when $0 < \beta_1 < 1$ and $0 < \beta_2 < 1$. We note that in the case of real parameters this result correctly yields a real expansion on the negative real axis $\arg \xi = 0$.

In the case $\beta_1 = \beta_2$, the above expansion simplifies to

$$F(z) \sim E(z) + E(ze^{\mp 2\pi i}) + 2\cos \pi(b_1 - b_2)\, E_*(\xi) \tag{4.3}$$

for $|z| \to \infty$ in $|\arg(-z)| \leq \pi - \epsilon$. The upper or lower signs in (4.2) and (4.3) are chosen according as $\arg z > 0$ or $\arg z < 0$, respectively. It is clearly sufficient for real parameters to consider values of z satisfying $0 \leq \arg z \leq \pi$ and this we do throughout this section. To simplify the presentation of the results we consider only the expansion (4.3) with $\beta_1 = \beta_2 = \beta$ and define

$$B = 2\cos \pi(b_1 - b_2). \tag{4.4}$$

We use the subscript o to denote the optimal truncation of the exponential expansions in (2.1) and (3.7) at the index $j = M_o$. The optimally truncated expansion is thus written as $E_o(z)$ and corresponds to truncation of the asymptotic series for $E(z)$ at or near the smallest term in absolute value. An algorithm for the computation of the normalised coefficients $c_j = A_j/A_0$, where the leading coefficient A_0 is given in (3.4), is described in Appendix A. We have employed up to a maximum of 50 coefficients in our computations; the first ten coefficients c_j for ${}_0\Psi_2(z)$ in (4.1) are listed in Table 1 for the particular case $b_1 = \frac{1}{2}$, $b_2 = \frac{1}{3}$ and $\beta = \frac{1}{2}$.

j	c_j	j	c_j
1	$\frac{1}{8}$	2	$\frac{755}{3456}$
3	$\frac{23995}{82944}$	4	$\frac{3779875}{23887872}$
5	$-\frac{707843675}{573308928}$	6	$-\frac{2003005174555}{247669456896}$
7	$-\frac{6799898722925}{220150628352}$	8	$-\frac{167659489967405975}{3423782572130304}$
9	$\frac{381059450758385918 75}{82170781731127296}$	10	$\frac{6969724897503427236 6775}{11832592569282330624}$

Table 1: The coefficients c_j for $1 \leq j \leq 10$ for the sum (4.1) when $b_1 = \frac{1}{2}$, $b_2 = \frac{1}{3}$ and $\beta = \frac{1}{2}$.

We have computed $F(z)$ for different parameters on the negative real axis $\arg z = \pi$ where, provided the parameters are real, we have from (4.3)

$$F(z) \sim 2\mathrm{Re}\, E(z) + BE_*(ze^{-\pi i}) \qquad (\arg z = \pi).$$

In Table 2 we show the absolute values of $F(z) - 2\mathrm{Re}\, E_o(z)$ compared with $BE_*(ze^{-\pi i})$ for different parameters b_1 and b_2. The results clearly confirm the presence of the expansion $BE_*(ze^{-\pi i})$ on the negative real axis and, furthermore, that when $|b_1 - b_2| = \frac{1}{2}$ the coefficient multiplying $E_*(ze^{-\pi i})$ is indeed zero as predicted by (4.4). This last conclusion follows from the fact that, when $b_1 = 1$, $b_2 = \frac{1}{2}$, the value of $|F(z) - 2\mathrm{Re}\, E_o(z)|$ is many orders of

b_1	b_2	M_o	$\lvert F(z)-2\mathrm{Re}\,E_o(z)\rvert$	$\lvert BE_*(ze^{-\pi i})\rvert$
			$\beta=\frac{1}{4}\quad z=50e^{\pi i}\quad \kappa=\frac{3}{2}$	
$\frac{1}{2}$	$\frac{1}{3}$	30	2.725323×10^{-15}	2.725323×10^{-15}
1	$\frac{3}{4}$	32	4.769551×10^{-16}	4.769551×10^{-16}
1	$\frac{1}{2}$	38	1.626169×10^{-33}	$^\dagger\ 1.0308\times10^{-15}$
			$\beta=\frac{1}{2}\quad z=100e^{\pi i}\quad \kappa=2$	
$\frac{1}{2}$	$\frac{1}{3}$	32	1.399376×10^{-13}	1.399374×10^{-13}
1	$\frac{3}{4}$	39	1.891199×10^{-14}	1.891198×10^{-14}
1	$\frac{1}{2}$	40	2.422766×10^{-26}	$^\dagger\ 4.3853\times10^{-14}$
			$\beta=\frac{2}{3}\quad z=100e^{\pi i}\quad \kappa=\frac{7}{3}$	
$\frac{1}{2}$	$\frac{1}{3}$	30	1.552472×10^{-10}	1.552475×10^{-10}
1	$\frac{3}{4}$	32	2.408476×10^{-11}	2.408719×10^{-11}
1	$\frac{1}{2}$	38	3.744984×10^{-19}	$^\dagger\ 5.3847\times10^{-11}$

Table 2: Values of the absolute error in $F(z)-2\mathrm{Re}\,E_o(z)$ in (4.1) compared with $|BE_*(ze^{-\pi i})|$ for different parameters on arg $z=\pi$. The expansion $E(z)$ is optimally truncated at index M_o and the dagger denotes the value of $|E_*(ze^{-\pi i})|$ *without* the coefficient B (which vanishes when $b_1=1$, $b_2=\frac{1}{2}$).

magnitude smaller than the value of $|E_*(ze^{-\pi i})|$ (without the coefficient B). Table 3 shows our computations for complex values of $z=|z|e^{i\theta}$ when $\frac{1}{2}\pi\leq\theta\leq\pi$. It can be seen that as we approach arg $z=\frac{1}{2}\pi$, the value in the second column begins to differ significantly from the corresponding value in the third column. This is not due to any imprecision in the expansion (4.3) or in our computations, but can be ascribed to the Stokes phenomenon which we discuss below.

θ/π	$\lvert F(z)-\mathcal{E}_o(z)\rvert$	$\lvert BE_*(ze^{-\pi i})\rvert$
1.00	4.295293×10^{-18}	4.295292×10^{-18}
0.95	5.951357×10^{-18}	5.951359×10^{-18}
0.90	1.574211×10^{-17}	1.574212×10^{-17}
0.85	7.818456×10^{-17}	7.818201×10^{-17}
0.80	7.108583×10^{-16}	7.093135×10^{-16}
0.70	3.021297×10^{-13}	3.031269×10^{-13}
0.60	7.334040×10^{-10}	8.131071×10^{-10}
0.50	4.268643×10^{-6}	7.993992×10^{-6}

Table 3: Values of the absolute error in $F(z)-\mathcal{E}_o(z)$ in (4.1), where $\mathcal{E}_o(z)\equiv E_o(z)+E_o(ze^{-2\pi i})$, compared with $|BE_*(ze^{-\pi i})|$ for $z=40e^{i\theta}$, $b_1=\frac{1}{2}$, $b_2=\frac{3}{4}$ and $\beta=\frac{1}{10}$ $(\kappa=\frac{6}{5})$.

