UNIVERSITY *of York*

This is a repository copy of *Can different quantum state vectors correspond to the same physical state? An experimental test*.

White Rose
university consortium
Universities of Leeds, Sheffield & York

eprints@whiterose.ac.uk
https://eprints.whiterose.ac.uk/

PAPER • OPEN ACCESS

Can different quantum state vectors correspond to the same physical state? An experimental test

To cite this article: Daniel Nigg *et al* 2016 *New J. Phys.* **18** 013007

View the article online for updates and enhancements.

Related content

- Towards optimal experimental tests on the reality of the quantum state
  George C Knee

- Contextuality under weak assumptions
  Andrew W Simmons, Joel J Wallman, Hakop Pashayan et al.

- Quantum state discrimination and its applications
  Joonwoo Bae and Leong-Chuan Kwek

Recent citations

- Information-reality complementarity in photonic weak measurements
  Luca Mancino *et al*

- Martin Ringbauer

- In search of multipath interference using large molecules
  Joseph P. Cotter *et al*

New Journal of Physics

The open access journal at the forefront of physics

PAPER

CrossMark

OPEN ACCESS

Can different quantum state vectors correspond to the same physical state? An experimental test

Daniel Nigg[1,6,7], Thomas Monz[1,6], Philipp Schindler[1], Esteban A Martinez[1], Markus Hennrich[1], Rainer Blatt[1,2], Matthew F Pusey[3,4], Terry Rudolph[3] and Jonathan Barrett[5,7]

1   Institut für Experimentalphysik, Universität Innsbruck, Technikerstr. 25, A-6020 Innsbruck, Austria
2   Institut für Quantenoptik und Quanteninformation, Österreichische Akademie der Wissenschaften, Technikerstr. 21A, A-6020 Innsbruck, Austria
3   Department of Physics, Imperial College London, Prince Consort Road, London SW7 2AZ, UK
4   Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, ON N2L 2Y5, Canada
5   Department of Computer Science, University of Oxford, Parks Road, Oxford OX1 3QD, UK
6   These authors contributed equally to this work.
7   Authors to whom any correspondence should be addressed.

E-mail: Daniel.Nigg@uibk.ac.at and jonathan.barrett@cs.ox.ac.uk

Abstract

A century after the development of quantum theory, the interpretation of a quantum state is still discussed. If a physicist claims to have produced a system with a particular quantum state vector, does this represent directly a physical property of the system, or is the state vector merely a summary of the physicist's information about the system? Assume that a state vector corresponds to a probability distribution over possible values of an unknown physical or 'ontic' state. Then, a recent no-go theorem shows that distinct state vectors with overlapping distributions lead to predictions different from quantum theory. We report an experimental test of these predictions using trapped ions. Within experimental error, the results confirm quantum theory. We analyse which kinds of models are ruled out.

1. Introduction

Does the quantum state correspond directly to physical reality, or does it instead represent an experimenter's knowledge or information? If it represents information, then what is this information about? Many of the controversies surrounding quantum theory are related to these basic questions. Reference [1] (following [2, 3]) provides a no-go theorem that shines light on these controversies. Under an assumption of realism, the theorem considers models in which two distinct quantum state vectors sometimes describe the same underlying physical reality. It is shown that if independently prepared systems have independent physical states, then any such model must make predictions different from quantum theory.

In case this conclusion seems obvious—say, because different quantum states lead to different experimental predictions via the Born rule—it is helpful to consider a classical example. Imagine a die shaken randomly in a container that has internal components which prevent the die from being removed unless the value showing upperside is even. The filtering action of the container ensures that the probability distribution p_{even} one would assign to the upperside value of a die (out of $[1, 2, 3, 4, 5, 6]$) taken from it is $p_{even} = [0, 1/3, 0, 1/3, 0, 1/3]$. Now imagine a similar container that contrives to ensure the value shown is prime, so the assigned probability distribution is $p_{prime} = [0, 1/3, 1/3, 0, 1/3, 0]$. Suppose that an agent is given the die, so that he can observe which face is uppermost, and his task is to determine which of the two preparation procedures was used. He is not certain to succeed, because at least some of the time, the value of the die will be 2, and this is consistent with either preparation procedure. Assuming equal *a priori* probabilities for the two procedures to be used, the agent's overall probability of guessing correctly is 5/6.

Now consider a quantum system that can be prepared in two different ways corresponding to non-orthogonal quantum states. Here again, given the quantum system and asked to determine which preparation

procedure was used, an agent cannot always succeed. As is well known, it is not possible to discriminate non-orthogonal quantum states with certainty.

In the example of the die, the reason why the agent cannot guess the preparation with certainty is because the same physical state (number 2 uppermost) is assigned non-zero probability by each of the two probability distributions, p_{even} and p_{prime}. The impossibility of distinguishing non-orthogonal quantum states would receive a natural explanation if something similar were true—that is if the two different quantum states correspond to different distributions over some set of underlying physical states, and these distributions overlap, so that some physical states are compatible with both preparations.

In order to derive a contradiction with the predictions of quantum theory, [1] considers a more involved experiment than the simple preparation and measurement of a single system. Instead, a number of systems are prepared independently, before being brought together so that an entangled measurement can be performed. This work reports an experimental implementation of the test from [1] using trapped ions. In the following, we introduce the class of models being considered in more detail, describing briefly the original proposal of [1], and we identify a natural subclass of models that our experiment is able to rule out. The experimental setup is described, along with the results and their interpretation. Finally, we include a discussion of experimental loopholes.

