DIAGONAL FORMS OF HIGHER DEGREE OVER FUNCTION FIELDS OF p-ADIC CURVES

S. PUMPLÜN

School of Mathematical Sciences, University of Nottingham, University Park
Nottingham NG7 2RD, United Kingdom
susanne.pumpluen@nottingham.ac.uk

Received (25 May 2018)
Accepted (25 June 2019)

We investigate diagonal forms of degree d over the function field F of a smooth projective p-adic curve: if a form is isotropic over the completion of F with respect to each discrete valuation of F, then it is isotropic over certain fields F_U, F_P and F_p. These fields appear naturally when applying the methodology of patching; F is the inverse limit of the finite inverse system of fields $\{F_U, F_P, F_p\}$. Our observations complement some known bounds on the higher u-invariant of diagonal forms of degree d.

We only consider diagonal forms of degree d over fields of characteristic not dividing $d!$.

Keywords: Forms of higher degree, diagonal forms, function fields, p-adic curves, u-invariant.

Mathematics Subject Classification 2010: 11E76

Introduction

The fact that Springer's Theorem holds for diagonal forms of higher degree over fields of characteristic not dividing $d!$ [9] guarantees that on occasion diagonal forms of higher degree defined over function fields behave similarly to quadratic forms. For a survey on the behaviour of (diagonal) forms of higher degree in general the reader is referred to [12].

In this note we consider diagonal forms of degree d over function fields $F = K(X)$ where X is a smooth, projective, geometrically integral curve over K and K is the fraction field of a complete discrete valuation ring with a residue field k of characteristic not dividing $d!$. Let v be a rank one discrete valuation of F, and F_v the completion of F with respect to v. It was shown by Colliot-Thélène, Parimala and Suresh [2, Theorem 3.1] that a quadratic form which is isotropic over F_v for each v is already isotropic over F, using the methodology of patching developed by Habater and Hartmann [4], i.e. viewing F as the inverse limit of a finite inverse system of certain fields $\{F_U, F_P, F_p\}$.

Given a nondegenerate diagonal form φ over F of degree $d > 2$ and dimension > 2, it is not clear, however, whether the isotropy of φ over F_v for each v implies that φ is isotropic.

Our main result proves that the isotropy of a nondegenerate diagonal form φ over F_v for each v implies that at least over the field extensions F_U, F_P and F_p of F, φ is isotropic as well (Theorem 3.1). These fields depend on the choice of the form $\varphi = \langle a_1, \dots, a_n \rangle$, more precisely on the choice of the regular proper model $\mathcal{X}$ (over the complete discrete valuation ring A) of the curve X over K, which depends on φ: $\mathcal{X}$ is selected such that there exists a reduced divisor D with strict normal crossings, which contains both the support of the divisor of all the entries a_i, $1 \leq i \leq n$, and the components of the special fibre of X/A. Since nondegenerate diagonal forms of degree $d \geq 3$ have finite automorphism groups [6, p. 137], we are not able to apply [5, Theorem 3.7] to conclude that the isotropy of φ over the F_U's and F_P's implies that φ is also isotropic over F, however. This is only possible for $d = 2$.

After collecting the terminology and some basic results in Section 1, in particular defining diagonal u-invariants of degree d over k, we consider diagonal forms of higher degree over valued fields in Section 2 and then study diagonal forms of higher degree over function fields of p-adic curves using some of the ideas of [2] in Section 3. Recall that a p-adic field is a finite field extension of $\mathbb{Q}_p$.

As a consequence of Springer's Theorem for diagonal forms, any diagonal form of degree d and dimension $> d^3 + 1$ over a function field in one variable $F = K(t)$, where K is a p-adic field with residue field k, $\mathrm{char}(k) \nmid d!$, is isotropic over F_v for every discrete valuation v with residue field either a function field in one variable over k or a finite extension of K. Moreover, it is isotropic over F_U for each reduced, irreducible component $U \subset Y$ of the complement of S in the special fibre $Y = \mathcal{X} \times_A k$ of X/A, and isotropic over F_P for each $P \in S$ (Corollary 3.4), and thus isotropic over F_p for each $p = (U, P)$. Here, S is the inverse image under a finite A-morphism $f : \mathcal{X} \to \mathbb{P}^1_A$ of the point at infinity of the special fibre $\mathbb{P}^1_k$.

1. Preliminaries

Let k be a field such that $\mathrm{char}(k)$ does not divide $d!$.

