Fock representations of multicomponent (particularly non-Abelian anyon) commutation relations

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Abstract

Let H be a separable Hilbert space and T be a self-adjoint bounded linear operator on $H^{\otimes 2}$ with norm ≤ 1, satisfying the Yang–Baxter equation. Bożejko and Speicher (1994) proved that the operator T determines a T-deformed Fock space $\mathcal{F}(H)=\bigoplus_{n=0}^\infty \mathcal{F}_n(H)$. We start with reviewing and extending the known results about the structure of the n-particle spaces $\mathcal{F}_n(H)$ and the commutation relations satisfied by the corresponding creation and annihilation operators acting on $\mathcal{F}(H)$. We then choose $H=L^2(X\to V)$, the L^2-space of V-valued functions on X. Here $X:=\mathbb{R}^d$ and $V:=\mathbb{C}^m$ with $m\ge 2$. Furthermore, we assume that the operator T acting on $H^{\otimes 2}=L^2(X^2\to V^{\otimes 2})$ is given by $(Tf^{(2)})(x,y)=C_{x,y}f^{(2)}(y,x)$. Here, for a.a. $(x,y)\in X^2$, $C_{x,y}$ is a linear operator on $V^{\otimes 2}$ with norm ≤ 1 that satisfies $C_{x,y}^*=C_{y,x}$ and the spectral quantum Yang–Baxter equation. The corresponding creation and annihilation operators describe a multicomponent quantum system. A special choice of the operator-valued function C_{xy} in the case $d=2$ determines non-Abelian anyons (also called plektons). For a multicomponent system, we describe its T-deformed Fock space and the available commutation relations satisfied by the corresponding creation and annihilation operators. Finally, we consider several examples of multicomponent quantum systems.

Keywords: Deformed commutation relations; deformed Fock space; multicomponent quantum system; non-Abelian anyons (plektons)

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1 Introduction

This paper deals with the deformations of the canonical commutation/anticommutation relations that describe multicomponent quantum systems.

The first rigorous construction of a deformation of the canonical (bosonic) commutation relations (CCR) and the canonical (fermionic) anticommutation relations (CAR) was given by Bożejko and Speicher [9], see also Fivel [13, 14], Greenberg [20], Zagier [49]. Let H be a separable Hilbert space and let $q \in (-1, 1)$. On a q-deformed Fock space $\mathcal{F}(H)$ over H, Bożejko and Speicher [9] constructed creation and annihilation operators $a^+(f)$ and $a^-(f) := (a^+(f))^*$, respectively, for $f \in H$, that satisfy the q-commutation relations:

$$a^-(f)a^+(g) = qa^+(g)a^-(f) + (f, g)_H, \quad f, g \in H. \tag{1}$$

Observe that the limiting values $q = 1$ and $q = -1$ correspond to the CCR and CAR, respectively. In this case, one additionally has the *creation-creation* and *annihilation-annihilation* commutation relations

$$\begin{aligned} &a^+(f)a^+(g) = qa^+(g)a^+(f), \\ &a^-(f)a^-(g) = qa^-(g)a^-(f), \quad f, g \in H,\ q = \pm 1. \end{aligned} \tag{2}$$

respectively.

The operators $a^+(f)$, $a^-(f)$ $(f \in H)$ from [9] form the Fock representation of the commutation relation (1). This means that there exists a vacuum vector $\Omega \in \mathcal{F}(H)$ that is cyclic for the operators $a^+(f)$ $(f \in H)$ and satisfies

$$a^-(f)\Omega = 0 \quad \text{for all } f \in H. \tag{3}$$

In fact, formulas (1) and (3) and the condition of cyclicity of Ω uniquely identify the inner product on $\mathcal{F}(H)$. More precisely, the q-deformed Fock space has the form $\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(H)$ and the inner product on each n-particle space $\mathcal{F}_n(H)$ is determined by a bounded linear operator $\mathcal{P}_n$ on $H^{\otimes n}$, depending on q. So one of the main achievements of [9] was the proof of the positivity of the operators $\mathcal{P}_n$ on $H^{\otimes n}$. Unlike the case of CCR and CAR, for $q \in (-1, 1)$ the kernel of $\mathcal{P}_n$ contains only zero, and so $\mathcal{F}_n(H)$ coincides as a set with $H^{\otimes n}$. This implies the absence of creation-creation and annihilation-annihilation commutation relations, compare with (2). Note also that the creation and annihilation operators are bounded in the case $q \in [-1, 1)$.

For studies of the C^*-algebras generated by the q-commutation relations, see e.g. [12, 22, 25]. The related von Neumann algebras were studied e.g. in [37, 40, 42, 43]. The case $q = 0$ corresponds to the creation and annihilation operators acting on the full Fock space; these operators are particularly important for models of free probability, see e.g. [2, 5, 35]. Various aspects of noncommutative probability related to the general q-commutation relations (1) were discussed e.g. in [1, 4, 9, 11].

An important generalization of the main result of [9] was obtained in [10]. Let T be a self-adjoint bounded linear operator on $H^{\otimes 2}$ with norm ≤ 1, and assume that T satisfies the Yang–Baxter equation on $H^{\otimes 3}$, see formula (13) below. Then, similarly to the q case, Bożejko and Speicher [10] defined a T-deformed Fock space $\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(H)$. To this end, they showed that, for each $n \in \mathbb{N}$, the corresponding operator $\mathcal{P}_n$ on $H^{\otimes n}$, depending on T, is positive. Furthermore, in the case $\|T\| < 1$, the kernel of $\mathcal{P}_n$ contains only zero, and so $\mathcal{F}_n(H)$ coincides as a set with $H^{\otimes n}$. If the operator T is given by $Tf \otimes g = qg \otimes f$ for $f, g \in H$, then one recovers the q-deformed Fock space from [9].

By using the T-deformed Fock space, Bożejko and Speicher [10] constructed a Fock representation of the following discrete commutation relations between creation operators $\partial_i^\dagger$ and annihilation operators ∂_i:

$$\partial_i \partial_j^\dagger = \sum_{k,l} T_{jl}^{ik}\, \partial_k^\dagger \partial_l + \delta_{i,j}, \quad i, j \in \mathbb{N}. \tag{4}$$

Here $(T_{ij}^{kl})_{i,j,k,l}$ is the matrix of the operator T in a fixed orthonormal basis[1]. In particular, for complex q_{ij} with $\overline{q_{ij}} = q_{ji}$ and $\sup_{i,j} |q_{ij}| \le 1$, one obtains the Fock representation of the q_{ij}-commutation relations:

$$\partial_i \partial_j^\dagger = q_{ij} \partial_j^\dagger \partial_i + \delta_{ij}, \tag{5}$$

see also [44].

Jørgensen, Schmitt and Werner [23] found sufficient conditions for the existence of the Fock representation of the commutation relations (4) without requiring T to satisfy the Yang–Baxter equation. For further results related to the commutation relations (4) or (5), see e.g. [26, 27, 30, 33, 36]. In the case $\|T\| = 1$, Jørgensen, Proskurin, and Samoĭlenko [21] found, for $n \ge 2$, the kernel of the operator $\mathcal{P}_n$ that determines the inner product on $\mathcal{F}_n(H)$.

Liguori and Mintchev [29] constructed the Fock representation of quantum fields with generalized statistics. Let $H = L^2(X)$, the complex L^2-space on $X := \mathbb{R}^d$. Fix a function $Q : X^2 \to \mathbb{C}$ satisfying $Q(x, y) = \overline{Q(y, x)}$ and $|Q(x, y)| = 1$. Then the Fock representation of the corresponding generalized statistics is the family of the creation and annihilation operators on the T-deformed Fock space with the operator T on $H^{\otimes 2} = L^2(X^2)$ given by

$$(Tf^{(2)})(x, y) = Q(x, y) f^{(2)}(y, x), \quad f^{(2)} \in H^{\otimes 2}. \tag{6}$$

Let us formally define creation operators $a^+(x)$ and annihilation operators $a^-(x)$ at points $x \in X$ that satisfy

$$a^+(f) = \int_X f(x) a^+(x)\, dx, \quad a^-(f) = \int_X \overline{f(x)}\, a^-(x)\, dx, \quad f \in H.$$

[1] Note, however, that the question of convergence of the series on the right-hand side of formula (4) was not discussed in [10]. So formula (4) was rigorously proved in [10] only in the case where, for any fixed i, j, only a finite number of T_{jl}^{ik} are not equal to zero

It is shown in [29] that these operators satisfy the Q-commutation relations

$$a^-(x)a^+(y) = Q(x,y)a^+(y)a^-(x) + \delta(x-y) \tag{7}$$

and

$$a^+(x)a^+(y) = Q(y,x)a^+(y)a^+(x), \quad a^-(x)a^-(y) = Q(y,x)a^-(y)a^-(x) \quad (x \neq y), \tag{8}$$

the formulas making rigorous sense after smearing with a function $f(x)g(y) \in H^{\otimes 2}$. Note that, in this construction, the function Q may be defined only for a.a. $(x,y) \in X^2$.

In physics, generalized (intermediate) statistics have been discussed since Leinass and Myrheim [28] conjectured their existence. The first mathematically rigorous prediction of intermediate statistics was done by Goldin, Menikoff and Sharp [16,17]. The name *anyon* was given to such statistics by Wilczek [47, 48]. Anyon statistics were used, in particular, to describe the quantum Hall effect, see e.g. [45].

Fix $q \in \mathbb{C}$ with $|q| = 1$. Define a function $Q : X^2 \to \mathbb{C}$ by

$$Q(x,y) := \begin{cases} q & \text{if } x^1 < y^1, \\ \bar{q}, & \text{if } x^1 > y^1, \end{cases} \tag{9}$$

where x^1 denotes the first coordinate of x. As shown by Goldin and Sharp [19], Goldin and Majid [15], Liguori and Mintchev [29], for $d = 2$, the corresponding commutation relations (7), (8) describe anyons—particles associated with one-dimensional unitary representations of the braid group.

Aspects of noncommutative probability related to anyons were discussed in [6, 7]. Lytvynov [31] constructed a class of non-Fock representations of the anyon commutation relations for which the corresponding vacuum state is gauge-invariant quasi-free.

Note that, for any generalized statistics, the operator T given by (6) is unitary. In fact, for any operator T that is additionally unitary, the corresponding operator $\mathcal{P}_n$ on $H^{\otimes n}$ is a multiple of an orthogonal projection. See Bożejko [3] for a much weaker condition on T that is sufficient for each operator $\mathcal{P}_n$ to be a multiple of an orthogonal projection.

Bożejko, Lytvynov and Wysoczański [8] discussed Fock representations of the deformed commutation relations in the case where the operator T is given by formula (6) in which the function Q satisfies $Q(x,y) = \overline{Q(y,x)}$ and $|Q(x,y)| \leq 1$. In this work, the n-particle subspaces $\mathcal{F}_n(H)$ were described explicitly, and it was proved that the corresponding creation and annihilation operators satisfy the commutation relation (7). Moreover, the creation-creation and annihilation-annihilation commutation relations (8) hold for $x \neq y$ such that $|Q(x,y)| = 1$:

$$\begin{aligned} &a^+(x)a^+(y) = Q(y,x)a^+(y)a^+(x), \\ &a^-(x)a^-(y) = Q(y,x)a^-(y)a^-(x) \quad \text{if } x \neq y \text{ and } |Q(x,y)| = 1. \end{aligned} \tag{10}$$

In the present paper, by a multicomponent quantum system we understand a family of creation and annihilation operators $a^+(f)$, $a^-(f)$ on a T-deformed Fock space $\mathcal{F}(H)$, where f belongs to $H = L^2(X \to V)$, the L^2-space of V-valued functions on X. Here $V := \mathbb{C}^m$ with $m \geq 2$. Furthermore, we assume that the operator T acting on $H^{\otimes 2} = L^2(X^2 \to V^{\otimes 2})$ is given by

$$(Tf^{(2)})(x, y) := C_{x,y} f^{(2)}(y, x), \quad f^{(2)} \in H^{\otimes 2}. \tag{11}$$

Here $C_{x,y}$ is a linear operator on $V^{\otimes 2}$ with norm ≤ 1, which is defined for a.a. $(x, y) \in X^2$ and satisfies the symmetry relation $C^*_{x,y} = C_{y,x}$ together with the spectral quantum Yang–Baxter equation, see formula (48) below. Under the assumption that, for a.a. $(x, y) \in X^2$, $C_{x,y}$ is a unitary operator on $V^{\otimes 2}$ (or, equivalently, T is a unitary operator on $H^{\otimes 2}$), the multicomponent quantum systems were discussed in [29], see also the references therein.

A multicomponent counterpart of an anyon system was originally called *plektons*, see e.g. [15]. The first publication pointing out the possibility of such a quantum system was the comment by Menikoff, Sharp, and Goldin [18]. Plektons are quasiparticles in dimension $d = 2$ that are associated with higher-dimensional (non-Abelian) unitary representations of the braid group. In view of this, more recently these quasiparticles have been mostly called *non-Abelian anyons*, the term that will be used in the present paper. Non-Abelian anyons form a central tool in topological quantum computation, see e.g. [38, 46].

According to [15], a non-Abelian anyon system is determined by a unitary operator C on $V^{\otimes 2}$, which defines $C_{x,y}$ in (11) via the formula

$$C_{xy} := \begin{cases} C, & \text{if } x^1 < y^1, \\ C^*, & \text{if } x^1 > y^1, \end{cases} \tag{12}$$

compare with (9). The operator T satisfies the Yang–Baxter equation on $H^{\otimes 3}$ if and only if the operator C satisfies the Yang–Baxter equation on $V^{\otimes 3}$, see Lemma 4.4 below. In the latter case, the operator C determines, for each $n \geq 2$, a (non-Abelian) unitary representation of the braid group B_n.

The paper is organized as follows. In Section 2, we review and extend the results of [10,21] regarding the general deformed commutation relations governed by a bounded linear operator T satisfying the assumptions of the paper [10]. Our man results in this section are as follows.

(i) In the case $\|T\| = 1$, we clarify the structure of the n-particle subspaces $\mathcal{F}_n(H)$ of the T-deformed Fock space $\mathcal{F}(H)$ (Theorem 2.2 and Corollary 2.4). Furthermore, we show that the orthogonal projection $\mathbb{P}_n$ of $H^{\otimes n}$ onto its subspace $\mathcal{F}_n(H)$ can be represented, for $n \geq 3$, as (a multiple of) the parallel sum of two explicitly given orthogonal projections, built with the help of $\mathbb{P}_2$ (Proposition 2.1).