2. Different models: ψ-ontic versus ψ-epistemic

In order to describe more formally the kinds of models that the experiment is concerned with, suppose that a quantum system has an objective physical state of some kind, denoted λ, where objective means that λ is independent of the experimenter and of other physical systems. We will refer to λ as the *ontic state* [4]. If a measurement M is performed on the system, the probability of getting the outcome a is determined by λ, and can be written $P_M(a|\lambda)$. Given an ensemble of systems, each prepared in such a way that quantum theory assigns a quantum state $|\psi\rangle$, it is not necessarily the case that λ is the same for each member of the ensemble, hence we assume that a quantum state $|\psi\rangle$ corresponds to a probability distribution $\mu_\psi(\lambda)$. Quantum predictions are recovered if for any measurement, and any preparation:

$$\langle\psi|Q_{M,a}|\psi\rangle = \int P_M(a|\lambda)\mu_\psi(\lambda)\mathrm{d}\lambda,$$

where $Q_{M,a}$ is the positive operator corresponding to the outcome a of measurement M. These assumptions are similar to those made in derivations of Bell's theorem, and are often referred to as 'realism'. Bell's theorem shows that if, additionally, local causality is imposed on the ontic state, then predictions different from quantum theory are obtained.

Here, local causality is not imposed. Instead, we wish to investigate the possibility raised above, which is that the distributions $\mu_0(\lambda)$ and $\mu_1(\lambda)$ overlap for distinct quantum states $|\phi_0\rangle$ and $|\phi_1\rangle$. It is a non-trivial question whether models of this form actually exist which reproduce the predictions of quantum theory. The question has been originally raised by Harrigan and Spekkens [2] and by Hardy [3]. Harrigan and Spekkens refer to any model in which the distributions $\mu_0(\lambda)$ and $\mu_1(\lambda)$ overlap for some distinct $|\psi_0\rangle$ and $|\psi_1\rangle$ as ψ-*epistemic*. A model in which $\mu_0(\lambda)$ and $\mu_1(\lambda)$ are disjoint for any distinct $|\psi_0\rangle$ and $|\psi_1\rangle$ is called ψ-*ontic*. For a single qubit, a ψ-epistemic model was provided some time ago by Kochen and Specker [5]. More recently, explicit ψ-epistemic models for a single quantum system of arbitrary dimension have been constructed [6, 7]. Other works, again by considering a single system of arbitrary dimension, have derived bounds on how much the distributions μ_0 and μ_1 can overlap for distinct quantum states [8–11].

Reference [1], on the other hand, shows that under an additional mild assumption, *preparation independence*, ψ-epistemic models are incompatible with quantum theory[8]. This work reports an experimental test, based on the argument described in [1]. Although the result of [1] rules out an overlap between any pair of distinct quantum states, our experiment, for reasons to be explained in the next section, tests a pair of states $|\phi_0\rangle$, $|\phi_1\rangle$ satisfying $|\langle\phi_0|\phi_1\rangle| = 1/\sqrt{2}$. Before describing the experimental setup, we give a brief summary of the theoretical argument of [1], as it applies to such a pair.

Suppose that two systems are independently prepared, each in one of the two states. Then the overall joint state is a product: $|\phi_0\rangle \otimes |\phi_0\rangle, |\phi_0\rangle \otimes |\phi_1\rangle, |\phi_1\rangle \otimes |\phi_0\rangle$ or $|\phi_1\rangle \otimes |\phi_1\rangle$. The assumption of preparation independence is that the joint distribution for the ontic states also has a product form: $\mu_0(\lambda_1)\mu_0(\lambda_2)$, $\mu_0(\lambda_1)\mu_1(\lambda_2)$, $\mu_1(\lambda_1)\mu_0(\lambda_2)$, or $\mu_1(\lambda_1)\mu_1(\lambda_2)$ respectively.

If μ_0 and μ_1 overlap then the above four distributions all overlap too. In the example with dice, this corresponds to the fact that preparing two dice independently according to p_{even} or p_{prime} can always result in both dice showing 2. When that occurs it is impossible to rule out any of the preparations (even, even), (even,

---

[8] For a different approach, which gives related conclusions, but under different assumptions, see [23–25].

prime), (prime, even), (prime, prime). The result of [1] hinges on the surprising fact that the analogous task *is* always possible according to quantum theory. That is, the theory predicts a joint measurement with four outcomes 00, 01, 10, 11 such that outcome 00 never occurs on the state $|\phi_0\rangle \otimes |\phi_0\rangle$, outcome 01 never occurs on $|\phi_0\rangle \otimes |\phi_1\rangle$ and so on. Since a model where $\mu_0(\lambda)$ and $\mu_1(\lambda)$ overlap could not reproduce such a measurement, a contradiction is obtained.

3. The noisy case

The argument presented above relies on certain probabilities being exactly zero, and this will never be reproduced entirely in a real experiment. It is therefore important to consider what can be concluded if these probabilities are measured and found to be merely close to zero within experimental bounds.

To this end, note that the classical trace distance between two probability distributions, $p(x)$ and $q(x)$ is given by

$$D(p, q) = \frac{1}{2} \sum_x |p(x) - q(x)|.$$

$D(p, q)$ is 1 if the supports of the distributions p and q are disjoint, and it is 0 if p and q are identical. If the distributions are defined on continuous sample spaces, then the sum is replaced by an integral. Reference [1] shows that for any pair of distinct quantum states $|\psi_0\rangle$ and $|\psi_1\rangle$, the quantum predictions, together with the assumption of preparation independence, imply that $D(\mu_0, \mu_1) = 1$. It is also shown that if the appropriate experiment is carried out, and experimental results obtained that are close to the quantum predictions, then a lower bound can be placed on $D(\mu_0, \mu_1)$. There is no single experiment that could rule out the entire class of ψ-epistemic models. Instead, for any fixed choice of $|\psi_0\rangle$ and $|\psi_1\rangle$, if the appropriate measurement is performed, then the more closely the data match quantum predictions, the closer $D(\mu_0, \mu_1)$ is to 1, and the closer the two distributions are to being disjoint.