1.1. *Forms of higher degree*

Let V be a finite-dimensional vector space over k of dimension n. A *d-linear form* over k is a k-multilinear map $\theta : V \times \cdots \times V \to k$ (d-copies) which is *symmetric*, i.e. $\theta(v_1, \dots, v_d)$ is invariant under all permutations of its variables. A *form of degree d* over k (and of dimension n) is a map $\varphi : V \to k$ such that $\varphi(av) = a^d \varphi(v)$ for all $a \in k$, $v \in V$ and such that the map $\theta : V \times \cdots \times V \to k$ defined by

$$\theta(v_1, \dots, v_d) = \frac{1}{d!} \sum_{1 \leq i_1 < \cdots < i_l \leq d} (-1)^{d-l} \varphi(v_{i_1} + \cdots + v_{i_l})$$

($1 \leq l \leq d$) is a d-linear form over k. By fixing a basis $\{e_1, \dots, e_n\}$ of V, any form φ of degree d can be viewed as a homogeneous polynomial of degree d in $n = \dim V$ variables $x_1, \dots, x_n$ via $\varphi(x_1, \dots, x_n) = \varphi(x_1 e_1 + \cdots + x_n e_n)$ and, vice versa, any homogeneous polynomial of degree d in n variables over k is a form of degree d and dimension n over k. Any d-linear form $\theta : V \times \cdots \times V \to k$ induces a form $\varphi : V \to k$ of degree d via $\varphi(v) = \theta(v, \ldots, v)$. We can hence identify d-linear forms and forms of degree d.

Two d-linear spaces (V_i, θ_i), $i = 1, 2$, are *isomorphic* (written $(V_1, \theta_1) \cong (V_2, \theta_2)$ or just $\theta_1 \cong \theta_2$) if there exists a bijective linear map $f : V_1 \to V_2$ such that $\theta_2(f(v_1), \ldots, f(v_d)) = \theta_1(v_1, \ldots, v_d)$ for all $v_1, \ldots, v_d \in V_1$. A d-linear space (V, θ) (or the d-linear form θ) is *nondegenerate* if $v = 0$ is the only vector such that $\theta(v, v_2, \ldots, v_d) = 0$ for all $v_i \in V$. A form of degree d is called *nondegenerate* if its associated d-linear form is nondegenerate. A form φ over k is called *anisotropic*, if it does not have any non-trivial zeroes, otherwise it is called *isotropic*.

The *orthogonal sum* $(V_1, \theta_1) \perp (V_2, \theta_2)$ of two d-linear spaces (V_i, θ_i), $i = 1, 2$, is the k-vector space $V_1 \oplus V_2$ together with the d-linear form

$$(\theta_1 \perp \theta_2)(u_1 + v_1, \ldots, u_d + v_d) = \theta_1(u_1, \ldots, u_d) + \theta_2(v_1, \ldots, v_d)$$

$(u_i \in V_1,\ v_i \in V_2)$ [7].

A d-linear space (V, θ) is called *decomposable* if $(V, \theta) \cong (V, \theta_1) \perp (V, \theta_2)$ for two non-zero d-linear spaces (V, θ_i), $i = 1, 2$. If φ is represented by the homogeneous polynomial $a_1 x_1^d + \ldots + a_m x_m^d$ $(a_i \in k^\times)$ we use the notation $\varphi = \langle a_1, \ldots, a_n \rangle$ and call φ *diagonal*. A diagonal form $\varphi = \langle a_1, \ldots, a_n \rangle$ over k is nondegenerate if and only if $a_i \in k^\times$ for all $1 \leq i \leq n$.

If $d \geq 3$, $a_i, b_j \in k^\times$, then $\langle a_1, \ldots, a_n \rangle \cong \langle b_1, \ldots, b_n \rangle$ if and only if there is a permutation $\pi \in S_n$ such that $\langle b_i \rangle \cong \langle a_{\pi(i)} \rangle$ for every i. This is a special case of [6, Theorem 2.3].

Note that for quadratic forms ($d = 2$), the automorphism group of φ is infinite, whereas for $d \geq 3$, the automorphism group of φ usually is finite, for instance if φ is is nonsingular in the sense of algebraic geometry [13]. In particular, nondegenerate diagonal forms of degree $d \geq 3$ have finite automorphism groups [6, p. 137], which creates a problem when trying to imitate patching arguments as it is not possible to apply [5, Theorem 3.7].

1.2. *Higher degree u-invariants*

The *u-invariant (of degree d)* of k is defined as $u(d, k) = \sup\{\dim_k \varphi\}$, where φ ranges over all the anisotropic forms of degree d over k. The *diagonal u-invariant (of degree d)* of k is defined as $u_{diag}(d, k) = \sup\{\dim \varphi\}$, where φ ranges over all the anisotropic diagonal forms over k.

