A new convergence analysis for the Volterra series representation of nonlinear systems

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Abstract

The convergence of the Volterra series representation of nonlinear systems is the fundamental requirement for the analysis of nonlinear systems in the frequency domain. In the present study, a new criterion is derived to determine the convergence of the Volterra series representation of nonlinear systems described by a NARX (Nonlinear Auto Regressive with eXegenous input) model. The analysis is performed based on a new function known as Generalized Output Bound Characteristic Function (GOBCF), which is defined in terms of the input, output and parameters of the NARX model of nonlinear systems. Compared to the existing results, the new criterion provides a much more rigorous and effective approach to the analysis of the convergence conditions and properties of the Volterra series representation of nonlinear systems. Two case studies have been used to demonstrate the effectiveness of the new convergence analysis criterion and the advantages of the new analysis over those produced by existing approaches.

Key words: Volterra series; NARX model; Nonlinear systems; Convergence criterion;

1. Introduction

The Volterra series has been widely applied in analysing nonlinear systems (Boaghe & Billings, 2003; Wang, & Mortazawi, 2016). In the frequency domain, the Volterra kernels are represented by the multi-dimensional Generalized Frequency Response Functions (GFRFs) (George, 1959), which can be determined by using a recursive algorithm from the Nonlinear Differential Equation (NDE) or Nonlinear Auto Regressive with eXegenous input (NARX) model of nonlinear systems (Peyton-Jones & Billings, 1989; Jing et al., 2009). Besides, one-dimensional functions such as the Nonlinear Output Response Functions (NOFRFs) (Lang & Billings, 2005) and the Output Frequency Response Function (OFRF) (Lang, Billings et al., 2007) have been proposed, based on the GFRFs, for the frequency domain analysis of nonlinear systems that can be represented by a Volterra series; the associated methods have been applied to the study of many nonlinear systems in engineering practice (Lang & Peng, 2008; Peng et al., 2010; Ho et al., 2014).

All these existing approaches require that the Volterra series representation for a nonlinear system is convergent. The assessment of the convergence can often be performed via numerical analyses to see whether the higher order terms of the Volterra series are degressive or not (Peng et al., 2008). Some analytical methods have also been derived to study the convergence of the Volterra series representation of relatively simple nonlinear systems (Barrett, 1965; Siu & Schetzen, 1991; Zhu & Lang, 2016). For example, the convergence of the Volterra series representation of the Duffing oscillator and quadratic nonlinear systems were discussed by using the ratio $|\sigma_{n+1}|/|\sigma_n|<1$, where $|\sigma_n|$ represents the magnitude of the nth term of the nonlinear output spectrum (Tomlinson et al., 1996; Li & Billings, 2011). Recently, by using the mathematical tools of analytic combinatorics, the Singular Inversion Theorem was used to discuss the convergence of a general Volterra series in the time domain (Hélie & Laroche, 2011; Xiao et al., 2013). The Analytic Inversion Theorem was also employed to evaluate the parameter convergence bound of a NARX model's Volterra series representation in the frequency domain (Xiao et al., 2014; Jing & Xiao, 2017).

However, using these available methods, the convergence criteria can only consider specific nonlinear systems subject to harmonic inputs (Tomlinson et al., 1996; Chatterjee & Vyas, 2000; Peng & Lang, 2007; Li & Billings, 2011) or produce, via complex operations, an over estimated bound on the system input which can ensure the convergence of the system's Volterra series representation (Hélie & Laroche, 2011; Xiao et al., 2013 and 2014; Jing & Xiao, 2017).

Therefore, it is necessary to develop a simpler and more efficient criterion for the analysis of the convergence of the Volterra series representation of a general class of non-linear systems subject to either harmonic or general input excitations. In the present study, a new convergence criterion for the Volterra series representation of the NARX model of nonlinear systems is derived to address the problems with existing methods. The new criterion is derived based on a new function known as the Generalized Output Bound Characteristic Function (GOBCF), and has the advantages of being independent of sampling frequency with the NARX model, applicable to nonlinear systems under general inputs, and having no need of carrying out complex mathematical computations. Two case studies including the analysis of an unplugged Van der Pol equation and a Duffing equation with nonlinear damping are used to demonstrate the effectiveness of the new criterion and its advantages over existing methods.

2. The analysis of nonlinear systems in the frequency domain

In practice, a large class of nonlinear systems can be described using a NARX model (Peyton-Jones & Billings, 1989):

$$y(k)=\sum_{m=1}^{M}\sum_{p=0}^{m}\sum_{k_1,k_{p+q}=1}^{K}\left[c_{p,q}\left(k_1,\cdots,k_{p+q}\right)\prod_{i=1}^{p}y(k-k_i)\times\prod_{i=p+1}^{p+q}u(k-k_i)\right] \tag{1}$$

where $y(k)$ and $u(k)$ are the system output and input at discrete time k, respectively, $p+q=m$, M and K are integers, $\sum_{k_1,k_{p+q}=1}^{K}=\sum_{k_{p+q}=1}^{K}\cdots\sum_{k_1=1}^{K}$, and $c_{p,q}\left(k_1,\cdots,k_{p+q}\right)$ are the coefficients of the system model.

If system (1) is asymptotically stable at the zero equilibrium, the output response of the system can be expressed by the discrete time Volterra series (Bayma et al., 2018)

$$y(k)=\sum_{n=1}^{+\infty}y_n(k)=\sum_{n=1}^{+\infty}\sum_{\tau_1=-\infty}^{+\infty}\cdots\sum_{\tau_n=-\infty}^{+\infty}h_n(\tau_1,\ldots,\tau_n)\prod_{i=1}^{n}u(k-\tau_i) \tag{2}$$

where $y_n(k)$ is the n th order nonlinear output of the system.

In (2), $h_n(\tau_1,\ldots,\tau_n)$ is known as the nth order discrete time Volterra kernel. The frequency domain representation of system (2) can be given by (Lang & Billings, 1996)

$$Y(\mathrm{j}\omega)=\sum_{n=1}^{+\infty}Y_n(\mathrm{j}\omega)=\sum_{n=1}^{+\infty}\frac{1}{\sqrt{n}(2\pi)^{n-1}}\times\int_{\omega_1+\cdots+\omega_n=\omega}H_n(\omega_1,\ldots,\omega_n)\prod_{i=1}^{n}U(\mathrm{j}\omega_i)\mathrm{d}\sigma_\omega \tag{3}$$

In (3), σ_ω represents the hyperplane $\omega_1+\cdots+\omega_n=\omega$ where $-\pi\le\omega\Delta t\le\pi$ and Δt represents the sampling period, $Y(\mathrm{j}\omega)$ and $U(\mathrm{j}\omega)$ are the output and input spectrum of the system, obtained by using the normalised Discrete Time Fourier Transform (DTFT) of $y(k)$ and $u(k)$, respectively. The normalised DTFT of a discrete time sequence $x(k)$ is here defined as $DF[x(k)]\Delta t$ with $DF[.]$ denoting DTFT.

$$H_n(\omega_1,\ldots,\omega_n)=\Delta t\sum_{\tau_1=-\infty}^{+\infty}\cdots\sum_{\tau_n=-\infty}^{+\infty}h_n(\tau_1,\ldots,\tau_n)\times\exp\left(-\mathrm{j}(\omega_1\tau_1\Delta t+\cdots+\omega_n\tau_n\Delta t)\right) \tag{4}$$

is the GFRFs. Given the NARX model (1) of a nonlinear system, the GFRFs of the system can be determined using a recursive algorithm (Peyton-Jones & Billings, 1989).

In order to evaluate the convergence of the Volterra series representation of a nonlinear system, a sufficient condition as described in the following Lemma can be applied.

Lemma 1 (Tomlinson et al., 1996)**:** *If there exists an integer N^* such that for all ω,*

$$|\sigma_{n+1}|/|\sigma_n|<1 \tag{5}$$

for all $n>N^*$, where $|\sigma_n|$ is the magnitude of the n th non zero term on the right hand side of equation (3), then the Volterra series representation of a nonlinear system is convergent.