The goal of our experiment is to achieve a sufficiently good lower bound on $D(\mu_0, \mu_1)$ that a natural class of models, which we refer to as *maximally ψ-epistemic* models can be ruled out. For an explanation, consider the operational interpretation of the classical trace distance. If a system such as an n-sided die is equally likely to have been prepared according to one of two probability distributions p and q, then on learning which face is uppermost, an agent's maximal probability of guessing correctly which preparation was used is $(1 + D(p, q))/2$. Similarly, given quantum states $|\psi_0\rangle$ and $|\psi_1\rangle$, with the preparation of each *a priori* equally likely, if an optimal measurement for discriminating the two is performed, the probability of guessing correctly is $(1 + D_Q(|\psi_0\rangle, |\psi_1\rangle))/2$, where the *quantum trace distance* for pure states is given by

$$D_Q\left(\left|\psi_0\right\rangle, \left|\psi_1\right\rangle\right) = \sqrt{1 - \left|\left\langle \psi_0 | \psi_1 \right\rangle\right|^2}. \tag{1}$$

Theorem. *Any ψ-epistemic model that reproduces the quantum predictions for a single system must satisfy*

$$D\left(\mu_0, \mu_1\right) \geqslant D_Q\left(\left|\psi_0\right\rangle, \left|\psi_1\right\rangle\right), \tag{2}$$

for all pairs of states $|\psi_0\rangle$ and $|\psi_1\rangle$.

Sketch proof: Consider a measurement device, set up to perform this optimal measurement, with two outputs, labelled 'guess $|\psi_0\rangle$' and 'guess $|\psi_1\rangle$'. Given a model in which measurement outcome probabilities are determined by an ontic state λ, this device is essentially in the same position as an agent, who is trying to distinguish the two distributions $\mu_0(\lambda)$ and $\mu_1(\lambda)$ by observing the value of λ. The device cannot do this with a success probability any greater than $1/2(1 + D(\mu_0, \mu_1))$, yet must succeed with probability $1/2(1 + D_Q(|\psi_0\rangle, |\psi_1\rangle))$.

With this theorem in mind, a *maximally ψ-epistemic* model is defined as one in which equality holds in equation (2) for all pairs of states [12]. The aim of the experiment is to rule out maximally ψ-epistemic models. This corresponds to a natural class of models, since if equality in equation (2) is satisfied, the probability of making an error when attempting to discriminate non-orthogonal quantum states does not need to be explained by any kind of quantum effect, but is in fact *entirely due* to the ordinary classical difficulty of distinguishing the corresponding distributions[9] μ_0 and μ_1. An example of such a model is that of Kochen and Specker, which reproduces the quantum predictions exactly for a single qubit.

---

[9] The idea that the impossibility of reliably distinguishing non-orthogonal quantum states is entirely due to overlapping μ_0 and μ_1 has also been considered in [12, 26, 27], but with alternative definitions of the distances between classical probability distributions.

Since the maximally ψ-epistemic hypothesis pertains to *every* pair of states, it suffices to refute it for a single pair of states. The reason for choosing $|\phi_0\rangle$ and $|\phi_1\rangle$ with $|\langle\phi_0|\phi_1\rangle| = 1/\sqrt{2}$ is that this is the largest overlap for which the argument of [1] requires just two systems. Probabilities inferred from the experimental data are indeed close to zero in relevant cases. It is then possible to derive a lower bound on the trace distance $D(\mu_0, \mu_1)$. Although the basic idea is to implement the experiment of [1], we will in fact use a slightly different lower bound to that derived in [1], which has been specially adapted to the present experimental context. This alternative lower bound is expressed in equation (9) below, and is derived in the appendix. Using this, our experimental results rule out maximally ψ-epistemic models.

4. Experimental implementation and results

The experiments described here were realized with a string of ^{40}Ca$^+$ ions which are confined in a linear Paul trap [13]. A single ion represents a qubit which is encoded in the electronic levels $S_{1/2}(m = -1/2) = |1\rangle$ and $D_{5/2}(m = -1/2) = |0\rangle$. Each experimental cycle consists of the initialization of the ions in their internal electronic and motional ground states followed by coherent manipulation of the qubits and finally detection of the quantum state. State initialization is realized by optical pumping into the $S_{1/2}(m = -1/2)$ state after cooling the axial centre-of-mass mode to the motional ground state. The manipulation of the qubits is implemented by coherently exciting the $S_{1/2} \leftrightarrow D_{5/2}$ quadrupole transition with laser pulses at a wavelength of 729 nm. Finally, the population of the qubit states is measured by exciting the $S_{1/2} \leftrightarrow P_{1/2}$ transition and detecting the fluorescence light, using electron shelving [14]. Our setup is capable of realizing collective qubit rotations

$$U(\theta, \phi) = \exp\left(-\mathrm{i}\frac{\theta}{2}\sum_i\left[\sin(\phi)\sigma_y^{(i)} - \cos(\phi)\sigma_x^{(i)}\right]\right) \tag{3}$$

via a laser beam addressing the entire register, and Mølmer–Sørenson entangling gate operations [15, 16]

$$MS(\theta, \phi) = \exp\left(-\mathrm{i}\frac{\theta}{4}\left[\sum_i\left(\sin(\phi)\sigma_y^{(i)} - \cos(\phi)\sigma_x^{(i)}\right)\right]^2\right). \tag{4}$$

Additionally we are able to perform single-qubit rotations on the ith ion of the form $U_z^{(i)}(\theta) = \exp\left(-\mathrm{i}\frac{\theta}{2}\sigma_z^{(i)}\right)$ using an off-resonant laser beam which addresses individual ions.