Thus the diagonal u-invariant $u_{diag}(d, k)$ is the smallest integer n such that all diagonal forms of degree d over k of dimension greater than n are isotropic, and the u-invariant $u(d, k)$ is the smallest integer n such that all forms of degree d over k of dimension greater than n are isotropic. Obviously, $u_{diag}(d, k) \leq u(d, k)$. If $u = u(d, k)$ then each anisotropic form of degree d over k of dimension u is universal. If $u = u_{diag}(d, k)$ then each diagonal anisotropic form of degree d over k

of dimension u is universal. We have

$$u_{diag}(d,k) \leq \min\{n \mid \text{all forms of degree } d \text{ over } k \text{ of dimension } \geq n \text{ are universal}\}$$

with the understanding that the "minimum" of an empty set of integers is ∞, cf. [12]. For $d = 2$, $u_{diag}(d,k) = u(d,k)$ is the u-invariant of quadratic forms.

For an algebraically closed field k, $|k^\times/k^{\times d}| = 1$ and hence $u_{diag}(d,k) = u(d,k) = 1$. For a formally real field k, the diagonal u-invariant is infinite for even d: since $-1 \notin \sum k^2$, also $-1 \notin \sum k^d$ for any even d. Thus the form $m \times \langle 1 \rangle$ of degree d is anisotropic for each integer m, implying $u_{diag}(d,k) = u(d,k) = \infty$.

The *strong diagonal* u-invariant of degree d of k, denoted $u_{diag,\,s}(d,k)$, is the smallest real number n such that

(1) every finite field extension E/k satisfies $u_{diag}(d,E) \leq n$, and
(2) every finitely generated field extension E/k of transcendence degree one satisfies $u_{diag}(d,E) \leq dn$.

If these u-invariants are arbitrarily large, put $u_{diag,\,s}(d,k) = \infty$.

Analogously as observed in [5] for $d = 2$, $u_{diag,\,s}(d,k) \leq n$ if and only if every finitely generated field extension E/k of transcendence degree $l \geq 1$ satisfies $u_{diag}(d,k) \leq d^l n$. Thus if $u_{diag,\,s}(d,k)$ is finite, it is at least 1 and lies in $\frac{1}{d}\mathbb{N}$.

1.3. C_r^0 *fields*

Let $r \geq 1$ be an integer. A field F is a C_r*-field* if for all $d \geq 1$ and $n > d^r$, every homogeneous form of degree d in n variables over F has a non-trivial solution in F. In particular, then F satisfies $u(d,F) \leq d^r$. Moreover, every finite extension of F is a C_r-field, and every one-variable function field over F a C_{r+1}-field [14, II.4.5]. Hence $u_{diag,\,s}(d,F) \leq d^r$ for a C_r-field F.

A field F is a C_r^0*-field* if the following holds: For any finite field extension F' of F and any integers $d \geq 1$ and $n > d^r$, for any homogeneous form over F' of degree d in n variables, the greatest common divisor of the degrees of finite field extensions F''/F' over which the form acquires a nontrivial zero is one. This amounts to requiring that the F'-hypersurface defined by the form has a zero-cycle of degree 1 over F'.

Assume $\mathrm{char}(F) = 0$. For each prime l, let F_l be the fixed field of a pro-l-Sylow subgroup of the absolute Galois group of F. Any finite subextension of F_l/F is of degree coprime to l. The field F is C_r^0 if and only if each of the fields F_l is C_r. A finite field extension of a C_r^0-field is C_r^0. If F is C_r^0 then a function field $E = F(x_1,\dots,x_s)$ in s indeterminates $x_1,\dots,x_s$ over F is C_{r+s}^0 [2, 2.1].

It is not known if p-adic fields have the C_2^0-property.

Remark 1.1. Assume that p-adic fields have the C_2^0-property. Let $K(X)$ be any function field of transcendence degree r over a p-adic field K (here we do not need to assume $p \neq 2,3$). Suppose that there is $\ell \neq 2$ such that there exists a finite subextension of $K_\ell(X)/K(X)$ of degree 2. Then any cubic form over $K(X)$ in strictly more than 3^{2+r} variables has a nontrivial zero: If the p-adic field K is C_3^0, then the function field $E = K(X)$ in r variables over K is C_{3+r}^0 [2, Lemma 2.1]. Thus a cubic form over $E = K(X)$ in strictly more than 3^{2+r} has a nontrivial zero in each of the fields $K_\ell(X)$, l a prime, hence in a finite extension of $K(X)$ of degree coprime to ℓ, for each ℓ prime. Pick $\ell \neq 2$, then $[K_\ell(X) : K(X)]$ is even. Moreover, pick $l \neq 2$ such that there exists a finite subextension of $E_\ell/K(X)$ of degree 2 then the cubic form has a zero over it. By Springer's Theorem for cubic forms and their behaviour under quadratic field extensions [8, VII], thus the cubic form has a nontrivial zero in $K(X)$. This is the analogue of [2, Proposition 2.2].