In practice, the use of Lemma 1 can be implemented by introducing a pre-specified threshold ρ and evaluating whether there exists a N^* such that when $\bar{N}\ge N^*$

$$\eta_N=\frac{\left|Y(\mathrm{j}\omega)-\sum_{n=1}^{\bar{N}}Y_n(\mathrm{j}\omega)\right|}{|Y(\mathrm{j}\omega)|}\times100\%\le\rho\quad\text{for all }\omega \tag{6}$$

This approach will be applied later on to validate the new convergence criterion proposed in the present study.

Remark 1: In order to study the issue of convergence using equation (5), $|\sigma_n|, n=1,2,\ldots$ have to be determined first. The Associate Linear Equations (ALEs) for the NARX model of nonlinear systems proposed in (Bayma et al., 2018) can be applied to accurately determine the system output contributed by different order nonlinearity.

3. The new convergence criterion for the Volterra series representation of nonlinear systems

In this section, a new convergence criterion for the Volterra series representation of the NARX model of nonlinear systems will be derived. This will be achieved using a newly defined function known as the Generalized Output Bound Characteristic Function (GOBCF), which is determined by the system input, output and model parameters. The new convergence analysis will be performed based on the zeros of the GOBCF.

3.1. The Generalized Output Bound Characteristic Function (GOBCF)

According to the condition of the existence of the Volterra series representation of a nonlinear system, it has been shown that if the system output spectrum is bounded and this bound continuously and smoothly changes against the input magnitude over the whole frequency range, then the system can be represented by a Volterra series (Hélie & Laroche, 2011; Xiao et al., 2013). In order to obtain this bound for the system output spectrum, the definition of the GOBCF is first introduced as follows.

Definition 1: *The GOBCF of the NARX model (1) is defined as*

$$f_{BC}(x)=x-\bar{L}_{W_{\mathrm{lin}}}\bar{C}_{0,1}[\![u]\!]-\sum_{m=2}^{M}\sum_{p=0}^{m}\bar{L}_W\bar{C}_{p,m-p}x^p[\![u]\!]^{m-p} \tag{7}$$

where x is the variable of the GOBCF,

$$\bar{C}_{p,m-p}=\sup_{\substack{\omega_1,\ldots,\omega_p\in W\\ \omega_{p+1},\ldots,\omega_m\in W_0}}\left|C_{p,m-p}^{sym}(\omega_1,\ldots,\omega_m)\right| \tag{8}$$

with

$$\left|C_{p,m-p}^{sym}(\omega_1,\ldots,\omega_m)\right|=\left|\frac{1}{m!}\sum_{\substack{v_1,\cdots,v_m\in\{1,\ldots,m\}\\ v_1\ne\cdots\ne v_m}}C_{p,m-p}(\omega_{v_1},\ldots,\omega_{v_m})\right|$$

$$\bar{L}_{W_{\mathrm{lin}}}=\sup_{\omega\in W_0}|L(\mathrm{j}\omega)|,\ \bar{L}_w=\sup_{\omega\in W}|L(\mathrm{j}\omega)| \tag{9}$$

where

$$L(\mathrm{j}\omega)=\left[1-\sum_{k_1=1}^{K}c_{1,0}(k_1)\exp(-\mathrm{j}k_1\omega\Delta t)\right]^{-1} \tag{10}$$

$$C_{p,m-p}(\omega_1,\ldots,\omega_m)=\sum_{k_1,k_m=1}^{K}c_{p,m-p}(k_1,\cdots,k_m)\times\exp\left(-\mathrm{j}(k_1\omega_1+\cdots+k_m\omega_m)\Delta t\right) \tag{11}$$

$\boldsymbol{W}_0$ is the input frequency range, $\boldsymbol{W}\subseteq[-\pi/\Delta t,\pi/\Delta t]$, and

$$[\![u]\!]=\max\left\{F^{-1}\left[|U(\mathrm{j}\omega)|\right]_{t\in(-\infty,+\infty)},|U(\mathrm{j}\omega)|_{\omega\in\boldsymbol{W}_0}\right\} \tag{12}$$

where $F^{-1}[.]$ denotes the inverse Fourier Transform (FT).

This new function of GOBCF is important as it is related to a bound on the output spectrum of the NARX model (1) of nonlinear systems as described in the following proposition.

Proposition 1: *One of the solutions to $f_{BC}(x)=0$ is*

$$[\![y]\!]=\sum_{n=1}^{+\infty}\bar{H}_n[\![u]\!]^n=\bar{H}_1[\![u]\!]+\cdots+\bar{H}_n[\![u]\!]^n+\cdots \tag{13}$$

where

$$\begin{cases}\bar{H}_1=\bar{L}_{w_{\text{lin}}}\bar{C}_{0,1}\\ \bar{H}_n=\bar{L}_w\left(\bar{C}_{0,n}+\sum_{m=2}^{M}\sum_{p=1}^{m}\left(\bar{C}_{p,m-p}\sum_{r_1,\ldots,r_p=1,\sum r_i=n-m+p}^{n-m+1}\prod_{i=1}^{p}\bar{H}_{r_i}\right)\right)\end{cases} \tag{14}$$

and $[\![y]\!]$ is a bound on the output spectrum of the NARX model (1) such that $[\![y]\!]\geq\max_{\omega\in\boldsymbol{W}}|Y(\mathrm{j}\omega)|$.

Proof: See Appendix A.

Remark 2: Generally speaking, the frequency range $\boldsymbol{W}$ contains all possible output frequencies, which can be obtained using the algorithm in (Lang & Billings, 1996) provided the sampling frequency $1/\Delta t$ with the NARX model is sufficiently high. However, in practice, if only the system nonlinearity up to $\hat{N}$ th order needs to be taken into account considering the effect of higher frequencies on the evaluation of $\max_{\omega\in\boldsymbol{W}}|Y(\mathrm{j}\omega)|$ is negligible, then

$$\boldsymbol{W}\subseteq\left[-\hat{N}\omega_{\max},\hat{N}\omega_{\max}\right] \tag{15}$$

where $\omega_{\max}$ represents the maximum frequency of the system input, and $2\hat{N}\omega_{\max}\leq 2\pi/\Delta t$. It is worthy to note that an appropriate choice of $\hat{N}$ is important for the analysis of the convergence of a Volterra series representation. This can be achieved by evaluating the system output response of concern using the priori known NARX model of the system

Remark 3: The value of $\bar{L}_w$ can be obtained from (9). $\bar{C}_{p,m-p}$ is the maximum of a known multi-variable symmetric function and can be determined using a numerical optimization approach such as, for example, the exhaustive search method (Miller & Thomson, 1994) or the gradient descent method (Tseng & Yun, 2009). In the present study, the exhaustive search method is applied to find $\bar{C}_{p,m-p}=\max\left|C^{sym}_{p,m-p}(\omega_1,\ldots,\omega_m)\right|$ with $\omega_1,\ldots,\omega_p\in\boldsymbol{W}$ and $\omega_{p+1},\ldots,\omega_m\in\boldsymbol{W}_0$, where all frequency variables are limited by the sampling frequency. In addition, as the nonlinear degree M in an identified NARX model is often selected to be low to reduce complexity and avoid possible unstable models (Napoli & Piroddi, 2010; Zhu et al., 2015), $C^{sym}_{p,m-p}(\omega_1,\ldots,\omega_m)$ can, in many cases, be reduced to a one-variable function such that its bound $\bar{C}_{p,m-p}$ can readily be determined.

Remark 4: Generally, following the ideas of evaluating the GFRFs for the NARX model (Peyton-Jones & Billings, 1989, Billings & Yusof, 1996), the specific form of function $\bar{C}_{p,m-p}$ can be determined using computer codes conducting symbolic computations.

In the following, the GOBCF will be used for the analysis of the convergence with the Volterra series representation of the NARX model of nonlinear systems.

3.2. Convergence analysis of the Volterra series representation of nonlinear systems

Proposition 1 implies that if there exist real positive solutions to equation $f_{BC}(x)=0$, then the output spectrum of the NARX model (1) is bounded by one of such solutions which satisfies (13). Consequently, the NARX model (1) can be described, around zero equilibrium, by a convergent Volterra series representation. In the following, Corollary 1 is introduced to reveal all possible situations about the solutions to equation $f_{BC}(x)=0$.