(ii) We find all possible commutation relations between the operators $a^{\pm}(f)$ and $a^{\pm}(g)$ (Theorem 2.8).

Note that previously the commutation relations between two creation operators and between two annihilation operators have only been found in the case where the operator T is given by formula (6), see [8, 15, 29].

In Section 3, we consider the general multicomponent quantum systems. We apply the results of Section 2 to the case where the operator T is given by formula (11). The main results of this section—Theorems 3.3, 3.11 and Corollaries 3.13, 3.14—describe the corresponding T-deformed Fock space and the available commutation relations between the creation/annihilation operators. In particular, we find a multicomponent counterpart of the commutation relations (7), (10).

Finally, in Section 4, we consider several examples of multicomponent quantum systems. These include examples when the operator-valued function $C_{x,y}$ in formula (11) is constant, i.e., $C_{x,y} = C$ for all x, y, examples of non-Abelian anyon quantum systems and other. In these examples, we give explicit description of the corresponding Fock space $\mathcal{F}(H)$ and the orthogonal projection $\mathbb{P}_2$ of $H^{\otimes n}$ onto $\mathcal{F}_2(H)$, and calculate the available commutation relations.

2 General T-deformed commutation relations

2.1 T-deformed tensor power of a Hilbert space

For a Hilbert space $\mathcal{H}$, let $\mathcal{L}(\mathcal{H})$ denote the space of all bounded linear operators on $\mathcal{H}$. We will denote by $\mathbf{1}_{\mathcal{H}}$ the identity operator on $\mathcal{H}$. However, where the Hilbert space in consideration is clear from the context, we will just use $\mathbf{1}$ for the identity operator on this space.

Let H be a separable complex Hilbert space, and let $T \in \mathcal{L}(H^{\otimes 2})$. We assume that T is self-adjoint, $\|T\| \leq 1$, and T satisfies the Yang–Baxter equation on $H^{\otimes 3}$:

$$(T \otimes \mathbf{1}_H)(\mathbf{1}_H \otimes T)(T \otimes \mathbf{1}_H) = (\mathbf{1}_H \otimes T)(T \otimes \mathbf{1}_H)(\mathbf{1}_H \otimes T). \tag{13}$$

For $i \in \mathbb{N}$, we denote by T_i the operator on $H^{\otimes n}$ with $n \geq i + 1$ given by

$$T_i := \mathbf{1}_{H^{\otimes(i-1)}} \otimes T \otimes \mathbf{1}_{H^{\otimes(n-i-1)}}.$$

Let S_n denote the symmetric group of degree n. We represent a permutation $\sigma \in S_n$ as an arbitrary product of adjacent transpositions,

$$\sigma = \sigma_{j_1} \cdots \sigma_{j_m}, \tag{14}$$

where $\sigma_j := (j, j + 1) \in S_n$ for $1 \leq j \leq n - 1$. A permutation $\sigma \in S_n$ can be represented (not in a unique way, in general) as a reduced product of a minimal number of adjacent transpositions, i.e., in the form (14) with a minimal m. Then the mapping $\sigma_k \mapsto T_{\sigma_k} := T_k \in \mathcal{L}(H^{\otimes n})$ can be multiplicatively extended to S_n by setting

$$S_n \ni \sigma \mapsto T_\sigma := T_{j_1} \cdots T_{j_m}. \tag{15}$$

(For the identity permutation $e \in S_n$, $T_e := \mathbf{1}$.) Although representation (14) of $\sigma \in S_n$ in a reduced form is not unique, formula (13) implies that the extension (15) is well-defined, i.e., it does not depend on the representation of σ.

For each $n \geq 2$, we define $\mathcal{P}_n \in \mathcal{L}(H^{\otimes n})$ by

$$\mathcal{P}_n := \sum_{\sigma \in S_n} T_\sigma.$$

By [10], the operator $\mathcal{P}_n$ is positive, i.e., $\mathcal{P}_n \geq 0$, and in the case $\|T\| < 1$, it is strictly positive. We denote

$$\mathcal{F}_n(H) := \ker(\mathcal{P}_n)^\perp = \overline{\operatorname{ran}(\mathcal{P}_n)}, \tag{16}$$

i.e., the orthogonal compliment of the kernel of $\mathcal{P}_n$ in $H^{\otimes n}$, or equivalently the closure of the range of $\mathcal{P}_n$. As easily seen, the operator $\mathcal{P}_n$ is strictly positive on $\mathcal{F}_n(H)$, so one can introduce a new inner product on $\mathcal{F}_n(H)$ by

$$(f^{(n)}, g^{(n)})_{\mathcal{F}_n(H)} := (\mathcal{P}_n f^{(n)}, g^{(n)})_{H^{\otimes n}}, \quad f^{(n)}, g^{(n)} \in \mathcal{F}_n(H),$$

which makes $\mathcal{F}_n(H)$ a Hilbert space. Note that, if $T = \mathbf{0}$, the Hilbert spaces $\mathcal{F}_n(H)$ and $H^{\otimes n}$ coincide. Thus, a non-zero operator T leads to a deformation of the Hilbert space $H^{\otimes n}$.

Let $\mathbb{P}_n$ denote the orthogonal projection of the Hilbert space $H^{\otimes n}$ onto its subspace $\mathcal{F}_n(H)$.

Assume in addition that the operator T is unitary. Then mapping (15) determines a unitary representation of S_n, hence in formula (15) σ should not necessarily be in a reduced form. This implies the equality $\mathbb{P}_n = \frac{1}{n!}\mathcal{P}_n$, which does not hold in the general case.

As already mentioned before, in the case $\|T\| < 1$, $H^{\otimes n}$ and $\mathcal{F}_n(H)$ coincide as sets. In the case where $\|T\| = 1$, the following result shown in [21] gives a description of $\ker(\mathcal{P}_n) = \ker(\mathbb{P}_n)$:

$$\ker(\mathcal{P}_n) = \overline{\sum_{i=1}^{n-1} \ker(\mathbf{1} + T_i)}, \tag{17}$$

i.e., the kernel of $\mathcal{P}_n$ is equal to the closure of the linear span of the subspaces $\ker(\mathbf{1}+T_i)$, $i = 1, \dots, n-1$. Note that formula (17) remains true when $\|T\| < 1$, in which case it gives $\ker(\mathcal{P}_n) = \{0\}$. Since $\ker(\mathbf{1} + T_i)^\perp = \overline{\operatorname{ran}(\mathbf{1} + T_i)}$, formulas (16) and (17) imply

$$\mathcal{F}_n(H) = \bigcap_{i=1}^{n-1} \overline{\operatorname{ran}(\mathbf{1} + T_i)}. \tag{18}$$

In view of formula (18), we can now give a representation of the orthogonal projection $\mathbb{P}_n$ onto $\mathcal{F}_n(H)$ by using the notion of a parallel sum of two projections, see e.g. [34]. Let us first recall this notion. Let $\mathcal{H}$ be a complex Hilbert space, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be closed subspaces of $\mathcal{H}$, and let P_1 and P_2 denote the orthogonal projections of $\mathcal{H}$ onto $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The parallel sum of P_1 and P_2, denoted by $(P_1 : P_2)$, is the self-adjoint bounded linear operator on $\mathcal{H}$ defined by its quadratic form

$$((P_1 : P_2)x, x)_{\mathcal{H}} = \inf_{y+z=x} \big((P_1 y, y)_{\mathcal{H}} + (P_2 z, z)_{\mathcal{H}}\big), \quad x \in \mathcal{H}. \tag{19}$$

The right-hand side of (19) is equal to $\frac{1}{2}\|x\|^2_{\mathcal{H}}$ for $x \in \mathcal{H}_1 \cap \mathcal{H}_2$ and equal to zero for $x \in (\mathcal{H}_1 \cap \mathcal{H}_2)^\perp$. Hence, $2(P_1 : P_2)$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_1 \cap \mathcal{H}_2$. Observe that $2(P_1 : P_2) = P_1 P_2$ if and only if P_1 and P_2 commute, or equivalently $\mathcal{H}_1 \perp \mathcal{H}_2$.

Denote $\mathfrak{P} := \mathbb{P}_2$ and analogously to operators T_i define operators $\mathfrak{P}_i$. Then, for $n \geq i+1$, $\mathfrak{P}_i$ is the orthogonal projection of $H^{\otimes n}$ onto $\overline{\operatorname{ran}(\mathbf{1}+T_i)}$.

Proposition 2.1. *Let $n \geq 3$. Define operators Q_1 and Q_2 on $H^{\otimes n}$ by*

$$Q_1 := \prod_{\substack{i \leq n-1 \\ i \text{ odd}}} \mathfrak{P}_i, \quad Q_2 := \prod_{\substack{i \leq n-1 \\ i \text{ even}}} \mathfrak{P}_i.$$

Then the operators Q_1 and Q_2 are orthogonal projections and $\mathbb{P}_n = 2(Q_1 : Q_2)$. Furthermore, for each $m \in \mathbb{N}$, $2 \leq m \leq n-1$,

$$\mathbb{P}_n = \mathbb{P}_n(\mathbb{P}_m \otimes \mathbf{1}_{H^{\otimes(m-n)}}) = \mathbb{P}_n(\mathbf{1}_{H^{\otimes(m-n)}} \otimes \mathbb{P}_m). \tag{20}$$

Proof. Observe that the projections $\mathfrak{P}_i$ with odd (respectively even) i mutually commute. This implies that Q_1 and Q_2 are orthogonal projections onto

$$\bigcap_{\substack{i \leq n-1 \\ i \text{ odd}}} \overline{\operatorname{ran}(\mathbf{1}+T_i)} \quad \text{and} \quad \bigcap_{\substack{i \leq n-1 \\ i \text{ even}}} \overline{\operatorname{ran}(\mathbf{1}+T_i)},$$

respectively. Formula (18) implies $\mathbb{P}_n = 2(Q_1 : Q_2)$.

Let us prove the first equality in (20), the second one being proved similarly. The operator $\mathbb{P}_m \otimes \mathbf{1}_{H^{\otimes(m-n)}}$ is the orthogonal projection of $H^{\otimes n}$ onto the subspace

$$\bigcap_{i=1}^{m-1} \overline{\operatorname{ran}(\mathbf{1}+T_i)}.$$

But $\mathcal{F}_n(H)$ is a subspace of this space (see (19)), which implies the statement. $\square$

Theorem 2.2. *For each $n \geq 2$, we have*

$$\mathcal{F}_n(H) = \big\{ f^{(n)} \in H^{\otimes n} \mid (\mathbf{1}-T_i) f^{(n)} \in \overline{\operatorname{ran}(\mathbf{1}-T_i^2)},\ i = 1, 2, \dots, n-1 \big\}. \tag{21}$$

Proof. Note that, when $\|T\| < 1$, formula (21) just states the known equality $\mathcal{F}_n(H) = H^{\otimes n}$. So we only need to prove formula (21) in the case $\|T\| = 1$. We start with the following lemma.

Lemma 2.3. *The kernel of the operator* $\mathbf{1} + T$ *has the following representation:*

$$\ker(\mathbf{1} + T) = (\mathbf{1} - T)\ker(\mathbf{1} - T^2). \tag{22}$$

Proof. If $f^{(2)} \in \ker(\mathbf{1} - T^2)$, then $(\mathbf{1} - T)f^{(2)} \in \ker(\mathbf{1} + T)$, which implies the inclusion

$$(\mathbf{1} - T)\ker(\mathbf{1} - T^2) \subset \ker(\mathbf{1} + T).$$

To prove the converse inclusion, take any $f^{(2)} \in \ker(\mathbf{1} + T)$ (i.e., $f^{(2)} = -Tf^{(2)}$). Then $\frac{1}{2} f^{(2)} \in \ker(\mathbf{1} - T^2)$ and

$$(\mathbf{1} - T)\tfrac{1}{2} f^{(2)} = \tfrac{1}{2} f^{(2)} + \tfrac{1}{2} f^{(2)} = f^{(2)}.$$

Thus, formula (22) is shown. □

In the case $n = 2$, formula (21) states

$$\mathcal{F}_2(H) = \left\{ f^{(2)} \in H^{\otimes 2} \mid (\mathbf{1} - T)f^{(2)} \in \overline{\operatorname{ran}(\mathbf{1} - T^2)} \right\}. \tag{23}$$

Let us now prove this formula. Observe that

$$H^{\otimes 2} = \ker(\mathbf{1} + T) \oplus \ker(\mathbf{1} - T) \oplus \overline{\operatorname{ran}(\mathbf{1} - T^2)}. \tag{24}$$

Thus, each $f^{(2)} \in H^{\otimes 2}$ can be represented as $f^{(2)} = f_1^{(2)} + f_2^{(2)} + f_3^{(2)}$, where $f_1^{(2)} \in \ker(\mathbf{1} + T)$, $f_2^{(2)} \in \ker(\mathbf{1} - T)$, $f_3^{(2)} \in \overline{\operatorname{ran}(\mathbf{1} - T^2)}$, and $f^{(2)} \in \mathcal{F}_2(H)$ if and only if $f_1^{(2)} = 0$. Note that the subspaces $\ker(\mathbf{1}+T)$, $\ker(\mathbf{1}-T)$, and $\overline{\operatorname{ran}(\mathbf{1} - T^2)}$ are invariant for the operator T, hence also for the operator $\mathbf{1} - T$. Therefore, for $f^{(2)} \in H^{\otimes 2}$, we get

$$(\mathbf{1} - T)f^{(2)} = (\mathbf{1} - T)f_1^{(2)} + (\mathbf{1} - T)f_3^{(2)}$$

with $(\mathbf{1} - T)f_1^{(2)} \in \ker(\mathbf{1} - T)$ and $(\mathbf{1} - T)f_3^{(2)} \in \overline{\operatorname{ran}(\mathbf{1} - T^2)}$. Hence, condition

$$(\mathbf{1} - T)f^{(2)} \in \overline{\operatorname{ran}(\mathbf{1} - T^2)} \tag{25}$$

is satisfied if and only if $(\mathbf{1} - T)f_1^{(2)} = 0$. But since $f_1^{(2)} \in \ker(\mathbf{1} + T)$, we have $(\mathbf{1} - T)f_1^{(2)} = 0$ if and only if $f_1^{(2)} = 0$. Thus, formula (23) is proved.