2. Diagonal forms over Henselian valued fields

2.1.

Let K be a valued field with valuation v, valuation ring R and maximal ideal m. Let Γ be the value group. Assume that $d!$ is not divisible by the characteristic of the residue field $k = R/m$. For $u \in R$, denote by $\bar{u}$ the image of u in k. For a polynomial $f \in R[X]$, $f = a_n x^n + \cdots + a_1 x + a_0$, define the polynomial $\overline{f} = \bar{a}_n x^n + \cdots + \bar{a}_1 x + \bar{a}_0$ over k. If $\varphi = \langle a_1, \ldots, a_n \rangle$ is a nondegenerate diagonal form with entries $a_i \in R$, define the diagonal form $\overline{\varphi} = \langle \bar{a}_1, \ldots, \bar{a}_n \rangle$ over k. φ is called a *unit form*, if $\overline{\varphi}$ is nondegenerate. Choose a set $\{\pi_\gamma \in R \mid \gamma \in I\}$ such that the values of the π_γ's represent the distinct cosets in $\Gamma/d\Gamma$. We may decompose a diagonal form φ as $\varphi = \perp \varphi'_\gamma$ by taking φ'_γ to be the diagonal form whose entries comprise all a_i with $v(a_i) = v(\pi_\gamma) \bmod d\,\Gamma$. By altering the slots by d-powers if necessary, we may then write $\varphi'_\gamma = \pi_\gamma \varphi_\gamma$ with each φ_γ a diagonal unit form. There are only finitely many non-trivial φ_γ [9]. If $\Gamma = \mathbb{Z}$, the set $\{\pi_\gamma \mid \gamma \in I\}$ can be chosen to be $\{\pi^i \mid i = 0, \ldots, d-1\}$ and $|\Gamma/d\Gamma| = d$ is finite.

If R satisfies Hensel's Lemma then (K, v) is called a *Henselian valued field* and R a *Henselian valuation ring*. Every complete discretely valued field is Henselian.

Let φ be a diagonal form over a Henselian valued field (K, v). Write $\varphi = \pi_1\varphi_1 \perp \cdots \perp \pi_r\varphi_r$ with each φ_i a diagonal unit form and the π_i having distinct values in $\Gamma/d\Gamma$. Then φ is isotropic if and only if some $\overline{\varphi_i}$ is isotropic [9, Proposition 3.1]. This is because for a diagonal unit form φ over a Henselian valued field (K, v), φ is isotropic if and only if $\overline{\varphi}$ is isotropic [9, Lemma 2.3].

Theorem 2.1. *([9] or [12, Theorem 4, Corollary 2]) Suppose that* $\mathrm{char}(k) \nmid d!$.
(i) Let (K, v) be a Henselian valued field. Then $u_{diag}(d, K) = |\Gamma/d\Gamma|\, u_{diag}(d, k)$.
(ii) Let (K, v) be a Henselian valued field. If every diagonal form of degree d of dimension $n + 1$ over k is isotropic, then every diagonal form of degree d and dimension $dn + 1$ over K is isotropic. If k has an anisotropic form of degree d and dimension n, then K has an anisotropic form of degree d and dimension dn.
(iii) Let K be a discretely valued field. Then $u_{diag}(d, K) \geq d\, u_{diag}(d, k)$.
(iv) Let F be a field extension of finite type over k of transcendence degree n. Then $u_{diag}(d, F) \geq d^n u_{diag}(d, k')$ *for a suitable finite field extension k'/k.*

The (in)equalities in (i), (iii), (iv) also hold when the values are infinite.

For $d = 2$, (ii) is Springer's Theorem for quadratic forms over Henselian valued fields [15]. Springer's Theorem does not hold for non-diagonal forms of higher degree than 2 [9, 2.7]. Theorem 2.1 is a major ingredient in our proofs, for instance we can show:

Proposition 2.2. *Let A be a discrete valuation ring with fraction field K and residue field k such that* $\mathrm{char}(k) \nmid d!$.
(i) $u_{diag}(d, K) \geq d\, u_{diag}(d, k)$ *and* $u_{diag,\, s}(d, K) \geq d\, u_{diag,\, s}(d, k)$.
(ii) If A is Henselian then $u_{diag}(d, K) = d\, u_{diag}(d, k)$.
(iii) If A is a complete discrete valuation ring then every finite extension of K has diagonal u-invariant at most $d\, u_{diag}(d, k)$.

The first assertions of (i) as well as (ii) and (iii) follow from Theorem 2.1. The proof of the second claim in (i) is analogous to the one of [5, 4.9], employing Theorem 2.1 instead of Springer's Theorem.

2.2.