Corollary 1: *Depending on $[\![u]\!]$ and the values of $\bar{L}_w$, $\bar{C}_{0,1}$ and $\bar{C}_{p,m-p}$, $p=0,\ldots,m$ with $m=2,\ldots,M$, which are associated with the NARX model parameters, there exist only three cases for the solutions to the equation*

$$f_{BC}(x)=x-\bar{L}_{w_{\text{lin}}}\bar{C}_{0,1}[\![u]\!]-\sum_{m=2}^{M}\sum_{p=0}^{m}\bar{L}_w\bar{C}_{p,m-p}x^p[\![u]\!]^{m-p}=0 \tag{16}$$

which are

(A) There are two real positive solutions $\bar{x}_{\min}$ and $\bar{x}_{\max}$ with $\bar{x}_{\max}>\bar{x}_{\min}$ or

(B) There is one real positive solution $\bar{x}_{\text{one}}$ or

(C) There is no real positive solution.

Proof: Evaluating the first and second derivative of the GOBCF $f_{BC}(x)$ yields

$$f'_{BC}(x)=1-\sum_{m=2}^{M}\sum_{p=1}^{m}p\bar{L}_w\bar{C}_{p,m-p}x^{p-1}[\![u]\!]^{m-p} \tag{17a}$$

$$f''_{BC}(x)=-\sum_{m=2}^{M}\sum_{p=2}^{m}p(p-1)\bar{L}_w\bar{C}_{p,m-p}x^{p-2}[\![u]\!]^{m-p} \tag{17b}$$

It is known from (17b) that $f''_{BC}(x)\leq 0$ when $x\geq 0$. Considering $f_{BC}(0)\leq 0$ is always satisfied since $\bar{L}_w$, $\bar{C}_{p,m-p}$, x and $[\![u]\!]$ are all positive, it is known that

(i) When $f'_{BC}(0)>0, f''_{BC}(x)<0$, for $x\geq 0$, $f_{BC}(x)$ will increase first and then decrease. Therefore, there exist cases (A)-(C) about the solutions to equation (16) which are illustrated in Fig.1 (a)-(d).

(ii) When $f'_{BC}(0)>0, f''_{BC}(x)=0$, for $x\geq 0$, $f_{BC}(x)$ will monotonically increase. Therefore, cases (B) and (C) exist about the solutions to equation (16) as illustrated in Fig.1 (f)-(g).

(iii) When $f'_{BC}(0)\leq 0$, for $x\geq 0$, equation (16) has no real positive solution. This is case (C) but in a different situation as shown in Fig.1 (e), (h) and (i).

By combining (i), (ii) and (iii), the conclusions of Corollary 1 are reached.

Based on Proposition 1 and Corollary 1, a sufficient condition on the convergence of the Volterra series representation of nonlinear systems can be derived. For the purpose of quantitatively evaluating the convergence issue, under condition (i), the extreme point of the GOBCF is denoted as $E\left(\bar{x}_{ex}, f_{BC}\left(\bar{x}_{ex}\right)\right)$ and $E\left(\bar{x}_0, f_{BC,[\![u]\!]=0}\left(\bar{x}_0\right)\right)$, which can be obtained by solving $f'_{BC}(x)=0$ for x in the two cases of $[\![u]\!]\neq 0$ and $[\![u]\!]=0$, respectively as shown in Fig.1.

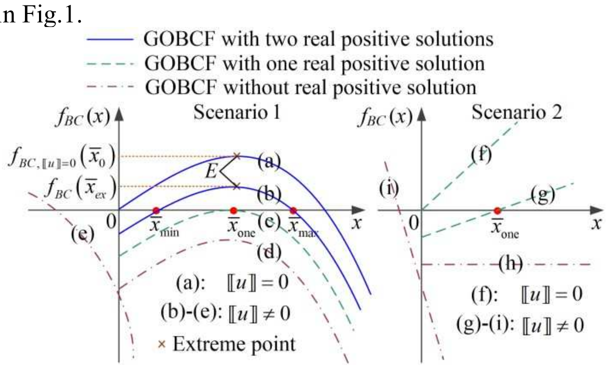


Fig.1. Illustration of the different situations of GOBCF with and without real positive solutions

A sufficient criterion for the NARX model to be described by a convergent Volterra series model can, consequently, be obtained as given in Proposition 2 below.

Proposition 2: *If GOBCF $f_{BC}(x)=0$ has real positive solutions, then, around zero equilibrium, the output response of the NARX model (1) can be described by a convergent Volterra series representation. In addition, the extent to which this representation is convergent can be quantified by*

$$\Gamma=\begin{cases}\dfrac{f_{BC}\left(\bar{x}_{ex}\right)}{f_{BC,[\![u]\!]=0}\left(\bar{x}_0\right)} & f'_{BC}(0)>0,\ f''_{BC}(x)<0,\ x\geq 0\\ f'_{BC}(0) & f'_{BC}(0)>0,\ f''_{BC}(x)=0,\ x\geq 0\\ -1 & \text{Otherwise}\end{cases} \tag{18}$$

such that when $[\![u]\!]$, $\bar{L}_w$, or $\bar{C}_{p,m-p}$ in the GOBCF decreases, $\Gamma\to 1$, the convergence is enhanced; and when $[\![u]\!]$, $\bar{L}_w$, or $\bar{C}_{p,m-p}$ increases, $\Gamma\to 0$, the convergence is weaken.

Proof: See Appendix B.

Corollary 2: *The Volterra series representation of the NARX model (1) under harmonic input*

$$u(k)=A\cos\left(\omega_F k\Delta t+\varphi\right) \tag{19}$$

is convergent if $\Gamma\in[0,1]$, where Γ is obtained from (18) using $f_{BC}\left(\bar{x}_{ex}\right)$, $f_{BC,[\![u]\!]=0}\left(\bar{x}_0\right)$, and $f'_{BC}(0)$ determined with $[\![u]\!]=A$,

$$\bar{C}_{p,m-p}=\sup_{\substack{\omega_1,\ldots,\omega_p\in W_H\\ \omega_{p+1},\ldots,\omega_m=\pm\omega_F}}\left|\frac{1}{m!}\sum_{\substack{v_1,\cdots,v_m\in\{1,\ldots,m\}\\ v_1\neq\cdots\neq v_m}}C_{p,m-p}\left(\omega_{v_1},\ldots,\omega_{v_m}\right)\right| \tag{20}$$

$$\bar{L}_w=\sup_{\omega\in W_H}\left|1-\sum_{k_1=1}^{K}c_{1,0}\left(k_1\right)\exp\left(-\mathrm{j}k_1\omega\Delta t\right)\right|^{-1} \tag{21}$$

where $W_H=\left\{\left|\pm\beta\omega_F\right|\leq\pi/\Delta t\,\middle|\,\beta\in Z^+,\beta\leq\hat{N}_H\right\}$ represents the system output frequencies over which the bound on the output spectrum can be evaluated and $\hat{N}_H\leq\pi/\left(\Delta t\omega_F\right)$.

Proof: Corollary 2 can be proven by using Proposition 2.

Remark 5: It can be shown that the above convergent analysis is independent of sampling frequency. This is due to the fact that the frequency domain descriptions of a continuous time system and its corresponding discrete time system are equivalent provided the dynamics of the continuous time system can be fully represented by its corresponding discrete time model. When $\Delta t\to 0$, for example, it is sufficient to say that the frequency domain descriptions of the continuous and discrete time models will be the same (Billings & Li, 2000). Therefore, the convergency analysis is independent of the sampling frequency.

Proposition 2 provides an efficient criterion to assess the convergence of the Volterra series representation by using a dimensionless criterion valued from 0 to 1. The results can also be applied to determine the convergence bound of a nonlinear system in terms of its parameters or input. The results are shown in Proposition 3 below.