For $n \geq 3$, formula (21) follows from (18) and (23). □

Corollary 2.4. *Assume additionally that the operator* T *is unitary. Then, for each* $n \geq 2$,

$$\mathcal{F}_n(H) = \left\{ f^{(n)} \in H^{\otimes n} \mid T_i f^{(n)} = f^{(n)},\ i = 1, 2, \dots, n-1 \right\}.$$

Proof. Since T is both self-adjoint and unitary, we have $T^{-1} = T$. Hence,

$$\operatorname{ran}(\mathbf{1} - T^2) = \{0\}.$$

Now the statement follows from Theorem 2.2. $\square$

Remark 2.5. In view of Corollary 2.4, in the case where T is additionally unitary, we can interpret $\mathcal{F}_n(H)$ as the nth T-symmetric tensor power of H.

2.2 Creation and annihilation operators on the full Fock space

We will now recall and extend the construction of creation and annihilation operators acting on the full Fock space, compare with e.g. [35, Lecture 7].

Let $H_{\mathbb{R}}$ be a real separable Hilbert space, and let H denote the complex Hilbert space constructed as the complexification of $H_{\mathbb{R}}$. More precisely, elements of H are of the form $f_1 + if_2$ with $f_1, f_2 \in H_{\mathbb{R}}$. For $f = f_1 + if_2$, $g = g_1 + ig_2 \in H$, we denote

$$\langle f, g\rangle := (f_1, f_2)_{H_{\mathbb{R}}} - (g_1, g_2)_{H_{\mathbb{R}}} + i\big((f_1, g_2)_{H_{\mathbb{R}}} + (f_2, g_1)_{H_{\mathbb{R}}}\big),$$

i.e., $\langle \cdot, \cdot\rangle$ is the extension of the inner product on $H_{\mathbb{R}}$ by linearity to $H \times H$. Then the inner product on H is given by $(f, g)_H := \langle f, Jg\rangle$, where

$$Jg = J(g_1 + ig_2) := g_1 - ig_2 \tag{26}$$

is the complex conjugation on the space H considered as the complexification of $H_{\mathbb{R}}$.

Let $\Gamma(H)$ denote the full Fock space over H:

$$\Gamma(H) := \bigoplus_{n=0}^{\infty} H^{\otimes n}.$$

Here $H^0 := \mathbb{C}$. The vector $\Omega := (1, 0, 0, 0, \dots) \in \Gamma(H)$ is called the vacuum.

For each $f \in H$, we denote by $l^+(f)$ the operator of left creation by f. This is the bounded linear operator on $\Gamma(H)$ satisfying $l^+(f)\Omega = f$ and $l^+(f)g^{(n)} = f \otimes g^{(n)}$ for $g^{(n)} \in H^{\otimes n}$, $n \in \mathbb{N}$. Note that $\|l^+(f)\| = \|f\|$. The operator of left annihilation by f is defined by

$$l^-(f) := l^+(Jf)^*.$$

This operator satisfies

$$\begin{aligned}
&l^-(f)\Omega = 0,\\
&l^-(f)g_1 \otimes g_2 \otimes \dots \otimes g_n = \langle f, g_1\rangle g_2 \otimes \dots \otimes g_n, \quad g_1, g_2, \dots, g_n \in H,\ n \in \mathbb{N}.
\end{aligned}$$

For $f_1, f_2, g \in H$, we denote $\langle f_1 \otimes f_2, g\rangle_2 := \langle f_2, g\rangle f_1$. As easily seen, $\langle f_1 \otimes f_2, \cdot\rangle_2$ determines a Hilbert–Schmidt operator on H. Extending this definition by linearity and continuity, we define, for an arbitrary $f^{(2)} \in H^{\otimes 2}$, a Hilbert–Schmidt operator $\langle f^{(2)}, \cdot \rangle_2$ on H with Hilbert–Schmidt norm $\|f^{(2)}\|_{H^{\otimes 2}}$.

For $f_1, f_2 \in H$ and $g^{(n)} \in H^{\otimes n}$, we have

$$l^+(f_1) l^-(f_2) g^{(n)} = \big(\langle f_1 \otimes f_2, \cdot \rangle_2\big) \otimes \mathbf{1}_{H^{\otimes(n-1)}} g^{(n)}. \tag{27}$$

Indeed, choosing $g^{(n)} = g_1 \otimes g_2 \otimes \cdots \otimes g_n$ with $g_1, \dots, g_n \in H$, we get

$$l^+(f_1) l^-(f_2) g^{(n)} = \langle f_2, g_1 \rangle f_1 \otimes g_2 \otimes \cdots \otimes g_n = \big(\langle f_1 \otimes f_2, g_1 \rangle_2\big) \otimes g_2 \otimes \cdots \otimes g_n,$$

which implies (27). In view of (27), for each $f^{(2)} \in H^{\otimes 2}$, we can define a bounded linear operator $l^{+-}(f^{(2)})$ on $\Gamma(H)$ by

$$\begin{aligned} &l^{+-}(f^{(2)})\Omega := 0, \\ &l^{+-}(f^{(2)}) g^{(n)} := \big(\langle f^{(2)}, \cdot \rangle_2\big) \otimes \mathbf{1}_{H^{\otimes(n-1)}}, \quad g^{(n)} \in H^{\otimes n},\ n \in \mathbb{N}. \end{aligned}$$

Let $(e_i)_{i \geq 1}$ be an orthonormal basis of $H_{\mathbb{R}}$, hence also an orthonormal basis of H. Then, for each $f^{(2)} \in H^{\otimes 2}$, we easily see that

$$l^{+-}(f^{(2)}) = \sum_{i,j} \langle f^{(2)}, e_i \otimes e_j \rangle l^+(e_i) l^-(e_j),$$

where the series converges in the operator norm. Here and below, for $f^{(2)}, g^{(2)} \in H^{\otimes 2}$, we use the notation $\langle f^{(2)}, g^{(2)} \rangle := (f^{(2)}, J g^{(2)})_{H^{\otimes 2}}$, where J denotes the complex conjugation on $H^{\otimes 2}$, cf. (26).

Similarly, for each $f^{(2)} \in H^{\otimes 2}$, we define bonded linear operators $l^{++}(f^{(2)})$ and $l^{--}(f^{(2)})$ on $\Gamma(H)$ that satisfy

$$\begin{aligned} l^{++}(f^{(2)}) &= \sum_{i,j} \langle f^{(2)}, e_i \otimes e_j \rangle l^+(e_i) l^+(e_j), \\ l^{--}(f^{(2)}) &= \sum_{i,j} \langle f^{(2)}, e_i \otimes e_j \rangle l^-(e_i) l^-(e_j), \end{aligned}$$

the series converging in the operator norm. Note that $l^{++}(f^{(2)})^* = l^{--}(\mathbb{S} f^{(2)})$, where $\mathbb{S}$ denotes the continuous antilinear operator on $H^{\otimes 2}$ satisfying

$$\mathbb{S} f \otimes g := J(g \otimes f), \quad f, g \in H.$$

2.3 Creation and annihilation operators on the T-deformed Fock space

Let an operator T and a Hilbert space H be as in Subsection 2.1 and 2.2, respectively. We define the T-deformed Fock space over H by

$$\mathcal{F}(H) := \bigoplus_{n=0}^{\infty} \mathcal{F}_n(H).$$

Here $\mathcal{F}_0(H) := \mathbb{C}$ and the vector $\Omega = (1, 0, 0, 0, \dots)$ is still called the vacuum. Note that the full Fock space $\Gamma(H)$ is the special case of $\mathcal{F}(H)$ for $T = \mathbf{0}$.

Let $\mathcal{F}_{\mathrm{fin}}(H)$ denote the subspace of $\mathcal{F}(H)$ that consists of all finite sequences $f = (f^{(0)}, f^{(1)}, \dots, f^{(n)}, 0, 0, \dots)$ with $f^{(i)} \in \mathcal{F}_i(H)$. We equip $\mathcal{F}_{\mathrm{fin}}(H)$ with the topology of the topological direct sum of the $\mathcal{F}^{(n)}(H)$ spaces. Thus, convergence of a sequence in $\mathcal{F}_{\mathrm{fin}}(H)$ means uniform finiteness and coordinate-wise convergence of nonzero coordinates. We denote by $\mathcal{L}(\mathcal{F}_{\mathrm{fin}}(H))$ the space of all continuous linear operators on $\mathcal{F}_{\mathrm{fin}}(H)$.

For $f \in H$, we define a creation operator $a^+(f)$ as the linear operator on $\mathcal{F}_{\mathrm{fin}}(H)$ given by

$$\begin{aligned} &a^+(f)\Omega := f, \\ &a^+(f)g^{(n)} := \mathbb{P}_{n+1} l^+(f) g^{(n)} = \mathbb{P}_{n+1}(f \otimes g^{(n)}), \quad g^{(n)} \in \mathcal{F}_n(H),\ n \in \mathbb{N}. \end{aligned}$$

Note that formula (20) implies that

$$a^+(f)\mathbb{P}_n g^{(n)} = \mathbb{P}_{n+1} l^+(f) g^{(n)}, \quad g^{(n)} \in H^{\otimes n},\ n \geq 2. \tag{28}$$

Next, for $f \in H$, we define an annihilation operator $a^-(f)$ on $\mathcal{F}_{\mathrm{fin}}(H)$ by

$$a^-(f) := a^+(Jf)^* \restriction \mathcal{F}_{\mathrm{fin}}(H).$$

By [10], one has the following explicit formula for the action of $a^-(f)$:

$$a^-(f)g^{(n)} = \mathbb{P}_{n-1} l^-(f) \mathbb{T}_n g^{(n)}, \quad g^{(n)} \in \mathcal{F}_n(H), \tag{29}$$

where $\mathbb{T}_n \in \mathcal{L}(H^{\otimes n})$ is defined by

$$\mathbb{T}_n := \mathbf{1} + T_1 + T_1T_2 + \dots + T_1T_2\cdots T_{n-1}. \tag{30}$$

Proposition 2.6. *For each $f \in H$, $a^+(f), a^-(f) \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(H))$.*

Proof. It is sufficient to prove that, for each $n \in \mathbb{N}$, the operators $a^+(f) : \mathcal{F}_n(H) \to \mathcal{F}_{n+1}(H)$ and $a^-(f) : \mathcal{F}_{n+1}(H) \to \mathcal{F}_n(H)$ are bounded. Since $a^-(Jf)$ is the adjoint of $a^+(f)$, both operators $a^+(f)$ and $a^-(f)$ are closed. Now the statement follows from the closed graph theorem. □

By analogy with Subsection 2.2, we will now introduce, for each $f^{(2)} \in H^{\otimes 2}$, operators $a^{+-}(f^{(2)})$, $a^{++}(f^{(2)})$, and $a^{--}(f^{(2)})$ on $\mathcal{F}_{\mathrm{fin}}(H)$. For any $f_1, f_2 \in H$ and $g^{(n)} \in \mathcal{F}_n(H)$ with $n \geq 1$, we get, by (28) and (29),

$$a^+(f_1)a^-(f_2)g^{(n)} = \mathbb{P}_n l^{+-}(f_1 \otimes f_2) \mathbb{T}_n g^{(n)}.$$

Hence, for each $f^{(2)} \in H^{\otimes 2}$, we define a linear operator $a^{+-}(f^{(2)})$ on $\mathcal{F}_{\mathrm{fin}}(H)$ by setting

$$a^{+-}(f^{(2)})\Omega := 0, \quad a^{+-}(f^{(2)})g^{(n)} := \mathbb{P}_n l^{+-}(f^{(2)}) \mathbb{T}_n g^{(n)}, \quad g^{(n)} \in \mathcal{F}_n(H),\ n \in \mathbb{N}.$$

Note that, for a fixed $G \in \mathcal{F}_{\mathrm{fin}}(H)$, the mapping

$$H^{\otimes 2} \ni f^{(2)} \mapsto a^{+-}(f^{(2)})G \in \mathcal{F}(H) \tag{31}$$

is continuous.

Let a sequence $(A_n)_{n=1}^{\infty}$ and an operator A be from $\mathcal{L}(\mathcal{F}_{\mathrm{fin}}(H))$. As usual, we will say that A_n strongly converges to A on $\mathcal{F}_{\mathrm{fin}}(H)$ if for each fixed $G \in \mathcal{F}_{\mathrm{fin}}(H)$, we have $\lim_{n\to\infty} A_n G = AG$ in the topology of $\mathcal{F}_{\mathrm{fin}}(H)$.