A field K is called an *m-local field* with *residue field k* if there is a sequence of fields $k_0, \dots, k_m$ with $k_0 = k$ and $k_m = K$, and such that k_i is the fraction field of an excellent Henselian discrete valuation ring with residue field k_{i-1} for $i = 1, \dots, m$. Recall that a discrete valuation ring R is called *excellent*, if the field extension $\widehat{K}/K$ is separable, where $\widehat{K}$ denotes the quotient field of R and K is its completion. (This condition is trivially satisfied if K has characteristic 0 or R is complete.)

Proposition 2.2 implies (compare the next two results with [5, Corollary 4.13, 4.14] for quadratic forms):

Corollary 2.3. *Suppose that K is an m-local field whose residue field k is a C_r-field with* $\mathrm{char}(k) \nmid d!$. *Let F be a function field over K in one variable.*
(i) $u_{diag}(d, k) = u_{diag,s}(d, k) = d^r$ *and* $u_{diag}(d, K) = d^{r+m}$.
Moreover, if some normal K-curve with function field F has a K-point, then $u_{diag}(d, F) \geq d^{r+m+1}$.
(ii) If $u_{diag}(d, k') = d^r$ *for every finite extension k'/k, then* $u_{diag}(d, F) \geq d^{r+m+1}$.

Proof. (i) Since k is a C_r-field, $u_{diag}(d, k) = d^r$, thus $u_{diag}(d, k) \leq u_{diag,s}(d, k) \leq d^r$ and the first two equations follow. Applying Proposition 2.2 and induction yields that $u_{diag}(d, K) \geq d^m u_{diag}(d, k) = d^{r+m}$. Let X be a normal K-curve with function field F and let ξ be a K-point on X. The local ring at ξ has fraction field F and residue field K. So Proposition 2.2 implies that $u_{diag}(d, K) = d^m u_{diag}(d, k)$ and $u_{diag}(d, F) \geq d u_{diag}(d, K) = d^{r+m+1}$.
(ii) Choose a normal or equivalently a regular K-curve X with function field F, and a closed point ξ on X. Let R be the local ring of X at ξ with residue field $\kappa(\xi)$. Then the fraction field of R is F, and $\kappa(\xi)$ is a finite extension of K. Hence $\kappa(\xi)$

is an m-local field whose residue field k' is a finite extension of k. By assumption, $u_{diag}(d, k') = d^r$ and k' is a C_r-field since k is. So applying part (ii) to k' and $\kappa(\xi)$, it follows that $u_{diag}(d, \kappa(\xi)) \geq d^{r+m}$. Proposition 2.2 yields $u_{diag}(d, F) \geq d^{r+m+1}$. □

Corollary 2.4. *(i) Let F be a one-variable function field over an m-local field K with residue field k such that* char$(k) \nmid d!$ *and k is algebraically closed. Then $u_{diag}(d, F) \geq d^{m+1}$.*
(ii) If k is a finite field and $u_{diag}(d, k) = d^r$ with $r \in \{0, 1\}$, then $u_{diag}(d, k) = u_{diag,s}(d, k) = d^r$ and $u_{diag}(d, K) \geq d^{r+m}$. Moreover, if some normal K-curve with function field F has a K-point, then $u_{diag}(d, F) \geq d^{r+m+1}$.

Proof. (i) This is a special case of Corollary 2.3 using that an algebraically closed field k is C_0, satisfies $u_{diag}(d, k) = 1$, and has no non-trivial finite extensions.
(ii) A finite field is C_1 [14, II.3.3(a)], hence $u_{diag}(d, k) \leq d$. Here $u_{diag}(d, k) = u_{diag,\,s}(d, k) = d^r$ and Corollary 2.3 yields the assumption. □

In general, for any finite field $k = \mathbb{F}_q$ we obviously do not have $u_{diag}(d, k) = d^r$, $r \in \{0, 1\}$: for instance, if $-1 \in \mathbb{F}_q^{\times d}$ and $d \geq 4$ then $u_{diag}(d, \mathbb{F}_q) \leq d - 1$ by [10]. Or, if $d^* = \gcd(d, q - 1)$, then $u_{diag}(d, \mathbb{F}_q) \leq d^*$. This implies that $u_{diag}(d, \mathbb{F}_q) = 1$, if d is relatively prime to $q - 1$ and that for $q > (d^* - 1)^4$, $u_{diag}(d, \mathbb{F}_q) = 2$ [12, 5.1].

3. The behaviour of diagonal forms of higher degree over function fields of p-adic curves

Whenever we write 'discrete valuation ring' and 'discrete valuation' we mean a discrete valuation ring of rank one and a valuation with value group $\mathbb{Z}$.

3.1.