Proposition 3: *The convergence bound on the characteristic parameters or input of the NARX model (1) can be obtained by solving the simultaneous equations*

$$\begin{cases}f_{BC}(\xi,x)=x-\bar{L}_{W_{\text{lin}}}\bar{C}_{0,1}[\![u]\!]-\sum_{m=2}^{M}\sum_{p=0}^{m}\bar{L}_w\bar{C}_{p,m-p}x^p[\![u]\!]^{m-p}=0\\ f'_{BC}(\xi,x)=1-\bar{L}_w\sum_{m=2}^{M}\sum_{p=1}^{m}p\bar{C}_{p,m-p}x^{p-1}[\![u]\!]^{m-p}=0\end{cases} \tag{22a}$$

under condition (i) or

$$f'_{BC}(\xi)=1-\bar{L}_w\sum_{m=2}^{M}\bar{C}_{1,m-1}[\![u]\!]^{m-1}=0 \tag{22b}$$

under condition (ii) for ξ where ξ, dependending on the need of analysis, can be $\bar{L}_w$, $\bar{C}_{p,m-p}$ or $[\![u]\!]$ representing the convergence bound on the system linear characteristic parameters, nonlinear characteristic parameters, or input.

Proof: Proposition 3 can be obtained from Fig.1 by evaluating the conditions under which function $f_{BC}(x)$ is in situations (c) and (h), respectively.

The effect of the characteristic parameters of nonlinear systems on the convergence of systems' Volterra series representation has been investigated using either numerical methods or analytical approaches in (Xiao et al., 2014; Jing & Xiao, 2017). However, more rigorous and less conservative results can be obtained using Proposition 3, which will be demonstrated in the case studies.

3.3. The procedure for the new analysis

A general procedure for analyzing the convergence of the Volterra series representation of the NARX model (1) is summarized in the following, where Propositions 2 and 3 are applied to assess the convergence and evaluate a convergence bound, respectively.

1) The procedure for the analysis when the system is subject to a general input

If the system subject to a general input with spectrum $U(\mathrm{j}\omega)$, the analysis procedure is summarised as follows:

Procedure of the convergence analysis

1: **Produce the system's NARX model:** Describe the system by a NARX model (1), such that all linear and nonlinear coefficients $c_{p,q}(.)$ can be determined. This can be achieved by using a data driven system identification approach (Billings, 2013) or discretising a continuous time model of the system under study.

2: **Determine the system's output frequency range of interest:** From the maximum input frequency $\omega_{\max}$ and the maximum order $\hat{N}$ of the system nonlinearity that needs to be taken into account for evaluating the bound on the output spectrum, find the output frequency range of interest $\boldsymbol{W} \subseteq \left[-\hat{N}\omega_{\max}, \hat{N}\omega_{\max}\right]$.

3 **Evaluate the input bound:** Compute $F^{-1}\left[\left|U\left(\mathrm{j}\omega\right)\right|\right]$, and the input bound

$$[\![u]\!] = \max\left\{F^{-1}\left[\left|U\left(\mathrm{j}\omega\right)\right|\right]_{t\in(-\infty,+\infty)}, \left|U\left(\mathrm{j}\omega\right)\right|_{\omega\in\boldsymbol{W}_0}\right\}.$$

4 **Compute the coefficients bound:** Calculate $\bar{L}_{w_{\text{lin}}}$ and $\bar{L}_w$ according to (9) over the frequency range of $\omega\in\boldsymbol{W}_0$ and $\omega\in\boldsymbol{W}$, respectively, and determine the value of $\bar{C}_{p,m-p}$ from (8), and the NARX model coefficients.

5: **Calculate the extreme point/slope of the GOBCF:** Compute the first and second derivatives of the GOBCF $f_{BC}\left(x\right)$ using equation (17). Then, solve equation $f'_{BC}\left(x\right)=0$ to obtain $E\left(\bar{x}_{ex}, f_{BC}\left(\bar{x}_{ex}\right)\right)$ and $E\left(\bar{x}_0, f_{BC,[\![u]\!]=0}\left(\bar{x}_0\right)\right)$ of the GOBCF in the cases of $[\![u]\!]\neq 0$ and $[\![u]\!]=0$, respectively.

6: **Convergence assessment:** Determine Γ from (18) by using $f_{BC}\left(\bar{x}_{ex}\right)$, $f_{BC,[\![u]\!]=0}\left(\bar{x}_0\right)$, and $f'_{BC}\left(0\right)$ obtained in Step 5. If $\Gamma\in[0,1]$, then it can be concluded that the Volterra series representation of the nonlinear system is convergent.

7: **Evaluation of the parameter bound:** Determine the convergence bound on the system characteristic parameters or input by solving equation (22) for ξ.

2) The procedure for the analysis when the system is subject to a harmonic input

If the input signal of the system is the harmonic input (19), the convergence of the system's Volterra series representation can be analysed following Steps 1 to 7 above but, a $\hat{N}_H \le \pi f_s/\omega_F$ is selected to determine the frequency range of $\left\{\beta\omega_F \le \pi/\Delta t \middle| \beta\in Z^+, \beta\le \hat{N}_H\right\}$ in Step 2, and the input bound $\|u\| = A$ in Step 3. In Steps 4 and 5, $\bar{L}_{w_{\text{lin}}}$, $\bar{L}_w$ and $\bar{C}_{p,m-p}$ are determined over the frequency range of $\omega\in\boldsymbol{W}_H$. Steps 6 and 7 are the same as in the general input case.

In comparison with the convergence criteria recently proposed in (Hélie & Laroche, 2011; Xiao et al., 2013 and 2014; Jing & Xiao, 2017), the newly proposed convergent analysis has the following advantages:

(I) The complex mathematical operations needed for evaluating both the Hélie's and Xiao's criterion (Hélie & Laroche, 2011; Xiao et al., 2013 and 2014) are avoided. In addition, Proposition 3 can provide a more rigorous and less conservative analysis than the analysis in Xiao's study.

(II) In the Xiao's criterion (Xiao et al., 2013 and 2014), the coefficient bound $\bar{C}_{p,m-p}$ in (20) is given as the summation of all absolute values of the model coefficients, producing overestimated results. In addition, the result is dependent on the sampling frequency of the NARX model, so the analysis may fail if the sampling frequency is inappropriately selected. On the contrary, the new criterion is independent of the sampling frequency. This will be discussed in details in Case study 1.

(III) It is worth pointing out that the convergence criterion under harmonic input proposed in previous works (Tomlinson et al., 1996; Chatterjee & Vyas, 2000; Peng & Lang, 2007; Xiao et al., 2013 and 2014) cannot be directly used for general inputs (Jing & Xiao, 2017). By using the new convergence criterion, however, the convergence analysis problems can be resolved for both harmonic and general input cases. The new convergence analysis under a general input will be demonstrated in Case study 2.

In the next section, an Unplugged Van der Pol equation and a damped Duffing oscillator will be used in two case studies, respectively, to demonstrate the application of the proposed criterion to the analysis of the convergence of the Volterra series representation of nonlinear systems.

4. Case studies

4.1. Case 1- Unplugged Van der Pol equation

Consider the unplugged Van der Pol equation (Mickens, 2001; Tacha et al., 2016)

$$\ddot{y}\left(t\right)+c\dot{y}\left(t\right)+ky\left(t\right)+c_e y^2\left(t\right)\dot{y}\left(t\right)=u(t) \tag{23}$$

under input $u\left(t\right)=A\cos\left(\omega_F t\right)$ with the parameters

$$c=50\ \text{N/ms},\ k=10^4\ \text{N/m},\ c_e=2\times10^7\ \text{N}^3/\text{m}^2\text{s} \tag{24}$$

Model (23) can be discretized under a sampling frequency $1/\Delta t=512$ Hz to produce a NARX model

$$\begin{aligned} y\left(k\right) &= c_{0,1}\left(1\right)u(k-1)+c_{1,0}\left(1\right)y(k-1)\\ &+c_{1,0}\left(2\right)y(k-2)+c_{3,0}\left(1,1,1\right)y^3(k-1)\\ &+c_{3,0}\left(1,1,2\right)y^2(k-1)y(k-2) \end{aligned} \tag{25}$$

where

$$\begin{aligned} &c_{0,1}\left(1\right)=\Delta t^2;\ c_{1,0}\left(1\right)=2-\Delta tc-\Delta t^2 k;\\ &c_{1,0}\left(2\right)=\Delta tc-1;\ c_{3,0}\left(1,1,1\right)=-\Delta tc_e;\ c_{3,0}\left(1,1,2\right)=\Delta tc_e \end{aligned} \tag{26}$$

To analyse the convergence of the Volterra series representation of system (23), Steps 1-6 proposed in Section 3.3 are followed as follows.