Then the continuity of the mapping (31) implies the decomposition

$$a^{+-}(f^{(2)}) = \sum_{i,j} \langle f^{(2)}, e_i \otimes e_j \rangle a^+(e_i) a^-(e_j),$$

where the series strongly converges on $\mathcal{F}_{\mathrm{fin}}(H)$. This also immediately implies

$$a^{+-}(f^{(2)})^* = a^{+-}(\mathbb{S} f^{(2)}). \tag{32}$$

Similarly, for each $f^{(2)} \in H^{\otimes 2}$, we define a linear operator $a^{++}(f^{(2)})$ on $\mathcal{F}_{\mathrm{fin}}(H)$ by

$$a^{++}(f^{(2)})\Omega := \mathbb{P}_2 f^{(2)}, \quad a^{++}(f^{(2)}) g^{(n)} = \mathbb{P}_{n+2}(l^{++}(f^{(2)}) g^{(n)}), \quad g^{(n)} \in \mathcal{F}_n(H),\ n \in \mathbb{N}. \tag{33}$$

Finally, to construct an operator $a^{--}(f^{(2)})$, we proceed as follows. For $f_1, f_2 \in H$ and $g^{(n)} \in \mathcal{F}_n(N)$, $n \geq 2$, we get

$$\begin{aligned}
a^-(f_1)a^-(f_2)g^{(n)} &= a^-(f_1)\mathbb{P}_{n-1} l^-(f_2)\mathbb{T}_n g^{(n)} \\
&= a^-(f_1)(l^-(f_2) \restriction_H \otimes \mathbb{P}_{n-1})\mathbb{T}_n g^{(n)} \\
&= \mathbb{P}_{n-2} l^-(f_1)\mathbb{T}_{n-1}(l^-(f_2) \restriction_H \otimes \mathbb{P}_{n-1})\mathbb{T}_n g^{(n)} \\
&= \mathbb{P}_{n-2} l^-(f_1)(l^-(f_2) \restriction_H \otimes (\mathbb{T}_{n-1}\mathbb{P}_{n-1}))\mathbb{T}_n g^{(n)} \\
&= \mathbb{P}_{n-2} l^-(f_1) l^-(f_2)(\mathbf{1}_H \otimes (\mathbb{T}_{n-1}\mathbb{P}_{n-1}))\mathbb{T}_n g^{(n)} \\
&= \mathbb{P}_{n-2} l^{--}(f_1 \otimes f_2)(\mathbf{1}_H \otimes (\mathbb{T}_{n-1}\mathbb{P}_{n-1}))\mathbb{T}_n g^{(n)}.
\end{aligned}$$

Thus, for $f^{(2)} \in H^{\otimes 2}$, we define a linear operator $a^{--}(f^{(2)})$ by

$$\begin{aligned}
&a^{--}(f^{(2)})\Omega := 0, \quad a^{--}(f^{(2)})g := 0, \quad g \in H, \\
&a^{--}(f^{(2)})g^{(n)} := \mathbb{P}_{n-2} l^{--}(f^{(2)})(\mathbf{1}_H \otimes (\mathbb{T}_{n-1}\mathbb{P}_{n-1}))\mathbb{T}_n g^{(n)}, \quad g^{(n)} \in \mathcal{F}_n(H), \quad n \geq 2.
\end{aligned}$$

We easily see that the above statements related to the operator $a^{+-}(f^{(2)})$ remain true (with obvious changes) for $a^{++}(f^{(2)})$ and $a^{--}(f^{(2)})$. In particular,

$$a^{++}(f^{(2)}) = \sum_{i,j} \langle f^{(2)}, e_i \otimes e_j \rangle a^+(e_i) a^+(e_j),$$

$$a^{--}(f^{(2)}) = \sum_{i,j} \langle f^{(2)}, e_i \otimes e_j \rangle a^-(e_i) a^-(e_j),$$

where the series strongly converge on $\mathcal{F}_{\mathrm{fin}}(H)$. Hence,

$$a^{++}(f^{(2)})^* = a^{--}(\mathbb{S}f^{(2)}), \quad f^{(2)} \in H^{\otimes 2}. \tag{34}$$

By using formulas (32) and (34), analogously to the proof of Proposition 2.6, we conclude the following proposition.

Proposition 2.7. *For each $f^{(2)} \in H^{\otimes 2}$, we have*

$$a^{+-}(f^{(2)}),\ a^{++}(f^{(2)}),\ a^{--}(f^{(2)}) \in \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(H)).$$

Assume that there exists an operator $\widetilde{T} \in \mathcal{L}(H^{\otimes 2})$ that satisfies the following identity:

$$\langle T f_1 \otimes f_2, f_3 \otimes f_4 \rangle = \langle \widetilde{T} f_3 \otimes f_1, f_4 \otimes f_2 \rangle, \quad f_1, f_2, f_3, f_4 \in H. \tag{35}$$

Observe that identity (35) does not necessarily identify a bounded linear operator $\widetilde{T}$, but in all known examples $\widetilde{T}$ indeed exists. Note also that if $\widetilde{T}$ exists, then it is obviously unique.

The following theorem states the commutation relations that the creation and annihilation operators satisfy on the T-deformed Fock space.

Theorem 2.8. *For any $f, g \in H$,*

$$a^-(f) a^+(g) = a^{+-}(\widetilde{T} f \otimes g) + \langle f, g \rangle. \tag{36}$$

Further let $f^{(2)} \in H^{\otimes 2}$. Then

$$a^{++}(f^{(2)}) = \mathbf{0} \ \Leftrightarrow\ f^{(2)} \in \ker(\mathbf{1} + T) \tag{37}$$

and

$$a^{--}(f^{(2)}) = \mathbf{0} \ \Leftrightarrow\ \mathbb{S}f^{(2)} \in \ker(\mathbf{1} + T). \tag{38}$$

Moreover, if $f^{(2)} \in \ker(\mathbf{1} - T^2)$, then

$$a^{++}(f^{(2)}) = a^{++}(T f^{(2)}), \tag{39}$$

and if $\mathbb{S}f^{(2)} \in \ker(\mathbf{1} - T^2)$, then

$$a^{--}(f^{(2)}) = a^{--}(\widehat{T} f^{(2)}), \tag{40}$$

where

$$\widehat{T} := \mathbb{S} T \mathbb{S}. \tag{41}$$

Proof. Formula (36) follows from [10] (where it is written through an orthonormal basis in $H^{\otimes 2}$), see also [27] for the definition of the operator $\widetilde{T}$ in the basis-free form.

By (20) and (33), we have

$$a^{++}(f^{(2)}) = a^{++}(\mathbb{P}_2 f^{(2)}).$$

Hence, if $f^{(2)} \in \ker(\mathbf{1}+T) = \ker(\mathbb{P}_2)$, then $a^{++}(f^{(2)}) = \mathbf{0}$, and if $f^{(2)} \notin \ker(\mathbf{1}+T)$, then (33) implies $a^{++}(f^{(2)})\Omega \neq 0$. Thus, (37) holds. Formula (38) follows from (34) and (37).

Formula (39) follows from (37) and (22). Finally, by (22) and (41),

$$\mathbb{S}\ker(\mathbf{1}+T) = \big\{ f^{(2)} - \widehat{T} f^{(2)} \mid \mathbb{S} f^{(2)} \in \ker(\mathbf{1}-T^2) \big\}.$$

Hence, formula (40) follows from (38). □

Remark 2.9. In view of (22) and (37), formulas (39) and (40) describe all possible commutation relations between two creation operators or two annihilation operators, respectively.

Remark 2.10. If the operator T is unitary, then $\ker(\mathbf{1}-T^2) = H^{\otimes 2}$, hence equalities (39), (40) hold for all $f^{(2)} \in H^{\otimes 2}$, in particular, for all $f^{(2)} \in \mathcal{F}_2(H)$.

For $A \in \mathcal{L}(H^{\otimes 2})$, we write

$$A_{ij}^{kl} := \langle A e_i \otimes e_j, e_k \otimes e_l \rangle. \tag{42}$$

Note that

$$\widetilde{T}_{ij}^{kl} = T_{jl}^{ik}, \quad \widehat{T}_{ij}^{kl} = \overline{T_{ji}^{lk}}. \tag{43}$$

The following corollary in immediate.

Corollary 2.11. *We have*

$$a^-(e_i)a^+(e_j) = \sum_{k,l} \widetilde{T}_{ij}^{kl} a^+(e_k) a^-(e_l) + \delta_{ij}, \tag{44}$$

where δ_{ij} is the Kronecker delta. Furthermore, if $e_i \otimes e_j \in \ker(\mathbf{1}-T^2)$, then

$$a^+(e_i)a^+(e_j) = \sum_{k,l} T_{ij}^{kl} a^+(e_k) a^+(e_l), \tag{45}$$

$$a^-(e_j)a^-(e_i) = \sum_{k,l} \widehat{T}_{ji}^{kl} a^-(e_k) a^-(e_l). \tag{46}$$

In formulas (44)–(46), the series converge strongly on $\mathcal{F}_{\mathrm{fin}}(H)$.

2.4 Wick algebras

We finish this section with a short discussion of Wick algebras. Assume that the operator $\widetilde{T}$ has the following property: for any i, j, only a finite number of $\widetilde{T}_{ij}^{kl}$ are not equal to zero. (We will say that the operator $\widetilde{T}$ has a finite matrix.) Let $\mathbf{A}$ denote the complex algebra generated by the operators $a^+(e_i)$, $a^-(e_j)$ $(i, j \geq 1)$ and the identity operator. Then the commutation relation (44) implies that each element of this algebra can be represented as a linear combination of the identity operator and products of creation and annihilation operators in the Wick form:

$$a^+(e_{i_1}) \cdots a^+(e_{i_k}) a^-(e_{j_1}) \cdots a^-(e_{j_l}), \quad k, l \geq 0, \ k + l \geq 1,$$

i.e., all creation operators are on the right and all annihilation operators are on the left. This is why one calls $\mathbf{A}$ a Wick algebra, see e.g. [21, 23, 26, 27].

In the case where the matrix of the operator $\widetilde{T}$ is not finite, one can proceed as follows. First, let us recall that if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, then $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, the space of all bounded linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$, is complete with respect to the strong convergence of bounded linear operators. Furthermore, an immediate consequence of the uniform boundedness principle states that, if $(A_n)_{n=1}^\infty$ and $(B_n)_{n=1}^\infty$ are sequences in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $\lim_{n\to\infty} A_n = A$, $\lim_{n\to\infty} B_n = B$, then $\lim_{n\to\infty} A_n B_n = AB$ (all limits being understood in the strong sense.) These statements immediately imply the following lemma.

Lemma 2.12. *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(H))$. Let $\overline{\mathcal{A}}$ denote the closure of $\mathcal{A}$ with respect to the strong convergence on $\mathcal{F}_{\mathrm{fin}}(H)$. Then $\overline{\mathcal{A}} \subset \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(H))$. Furthermore, if $\mathcal{A}$ is an algebra (with respect to addition and product of operators), then $\overline{\mathcal{A}}$ is also an algebra.*

Define the algebra $\mathbf{A}$ just as in the case where $\widetilde{T}$ had a finite matrix. Let $\mathbf{W}$ denote the subset of $\mathbf{A}$ that consists of all elements of $\mathbf{A}$ that can be represented as a linear combination of the identity operator and products of creation and annihilation operators in the Wick form. (Note that $\mathbf{W}$ is not anymore an algebra.) Let $\overline{\mathbf{A}}$ and $\overline{\mathbf{W}}$ denote the closures of $\mathbf{A}$ and $\mathbf{W}$ with respect to the strong convergence on $\mathcal{F}_{\mathrm{fin}}(H)$. Then, by Lemma 2.12, $\overline{\mathbf{A}} \subset \mathcal{L}(\mathcal{F}_{\mathrm{fin}}(H))$ and $\mathbf{A}$ is an algebra. By formula (44), we get $\overline{\mathbf{A}} = \overline{\mathbf{W}}$. Hence, in this case we may also think of $\mathbf{A}$ as a Wick algebra.

3 Multicomponent quantum systems

We will now discuss the commutation relations for multicomponent quantum systems, in particular, for non-Abelian anyons.

Let $X := \mathbb{R}^d$ $(d \in \mathbb{N})$ and let $V := \mathbb{C}^m$, where $m \in \mathbb{N}$, $m \geq 2$. We choose $H = L^2(X \to V)$, the L^2-space of V-valued functions on X. Here, as a reference measure, we chose the Lebesgue measure dx on the Borel σ-algebra of X. Note that the space H is the complexification of $L^2(X \to \mathbb{R}^m)$, the L^2-space of $\mathbb{R}^m$-valued functions on X. Note also that H can be naturally identified with the tensor product $L^2(X)\otimes V$, where $L^2(X)$ is the L^2-space of complex-valued functions on X.

Let

$$X^{(2)} := \{(x,y) \in X^2 \mid x^1 \neq y^1\}, \tag{47}$$

where x^i denotes the ith coordinate of x. Note that $X^2 \setminus X^{(2)} = \{(x,y) \in X^2 \mid x^1 = y^1\}$ is a set of zero $dx\,dy$-measure. Hence,

$$H^{\otimes 2} = L^2(X^{(2)} \to V^{\otimes 2}) = L^2(X^{(2)}) \otimes V^{\otimes 2}.$$

Similarly, for $n \geq 3$, we have

$$H^{\otimes n} = L^2(X^{(n)} \to V^{\otimes n}) = L^2(X^{(n)}) \otimes V^{\otimes n},$$

where $X^{(n)} := \{(x_1,\dots,x_n) \in X^n \mid x_i^1 \neq x_j^1 \text{ if } i \neq j\}$.

Below, for $(x,y) \in X^{(2)}$, we will write $x < y$ or $x > y$ if $x^1 < y^1$ or $x^1 > y^1$, respectively.

Consider $\mathcal{L}(V^{\otimes 2})$, the space of linear operators on $V^{\otimes 2}$, equivalently $m^2 \times m^2$ matrices with complex entries. Consider a measurable mapping

$$X^{(2)} \ni (x,y) \mapsto C_{x,y} \in \mathcal{L}(V^{\otimes 2})$$

that satisfies the following assumptions: for each $(x,y) \in X^{(2)}$, $\|C_{x,y}\| \leq 1$ and $C^*_{x,y} = C_{y,x}$. Define a linear operator T on $H^{\otimes 2}$ by (11). As easily seen, the operator T is bounded with $\|T\| \leq 1$ and self-adjoint.

Lemma 3.1. *The operator T satisfies the Yang–Baxter equation* (13) *if and only if the following equation holds on $V^{\otimes 3}$ for a.a. $(x,y,z) \in X^{(3)}$:*

$$C^{1,2}_{x,y} C^{2,3}_{x,z} C^{1,2}_{y,z} = C^{2,3}_{y,z} C^{1,2}_{x,z} C^{2,3}_{x,y} \tag{48}$$

Here $C^{k,k+1}_{v,w}$, $k = 1,2$, denotes the operator $C_{v,w}$ acting on the kth and $(k+1)$th components of the tensor product $V^{\otimes 3}$.

Remark 3.2. Equation (48) is a spectral quantum Young–Baxter equation, see e.g. [29, Section 6] and the references therein.