When $1<\kappa<2$, the function in (4.1) is exponentially large in the sector $|\arg z|<\frac{1}{2}\pi\kappa$ and exponentially small in the sector $|\arg(-z)|<\frac{1}{2}\pi(2-\kappa)$. The expansion $E(ze^{-2\pi i})$ in

(4.3) is subdominant in the upper half plane but combines with $E(z)$ on the negative real axis to give (for real b_1 and b_2) a real expansion. Since the exponential factors associated with $E(z)$ and $E(ze^{-2\pi i})$ are $\exp(|Z|e^{i\theta/\kappa})$ and $\exp(|Z|e^{i(\theta-2\pi)/\kappa})$, respectively, the greatest difference between these factors occurs when

$$\sin\left(\frac{\theta}{\kappa}\right) = \sin\left(\frac{\theta-2\pi}{\kappa}\right);$$

that is, when $\theta = \frac{1}{2}\pi(2-\kappa)$. Consequently, as $\arg z$ decreases in the upper half-plane from the value π, we expect that the expansion $E(ze^{-2\pi i})$ should switch off across the Stokes line $\arg z = \frac{1}{2}\pi(2-\kappa)$. An analogous reasoning shows that the greatest difference between the exponential factors associated with $E(z)$ and $E_*(ze^{-\pi i})$ in (4.3) occurs when

$$\sin\left(\frac{\theta}{\kappa}\right) = \sin\left(\frac{\pi-\theta}{\kappa}\right);$$

that is, on the ray $\theta = \frac{1}{2}\pi$. Thus, as $\arg z$ decreases in the upper half-plane from the value π we would expect the subdominant expansion $E_*(ze^{-\pi i})$ to switch off across the Stokes line $\arg z = \frac{1}{2}\pi$. Similar considerations apply to $E(ze^{2\pi i})$ and $E_*(ze^{\pi i})$ across the Stokes lines $\arg z = -\frac{1}{2}\pi(2-\kappa)$ and $\arg z = -\frac{1}{2}\pi$ in the lower half-plane.

To demonstrate the truth of this assertion, we choose $\beta = \frac{1}{10}$ (so that $\kappa = \frac{6}{5}$) in (4.1); the exponentially large sector is then $|\arg z| < \frac{3}{5}\pi$ and the Stokes lines in the upper half plane are $\theta = \frac{1}{2}\pi$ and $\theta = \frac{2}{5}\pi$. We define the Stokes multipliers $S_1(\theta)$ and $S_2(\theta)$ (at constant $|z|$) associated with the expansions $E_*(ze^{-\pi i})$ and $E(ze^{-2\pi i})$ as follows. For the Stokes multiplier associated with the expansion $E_*(ze^{-\pi i})$ we set

$$F(z) = E_{\rm o}(z) + E_{\rm o}(ze^{-2\pi i}) + BA_0X^{\vartheta}e^{-X}\,S_1(\theta),$$

where we recall that the subscript o denotes that the exponential expansions are optimally truncated and are therefore finite sums. To detect the switching-off of the expansion $E(ze^{-2\pi i})$, we choose $|b_1-b_2| = \frac{1}{2}$ so that $B=0$ (thereby eliminating the contribution from $E_*(ze^{-\pi i})$), and define the Stokes multiplier associated with the expansion $E(ze^{-2\pi i})$ when $B=0$ by

$$F(z) = E_{\rm o}(z) + A_0(Ze^{-2\pi i/\kappa})^{\vartheta}\exp(Ze^{-2\pi i/\kappa})\,S_2(\theta).$$

In Table 4 we present the variation of the real part[3] of the Stokes multipliers $S_1(\theta)$ and $S_2(\theta)$ in the neighbourhood of $\theta = \frac{1}{2}\pi$ and $\theta = \frac{2}{5}\pi$. It is seen that these multipliers exhibit the familiar smooth transition across their respective Stokes lines. This viewpoint can be confirmed for a particular case of (4.1), where it is possible to make a routine application of the saddle-point method applied to a Laplace-type integral representation; see Appendix B.

To conclude this section we present the expansions for the function

$$_0\Psi_q(z) \equiv F(z) = \sum_{n=0}^{\infty}\frac{z^n}{n!}\left\{\prod_{r=1}^{q}\Gamma(\beta_r n + b_r)\right\}^{-1} \tag{4.5}$$

in the cases $q=3$ and $q=4$. For simplicity in presentation, we shall assume that the $\beta_j = \beta$ and that the b_j are all real (the results are easily modified when these assumptions are not satisfied). This simplification results in the coefficients $\hat{B}_k$ in (3.5) satisfying

$$\hat{B}_k = \overline{\hat{B}}_{N-k} \qquad (1 \le k \le N/2;\ N = 2^q),$$

where the bar denotes the complex conjugate. When $q=3$ ($N=8$), we have from (3.5) $\omega_2 = \omega_3 = \omega_4 = \beta$, $\omega_5 = \omega_6 = \omega_7 = -\beta$ and $\hat{B}_k = -i\exp(\pi i {\sum}' b_r)$ $(2 \le k \le 4)$, where the

[3]The Stokes multipliers possess a small imaginary part that is not shown.

$\lvert z\rvert=40,\ b_1=\frac{1}{2},\ b_2=\frac{3}{4}$		$\lvert z\rvert=20,\ b_1=\frac{1}{4},\ b_2=\frac{3}{4}$	
θ/π	$\mathrm{Re}\,S_1(\theta)$	θ/π	$\mathrm{Re}\,S_2(\theta)$
0.80	0.9848	1.00	0.9908
0.70	0.9825	0.80	0.9896
0.60	0.8920	0.60	0.9862
0.56	0.7792	0.55	0.9575
0.54	0.7029	0.50	0.8679
0.52	0.6161	0.45	0.6893
0.50	0.5227	0.40	0.4477
0.48	0.4276	0.35	0.2230
0.46	0.3361	0.30	0.0797
0.40	0.1229	0.20	0.0020

Table 4: Variation of the real part of the Stokes multipliers $S_1(\theta)$ and $S_2(\theta)$ associated with (4.1) about the Stokes lines $\theta=\frac{1}{2}\pi$ and $\theta=\frac{2}{5}\pi$ when $\beta=\frac{1}{10}$.

prime denotes that the coefficient of b_{k-1} is replaced by -1 in the summation. Then

$$B_1=\sum_{r=2}^4 \hat{B}_r=-i\{e^{\pi i(b_1+b_2-b_3)}+e^{\pi ib_3}2\cos\pi(b_1-b_2)\},\qquad B_2=\sum_{r=5}^7 \hat{B}_r=\overline{B}_1.$$

From (3.9) we therefore obtain the expansion

$$F(z)\sim E(z)+E(ze^{\mp 2\pi i})+B_1E_*(\xi e^{\pi i\beta})+\overline{B}_1E_*(\xi e^{-\pi i\beta}) \tag{4.6}$$

when $q=3$ as $|z|\to\infty$ in $|\arg z|\leq\pi-\epsilon$.