Step 1: The NARX model is derived and given in (25);

Step 2: Because, in this case, the system output response at driven frequency ω_F is dominant so that

$$\max_{\omega\in\boldsymbol{W}}\left|Y\left(\mathrm{j}\omega\right)\right| = \max_{\omega=\pm\omega_F}\left|Y\left(\mathrm{j}\omega\right)\right|$$

$\hat{N}_H=1$, and the system output frequency range of interest is determined as $\boldsymbol{W}_H=\left\{\pm\omega_F\right\}$;

Step 3: Let $[\![u]\!]=A$;

Step 4: $\bar{L}_{w_{\text{lin}}}$ and $\bar{L}_w$ are obtained over $\omega\in\boldsymbol{W}_H$. For the NARX model (25), $M=3$ and $c_{p,m-p}\left(k_1,\cdots,k_m\right)=0$ except $c_{3,0}\left(1,1,1\right)$ and $c_{3,0}\left(1,1,2\right)$ as shown in (26).

$$\begin{aligned} \bar{C}_{p,m-p} &= \bar{C}_{3,0}\\ &= \sup_{\substack{\omega_1,\omega_2,\omega_3=\pm\omega_F;\\ \omega_1+\omega_2+\omega_3=\pm\omega_F}} \left|\frac{1}{6}\sum_{\substack{v_1,v_2,v_3\in\{1,2,3\}\\ v_1\neq v_2\neq v_3}} c_{3,0}\left(1,1,1\right)\exp\left(-\mathrm{j}\Delta t\sum_{i=1}^{3}\omega_{v_i}\right)\right.\\ &\quad\left.+c_{3,0}\left(1,1,2\right)\exp\left(-\mathrm{j}\left(\omega_{v_1}+\omega_{v_2}+2\omega_{v_3}\right)\Delta t\right)\right|\\ &= \sup_{\omega_1,\omega_2,\omega_3=\pm\omega_F}\Delta tc_e\left|\frac{1}{3}\sum_{i=1}^{3}\left[1-\exp\left(-\mathrm{j}\omega_{v_i}\Delta t\right)\right]\right| \end{aligned} \tag{27}$$

Step 5: Substituting $[\![u]\!]$, $\bar{L}_w$ and $\bar{C}_{p,m-p}$ in this specific case into (17) yields

$$f'_{BC}(x) = f'_{BC,[\![u]\!]=0}(x) = 1 - 3\bar{L}_w \bar{C}_{3,0} x^2 = 0 \tag{28}$$

producing $E(\bar{x}_{ex}, f_{BC}(\bar{x}_{ex}))$ and $E(\bar{x}_0, f_{BC,[\![u]\!]=0}(\bar{x}_0))$. In this case, $\bar{x}_{ex} = \bar{x}_0$ but the value of $f_{BC}(\bar{x}_{ex})$ and $f_{BC,[\![u]\!]=0}(\bar{x}_0)$ are different, as illustrated in Fig.1.

Step 6: Determine Γ from (18) by using $f_{BC}(\bar{x}_{ex})$ and $f_{BC,[\![u]\!]=0}(\bar{x}_0)$ obtained in Step 5.

Following Steps 1-6 above, the new criterion Γ was evaluated over the frequency range of $\omega_F \in [0, 300]$ rad/s for $A = 4$ N . The results are shown in Fig.2 where the results evaluated using the Xiao's criterion (Xiao et al., 2013 and 2014) are also provided for comparison. It can be observed in Fig.2 that at high frequencies such as $\omega_{F,1} = 200$ rad/s , the new and Xiao's criterion both show the Volterra series representation is convergent. However, in a wide range of low frequencies, such as $\omega_{F,2} = 100$ rad/s , the new criterion indicates that the Volterra representation is convergent, while the Xiao's criterion indicates it may be divergent.

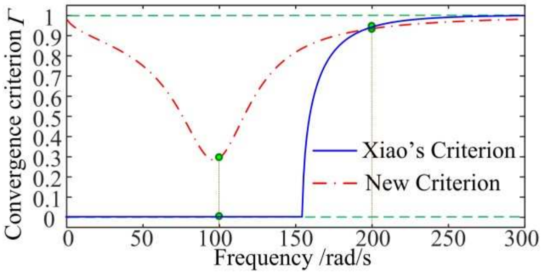


Fig.2. Convergence analysis for nonlinear system (25)

In order to validate the conclusion of the new criterion, the nonlinear output spectra up to the 5th order were calculated by using the ALEs over the frequency range of $\omega_F \in [0, 300]$ rad/s . The results are shown in Fig.3, indicating that $|Y_1(j\omega_{F,2})| > |Y_3(j\omega_{F,2})| > |Y_5(j\omega_{F,2})|$ where $\omega_{F,2} = 100$ rad/s . This observation from Fig.3 confirms that the conclusion from the new criterion is correct.

In addition, the values of the nonlinear output spectra up to 5th order at frequency $\omega_{F,2} = 100$ rad/s are shown in Tab.1. It is known from the convergence analysis using Lemma 1 that if the threshold ρ is taken as $\rho = 10^{-2}$, then Tab.1 implies that the system's Volterra series description can be convergent from $N = 5$ as

$$|Y_1(j\omega_{F,2})| > |Y_3(j\omega_{F,2})| > |Y_5(j\omega_{F,2})| \tag{29}$$

and $\eta_5 = 0.5952 \times 10^{-3} < 10^{-2}$.

It is worth noting that, by using the ALEs, the validation can be conducted by evaluating the output spectra up to an arbitrary high order. For example, it has also been observed that $|Y_1(j\omega_{F,2})| > |Y_3(j\omega_{F,2})| > \cdots > |Y_{13}(j\omega_{F,2})|$ but the details are omitted here due to space limit.

In the following, the effects of input magnitude and nonlinear parameters on the convergence of the Volterra series representation will first be discussed; Then, the effect of sampling frequency on the convergence analysis will be investigated; After that, the convergence boundary on the parameter and input amplitude of the unplugged Van der Pol system (23) will be evaluated.

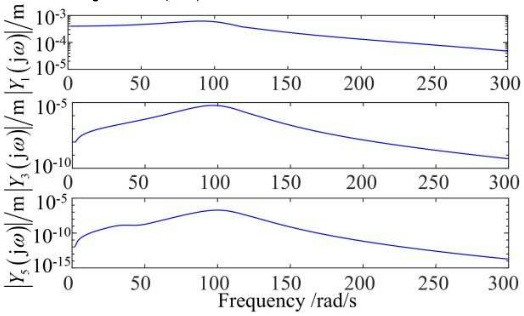


Fig.3. The nonlinear output spectra of system (25)

Tab.1. Evaluated nonlinear output spectra

	$\omega_{F,2} = 100$ rad/s	Relative error η_N
$\lvert Y_1(j\omega_{F,2})\rvert$	5.8533×10^{-4} m	$\eta_1 = 0.5944 \times 10^{-1}$
$\lvert Y_3(j\omega_{F,2})\rvert$	3.8892×10^{-5} m	$\eta_3 = 0.1094 \times 10^{-1}$
$\lvert Y_5(j\omega_{F,2})\rvert$	8.6722×10^{-6} m	$\eta_5 = 0.5952 \times 10^{-2}$

(1) Effect of the input magnitude and nonlinear parameters

It is obvious that if all system parameters are fixed, the output bound $[\![y]\!]$ will increase with the input magnitude. The criterion Γ is expected to decrease with the increase of the input magnitude, weakening the convergence of the system's Volterra series representation. The convergence criterion Γ under different input magnitudes of $A = [2, 4, 6]$ N is shown in Fig.4. Clearly, the results are consistent with the expectation.