Proof of Lemma 3.1. For the reader's convenience, we will prove this rather obvious lemma. For $g \in L^2(X^{(3)})$ and $v \in V^{\otimes 3}$, we have

$$(T \otimes \mathbf{1}_H)(g \otimes v)(x,y,z) = g(y,x,z) C^{1,2}_{x,y} v,$$
$$(\mathbf{1}_H \otimes T)(T \otimes \mathbf{1}_H)(g \otimes v)(x,y,z) = C^{2,3}_{y,z}(T \otimes \mathbf{1}_H)(g \otimes v)(x,z,y) = g(z,x,y) C^{2,3}_{y,z} C^{1,2}_{x,z} v,$$
$$(T \otimes \mathbf{1}_H)(\mathbf{1} \otimes T)(T \otimes \mathbf{1}_H)(g \otimes v)(x,y,z) = C^{1,2}_{x,y}(\mathbf{1}_H \otimes T)(T \otimes \mathbf{1}_H)(g \otimes v)(y,x,z)$$
$$= g(z,y,x) C^{1,2}_{x,y} C^{2,3}_{x,z} C^{1,2}_{y,z} v,$$

and analogously

$$(\mathbf{1}_H \otimes T)(T \otimes \mathbf{1}_H)(\mathbf{1}_H \otimes T)(g \otimes v)(x,y,z) = g(z,y,x) C^{1,2}_{x,y} C^{2,3}_{x,z} C^{1,2}_{y,z} v. \quad \square$$

Theorem 3.3. *Let $n \geq 2$ and let $f^{(n)} \in H^{\otimes n}$. Then $f^{(n)} \in \mathcal{F}_n(H)$ if and only if, for each $i \in \{1, \dots, n-1\}$ and a.a. $(x_1, \dots, x_n) \in X^{(n)}$, we have*

$$\begin{aligned} &f^{(n)}(x_1, \dots, x_n) - C^{i,i+1}_{x_i,x_{i+1}} f^{(n)}(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n) \\ &\quad \in V^{\otimes(i-1)} \otimes \operatorname{ran}(\mathbf{1}_{V^{\otimes 2}} - C_{x_i,x_{i+1}} C^*_{x_i,x_{i+1}}) \otimes V^{\otimes(n-i-1)}. \end{aligned} \tag{49}$$

Furthermore, if condition (49) is satisfied for some $i \in \{1, \dots, n-1\}$ and $(x_1, \dots, x_n) \in X^{(n)}$, then it is automatically satisfied for this i and $(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n) \in X^{(n)}$.

Remark 3.4. In view of the last statement of Theorem 3.3, in oder to check whether a given $f^{(n)} \in H^{\otimes n}$ belongs to $\mathcal{F}_n(H)$, it is sufficient to check condition (49) for all $i = 1, \dots, n-1$ and a.a. $(x_1, \dots, x_n) \in X^{(n)}$ with $x_1 < x_2 < \dots < x_n$.

In order to prove Theorem 3.3, we will need the following two lemmas.

Lemma 3.5. *Let $C \in \mathcal{L}(V^{\otimes 2})$ and let $w \in V^{\otimes 2}$. Then $w \in \ker(\mathbf{1} - CC^*)$ if and only if $C^*w \in \ker(\mathbf{1} - C^*C)$. Moreover, the mappings*

$$C^* : \ker(\mathbf{1} - CC^*) \to \ker(\mathbf{1} - C^*C), \quad C : \ker(\mathbf{1} - C^*C) \to \ker(\mathbf{1} - CC^*)$$

are bijective and inverse of each other.

Proof. Let $w \in \ker(\mathbf{1} - CC^*)$, $w \neq 0$. Then $w = CC^*w$, hence $C^*w \neq 0$. Furthermore, $C^*w = C^*CC^*w$, hence $C^*w \in \ker(\mathbf{1} - C^*C)$. Therefore,

$$C^* : \ker(\mathbf{1} - CC^*) \to \ker(\mathbf{1} - C^*C) \tag{50}$$

is an injective mapping. Swapping C and C^*, we conclude that

$$C : \ker(\mathbf{1} - C^*C) \to \ker(\mathbf{1} - CC^*) \tag{51}$$

is an injective mapping. Finally, for $w \in \ker(\mathbf{1} - CC^*)$, we have $w = CC^*w$ and for $v \in \ker(\mathbf{1} - C^*C)$, we have $v = C^*Cv$. Hence, both mappings (50) and (51) are bijective and inverse of each other. □

Lemma 3.6. *Let the conditions of Lemma 3.5 be satisfied. Then, for any $u, v \in V^{\otimes 2}$, we have $u - Cv \in \operatorname{ran}(\mathbf{1} - CC^*)$ if and only if $v - C^*u \in \operatorname{ran}(\mathbf{1} - C^*C)$.*

Proof. Assume $u - Cv \in \operatorname{ran}(\mathbf{1} - CC^*)$. Then,

$$(u, w) - (Cv, w) = (u - Cv, w) = 0 \quad \text{for all } w \in \ker(\mathbf{1} - CC^*).$$

Since $w = CC^*w$, we conclude:

$$0 = (u, CC^*w) - (Cv, w) = (C^*u - v, C^*w) \quad \text{for all } w \in \ker(\mathbf{1} - CC^*).$$

Hence, by Lemma 3.5,

$$(C^*u - v, w) = 0 \quad \text{for all } w \in \ker(\mathbf{1} - C^*C),$$

which implies $C^*u - v \in \operatorname{ran}(\mathbf{1} - C^*C)$. The converse implication is obvious. □

We can now proceed with the proof of Theorem 3.3.

Proof of Theorem 3.3. In view of Theorem 2.2, it is sufficient to prove the result for $n = 2$. By the definition of T, we have, for each $f^{(2)} \in H^{\otimes 2}$,

$$\big((\mathbf{1} - T^2)f^{(2)}\big)(x, y) = f^{(2)}(x, y) - C_{x,y}C_{y,x}f^{(2)}(x, y) = (\mathbf{1} - C_{x,y}C^*_{x,y})f^{(2)}(x, y).$$

From here we easily conclude that

$$\overline{\operatorname{ran}(\mathbf{1} - T^2)} = \big\{f^{(2)} \in H^{\otimes 2} \mid f(x, y) \in \operatorname{ran}(\mathbf{1} - C_{x,y}C^*_{x,y}) \text{ for a.a. } (x, y) \in X^{(2)}\big\}.$$

Theorem 2.2 now implies that, for each $f^{(2)} \in H^{\otimes 2}$, we have $f^{(2)} \in \mathcal{F}_2(H)$ if and only if

$$f(x, y) - C_{x,y}f(y, x) \in \operatorname{ran}(\mathbf{1} - C_{x,y}C^*_{x,y}) \quad \text{for a.a. } (x, y) \in X^{(2)}, \tag{52}$$

It follows from Lemma 3.6 that if condition (52) is satisfied for some $(x, y) \in X^{(2)}$, then it is automatically satisfied for the point $(y, x) \in X^{(2)}$. □

The following immediate corollary gives an explicit form of $\mathfrak{P}$, the orthogonal projection of $H^{\otimes 2}$ onto $\mathcal{F}_2(H)$ (compare with Proposition 2.1).

Corollary 3.7. *For $(x, y) \in X^{(2)}$, $x < y$, denote by $P_{x,y}$ the orthogonal projection of the space $V^{\otimes 2} \oplus V^{\otimes 2}$ onto the subspace*

$$\big\{(u, v) \in V^{\otimes 2} \oplus V^{\otimes 2} \mid u - C_{x,y}v \in \operatorname{ran}(\mathbf{1} - C_{x,y}C^*_{x,y})\big\}.$$

Further for $P_{x,y}(u, v) = (w_1, w_2)$, with $w_1, w_2 \in V^{\otimes 2}$, we denote $P^1_{x,y}(u, v) := w_1$ and $P^2_{x,y}(u, v) := w_2$, i.e., $P^1_{x,y}(u, v)$ and $P^2_{x,y}(u, v)$ are the first and second $V^{\otimes 2}$-coordinates of the vector $P_{x,y}(u, v)$. Then $\mathfrak{P}$, the orthogonal projection of $H^{\otimes 2}$ onto $\mathcal{F}_2(H)$, has the following form: for $(x, y) \in X^{(2)}$ with $x < y$,

$$(\mathfrak{P}f^{(2)})(x, y) = P^1_{x,y}\big(f^{(2)}(x, y), f^{(2)}(y, x)\big), \quad (\mathfrak{P}f^{(2)})(y, x) = P^2_{x,y}\big(f^{(2)}(x, y), f^{(2)}(y, x)\big).$$

Let us now describe $\ker(\mathbf{1} + T) = \ker(\mathfrak{P})$.

Proposition 3.8. *We have*

$$\begin{aligned}\ker(\mathbf{1} + T) &= \big\{f^{(2)} - Tf^{(2)} \mid f^{(2)} \in H^{\otimes 2} \text{ and } f^{(2)}(x, y) \in \ker(\mathbf{1} - C_{x,y}C^*_{x,y}) \\ &\qquad \text{for a.a. } (x, y) \in X^{(2)}\big\} &(53)\\ &= \big\{f^{(2)} \in H^{\otimes 2} \mid f^{(2)}(x, y) \in \ker(\mathbf{1} - C_{x,y}C^*_{x,y}) \text{ if } x < y \\ &\qquad \text{and } f^{(2)}(x, y) = -C_{x,y}f^{(2)}(y, x) \text{ if } x > y \text{ for a.a. } (x, y) \in X^{(2)}\big\}. &(54)\end{aligned}$$

Formula (54) *remains true if we swap the conditions $x < y$ and $x > y$.*

Remark 3.9. Formula (54) can be interpreted as follows: $\ker(\mathbf{1}+T)$ consists of all functions of the form $f^{(2)} - Tf^{(2)}$, where $f^{(2)} \in H^{\otimes 2}$ satisfies the following assumption: for a.a. $(x, y) \in X^{(2)}$, $f^{(2)}(x, y) \in \ker(\mathbf{1} - C_{x,y}C^*_{x,y})$ if $x < y$ and $f^{(2)}(x, y) = 0$ if $x > y$.

Proof of Proposition 3.8. Formula (53) follows immediately from (22) and the equality $(\mathbf{1} - T^2)f^{(2)}(x, y) = (\mathbf{1} - C_{x,y}C^*_{x,y})f^{(2)}(x, y)$.

Due to the inclusion $\ker(\mathbf{1} + T) \subset \ker(\mathbf{1} - T^2)$, we get

$$\ker(\mathbf{1} + T) = \ker(\mathbf{1} + T) \cap \ker(\mathbf{1} - T^2),$$

or equivalently

$$\begin{aligned}\ker(\mathbf{1} + T) = \big\{ f^{(2)} \in H^{\otimes 2} \mid f^{(2)}(x, y) \in \ker(\mathbf{1} - C_{x,y}C^*_{x,y}) \\ \text{and } f^{(2)}(x, y) = -C_{x,y}f^{(2)}(y, x), \text{ for a.a. } (x, y) \in X^{(2)} \big\}.\end{aligned} \tag{55}$$

By Lemma 3.5, if the relation

$$f^{(2)}(x, y) = C_{x,y}C^*_{x,y}f^{(2)}(x, y), \quad f^{(2)}(y, x) = -C^*_{x,y}f^{(2)}(x, y).$$

holds for $x < y$, then it also holds for $x > y$. Hence, formula (54) follows from (55). □

According to the general considerations in Subsection 2.3, we can now construct creation and annihilation operators in the T-deformed Fock space $\mathcal{F}(H)$. It should be noticed that the operator $\mathbb{T}_n$ given by formula (30) (and used for the annihilation operators in formula (29)) has now the following form:

$$\begin{aligned}(\mathbb{T}_n f^{(n)})(x_1, \dots, x_n) &= f^{(n)}(x_1, \dots, x_n) \\ &+ \sum_{k=2}^{n} C^{1,2}_{x_1,x_2}C^{2,3}_{x_1,x_3} \cdots C^{k-1,k}_{x_1,x_k} f^{(n)}(x_2, x_3, \dots, x_k, x_1, x_{k+1}, \dots, x_n), \quad f^{(n)} \in H^{\otimes n}.\end{aligned}$$

Recall that, in Subsection 2.3, for the given operator $T \in \mathcal{L}(H^{\otimes 2})$, we defined the operator $\widetilde{T}$ through equality (35) and the operator $\widehat{T}$ by (41). Similarly, for a linear operator $C \in \mathcal{L}(V^{\otimes 2})$, we define linear operators $\widetilde{C}, \widehat{C} \in \mathcal{L}(V^{\otimes 2})$. (Note that, in the finite-dimensional setting, the operator $\widetilde{C}$ always exists.)

Lemma 3.10. *For $f^{(2)} \in H^{\otimes 2}$, we have*

$$(\widetilde{T}f^{(2)})(x, y) = \widetilde{C}_{y,x}f^{(2)}(y, x), \tag{56}$$

$$(\widehat{T}f^{(2)})(x, y) = \widehat{C}_{y,x}f^{(2)}(y, x). \tag{57}$$

Proof. For $i = 1, 2, 3, 4$, let $f_i(x) = \varphi_i(x)u_i$, where $\varphi_i \in L^2(X)$ and $u_i \in V$. Then

$$\langle Tf_1 \otimes f_2, f_3 \otimes f_4 \rangle = \int_X f_1(y)f_2(x)f_3(x)f_4(y)\langle C_{x,y}u_1 \otimes u_2, u_3, \otimes u_4 \rangle \, dx \, dy$$

$$
\begin{aligned}
&= \int_X f_1(y)f_2(x)f_3(x)f_4(y)\langle \widetilde{C}_{x,y}u_3 \otimes u_1, u_4, \otimes u_2\rangle \, dx\, dy \\
&= \int_X f_1(x)f_2(y)f_3(y)f_4(x)\langle \widetilde{C}_{y,x}u_3 \otimes u_1, u_4, \otimes u_2\rangle \, dx\, dy \\
&= \int_X (f_3 \otimes f_1)(y,x)(f_4 \otimes f_2)(x,y)\langle \widetilde{C}_{y,x}u_3 \otimes u_1, u_4, \otimes u_2\rangle \, dx\, dy,
\end{aligned}
$$

which proves (56).