When $q=4$ ($N=16$), we obtain after some routine algebra $\omega_k=2\beta$ $(2\leq k\leq 5)$, $\omega_k=0$ $(6\leq k\leq 11)$, $\omega_k=-2\beta$ $(12\leq k\leq 15)$ and

$$\hat{B}_k=-\exp\big(\pi i{\textstyle\sum}' b_r\big)\quad (2\leq k\leq 5),\qquad \hat{B}_k=\exp\big(\pi i{\textstyle\sum}'' b_r\big)\quad (6\leq k\leq 8),$$

where the double prime denotes that the coefficients of b_{k-5} and b_4 are replaced by -1 in the summation. This yields the coefficients

$$B_1=\sum_{r=2}^5 \hat{B}_r=-e^{\pi i(b_1+b_2)}2\cos\pi(b_3-b_4)-e^{\pi i(b_3+b_4)}2\cos\pi(b_1-b_2),\quad B_3=\sum_{r=12}^{15}\hat{B}_r=\overline{B}_1,$$

$$B_2=\sum_{r=6}^{11}\hat{B}_r=2\cos\pi(b_1+b_2-b_3-b_4)+4\cos\pi(b_1-b_2)\cos\pi(b_3-b_4).$$

Then, from (3.9),

$$F(z)\sim E(z)+E(ze^{\mp 2\pi i})+B_1E_*(\xi e^{2\pi i\beta})+B_2E_*(\xi)+\overline{B}_1E_*(\xi e^{-2\pi i\beta}) \tag{4.7}$$

when $q=4$ as $|z|\to\infty$ in $|\arg z|\leq\pi-\epsilon$. In both (4.6) and (4.7) we recall that $\xi=ze^{\mp\pi i}$ and that the upper or lower signs are chosen according as $\arg z>0$ or $\arg z<0$, respectively. Both these expansions correctly produce a real result (for real b_j) on the negative real axis. When the β_j are not all equal it is clear from (3.9) that more subdominant expansions can appear.

In Tables 5 and 6 we display the results of computations for $q = 3$ and $q = 4$ on the negative real axis using (4.6) and (4.7). Table 5 shows the case $q = 3$ for a range of parameters yielding different values of B_1. In the case $B_1 = 0$, we have shown the absolute value of $E_*(\xi e^{\pi i\beta})$, which is many orders of magnitude greater than the error in $F(z)$ using an optimally truncated $2\mathrm{Re}\,E(z)$, thereby confirming its absence. Table 6 shows similar calculations for the case $q = 4$ with $\beta = \frac{1}{5}$. In the case when either B_1 or B_2 (or both) is zero we have shown only the absolute values of the associated exponential expansions. The first set of values (with $B_1 = 0$) confirms the absence of the expansion $E_*(\xi e^{2\pi i\beta})$, whereas the second set of values (with $B_1 = 0$ and $B_2 = 0$) confirms the absence of both the expansions $E_*(\xi e^{2\pi i\beta})$ and $E_*(\xi)$. In the final entry of Table 6, where $B_1,\ B_2 \neq 0$, it is seen that the third column confirms the presence of the expansion $E_*(\xi e^{2\pi i\beta})$, but that our calculations are not sufficiently precise to be able to confirm the presence of the sub-subdominant expansion $E_*(\xi)$ in this case.

	$\beta=\frac{1}{4},\ b_1=\frac{7}{12},\ b_2=\frac{5}{12},\ b_3=-\frac{1}{2}$; $B_1=1+\sqrt{3},\ z=\lvert z\rvert e^{\pi i}$		$\beta=\frac{1}{4},\ b_1=\frac{1}{3},\ b_2=\frac{2}{3},\ b_3=1$; $B_1=0,\ z=\lvert z\rvert e^{\pi i}$	
$\lvert z\rvert$	$\lvert F(z)-2\mathrm{Re}E_o(z)\rvert$	$\lvert 2\mathrm{Re}B_1E_*(\xi e^{\pi i\beta})\rvert$	$\lvert F(z)-2\mathrm{Re}E_o(z)\rvert$	$\lvert E_*(\xi e^{\pi i\beta})\rvert$
50	3.009092×10^{-12}	3.008661×10^{-12}	6.167912×10^{-30}	5.092623×10^{-14}
80	7.988609×10^{-16}	7.988522×10^{-16}	3.208815×10^{-36}	3.149887×10^{-18}
100	1.000311×10^{-17}	1.000326×10^{-17}	2.704118×10^{-33}	9.825482×10^{-20}
150	4.196439×10^{-23}	4.196644×10^{-23}	1.127316×10^{-41}	2.178074×10^{-24}
	$\beta=\frac{1}{3},\ b_1=\frac{1}{4},\ b_2=1,\ b_3=\frac{3}{4}$; $B_1=1,\ z=\lvert z\rvert e^{\pi i}$		$\beta=\frac{1}{3},\ b_1=\frac{1}{3},\ b_2=\frac{1}{2},\ b_3=\frac{3}{4}$; $B_1\doteq-1.48356-0.25882i,\ z=\lvert z\rvert e^{\pi i}$	
$\lvert z\rvert$	$\lvert F(z)-2\mathrm{Re}E_o(z)\rvert$	$\lvert 2\mathrm{Re}B_1E_*(\xi e^{\pi i\beta})\rvert$	$\lvert F(z)-2\mathrm{Re}E_o(z)\rvert$	$\lvert 2\mathrm{Re}B_1E_*(\xi e^{\pi i\beta})\rvert$
50	1.446717×10^{-12}	1.521726×10^{-12}	6.515015×10^{-11}	6.514059×10^{-11}
80	4.133266×10^{-15}	4.124276×10^{-15}	2.427855×10^{-13}	2.429876×10^{-13}
100	3.343169×10^{-15}	3.342621×10^{-15}	1.024392×10^{-15}	1.008831×10^{-15}
150	1.840213×10^{-18}	1.840202×10^{-18}	7.682549×10^{-18}	7.695989×10^{-18}

Table 5: Values of the absolute error in $F(z) - 2\mathrm{Re}\,E_o(z)$ in (4.5) with $q = 3$ on the negative real axis compared with $\lvert 2\mathrm{Re}B_1E_*(\xi e^{\pi i\beta})\rvert$ for different β and parameters b_j. The exponential expansions are optimally truncated.

5. Functions with a finite algebraic expansion

We now turn to consideration of the function ${}_p\Psi_q(z)$ in the special case when the algebraic expansion $H(z)$ in (2.6) is finite. From (2.8), we have the integral representation

$$ {}_p\Psi_q(z) = \frac{1}{2\pi i}\int_C \Gamma(s)g(-s)\,\xi^{-s}ds, \tag{5.1}$$

where $\xi = ze^{\mp\pi i}$ and C denotes a loop in the positive sense enclosing only the poles of $\Gamma(s)$ at $s = 0, -1, -2, \ldots$. A finite algebraic expansion will result when, in addition to the restriction (1.2), the parameters are such that the zeros of $g(-s)$ cancel all but a finite number of the poles at $s = (a_r + k)/\alpha_r$ $(k = 0, 1, 2, \ldots;\ 1 \le r \le p)$. The sum of the residues at the finite set of poles $s = (a_m + k)/\alpha_m$ $(0 \le k \le k_0(m);\ 1 \le m \le p)$ will be denoted