The first step of the protocol is the preparation of one of the input states $\{|\phi_0\rangle \otimes |\phi_0\rangle,\ |\phi_0\rangle \otimes |\phi_1\rangle,\ |\phi_1\rangle \otimes |\phi_0\rangle,\ |\phi_1\rangle \otimes |\phi_1\rangle\}$ with $|\phi_0\rangle = \cos\left(\frac{\pi}{8}\right)|1\rangle + \sin\left(\frac{\pi}{8}\right)|0\rangle$ and $|\phi_1\rangle = \cos\left(\frac{\pi}{8}\right)|1\rangle - \sin\left(\frac{\pi}{8}\right)|0\rangle$. The input states are generated by a global rotation $U\left(\frac{\pi}{4}, -\frac{\pi}{2}\right)$, which maps $|1\rangle \otimes |1\rangle$ to $|\phi_0\rangle \otimes |\phi_0\rangle$, followed by single-qubit rotations $U_z^{(1,2)}(\pi)$ on the first, second or both ions, which maps $|\phi_0\rangle$ to $|\phi_1\rangle$. The corresponding sequences are illustrated in figure 1(b).

The second part of the protocol is a joint measurement on the system with the property that each measurement outcome should have probability zero for one of the four input states. Such a measurement can, in general, be realized by rotations $Z_\beta = |0\rangle\langle0| + \exp(\mathrm{i}\beta)|1\rangle\langle1|$ followed by a conditional phase gate R_α, with $R_\alpha|11\rangle = \exp(\mathrm{i}\alpha)|11\rangle$, Hadamard gates H and finally a measurement in the computational basis, as shown in figure 1(a).

Our pulse sequence for the measurement procedure is shown in figure 1(c). For the input states used here, the appropriate values α and β are equal to $\alpha = \pi$ and $\beta = 0$, which reduces the rotation Z_β to the identity. Up to local phases the remaining two operations are equivalent to a global rotation $U\left(\frac{\pi}{2}, \pi\right)$ followed by a maximally entangling gate[10] $MS\left(\frac{\pi}{2}, 0\right)$.

In an ideal experiment, the pulses applied to the single ions used to select different input states would have no effect on the neighbouring ion. This requirement is in fact only fulfilled to a certain degree of accuracy due to residual light on the neighbouring ions (crosstalk). If, for example, the phase shift operation $U_z^{(2)}(\pi)$ is applied on the second qubit, a residual phase shift $U_z^{(1)}(\kappa\pi)$ occurs on the first ion, with κ on the order of 1%. Hence, instead of the ideal input state $|\phi_0\rangle \otimes |\phi_1\rangle$, we have $|\phi_0'\rangle \otimes |\phi_1\rangle$, where $|\phi_0'\rangle = \mathrm{e}^{\mathrm{i}\kappa\pi}\cos\left(\frac{\pi}{8}\right)|1\rangle + \sin\left(\frac{\pi}{8}\right)|0\rangle$. Due to the residual light in both directions, $|\phi_1\rangle \otimes |\phi_1\rangle$ becomes $|\phi_1'\rangle \otimes |\phi_1'\rangle$ where
$|\phi_1'\rangle = \mathrm{e}^{\mathrm{i}\kappa\pi}\cos\left(\frac{\pi}{8}\right)|1\rangle - \sin\left(\frac{\pi}{8}\right)|0\rangle$.

---

[10] The sequence was optimized with respect to the number of pulses, since local phases before the final measurement in the computational basis do not matter.
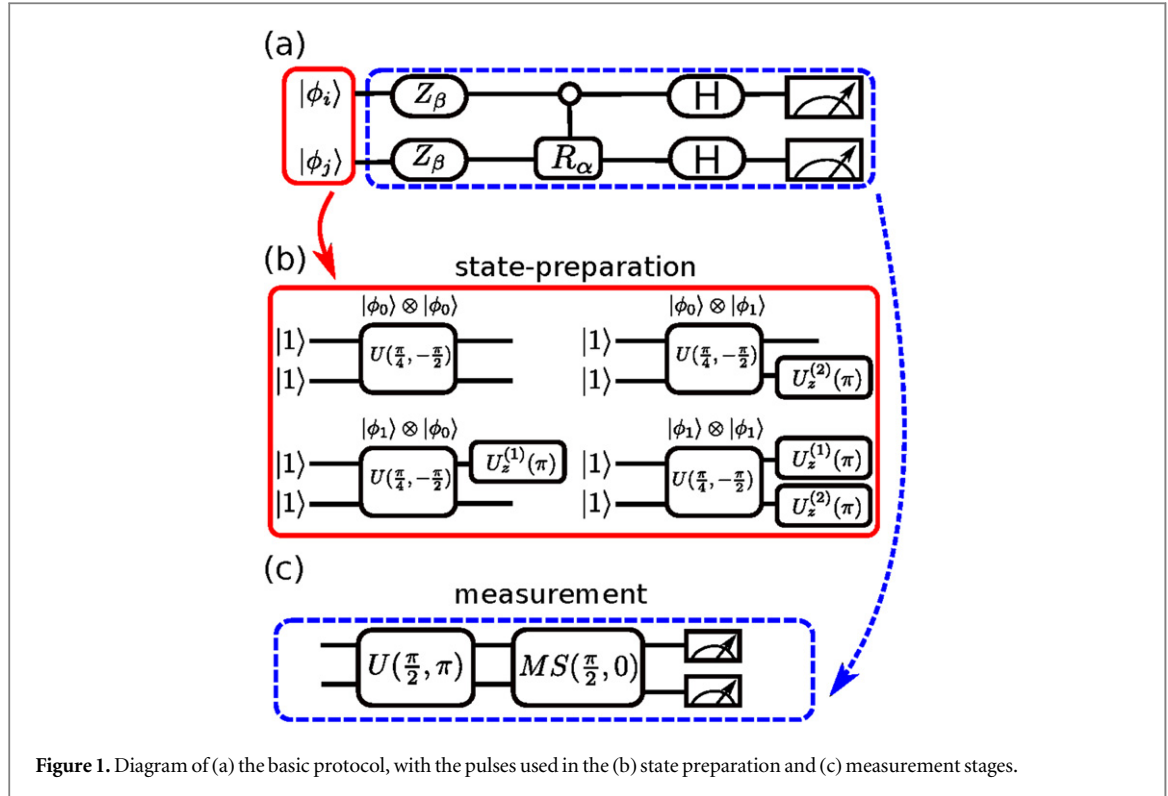
Figure 1. Diagram of (a) the basic protocol, with the pulses used in the (b) state preparation and (c) measurement stages.