To prove (57), we proceed as follows:

$$
\begin{aligned}
&(\mathbb{S}f_1 \otimes f_2)(x,y) = \overline{\varphi_1(y)\varphi_2(x)}\, J(u_2 \otimes u_1), \\
&(T\mathbb{S}f_1 \otimes f_2)(x,y) = \overline{\varphi_1(x)\varphi_2(y)}\, C_{x,y}J(u_2 \otimes u_1), \\
&(\mathbb{S}T\mathbb{S}f_1 \otimes f_2)(x,y) = \varphi_1(y)\varphi_2(x)\widehat{C}_{y,x}u_1 \otimes u_2,
\end{aligned}
$$

which implies (57). □

To specialize the result of Theorem 2.8 to our current setting, it will be convenient for us to introduce formal operators of creation and annihilation at point $x \in X$. Let $f \in H = L^2(X \to V)$. Then

$$
f(x) = (\varphi_1(x), \varphi_2(x), \dots, \varphi_m(x)), \tag{58}
$$

where $\varphi_1, \varphi_2, \dots, \varphi_m \in L^2(X)$. For $i = 1, 2, \dots, m$ and $\varphi \in L^2(X)$, we denote

$$
a_i^+(\varphi) := a^+(0, \dots, 0, \underbrace{\varphi}_{i\text{th place}}, 0, \dots, 0).
$$

Then, for $f \in L^2(X \to V)$ of the form (58), we get

$$
a^+(f) = \sum_{i=1}^m a_i^+(\varphi_i). \tag{59}
$$

For $i = 1, 2, \dots, m$ and $x \in X$, we formally define a creation operator $a_i^+(x)$ that satisfies

$$
a_i^+(\varphi) = \int_X \varphi(x) a_i^+(x)\, dx \quad \text{for all } \varphi \in L^2(X). \tag{60}
$$

Thus, $a_i^+(x)$ can be formally interpreted as an operator-valued distribution. Next, we define a vector $a^+(x)$ of operator-valued distributions by

$$
a^+(x) := (a_1^+(x), a_2^+(x), \dots, a_m^+(x)).
$$

In other words, $a^+(x)$ has n components, each of which is an operator-valued distribution.

We will formally operate with $a^+(x)$ as a usual vector from V. So, for a vector $v = (v_1, \dots, v_m) \in V$, the $\langle \cdot, \cdot \rangle$ product of v and $a^+(x)$ is given by

$$\langle v, a^+(x) \rangle = \sum_{i=1}^{m} v_i a_i^+(x).$$

Hence, for a fixed $x \in X$ and a function $f(x)$ of the form (58), we have

$$\langle f(x), a(x) \rangle = \sum_{i=1}^{m} \varphi_i(x) a_i^+(x). \tag{61}$$

In view of formulas (59)–(61), we get

$$a^+(f) = \int_X \langle f(x), a^+(x) \rangle \, dx.$$

We similarly define $a^-(x)$ satisfying

$$a^-(f) = \int_X \langle f(x), a^-(x) \rangle \, dx.$$

Next, for $x, y \in X$, we may formally use the tensor product of the 'vectors' $a^+(x)$ and $a^-(y)$:

$$a^+(x) \otimes a^-(y) = \big(a_i^+(x) a_j^-(y)\big)_{i,j=1,\dots,m}.$$

Hence, for $f \in L^2(X \to V)$ of the form (58) and $g \in L^2(X \to V)$ of the form

$$g(y) = (\psi_1(y), \psi_2(y), \dots, \psi_m(y)),$$

we get

$$\begin{aligned}
&\int_{X^2} \big\langle (f \otimes g)(x, y), a^+(x) \otimes a^-(y) \big\rangle \, dx \, dy \\
&\quad = \sum_{i,j=1}^{m} \int_{X^2} \varphi_i(x) \psi_j(y) a_i^+(x) a_j^-(y) \, dx \, dy \\
&\quad = \sum_{i,j=1}^{m} \int_X \varphi_i(x) a_i^+(x) \, dx \int_X \psi_j(y) a_j^-(y) \, dy \\
&\quad = \sum_{i,j=1}^{m} a_i^+(\varphi_i) a_j^-(\psi_j) = a^+(f) a^-(g).
\end{aligned}$$

Hence, for an arbitrary $f^{(2)} \in H^{\otimes 2}$, we will write

$$a^{+-}(f^{(2)}) = \int_{X^2} \big\langle f^{(2)}(x, y), a^+(x) \otimes a^-(y) \big\rangle \, dx \, dy.$$

We will use similar notations for $a^{++}(f^{(2)})$, $a^{--}(f^{(2)})$, and for a product $a^-(f)a^+(g)$ with $f, g \in H$.

Let

$$X^{(2)} \ni (x,y) \mapsto M_{x,y} \in \mathcal{L}(V^{\otimes 2})$$

be a measurable mapping with $\|M_{x,y}\| \le 1$. Then, we will write, for $f^{(2)} \in H^{\otimes 2}$,

$$\int_{X^2} \big\langle M_{x,y} f^{(2)}(x,y), a^+(x) \otimes a^-(y) \big\rangle \, dx\, dy = \int_{X^2} \big\langle f^{(2)}(x,y), M_{x,y}^T a^+(x) \otimes a^-(y) \big\rangle \, dx\, dy,$$

where A^T denotes the transpose of a matrix A.

Theorem 3.11. *For any $f, g \in H$, we have*

$$\int_{X^2} \big\langle (f \otimes g)(x,y), a^-(x) \otimes a^+(y) \big\rangle \, dx\, dy$$
$$= \int_{X^2} \big\langle (f \otimes g)(x,y), \widetilde{C}_{x,y}^T a^+(y) \otimes a^-(x) \big\rangle \, dx\, dy + \int_X \langle f(x), g(x) \rangle \, dx. \tag{62}$$

Further assume that $\ker(\mathbf{1} + T) \neq \{0\}$ *and let* $f^{(2)} \in H^{\otimes 2}$*. If for a.a.* $(x,y) \in X^{(2)}$, $f^{(2)}(x,y) \in \ker(\mathbf{1} - C_{x,y} C_{x,y}^*)$*, then*

$$\int_{X^2} \big\langle f^{(2)}(x,y), a^+(x) \otimes a^+(y) \big\rangle \, dx\, dy = \int_{X^2} \big\langle f^{(2)}(x,y), C_{y,x}^T a^+(y) \otimes a^+(x) \big\rangle \, dx\, dy, \tag{63}$$

and if for a.a. $(x,y) \in X^{(2)}$, $(\mathbb{S} f^{(2)})(x,y) \in \ker(\mathbf{1} - C_{x,y} C_{x,y}^*)$*, then*

$$\int_{X^2} \big\langle f^{(2)}(x,y), a^-(x) \otimes a^-(y) \big\rangle \, dx\, dy = \int_{X^2} \big\langle f^{(2)}(x,y), \widehat{C}_{x,y}^T a^-(y) \otimes a^-(x) \big\rangle \, dx\, dy. \tag{64}$$

Proof. The statement follows immediately from Theorem 2.8, formula (53) from Proposition 3.8 and Lemma 3.10. □

Remark 3.12. Let A be a measurable subset of X^2 and assume that a function $f^{(2)} \in H^{\otimes 2}$ vanishes outside the set A. Then it is natural to write

$$a^{++}(f^{(2)}) = \int_A f^{(2)}(x,y) a^+(x) a^+(y), \quad a^{--}(f^{(2)}) = \int_A f^{(2)}(x,y) a^+(x) a^+(y).$$

In view of (54) (see also Remark 3.9), formulas (63) and (64) can be *equivalently* written as follows. Let $f^{(2)} \in H^{\otimes 2}$ be such that $f^{(2)}(x,y) = 0$ for all $(x,y) \in X^{(2)}$ with $x > y$. If for a.a. $(x,y) \in X^{(2)}$ with $x < y$, we have $f^{(2)}(x,y) \in \ker(\mathbf{1} - C_{x,y} C_{x,y}^*)$, then

$$\int_{\{x<y\}} \big\langle f^{(2)}(x,y), a^+(x) \otimes a^+(y) \big\rangle \, dx\, dy = \int_{\{x<y\}} \big\langle f^{(2)}(x,y), C_{y,x}^T a^+(y) \otimes a^+(x) \big\rangle \, dx\, dy, \tag{65}$$

and if for a.a $(x,y) \in X^{(2)}$ with $x > y$, we have $\mathbb{S}f^{(2)}(x,y) \in \ker(\mathbf{1} - C_{x,y}C^*_{x,y})$, then

$$\int_{\{x<y\}} \big\langle f^{(2)}(x,y), a^-(x) \otimes a^-(y) \big\rangle \, dx\, dy = \int_{\{x<y\}} \big\langle f^{(2)}(x,y), \widehat{C}^T_{x,y} a^-(y) \otimes a^-(x) \big\rangle \, dx\, dy. \tag{66}$$

If we swap the conditions $x < y$ and $x > y$, the above results will remain true.

Theorem 3.11 (and formulas (65), and (66)) can be easily understood by using formal commutation relations between the creation and annihilation operators at point.

Corollary 3.13. *The following formal commutation relations hold.*

(i) *For all $(x,y) \in X^{(2)}$, we have*

$$a^-(x) \otimes a^+(y) = \widetilde{C}^T_{x,y} a^+(y) \otimes a^-(x) + \delta(x-y)\Delta.$$

Here $\Delta := (\delta_{ij})_{i,j=1,\dots,m}$ with δ_{ij} being the Kronecker delta, so that for any $f, g \in H$,

$$\int_{X^2} \big\langle (f \otimes g)(x,y), \delta(x-y)\Delta \big\rangle \, dx\, dy = \int_X \langle f(x), g(x) \rangle \, dx.$$

(ii) *For each $(x,y) \in X^{(2)}$ and a vector $v \in \ker(\mathbf{1} - C_{x,y}C^*_{x,y})$, we have*

$$\langle v, a^+(x) \otimes a^+(y) \rangle = \big\langle C_{y,x} v, a^+(y) \otimes a^+(x) \big\rangle = \big\langle v, C^T_{y,x} a^+(y) \otimes a^+(x) \big\rangle,$$

and for each $(x,y) \in X^{(2)}$ and a vector $v \in V^{\otimes 2}$ such that $\mathbb{S}v \in \ker(\mathbf{1} - C_{x,y}C^*_{x,y})$, we have

$$\langle v, a^-(x) \otimes a^-(y) \rangle = \big\langle \widehat{C}_{x,y} v, a^-(y) \otimes a^-(x) \big\rangle = \big\langle v, \widehat{C}^T_{x,y} a^-(y) \otimes a^-(x) \big\rangle.$$

Here $\mathbb{S}$ acts on the space $V^{\otimes 2}$.

In the case where the operator T is unitary, formulas (63), (64) hold for all $f^{(2)} \in H^{\otimes 2}$. Thus, the commutation relations take the following form.

Corollary 3.14. *Let T be unitary. Then, for all $(x,y) \in X^{(2)}$, we formally have:*

$$\begin{aligned} a^-(x) \otimes a^+(y) &= \widetilde{C}^T_{x,y} a^+(y) \otimes a^-(x) + \delta(x-y)\Delta, \\ a^+(x) \otimes a^+(y) &= C^T_{y,x} a^+(y) \otimes a^+(x), \\ a^-(x) \otimes a^-(y) &= \widehat{C}^T_{x,y} a^-(y) \otimes a^-(x). \end{aligned}$$

4 Examples

In this section, we will consider several particular examples of the operator T associated with a multicomponent quantum system and explicitly compute the corresponding Fock space and commutation relations between creation and annihilation operators. In all but the very last example, the operator T will be constructed through a single linear operator C on $V^{\otimes 2}$ which satisfies the (constant) Yang–Baxter equation on $V^{\otimes 3}$:

$$C^{1,2}C^{2,3}C^{1,2} = C^{2,3}C^{1,2}C^{2,3}. \tag{67}$$

We restrict ourselves to $V = \mathbb{C}^2$, in which case all solutions of the (equivalent form of the) Yang–Baxter equation are classified in [24], see also the earlier PhD thesis [32].

We will denote by (e_1, e_2) the standard orthonormal basis of $V = \mathbb{C}^2$, and by $(e_{11}, e_{12}, e_{21}, e_{22})$, with $e_{ij} := e_i \otimes e_j$, the corresponding orthonormal basis of $V^{\otimes 2}$. In this basis, we will identify linear operators on $V \otimes V$ with matrices acting on column vectors. By (43), if

$$\text{if } C = \begin{pmatrix} c_{11}^{11} & c_{11}^{12} & c_{11}^{21} & c_{11}^{22} \\ c_{12}^{11} & c_{12}^{12} & c_{12}^{21} & c_{12}^{22} \\ c_{21}^{11} & c_{21}^{12} & c_{21}^{21} & c_{21}^{22} \\ c_{22}^{11} & c_{22}^{12} & c_{22}^{21} & c_{22}^{22} \end{pmatrix}, \text{ then } \widetilde{C} = \begin{pmatrix} c_{11}^{11} & c_{12}^{11} & c_{11}^{12} & c_{12}^{12} \\ c_{21}^{11} & c_{22}^{11} & c_{21}^{12} & c_{22}^{12} \\ c_{11}^{21} & c_{12}^{21} & c_{11}^{22} & c_{12}^{22} \\ c_{21}^{21} & c_{22}^{21} & c_{21}^{22} & c_{22}^{22} \end{pmatrix}. \tag{68}$$

For a function $f^{(n)} \in H^{\otimes n}$, we will denote by $f^{(n)}_{i_1 i_2 \dots i_n} \in L^2(X^{(n)})$ the $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}$ coordinate of $f^{(n)}$, where $i_1, i_2, \dots, i_n \in \{1, 2\}$.

4.1 Spatially constant C

We start with the case where C_{xy} is independent of spatial variables x, y, i.e., $C_{xy} = C$ for a fixed matrix $C = C^*$, $\|C\| \le 1$. Then the operator T satisfies equation (13) if and only if the matrix C satisfies requation (67).