$\|z\|$	$\|F(z)-2\mathrm{Re}E_o(z)\|$	$\|E_*(\xi e^{2\pi i\beta})\|$	$\|B_2E_*(\xi)\|$
	$\beta=\frac{1}{5},\ b_1=\frac{1}{6},\ b_2=\frac{1}{3},\ b_3=\frac{2}{3},\ b_4=\frac{5}{6}$; $B_1=0,\ B_2=1,\ z=\|z\|e^{\pi i}$		
30	4.972565×10^{-13}	1.482075×10^{-10}	4.972569×10^{-13}
50	1.685216×10^{-16}	1.169696×10^{-13}	1.685216×10^{-16}
80	1.086265×10^{-20}	5.703438×10^{-17}	1.086265×10^{-20}
100	4.256578×10^{-23}	1.875656×10^{-18}	4.256578×10^{-23}
120	2.682152×10^{-25}	4.362680×10^{-20}	2.682152×10^{-25}
	$\beta=\frac{1}{5},\ b_1=\frac{5}{4},\ b_2=\frac{3}{4},\ b_3=\frac{1}{2},\ b_4=1$; $B_1=0,\ B_2=0,\ z=\|z\|e^{\pi i}$		
$\|z\|$	$\|F(z)-2\mathrm{Re}E_o(z)\|$	$\|E_*(\xi e^{2\pi i\beta})\|$	$\|E_*(\xi)\|$
30	3.432624×10^{-26}	1.822456×10^{-11}	1.154181×10^{-13}
50	4.796958×10^{-31}	4.832596×10^{-14}	2.535357×10^{-17}
80	4.709347×10^{-36}	1.973750×10^{-17}	1.098475×10^{-21}
100	2.278757×10^{-38}	2.358151×10^{-19}	3.566169×10^{-24}
120	4.541513×10^{-41}	4.049120×10^{-21}	1.927266×10^{-26}
	$\beta=\frac{1}{5},\ b_1=\frac{1}{6},\ b_2=\frac{1}{6},\ b_3=\frac{3}{4},\ b_4=1$; $B_1\doteq-2.12132+0.18947i,\ B_2\doteq2.31079,\ z=\|z\|e^{\pi i}$		
$\|z\|$	$\|F(z)-2\mathrm{Re}E_o(z)\|$	$\|2\mathrm{Re}B_1E_*(\xi e^{2\pi i\beta})\|$	$\|B_2E_*(\xi)\|$
30	6.160597×10^{-10}	5.986047×10^{-10}	1.105646×10^{-12}
50	6.506430×10^{-13}	6.542411×10^{-13}	3.620406×10^{-16}
80	3.349512×10^{-16}	3.353693×10^{-16}	2.266096×10^{-20}
100	8.317661×10^{-18}	8.318711×10^{-18}	8.762217×10^{-23}
120	1.851550×10^{-19}	1.850054×10^{-19}	5.462822×10^{-25}

Table 6: Values of the absolute error in $F(z)-2\mathrm{Re}\,E_o(z)$ in (4.5) with $q=4$ on the negative real axis compared with the other expansions for different parameters b_j. The exponential expansions are optimally truncated.

by $H_f(ze^{\mp\pi i})$, where the subscript f designates 'finite', and is given by (2.6) with the sums $S_{p,q}(z;m)$ in (2.7) replaced by finite sums over $0\le k\le k_0(m)$, viz.

$$S_{p,q}(z;m)=\sum_{k=0}^{k_0(m)}\frac{(-)^k}{k!}\Gamma\left(\frac{k+a_m}{\alpha_m}\right)\frac{\prod_{r=1}^{\prime\,p}\Gamma(a_r-\alpha_r(k+a_m)/\alpha_m)}{\prod_{r=1}^{q}\Gamma(b_r-\beta_r(k+a_m)/\alpha_m)}z^{-k/\alpha_m}. \tag{5.2}$$

The contour C is now displaced to the right over this finite set of poles; see Fig. 1(b). The displaced contour, which we shall call C', still has endpoints at infinity in $\mathrm{Re}(s)<0$. Then we have

$${}_p\Psi_q(z)=\frac{1}{2\pi i}\int_{C'}\Gamma(s)g(-s)\,\xi^{-s}ds+H_f(ze^{\mp\pi i}). \tag{5.3}$$

The integral in (5.3) can be dealt with in exactly the same manner as that discussed in §3. Use of the reflection formula for the gamma function leads to

$$\frac{1}{2\pi i}\int_{C'}\Gamma(s)g(-s)\xi^{-s}ds=\frac{(2\pi)^{p-q}}{2\pi i}\int_{C'}\Gamma(s)\frac{\prod_{r=1}^{q}\Gamma(1-b_r+\beta_r s)}{\prod_{r=1}^{p}\Gamma(1-a_r+\alpha_r s)}\,\Xi(s)\,\xi^{-s}ds,$$

where

$$\Xi(s) = 2^{q-p} \frac{\prod_{r=1}^{q} \sin \pi(b_r - \beta_r s)}{\prod_{r=1}^{p} \sin \pi(a_r - \alpha_r s)}. \tag{5.4}$$

From (A.2), we have the inverse factorial expansion

$$\Gamma(s) \frac{\prod_{r=1}^{q} \Gamma(1 - b_r + \beta_r s)}{\prod_{r=1}^{p} \Gamma(1 - a_r + \alpha_r s)} =$$

$$\frac{\kappa(h\kappa^\kappa)^{-s}}{(2\pi)^{p-q}} \left\{ \sum_{j=0}^{M-1} (-)^j A_j \Gamma(\kappa s + \vartheta - j) + O(1)\Gamma(\kappa s + \vartheta - M) \right\}$$

for $|s| \to \infty$ in $|\arg s| \leq \pi - \epsilon$, where M is a positive integer. The coefficients A_j (which depend on the parameters) are determined in particular cases by the algorithm described in Appendix A.

Since the integrand in (5.3) has no poles to the right of C', the sines in the numerator of $\Xi(s)$ in (5.4) must cancel with those in the numerator, with the consequence that $\Xi(s)$ must be expandable as a series of exponentials in the form

$$\Xi(s) = \sum_{k=1}^{N} \hat{B}_k e^{-\pi i \omega_k s}, \tag{5.5}$$

where N depends on the parameter values,

$$\omega_1 = \sum_{r=1}^{q} \beta_r - \sum_{r=1}^{p} \alpha_r = \kappa - 1, \qquad \omega_k \geq \omega_{k+1}, \qquad \omega_N = -\omega_1$$

and

$$\hat{B}_1 = e^{-\pi i \vartheta}, \qquad \hat{B}_N = e^{\pi i \vartheta}.$$

Then the same reasoning leading to (3.9) shows that (with $\xi = ze^{\mp\pi i}$)

$$_p\Psi_q(z) \sim E(z) + E(ze^{\mp 2\pi i}) + \sum_{k=2}^{N-1} \hat{B}_k E_*(\xi e^{\pi i \omega_k}) + H_f(ze^{\mp\pi i}) \tag{5.6}$$

for $|z| \to \infty$ in $|\arg(-z)| \leq \pi - \epsilon$, where the upper or lower signs are chosen according as $\arg z > 0$ or $\arg z < 0$, respectively.