Overall then, in place of the original protocol's preparations $|\phi_0\rangle \otimes |\phi_0\rangle, |\phi_0\rangle \otimes |\phi_1\rangle, |\phi_1\rangle \otimes |\phi_0\rangle,$ $|\phi_1\rangle \otimes |\phi_1\rangle$, we instead have $|\phi_0\rangle \otimes |\phi_0\rangle, |\phi_0'\rangle \otimes |\phi_1\rangle, |\phi_1\rangle \otimes |\phi_0'\rangle, |\phi_1'\rangle \otimes |\phi_1'\rangle$. Since the laser field can be viewed as a product of independent coherent states acting on each ion, it is reasonable to model these as independent preparations:

$$\mu_{00}(\lambda_1, \lambda_2) = \mu_0(\lambda_1)\mu_0(\lambda_2), \tag{5}$$

$$\mu_{0'1}(\lambda_1, \lambda_2) = \mu_{0'}(\lambda_1)\mu_1(\lambda_2), \tag{6}$$

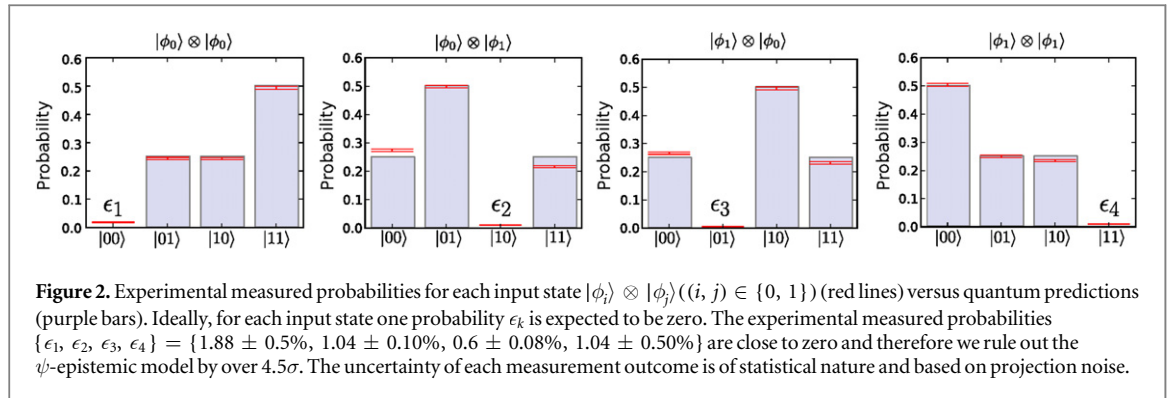$$\mu_{10'}(\lambda_1, \lambda_2) = \mu_1(\lambda_1)\mu_{0'}(\lambda_2), \tag{7}$$

$$\mu_{1'1'}(\lambda_1, \lambda_2) = \mu_{1'}(\lambda_1)\mu_{1'}(\lambda_2). \tag{8}$$

It is shown in the appendix that if this modified state preparation is implemented, and the outcomes that never occur in the ideal protocol happen with average probability ϵ, then the relevant classical trace distances satisfy

$$D(\mu_0, \mu_1) + D(\mu_0, \mu_{0'}) + D(\mu_1, \mu_{1'}) \geqslant 1 - 2\sqrt{\epsilon}. \tag{9}$$

Supposing that all three classical trace distances were equal to the corresponding quantum trace distances, we would obtain $\epsilon \geqslant 1.83\%$, and so an experiment with a smaller value rules out maximally ψ-epistemic models. For simplicity, we are treating the ions identically, whereas in fact the actions of the laser on each ion will be slightly different. In the appendix it is shown that this consideration does not significantly alter the lower bound on ϵ.

Ideally, each outcome would have probability zero for some input state. In fact these nonzero outcomes occur with small probabilities ϵ_k. For each of the input states the measurement was repeated up to 10000 times to gain significant statistics. Figure 2 shows the measured probabilities. We find $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\} = \{1.88\%, 1.04\%, 0.6\%, 1.04\%\}$, which yields a mean value of $\epsilon = (1.14 \pm 0.18)\%$. This violates the bound of equation (9) given by our trace distance hypothesis by over 4.5σ. Therefore the probability of this data to be consistent with a maximally ψ-epistemic model is less than 10^{-4}.

Figure 2. Experimental measured probabilities for each input state $|\phi_i\rangle \otimes |\phi_j\rangle$ $((i, j) \in \{0, 1\})$ (red lines) versus quantum predictions (purple bars). Ideally, for each input state one probability ϵ_k is expected to be zero. The experimental measured probabilities $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\} = \{1.88 \pm 0.5\%, 1.04 \pm 0.10\%, 0.6 \pm 0.08\%, 1.04 \pm 0.50\%\}$ are close to zero and therefore we rule out the ψ-epistemic model by over 4.5σ. The uncertainty of each measurement outcome is of statistical nature and based on projection noise.