Let A be a complete discrete valuation ring with fraction field K and residue field k with char$(k) \nmid d!$. Let X be a smooth, projective, geometrically integral curve over K and $F = K(X)$ be the function field of X. Let t denote a uniformizing parameter for A. For each (rank one) discrete valuation v of F, let F_v denote the completion of F with respect to v.

We will adapt some ideas from [2] to diagonal forms of higher degree: take a nondegenerate form $\varphi = \langle a_1, \ldots, a_n \rangle$ of degree d over F. Then choose a regular proper model $\mathcal{X}/A$ of X/K, such that there exists a reduced divisor D with strict normal crossings which contains both the support of the divisor of all the entries a_i, $1 \leq i \leq n$, and the components of the special fibre of X/A. (Note that this implies that the regular proper model $\mathcal{X}/A$ depends on the form φ, and thus so do Y, Y_i, S_0, S, $F_P, F_U, \ldots$ as defined in the following.)

Let $Y = \mathcal{X} \times_A k$ be the special fibre of X/A. Let x_i be the generic point of an irreducible component Y_i of Y. Then there is an affine Zariski neighbourhood $W_i \subset \mathcal{X}$ of x_i, such that the restriction of Y_i to W_i is a principal divisor. Let S_0

be a finite set of closed points of Y containing all singular points of D, and all the points that lie on some Y_i, but not in the corresponding W_i.

Choose a finite A-morphism $f : \mathcal{X} \to \mathbb{P}^1_A$ as in [4, Proposition 6.6]. Let S be the inverse image under f of the point at infinity of the special fibre $\mathbb{P}^1_k$. Then the set S_0 is contained in S. All the intersection points of two components Y_i are in S. Each component Y_i contains at least one point of S. Let $U \subset Y$ run through the reduced irreducible components of the complement of S in Y. Then each U is a regular affine irreducible curve over k and we define $k[U]$ to be its ring of regular functions and $k(U)$ to be its fraction field. $k[U]$ is a Dedekind domain and $U = \operatorname{Spec} k[U]$. Each U is contained in an open affine subscheme $\operatorname{Spec} R^U$ of $\mathcal{X}$ and is a principal effective divisor in $\operatorname{Spec} R^U$. Moreover, R_U is the ring of elements in F which are regular on U and also a regular ring, since it is the direct limit of regular rings. The ring R_U is a localisation of R^U and so U is a principal effective divisor on $\operatorname{Spec} R_U$ given by the vanishing of an element $s \in R_U$. The t-adic completion $\widehat{R}_U$ of R_U is a domain and coincides with the s-adic completion of R_U, since $t = us^r$ for some integer $r \geq 1$ and a unit $u \in R_U^\times$. By definition, F_U is the field of fractions of $\widehat{R}_U$. We have $k[U] = R_U/s = \widehat{R}_U/s$. For $P \in S$, the completion $\widehat{R}_P$ of the local ring R_P of $\mathcal{X}$ at P is a domain and F_P is the field of fractions of $\widehat{R}_P$. Let $p = (U, P)$ be a pair with $P \in S$ in the closure of an irreducible component U of the complement of S in Y. Then let R_p be the complete discrete valuation ring which is the completion of the localisation of $\widehat{R}_P$ at the height one prime ideal corresponding to U. Then F_p is the field of fractions of $\widehat{R}_p$ and F is the inverse limit of the finite inverse system of fields $\{F_U, F_P, F_p\}$ by [4, Proposition 6.3].

The following can be seen as a weak generalization of [2, Theorem 3.1] to diagonal forms of higher degree. Here we are not able to conclude that under the given assumptions, φ is isotropic over F, only over the F_U's and F_P's:

Theorem 3.1. *Let φ be a nondegenerate diagonal form of degree d over F. If φ is isotropic over the completion F_v of F with respect to each discrete valuation v of F with residue field either a function field in one variable over k or a finite extension of K, then:*
(i) φ is isotropic over F_U for each reduced irreducible component $U \subset Y$ of the complement of S in Y,
(ii) φ is isotropic over F_P for each $P \in S$.

Proof. Suppose $\varphi = \langle a_1, \dots, a_n \rangle$.
(i) Each entry a_i of φ is supported only along U in $\operatorname{Spec} R_U$, thus has the form us^j where $u \in R_U^\times$. We sort the entries $a_i = u_i s^j$ by the power j of s and use them to define new diagonal forms ρ_j which have all the u_i's belonging to those a_i where s occurred in the jth power as their diagonal entries. Hence φ is isomorphic to the diagonal form

$$\rho_0 \perp s\rho_1 \perp \cdots \perp s^{d-1}\rho_{d-1}$$

over F, where the ρ_i are nondegenerate diagonal forms of degree d over R_U. Note that if for some $j \in \{0, 1, \dots, d-1\}$ there is no a_i with $a_i = u_i s^j$, then there is no corresponding form ρ_j and a ρ_j does not appear as a component in the sum.