In the simple case $p = q = 1$, corresponding to the function

$$_1\Psi_1(z) \equiv F(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\Gamma(\alpha n + a)}{\Gamma(\beta n + b)} = \frac{1}{2\pi i} \int_C \Gamma(s) \frac{\Gamma(a - \alpha s)}{\Gamma(b - \beta s)} \xi^{-s} ds, \tag{5.7}$$

the coefficients $\hat{B}_k$ can be determined explicitly. For a finite algebraic expansion, it is easily shown that, for positive integer M, we require

$$\beta = M\alpha, \qquad \gamma := b - Ma \in \mathrm{N}. \tag{5.8}$$

If $\gamma = 0, -1, -2, \ldots$, there are no poles on the right of C and accordingly $H(z) \equiv 0$, whereas if $\gamma = 1, 2, \ldots$ there is a finite number $k_0(1) \equiv k_0 = \lceil \gamma/M \rceil$ of such poles, where $\lceil x \rceil$ denotes the smallest integer not less than x. The algebraic expansion in this last case is therefore

$$H_f(ze^{\mp\pi i}) = \frac{1}{\alpha} \sum_{k=0}^{k_0} \frac{(-)^k}{k!} \frac{\Gamma((k+a)/\alpha)}{\Gamma(\gamma - Mk)} \xi^{-(k+a)/\alpha}. \tag{5.9}$$

In terms of the new variable $u := \alpha s - a$, the function $\Xi(s)$ is given by

$$\begin{aligned}\Xi(s) &= \frac{\sin\pi(b-\beta s)}{\sin\pi(a-\alpha s)} = (-)^\gamma\,\frac{\sin\pi M u}{\sin\pi u}\\ &= (-)^\gamma\biggl\{\sum_{r=1}^{\lfloor M/2\rfloor} 2\cos\pi(M-2r+1)u+\delta_M\biggr\},\end{aligned}$$

where $\delta_M = 0$ (M even), 1 (M odd). It is then easily seen in (5.5) that $N = M$ and[4]

$$\omega_k = (M-2k+1)\alpha, \qquad \hat{B}_k = (-)^\gamma\, e^{\pi i(M-2k+1)a} \qquad (1\leq k\leq M).$$

An example closely related to (5.7), with $\alpha = 1/M$, $\beta = 1$, $a = 1/(2M)$ and $b = \frac{1}{2}$, has been studied in [17] in connection with the asymptotics of a generalised incomplete gamma function.

In Table 7 we show computations for the case $M = 3$ and integer $\gamma = b - 3a$ for different parameter values on the negative real axis, where from (5.6) we obtain the expansion

$$F(z) \sim E(z) + E(ze^{\mp 2\pi i}) + \cos\pi\gamma\, E_*(\xi) + H_f(ze^{\mp\pi i}), \tag{5.10}$$

with $H_f(ze^{\mp\pi i})$ defined in (5.9). We remark that the results when $\kappa = 2$ are more precise than when $\kappa < 2$. This results from the fact that when $\kappa = 2$ the algebraic expansion and $\mathrm{Re}\,E(z)$ are both of algebraic order with $E_*(\xi)$ being subdominant, whereas when $\kappa < 2$, $\mathrm{Re}\,E(z)$ is subdominant and $E_*(\xi)$ is sub-subdominant. It is therefore more difficult to detect numerically the expansion $E_*(\xi)$ when $\kappa < 2$.

As a second example, we take $p = 2$, $q = 1$ and consider the sum

$$F(z) = \sum_{n=0}^\infty \frac{z^n}{n!}\,\frac{\Gamma(\frac{1}{4}n+b-\frac{1}{6})\Gamma(\frac{1}{4}n+b-\frac{1}{3})}{\Gamma(\frac{3}{2}n+6b)}, \tag{5.11}$$

for which $\kappa = 2$, $\vartheta = -4b-1$ and the integer M in (5.8) satisfies $M = 6$. The integral representation in (5.1) for this function is associated with two poles on the right of the (possibly indented) contour C at $s = 4b - \frac{4}{3}$ and $s = 4b - \frac{2}{3}$. From (5.2) with $k_0(1) = k_0(2) = 1$, we then have the finite algebraic expansion given by

$$H_f(ze^{\mp\pi i}) = 4(ze^{\mp\pi i})^{-4b+\frac{4}{3}}\left\{\Gamma(4b-\tfrac{4}{3})\Gamma(\tfrac{1}{6}) + \Gamma(4b-\tfrac{2}{3})\Gamma(-\tfrac{1}{6})(ze^{\mp\pi i})^{-\frac{2}{3}}\right\}.$$

To deal with the function $\Xi(s)$ in this case we make use of the multiple-angle expansion of $\sin n\theta$ for positive integer n [18, p. 119]

$$\sin n\theta = 2^{n-1}\prod_{j=0}^{n-1}\sin\biggl(\theta+\frac{\pi j}{n}\biggr).$$

Then, from (5.4),

$$\begin{aligned}-\Xi(s) &= \frac{2^{-1}\sin\pi(\frac{3}{2}s-6b)}{\sin\pi(\frac{1}{4}s-b+\frac{1}{6})\,\sin\pi(\frac{1}{4}s-b+\frac{1}{3})} = 2^4\sin\pi(\tfrac{1}{4}s-b)\prod_{j=3}^{5}\sin\pi(\tfrac{1}{4}s-b+\tfrac{1}{6}j)\\ &= e^{-\pi is+4\pi ib} + i\sqrt{3}e^{-\frac{1}{2}\pi is+2\pi ib} - 2 - i\sqrt{3}e^{\frac{1}{2}\pi is-2\pi ib} + e^{\pi is-4\pi ib}.\end{aligned}$$

[4] Since $\kappa = 1+\beta-\alpha$ and $\vartheta = a-b = a(1-M)-\gamma$, this correctly produces $\omega_1 = (M-1)\alpha = \kappa-1$ and $\hat{B}_1 = (-)^\gamma\, e^{\pi i(M-1)a} = e^{-\pi i\vartheta}$.

	$\beta=\frac{3}{2}$ $(\kappa=2)$, $a=\frac{1}{2}$, $b=\frac{11}{2}$ $\gamma=4$, $z=\vert z\vert e^{\pi i}$		$\beta=\frac{3}{2}$ $(\kappa=2)$, $a=\frac{1}{2}$, $b=\frac{13}{2}$ $\gamma=5$, $z=\vert z\vert e^{\pi i}$	
$\vert z\vert$	$\hat{F}(z)-2\mathrm{Re}E_o(z)$	$\cos\pi\gamma\, E_*(\xi)$	$\hat{F}(z)-2\mathrm{Re}E_o(z)$	$\cos\pi\gamma\, E_*(\xi)$
100	4.755780×10^{-11}	4.753537×10^{-11}	-5.620471×10^{-12}	-5.615518×10^{-12}
200	4.739104×10^{-14}	4.739146×10^{-14}	-3.859942×10^{-15}	-3.859482×10^{-15}
300	3.274243×10^{-16}	3.274256×10^{-16}	-2.151689×10^{-17}	-2.151612×10^{-17}
400	5.678609×10^{-18}	5.678610×10^{-18}	-3.208375×10^{-19}	-3.208393×10^{-19}
500	1.724549×10^{-19}	1.724549×10^{-19}	-8.671458×10^{-21}	-8.671433×10^{-21}
	$\beta=1$ $(\kappa=\frac{5}{3})$, $a=\frac{1}{4}$, $b=\frac{15}{4}$ $\gamma=3$, $z=\vert z\vert e^{\pi i}$		$\beta=1$ $(\kappa=\frac{5}{3})$, $a=\frac{1}{4}$, $b=\frac{19}{4}$ $\gamma=4$, $z=\vert z\vert e^{\pi i}$	
$\vert z\vert$	$\hat{F}(z)-2\mathrm{Re}E_o(z)$	$\cos\pi\gamma\, E_*(\xi)$	$\hat{F}(z)-2\mathrm{Re}E_o(z)$	$\cos\pi\gamma\, E_*(\xi)$
100	-8.950444×10^{-14}	-8.984340×10^{-14}	7.761350×10^{-15}	7.648353×10^{-15}
200	-3.679767×10^{-19}	-3.674985×10^{-19}	1.994065×10^{-20}	2.010004×10^{-20}
300	-2.229504×10^{-23}	-2.229400×10^{-23}	9.453819×10^{-25}	9.452937×10^{-25}
400	-5.372028×10^{-27}	-5.372107×10^{-27}	1.904813×10^{-28}	1.904202×10^{-28}
500	-3.123141×10^{-30}	-3.123180×10^{-30}	9.640370×10^{-32}	9.640878×10^{-32}