5. Discussion of assumptions and experimental loopholes

The result of [1] can be considered as a no-go theorem for interpretations of quantum theory, analogous to Bell's theorem. Each theorem states that a certain class of theories must make different predictions from quantum theory—*locally causal* theories in the case of Bell's theorem and ψ-*epistemic* theories in the case of [1].

What assumptions are needed? First, both Bell's theorem and [1] assume that a system has an objective physical state λ, independently of other systems, and that probabilities for measurement outcomes are determined by λ. This assumption is often referred to as 'realism' in presentations as Bell's theorem. The response of many physicists to Bell's theorem, and no doubt to the theorem of [1] too, is that this assumption should be rejected. Indeed one possible attitude to no-go results such as Bell's theorem, that of [1], and the corresponding experimental tests such as that reported here, is that they provide evidence that 'realism' should be rejected because the other possibilities are too unpalatable. However, this is not the place for an extended discussion about realism. But it's important to be aware of what is being given up: neither Bell's theorem nor that of [1] need to assume that observables have underlying definite values, or that familiar classical properties such as position and momentum are well defined for all systems. No assumptions are made about the ontic state λ at all except that it is objective and able to take on different values within some set. Nor does either result assume determinism—measurement outcomes can depend only probabilistically on λ.

Second, the theorem of [1] needs to assume *preparation independence*. An ideal implementation of the protocol of [1] would involve preparations of quantum systems that are independent by any reasonable scientific judgement (say, one on Earth and one on Mars, spacelike separated, using apparatuses manufactured in separate factories, each unaware of the other...). In this case, we would not regard the assumption of preparation independence as strong: it is a basic tenet of physical science that one can perform independent experiments, and when this is done, the relevant systems are uncorrelated. An assumption needed by Bell's theorem, that experimenters can freely choose measurements, has a similar character. In each case, models can be constructed that violate the assumption and evade the theorem [17], but they are highly contrived [6, 18, 19].

Real experiments, however, deviate from the ideal in various ways. In our experiment, the quantum systems are ions in close proximity. The judgement that these are independent systems, derives from the fact that quantum theory itself assigns a product pure state (i.e., the judgement does *not* come from more general background considerations, as in the ideal case). Furthermore, the decision whether to prepare one ion in the state $|\psi_0\rangle$ or $|\psi_1\rangle$ has a measurable effect on the quantum state of the other ion, hence these cannot be regarded as truly independent preparations and in that sense corresponds to a potential state preparation loophole.

This loophole is partially closed off by the use of the inequality (9) of the main text, which is designed to allow for the fact that when one ion is rotated from the state $|\psi_0\rangle$ to the state $|\psi_1\rangle$, a small crosstalk effect changes the state of the other ion from $|\psi_0\rangle$ to $|\psi_1'\rangle$. The inequality (9) allows the corresponding distributions $\mu_0(\lambda)$ and $\mu_1(\lambda)$ to be different. But the derivation of inequality (9) does still assume that a preparation of a quantum state which is (close to) a direct product, e.g., $|\psi_0\rangle \otimes |\psi_1\rangle$, corresponds to a distribution which is also (close to) a product, e.g., $\mu_0(\lambda_1)\mu_1(\lambda_2)$. Given the crosstalk effect, a model which violates this assumption is not as farfetched as such a model would be, were the preparations completely independent at the quantum level. Future experiments may be able to achieve this.

Finally, the quantum states prepared will not in fact have been completely pure. This will not affect the bound on classical trace distances that we obtained, but it is not clear exactly what a natural hypothesis for these classical trace distances would be. One certainly cannot ask that the classical trace distance is equal to the quantum trace distance in the case of mixed states, because sometimes even identical mixed quantum states cannot be represented by the same classical probability distribution [20]. This opens the possibility for a 'mixed states' preparations loophole.

6. Conclusions

The aim of this work has been to investigate experimentally the fundamental question of the interpretation of the quantum state vector. This was done by implementing, with trapped ions, a protocol described theoretically in [1]. The idea of [1] is that if two distinct quantum states can sometimes represent the same underlying physical state of a system, then different predictions are obtained from those of quantum theory. While a real experiment cannot rule out the entire class of ψ-epistemic models, it is possible to obtain a lower bound on the trace distance between the probability distributions μ_0 and μ_1 corresponding to distinct quantum states. We have suggested that a natural threshold is defined by quantum state discrimination. Given our assumptions, the experiment reported here rules out maximally ψ-epistemic models, i.e., models in which errors in quantum state discrimination are explained entirely in terms of overlapping classical probability distributions.

Note

Since this work was completed, related experimental tests have been reported, using coherent states in optical fibres [21], and single photons [22].

Acknowledgments

Appendix

Theorem. *Let* $|\psi_0\rangle$, $|\psi_0'\rangle$, $|\psi_1\rangle$, $|\psi_1'\rangle$ *be qubit states. Suppose 2 qubits are independently prepared in one of the four states*

$$\left|\psi_0\right\rangle \otimes \left|\psi_0'\right\rangle, \left|\psi_0'\right\rangle \otimes \left|\psi_1\right\rangle, \left|\psi_1\right\rangle \otimes \left|\psi_0'\right\rangle, \left|\psi_1'\right\rangle \otimes \left|\psi_1'\right\rangle \tag{10}$$

and a measurement is performed where outcome k has probability ϵ_k *on the kth quantum state. Let* $\epsilon = \frac{1}{4}\sum_{i=1}^4 \epsilon_k$. *A model that reproduces these results must satisfy*

$$D\left(\mu_0, \mu_1\right) + D\left(\mu_0, \mu_{0'}\right) + D\left(\mu_1, \mu_{1'}\right) \geqslant 1 - 2\sqrt{\epsilon}. \tag{11}$$