Example 4.1. Let us consider the operator C given by the matrix

$$C = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & \bar{q} & 0 & 0 \\ 0 & 0 & 0 & k \end{pmatrix}.$$

Here $k \in (-1, 1)$ and $q \in \mathbb{C}$, $|q| = 1$. Then

$$\mathbf{1} - C^2 = \begin{pmatrix} 1-k^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-k^2 \end{pmatrix},$$

which implies

$$\operatorname{ran}(\mathbf{1}-C^2) = \text{l.s.}\{e_{11}, e_{22}\}, \tag{69}$$

$$\ker(\mathbf{1}-C^2) = \text{l.s.}\{e_{12}, e_{21}\}. \tag{70}$$

Here l. s. denotes the linear span. For $f^{(2)} \in H^{\otimes 2}$,

$$\begin{aligned} f^{(2)}(x,y) - Cf^{(2)}(y,x) = \big(f_{11}^{(2)}(x,y) - kf_{11}^{(2)}(y,x)\big)e_{11} + \big(f_{12}^{(2)}(x,y) - \bar{q}f_{21}^{(2)}(y,x)\big)e_{12} \\ + \big(f_{21}^{(2)}(x,y) - qf_{12}^{(2)}(y,x)\big)e_{21} + \big(f_{22}^{(2)}(x,y) - kf_{221}^{(2)}(y,x)\big)e_{22}. \end{aligned}$$

Hence, by (69), the condition $f^{(2)}(x,y) - Cf^{(2)}(y,x) \in \operatorname{ran}(\mathbf{1}-C^2)$ is equivalent to

$$f_{21}^{(2)}(x,y) = qf_{12}^{(2)}(y,x). \tag{71}$$

By Theorem 3.3, we now get the following explicit description of $\mathcal{F}_n(H)$. Define

$$Q(1,2) := \bar{q}, \quad Q(2,1) := q. \tag{72}$$

Then for $n \geq 2$, $\mathcal{F}_n(H)$ consists of all functions $f^{(n)} \in H^{\otimes n}$ that satisfy a.e. the following symmetry condition:

$$f_{i_1\dots i_n}^{(n)}(x_1,\dots,x_n) = Q(i_k,i_{k+1}) f_{i_1\dots i_{k-1}i_{k+1}i_k i_{k+2}\dots i_n}^{(n)}(x_1,\dots,x_{k-1},x_{k+1},x_k,x_{k+2},\dots,x_n) \tag{73}$$

for $k \in \{1,\dots,n-1\}$ and $i_1,\dots,i_n \in \{1,2\}$ with $i_k \neq i_{k+1}$. In particular, the function $f^{(n)} \in \mathcal{F}_n(H)$ is completely identified by its coordinates $f_{i_1\dots i_n}^{(n)}$ with $i_1 \leq i_2 \leq \dots \leq i_n$.

By using Corollary 3.7 and (69), one can easily calculate $\mathfrak{P}$, the orthogonal projection of $H^{\otimes 2}$ onto $\mathcal{F}_2(H)$:

$$\begin{aligned} (\mathfrak{P}f^{(2)})(x,y) = f_{11}^{(2)}(x,y)e_{11} + f_{22}^{(2)}(x,y)e_{22} + \frac{1}{2}\big(f_{12}^{(2)}(x,y) + \bar{q}f_{21}^{(2)}(y,x)\big)e_{12} \\ + \frac{1}{2}\big(f_{21}^{(2)}(x,y) + qf_{12}^{(2)}(y,x)\big)e_{21}. \end{aligned} \tag{74}$$

To obtain the commutation relations between creation and annihilation operators, we use Theorem 3.11 and (70). Additionally to (72), set also

$$Q(1,1) = Q(2,2) := k.$$

By (68), we get $\widetilde{C} = C^*$. Hence, for all $\varphi, \psi \in L^2(X)$,

$$\begin{aligned} &a_i^-(\varphi)a_i^+(\psi) = Q(i,j)a_i^+(\psi)a_i^-(\varphi) + \delta_{ij}\langle\varphi,\psi\rangle, \quad i,j \in \{1,2\}, \\ &a_i^+(\varphi)a_i^+(\psi) = Q(j,i)a_i^+(\psi)a_i^+(\varphi), \quad i \neq j, \\ &a_i^-(\varphi)a_i^-(\psi) = Q(j,i)a_i^-(\psi)a_i^-(\varphi), \quad i \neq j. \end{aligned} \tag{75}$$

It can be shown that in this case there exists the universal C^*-algebra $\mathbf{A}$ generated by $a_i^+(\varphi)$, $a_i^-(\varphi)$, $i=1,2$, $\varphi\in L^2(X)$. Let also $\mathbf{A}_1$, $\mathbf{A}_2$ be the C^*-subalgebras of $\mathbf{A}$ generated by $a_1^+(\varphi)$, $a_1^-(\varphi)$ and $a_2^+(\varphi)$, $a_2^-(\varphi)$, respectively. Note that each $\mathbf{A}_i$ (i=1,2) is the C^*-algebras generated by the k-deformed commutation relations with $k\in(-1,1)$, see [9].

One can construct the tensor product $\mathbf{A}_1\otimes\mathbf{A}_2$ and consider its Rieffel deformation, denoted by $\mathbf{A}_1\otimes_q\mathbf{A}_2$, see [41]. Then it turns out that the Fock representation of $\mathbf{A}$ can be realized as the composition of the canonical surjection $\Phi\colon\mathbf{A}\to\mathbf{A}_1\otimes_q\mathbf{A}_2$ and the Fock representation of $\mathbf{A}_1\otimes_q\mathbf{A}_2$. This approach will give us a deeper insight into the structure of the Fock representation of $\mathbf{A}$.

Below we will use the fact that any irreducible representation of $\mathbf{A}$ that possesses a vacuum vector annihilated by $a_i^-(\varphi)$, $i=1,2$, $\varphi\in L^2(X)$, is unitarily equivalent to the Fock representation, see [23].

Let $K:=L^2(X)$. Construct the Fock space $\mathcal{F}(K)=\bigoplus_{n=0}^\infty\mathcal{F}_n(K)$ corresponding to the Fock representation of the k-deformed commutation relations, and denote by Ψ the vacuum vector in $\mathcal{F}(K)$, see [9] for details. Let $a^+(\varphi)$, $a^-(\varphi)$ ($\varphi\in K$) be the corresponding creation and annihilation operators on $\mathcal{F}(K)$. We construct a unitary operator $U\colon\mathcal{F}(K)\to\mathcal{F}(K)$ by

$$U\Psi=\Psi,\quad U\varphi^{(n)}=q^n\varphi^{(n)},\quad \varphi^{(n)}\in\mathcal{F}_n(K),\ n\in\mathbb{N}.$$

Obviously,

$$Ua^+(\varphi)=qa^+(\varphi)U,\quad Ua^-(\varphi)=\bar{q}a^-(\varphi)U.$$

Define the space $\mathcal{F}:=\mathcal{F}(K)\otimes\mathcal{F}(K)$ and bounded linear operators operators $a_i^+(\varphi)$, $a_i^-(\varphi)$ ($\varphi\in K$) on $\mathcal{F}$ by

$$\begin{aligned} &a_1^+(\varphi):=a^+(\varphi)\otimes\mathbf{1}_{\mathcal{F}(K)},\quad a_2^+(\varphi):=U\otimes a^+(\varphi),\\ &a_1^-(\varphi):=a^-(\varphi)\otimes\mathbf{1}_{\mathcal{F}(K)},\quad a_2^-(\varphi):=U^*\otimes a^-(\varphi). \end{aligned}\tag{76}$$

It is easy to verify that these operators satisfy the commutation relation (75). The family $\big(a^+(\varphi),\,a^-(\varphi)\big)_{\varphi\in K}$ is irreducible on $\mathcal{F}(K)$, see [23]. Then the Schur Lemma implies that the family $\big(a_i^+(\varphi),\,a_i^-(\varphi)\big)_{\varphi\in K,\,i=1,2}$ is irreducible on $\mathcal{F}$. Evidently, for $\Omega=\Psi\otimes\Psi$, we have $a_i^-(\varphi)\Omega=0$ for all $\varphi\in K$ and $i=1,2$. Thus, as noted above, the operators defined by (76) determine the Fock representation of the commutation relations (75).

As a corollary of our description we get the boundedness of the Fock representation of (75). Indeed, as follows from [9], for each $\varphi\in K$, the operator $a^+(\varphi)$ is bounded and $\|a^+(\varphi)\|=\frac{\|f\|}{\sqrt{1-k}}$. Hence,

$$\|a_i^+(\varphi)\|=\frac{\|\varphi\|}{\sqrt{1-k}},\quad i=1,2.$$

Example 4.2. Consider the following operator C related to the Pusz–Woronowicz twisted canonical commutation relations [39] (see also [3]),

$$C = \begin{pmatrix} \mu^2 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & \mu & \mu^2-1 & 0 \\ 0 & 0 & 0 & \mu^2 \end{pmatrix}, \quad \mu \in (0,1). \tag{77}$$

We then get

$$\mathbf{1} - C^2 = \begin{pmatrix} 1-\mu^4 & 0 & 0 & 0 \\ 0 & 1-\mu^2 & -\mu(\mu^2-1) & 0 \\ 0 & -\mu(\mu^2-1) & 1-\mu^2-(\mu^2-1)^2 & 0 \\ 0 & 0 & 0 & 1-\mu^4 \end{pmatrix}.$$

An easy calculation then shows that

$$\ker(\mathbf{1} - C^2) = \mathrm{l.\,s.}\{-\mu e_{12} + e_{21}\}, \tag{78}$$

$$\mathrm{ran}(\mathbf{1} - C^2) = \mathrm{l.\,s.}\big\{e_{11}, e_{22}, e_{12} + \mu e_{21}\big\}. \tag{79}$$

By (79), for a function $f^{(2)} \in H^{\otimes 2}$, we have

$$f^{(2)}(x,y) - Cf^{(2)}(y,x) \in \mathrm{ran}(\mathbf{1} - C^2)$$

if and only if $\mu \tilde f^{(2)}_{12} = \tilde f^{(2)}_{21}$. Here,

$$\tilde f^{(2)}_{ij}(x,y) := \frac{1}{2}\big(f^{(2)}_{ij}(x,y) + f^{(2)}_{ij}(y,x)\big).$$

Hence, by Theorem 3.3, for $n \ge 2$, $\mathcal{F}_n(H)$ consists of all functions $f^{(n)} \in H^{\otimes n}$ that satisfy a.e. the following symmetry condition:

$$\mu S^{(n)}_k f^{(n)}_{i_1 \dots i_{k-1} 1 2 i_{k+2} \dots i_n} = S^{(n)}_k f^{(n)}_{i_1 \dots i_{k-1} 2 1 i_{k+2} \dots i_n}$$

for $k \in \{1, \dots, n-1\}$ and $i_1, \dots, i_{k-1}, i_{k+2}, \dots, i_n \in \{1,2\}$. Here, for $f^{(n)} \in H^{\otimes n}$ and $i_1, \dots, i_n \in \{1,2\}$,

$$\begin{aligned} &(S^{(n)}_k f^{(n)}_{i_1,\dots,i_n})(x_1,\dots,x_n) \\ &\quad := \frac{1}{2}\big(f^{(n)}_{i_1,\dots,i_n}(x_1,\dots,x_n) + f^{(n)}_{i_1,\dots,i_n}(x_1,\dots,x_{k-1},x_{k+1},x_k,x_{k+2},\dots,x_n)\big). \end{aligned}$$

By using Corollary 3.7 and (79), we get, for $f^{(2)} \in H^{\otimes 2}$

$$\mathfrak{P} f^{(2)} = f^{(2)}_{11} e_{11} + f^{(2)}_{22} e_{22} + \big(f^{(2)}_{12} + \tfrac{\mu}{1+\mu^2}(-\mu \tilde f^{(2)}_{12} + \tilde f^{(2)}_{21})\big) e_{12}$$

$$+\big(f^{(2)}_{21}-\tfrac{1}{1+\mu^2}(-\mu\tilde f^{(2)}_{12}+\tilde f_{21})\big)e_{21}.$$

By (68) and (77),

$$\widetilde{C}=\begin{pmatrix}\mu^2 & 0 & 0 & 0\\ 0 & 0 & \mu & 0\\ 0 & \mu & 0 & 0\\ \mu^2-1 & 0 & 0 & \mu^2\end{pmatrix}.$$

Hence, by Theorem 3.11 and (78), we have, for all $\varphi,\psi\in L^2(X)$,

$$\begin{aligned}
&a_1^-(\varphi)a_1^+(\psi)=\mu^2a_1^+(\psi)a_1^-(\varphi)+\langle\varphi,\psi\rangle,\\
&a_2^-(\varphi)a_2^+(\psi)=(\mu^2-1)a_1^+(\psi)a_1^-(\varphi)+\mu^2a_2^+(\psi)a_2^-(\varphi)+\langle\varphi,\psi\rangle,\\
&a_1^-(\varphi)a_2^+(\psi)=\mu a_2^+(\psi)a_1^-(\varphi),\\
&a_2^-(\varphi)a_1^+(\psi)=\mu a_1^+(\psi)a_2^-(\varphi),\\
&a_2^+(\varphi)a_1^+(\psi)+a_2^+(\psi)a_1^+(\varphi)=\mu\big(a_1^+(\varphi)a_2^+(\psi)+a_1^+(\psi)a_2^+(\varphi)\big),\\
&a_1^-(\varphi)a_2^-(\psi)+a_1^-(\psi)a_2^-(\varphi)=\mu\big(a_2^-(\varphi)a_1^-(\psi)+a_2^-(\psi)a_1^-(\varphi)\big).
\end{aligned}$$

Example 4.3. Consider the operator C given by the matrix

$$C=\begin{pmatrix}0 & 0 & 0 & q\\ 0 & k & 0 & 0\\ 0 & 0 & k & 0\\ \bar q & 0 & 0 & 0\end{pmatrix}$$

where $q\in\mathbb{C}$, $|q|=1$ and $k\in[-1,1]$. Then

$$\mathbf{1}-C^2=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1-k^2 & 0 & 0\\ 0 & 0 & 1-k^2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

First assume $|k|<1$. Then

$$\operatorname{ran}(\mathbf{1}-C^2)=\mathrm{l.\,s.}\{e_{12},e_{21}\},\tag{80}$$

$$\ker(\mathbf{1}-C^2)=\mathrm{l.\,s.}\{e_{11},e_{22}\}.\tag{81}$$

Just as in Example 4.1, let $Q(1,2):=\bar q$ and $Q(2,1):=q$. By Theorem 3.3 and (80), $\mathcal{F}_n(H)$ consists of all functions $f^{(n)}\in H^{\otimes n}$ that satisfy a.e. the following symmetry condition:

$$\begin{aligned}
&f^{(n)}_{i_1\dots i_{k-1}i_ki_ki_{k+2}\dots i_n}(x_1,\dots,x_n)\\
&\quad=Q(i_k,j_k)f^{(n)}_{i_1\dots i_{k-1}j_kj_ki_{k+2}\dots i_n}(x_1,\dots,x_{k-1},x_{k+1},x_k,x_{k+2},\dots,x_n)
\end{aligned}\tag{82}$$

for $k \in \{1, \ldots, n-1\}$ and $i_1, \ldots, i_{k-1}, i_k, j_k, i_{k+2}, \ldots, i_n \in \{1,2\}$, $i_k \neq j_k$, compare with (73). Similarly to (74), we can easily find the explicit form of $\mathfrak{P}$.