Table 7: Values of the error in $\hat{F}(z)-2\mathrm{Re}\,E_o(z)$ in (5.7) on the negative real axis, where $\hat{F}(z)\equiv F(z)-H_f(ze^{-\pi i})$, compared with $\cos\pi\gamma\,E_*(\xi)$ for $M=3$ and different a, b and β. The exponential expansions are optimally truncated.

It follows that in (5.5) we have $N=5$, $\omega_1=-\omega_5=1$, $\omega_2=-\omega_4=\frac{1}{2}$, $\omega_3=0$ and

$$\hat{B}_1=e^{-\pi i\vartheta},\quad \hat{B}_2=-\sqrt{3}e^{-\frac{1}{2}\pi i\vartheta},\quad \hat{B}_3=2,\quad \hat{B}_4=-\sqrt{3}e^{\frac{1}{2}\pi i\vartheta},\quad \hat{B}_5=e^{\pi i\vartheta}.$$

Then, from (5.6), we obtain the expansion (when b is assumed to be real)

$$F(z)\sim E(z)+E(ze^{\mp 2\pi i})+\mathrm{Re}\,[B'E_*(\xi e^{\frac{1}{2}\pi i})]+2E_*(\xi)+H_f(ze^{\mp\pi i}) \tag{5.12}$$

for $|z|\to\infty$ in $|\arg(-z)|\leq\pi-\epsilon$, where $B'=-2\sqrt{3}e^{-\frac{1}{2}\pi i\vartheta}$. Since $\kappa=2$, the contribution $2\mathrm{Re}\,E(z)$ is of algebraic order on the negative real axis, whereas $E_*(i\xi)=O(\exp(-Xe^{\frac{1}{4}\pi i}))$ and $E_*(\xi)=O(\exp(-X))$, where $X=(2/3)^{3/4}(2|z|)^{1/2}$ by (3.6). In Table 8 we show the absolute error in the computation of $F(z)-H_f(ze^{-\pi i})-2\mathrm{Re}\,E_o(z)$ compared with the subdominant contribution $\mathrm{Re}\,B'E_*(i\xi)$ on $\arg z=\pi$. As $E_*(\xi)$ is sub-subdominant on $\arg z=\pi$, it was not possible to detect this second exponential expansion. To do this would require a hyperasymptotic evaluation of $F(z)$ on the lines of that described for the generalised Bessel function in [16].

$\|z\|\times 10^3$	$b=\frac{4}{3},\quad z=\|z\|e^{\pi i}$ $\|\hat{F}(z)-2\mathrm{Re}E_o(z)\|$	$\|\mathrm{Re}\,B'E_*(i\xi)\|$	$b=\frac{3}{2},\quad z=\|z\|e^{\pi i}$ $\|\hat{F}(z)-2\mathrm{Re}E_o(z)\|$	$\|\mathrm{Re}\,B'E_*(i\xi)\|$
0.50	7.615899×10^{-16}	7.654058×10^{-16}	4.076785×10^{-17}	4.082330×10^{-17}
1.00	1.050252×10^{-19}	1.052048×10^{-19}	5.934098×10^{-21}	5.928711×10^{-21}
2.00	7.092803×10^{-25}	7.094341×10^{-25}	3.322751×10^{-26}	3.338537×10^{-26}
4.00	4.900488×10^{-32}	4.900787×10^{-32}	3.275716×10^{-33}	3.274615×10^{-33}
5.00	1.689375×10^{-34}	1.689422×10^{-34}	6.472057×10^{-36}	6.471997×10^{-36}

Table 8: Values of the absolute error in $\hat{F}(z)-2\mathrm{Re}\,E_o(z)$ in (5.11) on the negative real axis, where $\hat{F}(z)\equiv F(z)-H_f(ze^{-\pi i})$, compared with $\mathrm{Re}\,B'E_*(i\xi)$ for different b. The exponential expansions are optimally truncated.

6. Discussion and concluding remarks

We have examined the generalised hypergeometric function, or Wright function, ${}_p\Psi_q(z)$ defined in (1.1) and determined exponentially small expansions present in its asymptotic description for $|z|\to\infty$. Such contributions are of relevance in high-precision evaluation, particularly when $\kappa\leq 2$ and the algebraic expansion $H(z)$ in a sector surrounding the negative real axis either vanishes or is finite. In such situations, it is then possible to detect numerically certain exponentially small series without the need for hyperasymptotics. In situations corresponding to $\kappa>2$, the function ${}_p\Psi_q(z)$ is exponentially large throughout the z plane and exponentially small expansions are, generally speaking, of less significance and *a fortiori* are more difficult to detect numerically.

The expansions we have developed primarily had parameter values corresponding to $\kappa\leq 2$. However, these expansions remain valid for $\kappa>2$ but, as mentioned above, they are of less importance in this domain. For example, the function in (4.1) when $\beta_1=\beta_2=1$ reduces to a multiple of the standard generalised hypergeometric function ${}_0F_2(b_1,b_2;z)$ with $\kappa=3$. Application of Theorem 3 (with $P=1$), combined with the fact that, from (2.1) and (3.7), $E(ze^{\pm 2\pi i})=e^{\pm\pi i\vartheta}E_*(\xi)$ (when $\kappa=3$), consequently yields

$$\frac{{}_0F_2(b_1,b_2;z)}{\Gamma(b_1)\Gamma(b_2)}\sim E(z)+E(ze^{\mp 2\pi i})+e^{\pm\pi i\vartheta}E_*(\xi)\qquad (|\arg z|\leq\pi). \tag{6.1}$$

It is clear that the coefficient of the exponentially small expansion $E_*(\xi)$ in (6.1) cannot be correct since, for real parameters, it does not yield a real expansion on the negative real z axis. Our result in (4.3) has the coefficient of $E_*(\xi)$ replaced by $2\cos\pi(b_1-b_2)$ and so yields

$$\frac{{}_0F_2(b_1,b_2;z)}{\Gamma(b_1)\Gamma(b_2)}\sim E(z)+E(ze^{\mp 2\pi i})+2\cos\pi(b_1-b_2)E_*(\xi)\qquad (|\arg z|\leq\pi)$$

as $|z|\to\infty$, in agreement with the expansion of ${}_0F_2(b_1,b_2;z)$ given in [19, p. 200]. A similar remark applies to the expansion (4.6) when $\beta_1=\beta_2=\beta_3=1$ ($\kappa=4$), which agrees with the expansion for ${}_0F_3(b_1,b_2,b_3;z)$ in [19, p. 201].