To prove this we define the k-overlap of probability distributions μ_1 ,..., μ_k, as in [1], by

$$\omega\left(\mu_1, \ldots, \mu_k\right) = \int_\Lambda \min_i \mu_i(\lambda)\mathrm{d}\lambda. \tag{12}$$

We will need the following link between four-overlaps and two-overlaps:

Lemma.

$$\omega\left(\mu_A, \mu_B, \mu_C, \mu_D\right)$$
$$\geqslant \omega\left(\mu_A, \mu_B\right) + \omega\left(\mu_B, \mu_C\right) + \omega\left(\mu_C, \mu_D\right) - 2. \tag{13}$$

Proof. First notice that for real numbers a, b, c

$$\min(a, b, c) = \min(a, b) + \min(b, c) - f(a, b, c), \tag{14}$$

where

$$f(a, b, c) = \begin{cases} \min(b, c) & a \leqslant b, c \\ b & b \leqslant a, c \\ \min(a, b) & c \leqslant a, b. \end{cases} \tag{15}$$

Since $f(a, b, c) \leqslant b$ we have

$$\min(a, b, c) \geqslant \min(a, b) + \min(b, c) - b. \tag{16}$$

By two applications of the above we obtain that for real numbers a, b, c, d

$$\begin{aligned}
\min(a, b, c, d) = \min(a, b, \min(c, d)) &\geqslant \\
\min(a, b) + \min(b, \min(c, d)) - b \\
= \min(a, b) + \min(b, c, d) - b \\
\geqslant \min(a, b) + \min(b, c) + \min(c, d) - b - c.
\end{aligned} \tag{17}$$

Let $a = \mu_A(\lambda)$, $b = \mu_B(\lambda)$, $c = \mu_C(\lambda)$, $d = \mu_D(\lambda)$, integrate each side with respect to λ, and recall that μ_B and μ_C are normalized to obtain the result. □

Proof of theorem. Since the systems are prepared independently, the resulting physical states are distributed according to

$$\mu_A(\lambda_1, \lambda_2) = \mu_0(\lambda_1) \times \mu_0(\lambda_2), \tag{18}$$

$$\mu_B(\lambda_1, \lambda_2) = \mu_{0'}(\lambda_1) \times \mu_1(\lambda_2), \tag{19}$$

$$\mu_C(\lambda_1, \lambda_2) = \mu_1(\lambda_1) \times \mu_{0'}(\lambda_2), \tag{20}$$

$$\mu_D(\lambda_1, \lambda_2) = \mu_{1'}(\lambda_1) \times \mu_{1'}(\lambda_2). \tag{21}$$

Hence

$$\begin{aligned}
\min_{k \in \{A,B,C,D\}} \mu_k(\lambda_1, \lambda_2) \\
\geqslant \min\left\{ \mu_0(\lambda_1), \mu_{0'}(\lambda_1), \mu_1(\lambda_1), \mu_{1'}(\lambda_1) \right\} \\
\times \min\left\{ \mu_0(\lambda_2), \mu_{0'}(\lambda_2), \mu_1(\lambda_2), \mu_{1'}(\lambda_2) \right\}.
\end{aligned} \tag{22}$$

Integrating both sides

$$\omega\left(\{\mu_k\} \right) \geqslant \left(\omega\left(\mu_0, \mu_{0'}, \mu_1, \mu_{1'} \right) \right)^2, \tag{23}$$

which square roots to

$$\sqrt{\omega\left(\{\mu_k\} \right)} \geqslant \omega\left(\mu_0, \mu_{0'}, \mu_1, \mu_{1'} \right). \tag{24}$$

Applying the lemma (notice that we can freely re-order the arguments of ω) we obtain

$$\sqrt{\omega\left(\{\mu_k\} \right)} \geqslant \omega\left(\mu_0, \mu_1 \right) + \omega\left(\mu_0, \mu_{0'} \right) + \omega\left(\mu_1, \mu_{1'} \right) - 2. \tag{25}$$

Recalling that the classical trace distance $D(\mu_0, \mu_1) = 1 - \omega(\mu_0, \mu_1)$ this becomes

$$\sqrt{\omega\left(\{\mu_k\} \right)} \geqslant 1 - D\left(\mu_0, \mu_1 \right) - D\left(\mu_0, \mu_{0'} \right) - D\left(\mu_1, \mu_{1'} \right). \tag{26}$$

Meanwhile, from the observed probabilities we have that

$$\int_{\Lambda^n} \xi_{M,k}(\vec{\lambda}) \mu_k(\vec{\lambda}) d\vec{\lambda} = \epsilon_k. \tag{27}$$

Since $\min_i \mu_i(\vec{\lambda}) \leqslant \mu_k(\vec{\lambda})$, and both $\xi_{M,k}(\vec{\lambda})$ and $\mu_k(\vec{\lambda})$ are non-negative

$$\int_{\Lambda^n} \xi_{M,k}(\vec{\lambda}) \min_i \mu_i(\vec{\lambda}) d\vec{\lambda} \leqslant \epsilon_k. \tag{28}$$

Finally, sum over k and use the normalization $\sum_k \xi_{M,k}(\vec{\lambda}) = 1$ to obtain

$$\omega\left(\{\mu_k\} \right) \leqslant 4\epsilon. \tag{29}$$

Square rooting each side gives

$$\sqrt{\omega\left(\{\mu_k\} \right)} \leqslant 2\sqrt{\epsilon}. \tag{30}$$

Combining this with (26) gives the desired result. □

Non-identical qubits

In the above it is assumed that the four quantum states $|\psi_0\rangle$, $|\psi_0'\rangle$, $|\psi_1\rangle$ and $|\psi_1'\rangle$ are the same for Alice and Bob, and the same underlying model of physical states $\lambda \in \Lambda$ is applied to both qubits. In reality the conditions of the qubits might not be identical, and so we should consider quantum states

$$\left| \psi_0 \right\rangle \otimes \left| \phi_0 \right\rangle, \left| \psi_0' \right\rangle \otimes \left| \phi_1 \right\rangle, \left| \psi_1 \right\rangle \otimes \left| \phi_0' \right\rangle, \left| \psi_1' \right\rangle \otimes \left| \phi_1' \right\rangle. \tag{31}$$