By hypothesis, φ is isotropic over the field of fractions of the completed local ring of $\mathcal{X}$ at the generic point of U. By Theorem 2.1, this implies that the image of at least one of the forms ρ_0, ρ_1 or ρ_{d-1} under the composite homomorphism $R_U \to k[U] \to k(U)$ is isotropic over $k(U)$. Since the residue characteristic p does not divide $d!$, the forms $\rho_0, \rho_1, \dots, \rho_{d-1}$ define a smooth projective variety over R_U. In particular, all of them define a smooth variety over $k[U]$. Since $k[U]$ is a Dedekind domain, if such a projective variety has a point over $k(U)$, it has a point over $k[U]$. Since the variety is smooth over R_U, a $k[U]$-point lifts to an $\widehat{R}_U$-point (cf. the discussion after [5, Lemma 4.5]). Thus φ has a nontrivial zero over F_U.
(ii) Let $P \in S$. The local ring R_P of $\mathcal{X}$ at P is regular. Its maximal ideal is generated by two elements (x, y) with the property that any a_i is the product of a unit, a power of x and a power of y. Thus over F, the fraction field of R_P, φ is isomorphic to

$$\varphi_1 \perp x\varphi_2 \perp y\varphi_3 \perp xy\varphi_4 \perp x^2\varphi_5 \perp y^2\varphi_6 \perp x^2y^2\varphi_7 \perp x^2y\varphi_8 \perp xy^2\varphi_9 \perp$$
$$\cdots \perp x^{d-1}y^{d-1}\varphi_{d^2},$$

where each φ_i is a nondegenerate diagonal form over R_P. Let R_y be the localization of R_P at the prime ideal (y). R_y is a discrete valuation ring with fraction field F. The residue field E of R_y is the field of fractions of the discrete valuation ring $R_P/(y)$. By hypothesis, the form

$$(\varphi_1 \perp x\varphi_2 \perp x^2\varphi_5 \perp \dots) \perp y(\varphi_3 \perp x\varphi_4 \perp \dots)$$
$$\perp y^2(\varphi_6 \perp \dots) \perp \cdots \perp y^{d-1}(\cdots \perp x^{d-1}\varphi_{d^2})$$

is isotropic over the field of fractions of the completion of R_y. By Theorem 2.1, the reduction of one of the forms

$$(\varphi_1 \perp x\varphi_2 \perp x^2\varphi_5 \perp \dots), (\varphi_3 \perp x\varphi_4 \perp \dots), \cdots,$$

is isotropic over E. Since x is a uniformizing parameter for $R_P/(y)$, by Theorem 2.1 this implies that over the residue field of $R_P/(y)$, the reduction of one of the forms $\varphi_1, \varphi_2, \varphi_3 \dots, \varphi_{d^2}$ is isotropic. But then one of these forms is isotropic over $\widehat{R}_P$, hence over the field F_P which is the fraction field of $\widehat{R}_P$. □

Remark 3.2. (i) In the proof of Theorem 3.1, one of the forms is isotropic over $\widehat{R}_P$, and since R_p is the complete discrete valuation ring which is the completion of the localisation of $\widehat{R}_P$ at the height one prime ideal corresponding to U when $p = (U, P)$, this form is also isotropic over R_p and therefore over the field of fractions F_p of $\widehat{R}_p$. This implies that if φ is isotropic over the completion of F with respect to each discrete valuation of F, then φ is isotropic over F_U for each reduced irreducible component $U \subset Y$ of the complement of S in Y, over F_P for each $P \in S$ and over R_p for each $p = (U, P)$. Since F is the inverse limit of the finite inverse system of

fields $\{F_U, F_P, F_p\}$, φ is isotropic over all overfields used in the inverse limit.
(ii) The discrete valuation rings used in the above proof are the local rings at a point of codimension 1 on a suitable regular proper model $\mathcal{X}$ of X determined by the choice of φ (analogously as noted in [2, Remark 3.2]).

Given a nondegenerate diagonal form φ of degree d and dimension greater than two over F, it is not clear whether the isotropy of φ over F_v for each v (respectively, of φ over all F_U, F_P and F_p) implies that φ is isotropic (the fact that $\dim \varphi > 2$ is necessary: it is easy to adjust the example in [2, Appendix] to two-dimensional diagonal forms of even degree).