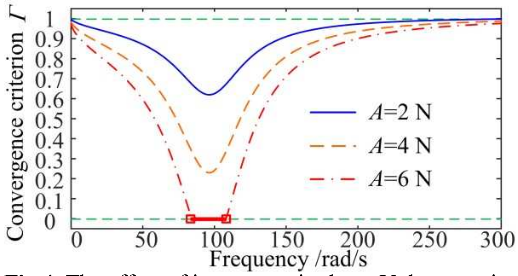


Fig.4. The effect of input magnitude on Volterra series convergence with $c_e = 2 \times 10^7$ N^3/m^{2}s and $1/\Delta t = 512$ Hz

Moreover, the effects of different nonlinear parameter of $c_e = [1\times10^7, 2\times10^7, 4\times10^7]$ N^3/m^{2}s on the results of the new criterion Γ are shown in Fig.5, which is again as expected.

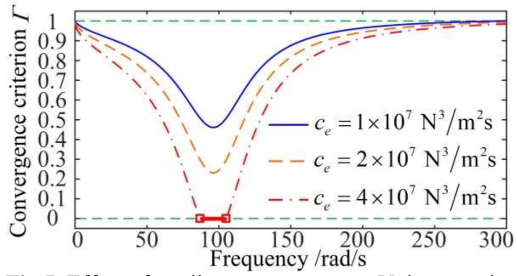


Fig.5. Effect of nonlinear parameter on Volterra series convergence with $A = 4$ N and $1/\Delta t = 512$ Hz

In Figs.4 and 5, the thick red line represents the possibly divergent range where $\Gamma < 0$ and the two red squares represent the point with $\Gamma = 0$ where the Volterra series is convergent according to the new criterion.

(2) Effect of the sampling frequency

The new convergence criterion Γ under three different sampling frequencies of $1/\Delta t = [512, 1024, 2048]$ Hz was evaluated. The results are shown in Fig.6. The results evaluated by using the Xiao's criterion (Xiao et al., 2013 and 2014) are also shown in Fig.6. These results are all obtained when $A = 2$ N and $c_e = 1\times10^7$ $\text{N}^3/\text{m}^2\text{s}$.

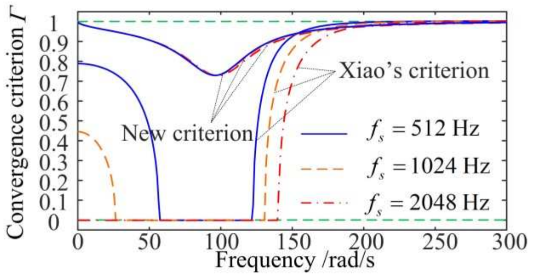


Fig.6. Effect of sampling frequency on new and Xiao's criterion

Fig.6 indicates that the new criterion is not sensitive to the sampling frequency while the Xiao's criterion (Xiao et al., 2013 and 2014) is, and may therefore fail to validate the convergence of the Volterra series representation of a NARX model if an inappropriate sampling frequency is used. The results confirm the statement in Remark 5 in Section 3.2 about the sampling frequency independent property of the newly proposed convergence analysis.

(3) The convergence boundary

In the case of the NARX model (25), equation (22) in Proposition 3 becomes

$$\begin{cases} f_{BC}(\xi, x) = x - \bar{L}_{w_{\text{lin}}}\bar{C}_{0,1}A - \bar{L}_w\bar{C}_{3,0}x^3 = 0 \\ f'_{BC}(\xi, x) = 1 - 3\bar{L}_w\bar{C}_{3,0}x^2 = 0 \end{cases} \tag{30}$$

By taking $\xi = A$ in (30), the convergence boundary of the input magnitude was calculated for $\omega_F \in [0, 300]$ rad/s when $c_e = 2\times10^7$ $\text{N}^3/\text{m}^2\text{s}$. The results are shown in Fig.7. Moreover, by taking $\xi = c_e$ in (30), the convergence boundary of the nonlinear parameter c_e was also evaluated for $\omega_F \in [0, 300]$ rad/s in the case of $A = 4$ N. The results are shown in Fig.8.

In Figs.7 and 8, a numerical boundary is also provided and referred to as the "true" convergence boundary to justify the accuracy of the boundary determined using the new criterion. The numerical boundary is obtained based on Lemma 1 in Section 2.1, by finding a boundary for A or c_e such that when A or c_e is below this boundary.

$$|Y_1(j\omega_F)| > |Y_3(j\omega_F)| > \cdots > |Y_{13}(j\omega_F)| \tag{31}$$

with $\eta_{13} < 10^{-3}$, and $Y_n(j\omega_F), n = 1, \ldots, 13$ are calculated by using the ALEs of system (25).

The results in Figs.7 and 8 indicate that the new criterion provides a more accurate convergence boundary than the Xiao's criterion (Xiao et al., 2013 and 2014).

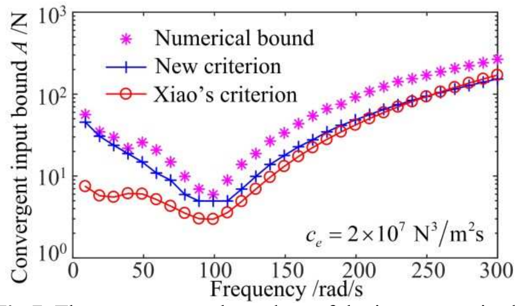


Fig.7. The convergence boundary of the input magnitude

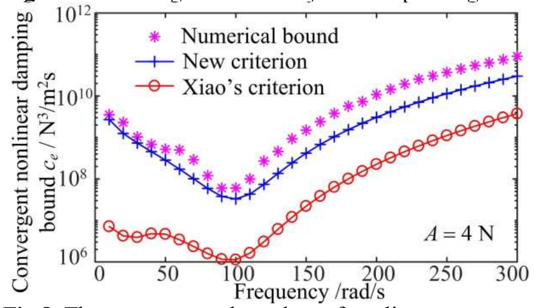


Fig.8. The convergence boundary of nonlinear parameter

4.2. *Case 2- Duffing oscillator with cubic damping*

The Duffing oscillator with cubic damping is described by

$$\ddot{y}(t) + c\dot{y}(t) + ky(t) + k_3y^3(t) + c_3\dot{y}^3(t) = u(t) \tag{32}$$

where

$$c = 50 \text{ N/ms}, k = 10^4 \text{ N/m}, \\ k_3 = 5\times10^8 \text{ N}^3/\text{m}^3, c_3 = 2\times10^3 \text{ N}^3/\text{m}^3\text{s}^3 \tag{33}$$

Under the sampling frequency $1/\Delta t = 512$ Hz, system (32) is discretized (using the forward difference for first order derivatives and the backward difference for second order derivatives (Worden & Tomlinson, 2000)), producing a NARX model as

$$\begin{aligned} y(k) &= c_{0,1}(1)u(k-1) + c_{1,0}(1)y(k-1) + c_{1,0}(2)y(k-2) \\ &+ c_{3,0}(1,1,1)y^3(k-1) + c_{3,0}(1,1,2)y^2(k-1)y(k-2) \\ &+ c_{3,0}(1,2,2)y(k-1)y^2(k-2) + c_{3,0}(2,2,2)y^3(k-2) \end{aligned} \tag{34}$$

where

$$\begin{aligned} &c_{0,1}(1) = \Delta t^2; c_{1,0}(1) = 2 - \Delta tc - \Delta t^2k; \\ &c_{1,0}(2) = \Delta tc - 1; c_{3,0}(1,1,1) = -k_3\Delta t^2 - c_3/\Delta t; \\ &c_{3,0}(1,1,2) = 3c_3/\Delta t; c_{3,0}(1,2,2) = -3c_3/\Delta t; \\ &c_{3,0}(2,2,2) = c_3/\Delta t \end{aligned} \tag{35}$$

In this case study, the system input is taken as

$$u(k) = \frac{A}{\pi}\frac{\sin(150(k-\tau_0)) - \sin(50(k-\tau_0))}{k-\tau_0} \tag{36}$$

where $k \in [0, 2\tau_0]$, $\tau_0\Delta t = 3$ sec, which has a frequency range of $\omega \in [50, 150]$ rad/s. In Step 2 of the analysis procedure, the output frequency range of interest is taken as $\boldsymbol{W} = [-\hat{N}\omega_{\max}, \hat{N}\omega_{\max}]$ with $\hat{N} = 1$ because the system output response over the input frequency range $\omega \in [50, 150]$ rad/s is dominant. In Step 3, it is determined that $[\![u]\!]=1.9316$ and $[\![u]\!]=5.7947$ in the two cases of $A=0.06$ N and $A=0.18$ N, respectively.