Corresponding to probability distributions over $\Lambda^a \times \Lambda^b$

$$\mu_A\left(\lambda_1, \lambda_2\right) = \mu_0^a\left(\lambda_1\right) \times \mu_0^b\left(\lambda_2\right), \tag{32}$$

$$\mu_B\left(\lambda_1, \lambda_2\right) = \mu_{0'}^a\left(\lambda_1\right) \times \mu_1^b\left(\lambda_2\right), \tag{33}$$

$$\mu_C\left(\lambda_1, \lambda_2\right) = \mu_1^a\left(\lambda_1\right) \times \mu_0^b\left(\lambda_2\right), \tag{34}$$

$$\mu_D\left(\lambda_1, \lambda_2\right) = \mu_{1'}^a\left(\lambda_1\right) \times \mu_{1'}^b\left(\lambda_2\right). \tag{35}$$

Hence in place of equation (23) we obtain

$$\omega\left(\left\{\mu_k\right\}\right) \geqslant \omega\left(\mu_0^a, \mu_{0'}^a, \mu_1^a, \mu_{1'}^a\right) \times \omega\left(\mu_0^b, \mu_{0'}^b, \mu_1^b, \mu_{1'}^b\right). \tag{36}$$

Applying the lemma and converting two-overlaps to classical trace distances then gives

$$\omega\left(\left\{\mu_k\right\}\right) \geqslant \left(1 - D\left(\mu_0^a, \mu_1^a\right) - D\left(\mu_0^a, \mu_{0'}^a\right) - D\left(\mu_1^a, \mu_{1'}^a\right)\right)$$
$$\times \left(1 - D\left(\mu_0^b, \mu_1^b\right) - D\left(\mu_0^b, \mu_{0'}^b\right) - D\left(\mu_1^b, \mu_{1'}^b\right)\right). \tag{37}$$

Combining this with (29) gives

$$\left(1 - D\left(\mu_0^a, \mu_1^a\right) - D\left(\mu_0^a, \mu_{0'}^a\right) - D\left(\mu_1^a, \mu_{1'}^a\right)\right)$$
$$\times \left(1 - D\left(\mu_0^b, \mu_1^b\right) - D\left(\mu_0^b, \mu_{0'}^b\right) - D\left(\mu_1^b, \mu_{1'}^b\right)\right) \leqslant 4\epsilon. \tag{38}$$

If we suppose that, for both systems, the classical trace distance equals the quantum trace distance, and that (to good approximation) the quantum trace distances between the relevant states are as before, we obtain exactly the same bound on ϵ.

References

[1] Pusey M F, Barrett J and Rudolph T 2012 *Nat. Phys.* **8** 476
[2] Harriganand N and Spekkens R W 2010 *Found. Phys.* **40** 125
[3] Hardy L 2005 private communication
[4] Leifer M S 2014 *Quanta* **3** 67
[5] Kochenand S and Specker E 1968 *Indiana Univ. Math. J.* **17** 59
[6] Lewis P G, Jennings D, Barrett J and Rudolph T 2012 *Phys. Rev. Lett.* **109** 150404
[7] Aaronson S, Bouland A, Chua L and Lowther G 2013 *Phys. Rev. A* **88** 032111
[8] Barrett J, Cavalcanti E G, Lal R and Maroney O J E 2014 *Phys. Rev. Lett.* **112** 250403
[9] Branciard C 2014 *Phys. Rev. Lett.* **113** 020409
[10] Leifer M S 2014b *Phys. Rev. Lett.* **112** 160404
[11] Allen J-M 2015 arXiv:1501.05969
[12] Maroney O 2012 How statistical are quantum states? arXiv:1207.6906
[13] Schindler P *et al* 2013 *New J. Phys.* **15** 123012
[14] Nagourney W, Sandberg J and Dehmelt H 1986 *Phys. Rev. Lett.* **56** 2797
[15] Sørenson A and Mølmer K 2000 *Phys. Rev. A* **62** 022311
[16] Kirchmair G, Benhelm J, Zähringer F, Gerritsma R, Roos C F and Blatt R 2009 *New J. Phys.* **11** 023002
[17] Emerson J, Serbin D, Sutherland C and Veitch V 2013 arXiv:1312.1345
[18] Hall M J W 2010 *Phys. Rev. Lett.* **105** 250404
[19] Barrettand J and Gisin N 2011 *Phys. Rev. Lett.* **106** 100406
[20] Spekkens R W 2005 *Phys. Rev. A* **71** 052108
[21] Patra M, Olislager L, Duport F, Safioui J, Pironio S and Massar S 2013 *Phys. Rev. A* **88** 032112
[22] Ringbauer M, Duffus B, Branciard C, Cavalcanti E G, White A and Fedrizzi A 2015 *Nat. Phys.* **11** 249–54
[23] Colbeckand R and Renner R 2012 *Phys. Rev. Lett.* **108** 150402
[24] Colbeckand R and Renner R 2013 arXiv:1312.7353v1
[25] Miller D J 2013 *Phys. Rev. A* **87** 014103
[26] Morris R 2009 *Master's Thesis* University of Waterloo
[27] Leiferand M S and Maroney O J E 2013 *Phys. Rev. Lett.* **110** 120401