Finally, we mention that it is possible to modify the procedure in §5 to deal with the case when the parameters of ${}_p\Psi_q(z)$ are such that an infinite sequence of poles exists on the right of the integration path C and the algebraic expansion $H(z)$ then becomes an asymptotic sum. The method closely follows that given in [10, pp. 186–189] in the treatment of the asymptotics of the Mittag-Leffler function and is given in [12]. The result is an expansion of the form (5.6), with $H_f(ze^{\mp\pi i})$ replaced by $H(ze^{\mp\pi i})$ in (2.6), but valid as $|z|\to\infty$ in a sector containing the negative real axis. The expansion in the rest of the plane (when

$\kappa < 2$) is given by Theorem 1. However, when $\kappa \leq 2$, the algebraic expansion is the dominant expansion in the sector $|\arg(-z)| < \frac{1}{2}\pi\kappa$ and the presence of sub-subdominant expansions of type $E_*(\xi)$ is then, in general, of less numerical importance.

Appendix A: An algorithm for the computation of the coefficients $c_j = A_j/A_0$

We describe an algorithm for the computation of the normalised coefficients A_j/A_0 appearing in the exponential expansions $E(z)$ in (2.1) and $E_*(\xi)$ in (3.7). Methods of computing these coefficients by recursion in the case when $\alpha_r = \beta_r = 1$ have been given by Riney [20] and Wright [21]; see [10, §2.2.2] for details. Here we describe an algebraic method valid for arbitrary $\alpha_r > 0$ and $\beta_r > 0$.

By application of Stirling's formula for the gamma function we have the two important inverse-factorial expansions [6, §3; 10, p. 39; 19, p. 36]

$$\frac{1}{\Gamma(s+1)}\,\frac{\prod_{r=1}^p \Gamma(\alpha_r s+a_r)}{\prod_{r=1}^q \Gamma(\beta_r s+b_r)} = \kappa(h\kappa^\kappa)^s\biggl\{\sum_{j=0}^{M-1}\frac{A_j}{\Gamma(\kappa s+\vartheta'+j)}+\frac{\sigma_M(s)}{\Gamma(\kappa s+\vartheta'+M)}\biggr\} \tag{A.1}$$

and

$$\Gamma(s)\,\frac{\prod_{r=1}^q \Gamma(1-b_r+\beta_r s)}{\prod_{r=1}^p \Gamma(1-a_r+\alpha_r s)} =$$

$$\frac{\kappa(h\kappa^\kappa)^{-s}}{(2\pi)^{p-q}}\biggl\{\sum_{j=0}^{M-1}(-)^j A_j\Gamma(\kappa s+\vartheta-j)+\rho_M(s)\Gamma(\kappa s+\vartheta-M)\biggr\} \tag{A.2}$$

for $|s|\to\infty$ uniformly in $|\arg s| \leq \pi-\epsilon$ ($\epsilon > 0$), where M denotes a positive integer,

$$A_0 = (2\pi)^{\frac{1}{2}(p-q)}\kappa^{-\frac{1}{2}-\vartheta}\prod_{r=1}^p \alpha_r^{a_r-\frac{1}{2}}\prod_{r=1}^q \beta_r^{\frac{1}{2}-b_r} \tag{A.3}$$

and the parameters κ, h, ϑ and ϑ' are defined in (1.3). The remainder functions $\sigma_M(s)$ and $\rho_M(s)$ are analytic in s except at the poles of the corresponding gamma function ratios and are such that $\sigma_M(s) = O(1)$ and $\rho_M(s) = O(1)$ as $|s| \to \infty$ uniformly in $|\arg s| \leq \pi - \epsilon$. These expansions play an important role in the determination of the exponential expansions associated with ${}_p\Psi_q(z)$. The coefficients A_j appearing in these two expansions are the same; hence, it is sufficient to present our algorithm for the gamma function ratio in (A.1).

We rewrite the expansion (A.1) in the form

$$\frac{g(s)\Gamma(\kappa s+\vartheta')}{\Gamma(s+1)} = \kappa A_0(h\kappa^\kappa)^s\biggl\{\sum_{j=0}^{M-1}\frac{c_j}{(\kappa s+\vartheta')_j}+\frac{O(1)}{(\kappa s+\vartheta')_M}\biggr\}, \tag{A.4}$$

for $|s| \to \infty$ uniformly in $|\arg s| \leq \pi-\epsilon$, where $g(s)$ is the ratio of gamma functions defined in (1.1), $(a)_j = \Gamma(a+j)/\Gamma(a)$ and $c_j = A_j/A_0$. Introduction of the scaled gamma function $\Gamma^*(z)$ defined by

$$\Gamma^*(z) = \Gamma(z)(2\pi)^{-\frac{1}{2}}e^z z^{\frac{1}{2}-z}$$

leads to the representation

$$\Gamma(\alpha s+a) = \Gamma^*(\alpha s+a)(2\pi)^{\frac{1}{2}}e^{-\alpha s}(\alpha s)^{\alpha s+a-\frac{1}{2}}e(\alpha s;a),$$

where

$$e(\alpha s;a) = \exp\biggl\{(\alpha s+a-\tfrac{1}{2})\log\Bigl(1+\frac{a}{\alpha s}\Bigr)-a\biggr\}.$$

After some straightforward algebra we then find that

$$\frac{g(s)\Gamma(\kappa s+\vartheta')}{\Gamma(s+1)}=\kappa A_0(h\kappa^\kappa)^s R(s)\Upsilon(s), \tag{A.5}$$

where

$$\Upsilon(s)=\frac{\prod_{r=1}^p \Gamma^*(\alpha_r s+a_r)}{\prod_{r=1}^q \Gamma^*(\beta_r s+b_r)}\,\frac{\Gamma^*(\kappa s+\vartheta')}{\Gamma^*(s+1)}$$

and

$$R(s)=\frac{\prod_{r=1}^p e(\alpha_r s;a_r)}{\prod_{r=1}^q e(\beta_r s;b_r)}\,\frac{e(\kappa s;\vartheta')}{e(s;1)}.$$

Substitution of (A.5) into (A.4) then yields

$$R(s)\Upsilon(s)=\sum_{j=0}^{M-1}\frac{c_j}{(\kappa s+\vartheta')_j}+\frac{O(1)}{(\kappa s+\vartheta')_M} \tag{A.6}$$

as $|s|\to\infty$ in $|\arg s|\leq\pi-\epsilon$.