In Step 4, $\bar{L}_w=0.0188$ and in this case study, only $c_{3,0}(.)$ is involved in the NARX model (34), so that $\bar{C}_{p,m-p}=\bar{C}_{3,0}$ is calculated as

$$\begin{aligned}\bar{C}_{3,0} &= \sup_{\substack{\omega_i\in W;\\ i=1,\dots,m}}\left|\begin{array}{l}-k_3\Delta t^2-\dfrac{1}{6}\displaystyle\sum_{\substack{v_1,v_2,v_3\in\{1,2,3\}\\ v_1\neq v_2\neq v_3}}\left\{\dfrac{c_3}{\Delta t}\right.\\ \times\left[1-\exp\left(-\mathrm{j}\left(\omega_{v_1}+\omega_{v_2}+\omega_{v_3}\right)\Delta t\right)\right]+3\dfrac{c_3}{\Delta t}\\ \left.\times\left[\exp\left(-\mathrm{j}\omega_{v_3}\Delta t\right)-\exp\left(-\mathrm{j}\left(\omega_{v_2}+\omega_{v_3}\right)\Delta t\right)\right]\right\}\end{array}\right|\\ &= \sup_{\substack{\omega_i\in W;\\ i=1,\dots,m}}\left|-k_3\Delta t^2-\frac{c_3}{\Delta t}\prod_{i=1}^{3}\left[1-\exp\left(-\mathrm{j}\omega_{v_i}\Delta t\right)\right]\right| \\ &= \sup_{\omega\in W}\left|-k_3\Delta t^2-\frac{c_3}{\Delta t}\left[1-\exp\left(-\mathrm{j}\omega\Delta t\right)\right]^3\right|\end{aligned}\tag{37}$$

Then in Steps 5 and 6, the criterion Γ is calculated as

$$\Gamma_{A=0.06}=0.5503\in(0,1] \text{ and } \Gamma_{A=0.18}=-0.3488<0 \tag{38}$$

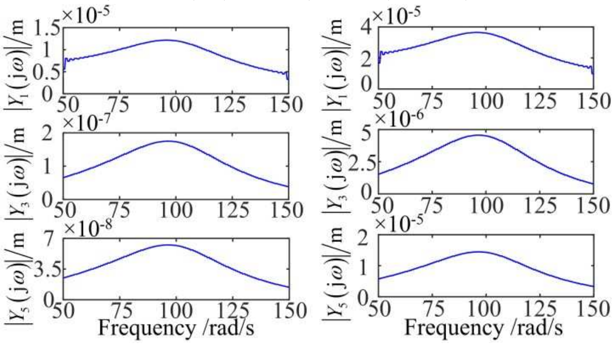
indicating that the Volterra series representation of system (32) is convergent when $A=0.06$ N while the representation may be divergent when $A=0.18$ N.

The nonlinear output spectra up to the 13th order were calculated by using the ALEs over the frequency range of $\omega\in[50,150]$ rad/s and the results up to the fifth order of nonlinearity are provided in Fig.9. The results show that the Volterra series representation of system (34) can be convergent at $A=0.06$ as

$$|Y_1(\mathrm{j}\omega)|>|Y_3(\mathrm{j}\omega)|>\cdots>|Y_{13}(\mathrm{j}\omega)| \tag{39}$$

while divergent at $A=0.18$ because, in this case

$$|Y_{13}(\mathrm{j}\omega)|>|Y_{11}(\mathrm{j}\omega)|>\cdots>|Y_1(\mathrm{j}\omega)| \tag{40}$$



(a) Input magnitude $A=0.06$ N (b) Input magnitude $A=0.18$ N

Fig.9. The nonlinear output spectra up to the 5th order

Moreover, the convergence boundary of the system input can be obtained by solving equation (22) in the specific case of system (34), which is,

$$\begin{cases} f_{BC}(\xi,x)=x-\bar{L}_{W_{\text{lin}}}\bar{C}_{0,1}[\![u]\!]-\bar{L}_w\bar{C}_{3,0}x^3=0\\ f'_{BC}(\xi,x)=1-3\bar{L}_w\bar{C}_{3,0}x^2=0\end{cases}\tag{41}$$

for $[\![u]\!]$, yielding $[\![u]\!]=4.295$ and the corresponding boundary on parameter A of the input signal (36) is 0.133. This boundary is between $A=0.06$ N and $A=0.18$ N, which further confirms the effectiveness of the proposed new convergence analysis.

5. Conclusions

In the present study, a new convergence analysis for the Volterra series representation of nonlinear systems has been studied. From the frequency domain representation of the NARX model of nonlinear systems, a function known as the Generalized Output Bound Characteristic Function (GOBCF) is defined. Moreover, a new criterion for the analysis of the convergence of the Volterra series representation is derived based on the GOBCF, producing a novel sufficient condition for a convergent Volterra series representation of the NARX model.

Compared to existing approaches, the new criterion provides a more rigorous and less conservative analysis result, and is applicable to nonlinear systems subject to either harmonic or general inputs. Two case studies have been used to demonstrate the effectiveness of the new analysis and its advantages over available methods. The results provide an important basis for the application of the frequency domain theories and methods of nonlinear systems to address the analysis and design problems with a wide range engineering systems.

Similar results can also be obtained for the convergence analysis of the rational NARX model as well as the NDE model of nonlinear systems, which will be reported in future publications.

Appendix A: Proof of Proposition 1

By taking the DTFT on both sides of the NARX model (1), taking into account the relationship between DTFT and FT, and using the frequency domain convolution theorem for n discrete time signals (Kamen & Heck, 2000), a frequency domain representation of the NARX model (1) can be obtained as

$$Y(\mathrm{j}\omega)=\sum_{m=1}^{M}\hat{Y}_m(\mathrm{j}\omega) \tag{A1}$$

where

$$\begin{cases}\hat{Y}_1(\mathrm{j}\omega)=L(\mathrm{j}\omega)C_{0,1}(\omega)U(\mathrm{j}\omega) & m=1\\ \hat{Y}_m(\mathrm{j}\omega)=\dfrac{L(\mathrm{j}\omega)}{\sqrt{m}(2\pi)^{m-1}}\\ \quad\times\displaystyle\sum_{p=0}^{m}\int_{\omega_1+\cdots+\omega_m=\omega}C_{p,m-p}(\omega_1,\dots,\omega_m) & m\geq 2\\ \quad\times\displaystyle\prod_{i=1}^{p}Y(\mathrm{j}\omega_i)\prod_{i=p+1}^{m}U(\mathrm{j}\omega_i)\mathrm{d}\sigma_\omega\end{cases}\tag{A2}$$

and $L(\mathrm{j}\omega)$ and $C_{p,m-p}(\omega_1,\dots,\omega_m)$ are as defined by (10) and (11).

It is known from (3) that

$$\begin{aligned}|Y(\mathrm{j}\omega)| &\leq \sum_{n=1}^{+\infty}\bar{H}_n\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}\prod_{i=1}^{n}|U(\mathrm{j}\omega_i)|\mathrm{d}\sigma_\omega\\ &=\sum_{n=1}^{+\infty}\bar{H}_nF\left[\bar{u}^n(t)\right]=\sum_{n=1}^{+\infty}\bar{H}_n\int_{-\infty}^{+\infty}\bar{u}^n(t)\exp(-\mathrm{j}\omega t)\mathrm{d}t \\ &\leq\sum_{n=1}^{+\infty}\bar{H}_n[\![u]\!]^{n-1}|U(\mathrm{j}\omega)|\leq\sum_{n=1}^{+\infty}\bar{H}_n[\![u]\!]^n=[\![y]\!]\end{aligned}\tag{A3}$$

where $\bar{u}(t)=F^{-1}\left[|U(\mathrm{j}\omega)|\right]$ with $\omega\in W_0$.