Corollary 3.3. *Let $r \geq 1$ be an integer and $d \geq 3$. Assume that any diagonal form in strictly more than dr variables over any function field in one variable over k is isotropic. Then:*
(i) Any diagonal form of degree d and dimension $> d^2r$ over the function field $F = K(X)$ of a curve X/K is isotropic over F_v, for every discrete valuation v with residue field either a function field in one variable over k or a finite extension of K.
(ii) Any diagonal form of degree d and dimension $> d^2r$ over the function field $F = K(X)$ of a curve X/K is isotropic over F_U for each reduced, irreducible component $U \subset Y$ of the complement of S in Y and is isotropic over F_P for each $P \in S$.

Note that Y and S depend on φ.

Proof. (i) Let L be a finite field extension of K. This is a complete discretely valued field with residue field a finite extension ℓ of k. The assumption made on diagonal forms of degree d over functions fields in one variable over k, in particular diagonal forms of degree d over the field $\ell(t)$, and Theorem 2.1 applied to $\ell((t))$ show that any diagonal form of dimension $> r$ over ℓ has a zero. A second application of Theorem 2.1 yields that any diagonal form of degree d of dimension $> dr$ over L is isotropic. Let φ be a diagonal form of dimension n over F with $n > d^2r$. By the assumption and Theorem 2.1, φ is isotropic over F_v for every discrete valuation v with residue field either a function field in one variable over k or a finite extension of K.
(ii) follows from Theorem 3.1. □

This shows that trying to extend [2, Corollary 3.4] from quadratic to diagonal forms of higher degree results in a much weaker version.

3.2.

Let K be a p-adic field with residue field k such that $\mathrm{char}(k) \nmid d!$.

Corollary 3.4. *Any diagonal form of degree d and dimension $> d^3 + 1$ over a function field in one variable $F = K(t)$ is*

(i) isotropic over F_v, for every discrete valuation v with residue field either a function field in one variable over k or a finite extension of K;
(ii) isotropic over F_U for each reduced, irreducible component $U \subset Y$ of the complement of S in Y and isotropic over F_P for each $P \in S$.

Proof. Every finite field k is C_1 and so every diagonal form of degree d and dimension $> d^2$ over any function field in one variable over k (which is C_2) is isotropic. Assertion (i) is a direct consequence of Theorem 2.1 and (ii) follows from Corollary 3.3 (ii). □

So if φ is a diagonal form of degree d in at least $d^3 + 1$ variables over $\mathbb{Q}(t)$ then φ is isotropic over $(\mathbb{Q}_p(t))_U$ for any $p \nmid d!$, for each reduced, irreducible component $U \subset Y$ of the complement of S in Y, and isotropic over $(\mathbb{Q}_p(t))_P$ for each $P \in S$.

Remark 3.5. Let us compare Corollary 3.4 with the Ax-Kochen-Ersov Transfer Theorem [1]: given a degree d, for almost all primes p, a form of degree d over $\mathbb{Q}_p$ of dimension greater than or equal to $d^2 + 1$ is isotropic [G, (7.4)]. Moreover, for any form φ of degree $d \geq 2$ and dimension greater than d^3 over $\mathbb{Q}(t)$, for almost all primes p the form φ is isotropic over $\mathbb{Q}_p(t)$ ([16] for $d = 2$, [12] for $d \geq 3$). The model-theoretic proofs of both results do not allow for a more concrete observation on which primes exactly are included here, nor can they be extended to other base fields.

Stronger upper bounds on $u_{diag}(d, \mathbb{F}_q(t))$ will yield stronger results on its dimension, since we only used the upper bound in the well known inequality $d \cdot u_{diag}(d, \mathbb{F}_q) \leq u_{diag}(d, \mathbb{F}_q(t)) \leq d^2$ to prove Corollary 3.4, for instance we obtain:

Corollary 3.6. *Assume that $u_{diag}(d, k(t)) = dr < d^2$ for some $r \in \{1, \dots, d-1\}$. Let φ be a diagonal form of degree d and dimension $> d^2 r + 1$ over a function field in one variable $F = K(t)$. Then:*
(i) φ is isotropic over F_v, for every discrete valuation v with residue field either a function field in one variable over k or a finite extension of K;
(ii) φ is isotropic over F_U for each reduced, irreducible component $U \subset Y$ of the complement of S in Y and over F_P for each $P \in S$.

It is well known that $u_{diag}(d, K) \leq d^2$ for a p-adic field K with residue field $k = \mathbb{F}_q$ [3]. Indeed, $u_{diag}(d, K) = d\, u_{diag}(d, \mathbb{F}_q)$ by Theorem 2.1, assuming that $\operatorname{char} \mathbb{F}_q = p \nmid d!$ as before, which shows that clearly $u_{diag}(d, K)$ can be smaller than d^2. On the other hand, Artin's conjecture that $\mathbb{Q}_p$ is a C_2-field is false for instance for forms of degree 4.

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