Furthermore, by (68),

$$\widetilde{C} = \begin{pmatrix} 0 & 0 & 0 & k \\ 0 & \bar{q} & 0 & 0 \\ 0 & 0 & q & 0 \\ k & 0 & 0 & 0 \end{pmatrix},$$

which, by Theorem 3.11 and (81), implies the commutation relations, for any $\varphi, \psi \in L^2(X)$,

$$\begin{aligned} a_1^-(\varphi)a_1^+(\psi) &= ka_2^+(\psi)a_2^-(\varphi) + \langle \varphi, \psi \rangle, \\ a_2^-(\varphi)a_2^+(\psi) &= ka_1^+(\psi)a_1^-(\varphi) + \langle \varphi, \psi \rangle, \\ a_1^-(\varphi)a_2^+(\psi) &= \bar{q}a_1^+(\psi)a_2^-(\varphi), \\ a_2^-(\varphi)a_1^+(\psi) &= qa_2^+(\psi)a_1^-(\varphi), \\ a_1^+(\varphi)a_1^+(\psi) &= qa_2^+(\psi)a_2^+(\varphi), \\ a_1^-(\varphi)a_1^-(\psi) &= \bar{q}a_2^-(\psi)a_2^-(\varphi). \end{aligned} \tag{83}$$

Note that the commutation relations (83) have a more complex structure than the commutation relations (75).

In the case $k = \pm 1$, the matrix C is unitary, and so $\mathbf{1} - C^2 = \mathbf{0}$. To describe $\mathcal{F}_n(H)$, in addition to (82), the following symmetry condition must be satisfied:

$$f^{(n)}_{i_1 \ldots i_n}(x_1, \ldots, x_n) = k f^{(n)}_{i_1 \ldots i_{k-1} i_{k+1} i_k i_{k+2} \ldots i_n}(x_1, \ldots, x_{k-1}, x_{k+1}, x_k, x_{k+2}, \ldots, x_n)$$

for $k \in \{1, \ldots, n-1\}$ and $i_1, \ldots, i_n \in \{1,2\}$ with $i_k \neq i_{k+1}$. Recall also that $\mathbb{P}_n = \frac{1}{n!}\mathcal{P}_n$ in this case. Additionally to the commutation relations (83), it also holds that

$$\begin{aligned} a_1^+(\varphi)a_2^+(\psi) &= ka_2^+(\psi)a_1^+(\varphi), \\ a_1^-(\varphi)a_2^-(\psi) &= ka_2^-(\psi)a_1^-(\varphi). \end{aligned}$$

4.2 Non-Abelian anyon quantum systems

In this section, we will discuss the case where the operator C depends on spatial variables $(x, y) \in X^{(2)}$ in a special way and determines a non-Abelian anyon quantum system when $d = 2$, see [15].

Recall (47). For $x, y \in X^{(2)}$, we will write $x < y$ and $x > y$ if $x^1 < y^1$ and $x^1 > y^1$, respectively. Let C be a unitary operator on $V \otimes V$ and we define $C_{x,y}$ by formula (12). By (11), we get $T^2 = \mathbf{1}$, hence T is a unitary operator.

Lemma 4.4. *The operator T satisfies the Yang–Baxter equation* (13) *on $H^{\otimes 3}$ if and only if the operator C satisfies the Yang–Baxter equation* (67) *on $V^{\otimes 3}$.*

Proof. Recall Lemma 3.1. In view of (12), for $x < y < z$, formula (48) becomes (67). If $x < z < y$, (12) obtains the form

$$C^{1,2}C^{2,3}(C^{1,2})^* = (C^{2,3})^* C^{1,2}C^{2,3}. \tag{84}$$

Multiplying equality (84) by C^{23} from the left and by C^{12} from the right, we arrive at (67). The other remaining cases are similar. □

Remark 4.5.

The next statement is Corollary 2.4 applied to our case.

Proposition 4.6. *For each $n \geq 2$, the space $\mathcal{F}_n(H)$ consists of all functions $f^{(n)} \in H^{\otimes n}$ that satisfy a.e. the following symmetry condition:*

$$f^{(n)}(x_1, \dots, x_n) = C^{k,k+1}_{x_k, x_{k+1}} f(x_1, \dots, x_{k-1}, x_{k+1}, x_k, x_{k+2}, \dots, x_n) \tag{85}$$

for each $k \in \{1, \dots, n-1\}$.

Also recall that, in this case, the orthogonal projection of $H^{\otimes n}$ onto $\mathcal{F}_n(H)$ satisfies $\mathbb{P}_n = \frac{1}{n!}\mathcal{P}_n$.

Example 4.7. Consider C of the form

$$C = \begin{pmatrix} q_1 & 0 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & 0 & q_4 \end{pmatrix}, \tag{86}$$

where $q_1, q_2, q_3, q_4 \in \mathbb{C}$ are of modulus 1. Define a complex-valued function Q a.e. on $(\{1,2\} \times X)^2$ by

$$Q(1,x,1,y) := \begin{cases} q_1, & \text{if } x < y, \\ \bar{q}_1, & \text{if } x > y, \end{cases} \qquad Q(2,x,2,y) := \begin{cases} q_4, & \text{if } x < y, \\ \bar{q}_4, & \text{if } x > y, \end{cases}$$

$$Q(1,x,2,y) := \begin{cases} q_3, & \text{if } x < y, \\ \bar{q}_2, & \text{if } x > y, \end{cases} \qquad Q(2,x,1,y) := \begin{cases} q_2, & \text{if } x < y, \\ \bar{q}_3, & \text{if } x > y. \end{cases}$$

Note that the function Q Hermitian:

$$Q(i,x,j,y) = \overline{Q(j,y,i,y)}.$$

Then, by Proposition 4.6 and (86), for each $n \geq 2$, the space $\mathcal{F}_n(H)$ consists of all functions $f^{(n)} \in H^{\otimes n}$ that satisfy a.e. the following symmetry condition:

$$f^{(n)}_{i_1 \dots i_n}(x_1, \dots, x_n)$$

$$= Q(i_k, x_k, i_{k+1}, x_{k+1}) f^{(n)}_{i_1 \dots i_{k-1} i_{k+1} i_k i_{k+2} \dots i_n}(x_1, \dots, x_{k-1}, x_{k+1}, x_k, x_{k+2}, \dots, x_n)$$

for all $i_1, \dots, i_n \in \{1, 2\}$ and $k \in \{1, \dots, n-1\}$.

By (68) and (86), we get $C = \widetilde{C}^T$ and $C^* = \big(\widetilde{C}^*\big)^T$. Hence, by Corollary 3.14, we obtain the following formal commutation relations:

$$\begin{aligned} a_i^-(x) a_j^+(y) &= Q(i, x, j, y) a_j^+(y) a_i^-(x) + \delta(x - y)\delta_{ij}, \\ a_i^+(x) a_j^+(y) &= Q(j, y, i, x) a_j^+(y) a_i^+(x), \\ a_i^-(x) a_j^-(y) &= Q(j, y, i, x) a_j^-(y) a_i^-(x). \end{aligned}$$

Remark 4.8. Note that the commutation relations in Examples 4.1 and 4.7 are governed by a single Hermitian function, $Q(i, j)$ in Example 4.1 and $Q(i, x, j, y)$ in Example 4.7. Therefore, to construct these examples, one could use the theory of commutation relations deformed with a Hermitian, complex-valued function Q, whose modulus is bounded by 1, see [8].

Another example of a non-Abelian anyon quantum system will be discussed below as a special case of Example 4.9.

4.3 General spatial dependence

We will now consider an example of a matrix $C_{x,y}$ with somewhat more complicated dependence on spatial variables $x, y \in X$.

Example 4.9. Let $Q_1, Q_2 : X^{(2)} \to \mathbb{C}$ satisfy

$$Q_i(x, y) = \overline{Q_i(y, x)}, \quad i = 1, 2, \quad |Q_1(x, y)| \le 1, \quad |Q_2(x, y)| = 1, \qquad (x, y) \in X^2.$$

Let matrix $C_{x,y}$ have the form

$$C_{x,y} = \begin{pmatrix} 0 & 0 & 0 & Q_1(x, y) \\ 0 & Q_2(x, y) & 0 & 0 \\ 0 & 0 & Q_2(x, y) & 0 \\ Q_1(x, y) & 0 & 0 & 0 \end{pmatrix}.$$

Note that $C_{x,y} = C^*_{y,x}$. A direct calculation shows that $C_{x,y}$ satisfies the Yang–Baxter equation (48). For $x, y \in X^{(2)}$, we have

$$\mathbf{1} - C_{x,y} C^*_{x,y} = \begin{pmatrix} 1 - |Q_1(x, y)|^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 - |Q_1(x, y)|^2 \end{pmatrix}.$$

We denote

$$Y := \big\{(x, y) \in X^2 \mid Q_1(x, y)|^2 = 1\big\}, \quad Z := \big\{(x, y) \in X^2 \mid Q_1(x, y)|^2 < 1\big\}.$$

Then, for $(x,y)\in Y$, $\mathbf{1}-C_{x,y}C_{x,y}^*=\mathbf{0}$, and for all $(x,y)\in Z$,

$$\ker(\mathbf{1}-C_{x,y}C_{x,y}^*)=\mathrm{l.\,s.}\{e_{12},e_{21}\},\quad \operatorname{ran}(\mathbf{1}-C_{x,y}C_{x,y}^*)=\mathrm{l.\,s.}\{e_{11},e_{22}\}.$$

Hence, by Theorem 3.3, for $n\ge 2$, $\mathcal{F}_n(H)$ consists of all functions $f^{(n)}\in H^{\otimes n}$ that satisfy a.e. the following symmetry conditions:

$$\begin{aligned}
&f^{(n)}_{i_1\dots i_{k-1}11i_{k+2}\dots i_n}(x_1,\dots,x_n)\\
&\quad =Q_1(x_k,x_{k+1})f^{(n)}_{i_1\dots i_{k-1}22i_{k+2}\dots i_n}(x_1,\dots,x_{k-1},x_{k+1},x_k,x_{k+2},\dots,x_n)\quad \text{if } (x_k,x_{k+1})\in Y,\\
&f^{(n)}_{i_1\dots i_{k-1}12i_{k+2}\dots i_n}(x_1,\dots,x_n)\\
&\quad =Q_2(x_k,x_{k+1})f^{(n)}_{i_1\dots i_{k-1}12i_{k+2}\dots i_n}(x_1,\dots,x_{k-1},x_{k+1},x_k,x_{k+2},\dots,x_n),\\
&f^{(n)}_{i_1\dots i_{k-1}21i_{k+2}\dots i_n}(x_1,\dots,x_n)\\
&\quad =Q_2(x_k,x_{k+1})f^{(n)}_{i_1\dots i_{k-1}21i_{k+2}\dots i_n}(x_1,\dots,x_{k-1},x_{k+1},x_k,x_{k+2},\dots,x_n),
\end{aligned}$$

for all $i_1,\dots,i_{k-1},i_{k+2},\dots,i_n\in\{1,2\}$ and $k\in\{1,\dots,n-1\}$.

In the case where the set Z is empty (or of zero measure), the corresponding operator T is unitary, hence $\mathbb{P}_n=\frac{1}{n!}\mathcal{P}_n$. If the set Z is of positive measure, the form $(\mathfrak{P}f^{(2)})(x,y)$ will depend on whether (x,y) is a point of Y or Z. In both cases, the explicit form of $(\mathfrak{P}f^{(2)})(x,y)$ can be easily calculated by using Corollary 3.7. We leave the details to the interested reader.

By (68), we get

$$\widetilde{C}_{x,y}=\begin{pmatrix}0&0&0&Q_2(x,y)\\0&Q_1(x,y)&0&0\\0&0&Q_1(x,y)&0\\Q_2(x,y)&0&0&0\end{pmatrix}.$$

Hence, by Corollary 3.13, we get the following formal commutation relations:

$$\begin{aligned}
a_1^-(x)a_1^+(y)&=Q_2(x,y)a_2^+(y)a_2^-(x)+\delta(x-y),\\
a_2^-(x)a_2^+(y)&=Q_2(x,y)a_1^+(y)a_1^-(x)+\delta(x-y),\\
a_1^-(x)a_2^+(y)&=Q_1(x,y)a_1^+(y)a_2^-(x),\\
a_2^-(x)a_1^+(y)&=Q_1(x,y)a_2^+(y)a_1^-(x),\\
a_1^+(x)a_2^+(y)&=Q_2(x,y)a_1^+(y)a_2^+(x),\\
a_2^+(x)a_1^+(y)&=Q_2(x,y)a_2^+(y)a_1^+(x),\\
a_1^+(x)a_1^+(y)&=Q_1(x,y)a_2^+(y)a_2^+(x)\quad \text{if } (x,y)\in Y,\\
a_1^-(x)a_2^-(y)&=Q_2(x,y)a_1^-(y)a_2^-(x),\\
a_2^-(x)a_1^-(y)&=Q_2(x,y)a_2^-(y)a_1^-(x),\\
a_1^-(x)a_1^-(y)&=Q_1(x,y)a_2^-(y)a_2^-(x)\quad \text{if } (x,y)\in Y.
\end{aligned}\tag{87}$$

Let us consider a special case of such a construction. Fix any $q_1, q_2 \in \mathbb{C}$ with $|q_1| = |q_2| = 1$ and define

$$Q_i(x,y) = \begin{cases} q_i, & \text{if } x < y, \\ \bar{q}_i, & \text{if } x > y, \end{cases} \qquad i = 1, 2.$$

With such a choice of functions Q_1, Q_2 and $d = 2$, the above construction gives an example of a non-Abelian anyon quantum system with the operator

$$C = \begin{pmatrix} 0 & 0 & 0 & q_1 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_2 & 0 \\ q_1 & 0 & 0 & 0 \end{pmatrix}.$$

Note that, in this case, the commutation relations (87) hold for all $(x,y) \in X^{(2)}$.

Further examples of such a construction can be achieved by choosing

$$Q_1(x,y) = ke^{i\alpha(x-y)}, \quad Q_2(x,y) = e^{i\beta(x-y)},$$

where $k \in [-1,1]$ and $\alpha, \beta \in \mathbb{R}$.

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