Now let $\chi=(\kappa s)^{-1}$ and expand $R(s)$ and $\Upsilon(s)$ for $\chi\to 0$ making use of the well-known expansion [22, p. 71; 10, p. 32]

$$\Gamma^*(z)\sim\sum_{k=0}^\infty (-)^k\gamma_k z^{-k}\qquad (|z|\to\infty;\ |\arg z|\leq\pi-\epsilon),$$

where γ_k are the Stirling coefficients. The first few coefficients are given by $\gamma_0=1$, $\gamma_1=-\frac{1}{12}$, $\gamma_2=\frac{1}{288}$, $\gamma_3=\frac{139}{51840},\ldots$. Some routine algebra then yields the expansions

$$\Gamma^*(\alpha s+a)=1-\frac{\gamma_1\kappa\chi}{\alpha}+O(\chi^2),\qquad e(\alpha s;a)=1+\frac{\kappa\chi}{2\alpha}a(a-1)+O(\chi^2),$$

whence

$$R(s)=1+\frac{\kappa\chi}{2}\left\{\sum_{r=1}^p\frac{a_r(a_r-1)}{\alpha_r}-\sum_{r=1}^q\frac{b_r(b_r-1)}{\beta_r}-\frac{\vartheta}{\kappa}(1-\vartheta)\right\}+O(\chi^2),$$

$$\Upsilon(s)=1+\frac{\kappa\chi}{12}\left\{\sum_{r=1}^p\frac{1}{\alpha_r}-\sum_{r=1}^q\frac{1}{\beta_r}+\frac{1}{\kappa}-1\right\}+O(\chi^2).$$

Upon equating coefficients of χ in (A.6) we obtain

$$c_1=\tfrac{1}{2}\kappa(\mathcal{A}+\tfrac{1}{6}\mathcal{B}), \tag{A.7}$$

where

$$\mathcal{A}=\sum_{r=1}^p\frac{a_r(a_r-1)}{\alpha_r}-\sum_{r=1}^q\frac{b_r(b_r-1)}{\beta_r}-\frac{\vartheta}{\kappa}(1-\vartheta),$$

$$\mathcal{B}=\sum_{r=1}^p\frac{1}{\alpha_r}-\sum_{r=1}^q\frac{1}{\beta_r}+\frac{1}{\kappa}-1.$$

The higher coefficients are the obtained by continuation of this process with the help of *Mathematica*. In specific cases (i.e., with numerical values for the various parameters) it is possible to generate the coefficients in this manner quite easily. In our computations we have used up to a maximum of 50 coefficients.

Appendix B: A specific example of (4.1) by the saddle-point method

If we take $\beta_1 = \beta_2 = \frac{1}{3}$ and $b_1 = \frac{1}{2}$, $b_2 = \frac{5}{6}$ in (4.1) then, upon use of the multiplication formula for the gamma function in (3.10), we find

$$F(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\frac{1}{3}n+\frac{1}{2})\Gamma(\frac{1}{3}n+\frac{5}{6})} = \tfrac{1}{2}\pi^{-3/2} \sum_{n=0}^{\infty} \frac{(12z)^n}{(2n)!}\,\Gamma(\tfrac{1}{3}n+\tfrac{1}{6})$$
$$= \tfrac{3}{2}\pi^{-3/2} \int_{-\infty}^{\infty} \exp\{-t^6 + (12z)^{\frac{1}{2}} t\}\, dt. \qquad \text{(B.1)}$$

This integral representation may be readily verified by expansion of the factor $\exp(12z)^{1/2}t$ followed by term-by-term integration.

The integrand in (B.1) has 5 saddle points at

$$t_{sr} = (z/3)^{1/10} e^{2\pi ir/5} \qquad (0 \le r \le 4).$$

When $0 \le \arg z \le \pi$, the integration path $(-\infty, \infty)$ in (B.1) can be deformed into paths of steepest descent passing through the saddles t_{s0}, t_{s1} and t_{s2} in the upper half-plane as shown in Fig. 2. If we let $\psi(t) = (12z)^{1/2}t - t^6$, then

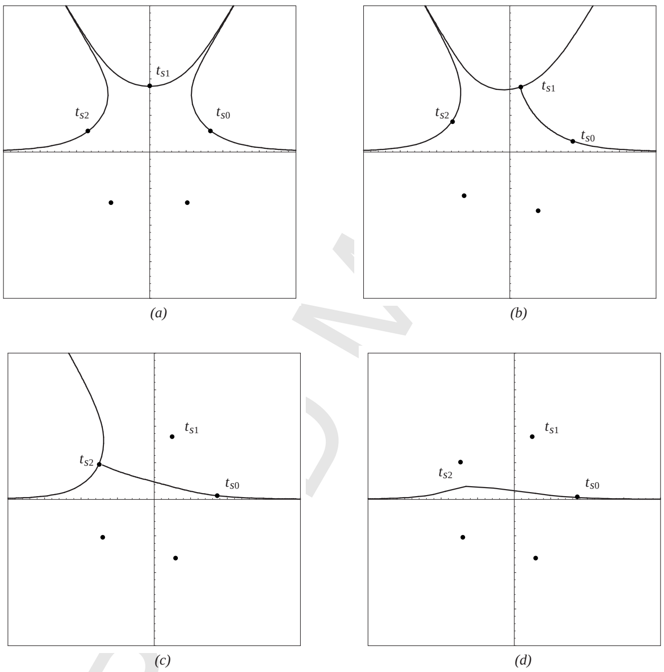

Figure 2: The paths of steepest descent connecting $-\infty$ to $+\infty$ for the integral in (B.1) for different values of $\theta = \arg z$: (a) $\theta = \pi$, (b) $\theta = \pi/2$, (c) $\theta = \pi/6$ and (d) $\theta = \pi/12$. Only the relevant paths are shown and the heavy dots denote the saddle points.

$$\psi(t_{sr}) = 5(z/3)^{3/5} e^{2\pi ir/5} = Ze^{2\pi ir/5},$$

where Z is defined in (2.1) with $\kappa = \frac{5}{3}$ and $h = 3^{2/3}$. The contributions from the saddles t_{sr}, $r = 0, 1, 2$ are then easily shown to correspond to the expansions $E(z)$, $E_*(ze^{-\pi i})$ and $E(ze^{-2\pi i})$, respectively. It is clear from Fig. 2 that there is a Stokes phenomenon on $\arg z = \frac{1}{2}\pi$, where (in the sense of decreasing $\arg z$) the saddle t_{s1} disconnects, and also on $\arg z = \frac{1}{6}\pi$, where the saddle t_{s2} disconnects. Consequently, we have either one, two or three saddles contributing to the integral as $|z| \to \infty$ in the sectors $0 \leq \arg z < \frac{1}{6}\pi$, $\frac{1}{6}\pi \leq \arg z < \frac{1}{2}\pi$ and $\frac{1}{2}\pi \leq \arg z \leq \pi$, respectively. Thus, we have the more precise asymptotic description of $F(z)$ given by

$$F(z) \sim \begin{cases} E(z) + E(ze^{-2\pi i}) + BE_*(ze^{-\pi i}) & (\frac{1}{2}\pi < \arg z \leq \pi) \\ \\ E(z) + E(ze^{-2\pi i}) & (\frac{1}{6}\pi < \arg z < \frac{1}{2}\pi) \\ \\ E(z) & (0 \leq \arg z < \frac{1}{6}\pi) \end{cases} \tag{B.2}$$

as $|z| \to \infty$, where B is defined in (4.4). A similar structure holds in the lower half-plane. Observe that the expansion $E_*(ze^{-\pi i})$, which becomes exponentially large near $\arg z = 0$ (but subdominant with respect to $E(z)$), switches off as $|\arg z|$ decreases across the Stokes lines $\arg z = \pm\frac{1}{2}\pi$ and so is not present in the expansion of $F(z)$ near the positive real axis. This example confirms the discussion in §4 of the Stokes phenomenon related to the expansion of (4.1).

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