From $\hat{Y}_m(\mathrm{j}\omega)$ given in (A2), it is known that

$$|\hat{Y}_1(\mathrm{j}\omega)|\leq\bar{L}_{W_{\text{lin}}}\bar{C}_{0,1}[\![u]\!] \tag{A4}$$

and

$$
\begin{aligned}
&\left|\hat{Y}_m(\mathrm{j}\omega)\right| \le \bar{L}_w \bar{C}_{p,m-p} \frac{1}{\sqrt{m}(2\pi)^{m-1}} \\
&\times \sum_{p=0}^{m} \int_{\omega_1+\cdots+\omega_m=\omega} \prod_{i=1}^{p}\left(\sum_{n=1}^{+\infty} \bar{H}_n [\![u]\!]^{n-1} \left|U(\mathrm{j}\omega_i)\right|\right) \prod_{i=p+1}^{m} \left|U(\mathrm{j}\omega_i)\right| \mathrm{d}\sigma_\omega \\
&\le \bar{L}_w \bar{C}_{p,m-p} \sum_{p=0}^{m} \prod_{i=1}^{p} \left(\sum_{n=1}^{+\infty} \bar{H}_n [\![u]\!]^n\right) [\![u]\!]^{m-p} \\
&= \sum_{p=0}^{m} \bar{L}_w \bar{C}_{p,m-p} [\![y]\!]^p [\![u]\!]^{m-p}
\end{aligned} \tag{A5}
$$

Consequently,

$$
\begin{aligned}
&\left|Y(\mathrm{j}\omega)\right| \le \sum_{m=1}^{M} \left|\hat{Y}_m(\mathrm{j}\omega)\right| \le \bar{L}_{w_{\mathrm{lin}}} \bar{C}_{0,1} [\![u]\!] \\
&+ \sum_{m=2}^{M} \sum_{p=0}^{m} \bar{L}_w \bar{C}_{p,m-p} [\![y]\!]^p [\![u]\!]^{m-p} = \bar{L}_{w_{\mathrm{lin}}} \bar{C}_{0,1} [\![u]\!] \\
&+ \sum_{n=2}^{+\infty} \bar{L}_w \left(\bar{C}_{0,n} + \sum_{m=2}^{n} \sum_{p=1}^{m} \left(\bar{C}_{p,m-p} \sum_{\substack{r_1,\ldots,r_p=1,\\ \sum r_i = n-m+p}}^{n-m+1} \prod_{i=1}^{p} \bar{H}_{r_i} \right) \right) [\![u]\!]^n
\end{aligned} \tag{A6}
$$

Considering that

$$
\begin{cases}
\bar{H}_1 = \bar{L}_{w_{\mathrm{lin}}} \bar{C}_{0,1} \\
\bar{H}_n = \bar{L}_w \left(\bar{C}_{0,n} + \sum_{m=2}^{n} \sum_{p=1}^{m} \left(\bar{C}_{p,m-p} \sum_{\substack{r_1,\ldots,r_p=1,\\ \sum r_i = n-m+p}}^{n-m+1} \prod_{i=1}^{p} \bar{H}_{r_i} \right) \right)
\end{cases} \tag{A7}
$$

are the bound on the system GFRFs over all possible frequency ranges (Peyton-Jones & Billings, 1989; Xiao et al., 2014) such that

$$
\bar{H}_n \ge \sup_{\omega_1,\ldots,\omega_n \in W_0} \left| H_n(\omega_1,\ldots,\omega_n) \right| \tag{A8}
$$

the right hand side of (A6) is equal to the right hand side of (A3), that is

$$
\begin{aligned}
&\bar{L}_{w_{\mathrm{lin}}} \bar{C}_{0,1} [\![u]\!] + \sum_{m=2}^{M} \sum_{p=0}^{m} \bar{L}_w \bar{C}_{p,m-p} [\![y]\!]^p [\![u]\!]^{m-p} \\
&= \sum_{n=1}^{\infty} \bar{H}_n [\![u]\!]^n = [\![y]\!]
\end{aligned} \tag{A9}
$$

Consequently, $[\![y]\!]$ is a solution to $f_{BC}([\![y]\!]) = 0$. Thus Proposition 1 is proven.

Appendix B: Proof of Proposition 2

If equation $f_{BC}(x) = 0$ has real positive solutions, according to Proposition 1, the output bound $[\![y]\!]$ given by (13) exists, indicating that output response of the NARX model can be represented by a convergent Volterra series.

Considering $\Gamma \in [0,1]$ when, the GOBCF has real and positive solutions and $f'_{BC}(0) > 0,\ f''_{BC}(x) < 0,\ x \ge 0$, the GOBCF's extreme point $E(x_{ex}, f_{BC}(x_{ex}))$ can be obtained by solving $f'_{BC}(x) = 0$ to find x_{ex} and

$$
f_{BC}(x_{ex}) = x_{ex} - \bar{L}_{w_{\mathrm{lin}}} \bar{C}_{0,1} [\![u]\!] - \sum_{m=2}^{M} \sum_{p=0}^{m} \bar{L}_w \bar{C}_{p,m-p} x_{ex}{}^p [\![u]\!]^{m-p} \tag{B1}
$$

Evaluating the first derivative with respect to $[\![u]\!]$ on both sides of (B1) yields

$$
\begin{aligned}
&\frac{\mathrm{d}f_{BC}(x_{ex}, [\![u]\!])}{\mathrm{d}[\![u]\!]} = -\bar{L}_{w_{\mathrm{lin}}} \bar{C}_{0,1} - \sum_{m=2}^{M} \sum_{p=1}^{m} (m-p) \bar{L}_w \bar{C}_{p,m-p} x_{ex}{}^p \\
&\times [\![u]\!]^{m-p-1} + \left(1 - \sum_{m=2}^{M} \sum_{p=1}^{m} p \bar{L}_w \bar{C}_{p,m-p} x_{ex}{}^{p-1} [\![u]\!]^{m-p}\right) \frac{\mathrm{d}x_{ex}}{\mathrm{d}[\![u]\!]}
\end{aligned} \tag{B2}
$$

Considering $f'_{BC}(x_{ex}) = 0$, (B2) can be rewritten as

$$
\begin{aligned}
&\frac{\mathrm{d}f_{BC}(x_{ex}, [\![u]\!])}{\mathrm{d}[\![u]\!]} = -\bar{L}_{w_{\mathrm{lin}}} \bar{C}_{0,1} - \sum_{m=2}^{M} \sum_{p=1}^{m} (m-p) \\
&\times \bar{L}_w \bar{C}_{p,m-p} x_{ex}{}^p [\![u]\!]^{m-p-1} < 0
\end{aligned} \tag{B3}
$$

So $f_{BC}(x_{ex}, [\![u]\!])$ monotonically decreases with the increase of $[\![u]\!]$, that is, $\Gamma \to 0$ when $[\![u]\!]$ increases and $\Gamma \to 1$ when $[\![u]\!]$ decreases.

If $f'_{BC}(0) > 0,\ f''_{BC}(x) = 0,\ x \ge 0$,

$$
f'_{BC}(0) = 1 - \sum_{m=2}^{M} \bar{L}_w \bar{C}_{1,m-1} [\![u]\!]^{m-1} \tag{B4}
$$

indicating that $f'_{BC}(0),\ x \ge 0$ is a constant and monotonically decreases with the increase of $[\![u]\!]$. Therefore, $\Gamma = f'_{BC}(0) \to 0$ when $[\![u]\!]$ increases and $\Gamma = f'_{BC}(0) \to 1$ when $[\![u]\!]$ decreases.

Considering that when the input bound $[\![u]\!]$ increases, the convergence of the Volterra series representation is weaken and when the input bound $[\![u]\!]$ decreases, the convergence is enhanced, one can reach to the conclusion of the second part of Proposition 2 with regard to the effect of $[\![u]\!]$ on the convergence of the system Volterra series representation.

The analysis can readily be extended to reveal similar effects of the system linear and nonlinear characteristic parameters $\bar{L}_w$ and $\bar{C}_{p,m-p}$ on the convergence of the system's Volterra series representation, and indicate that Γ can be used to quantify the extent to which the Volterra series representation is convergent. Thus, Proposition 2 is proven.

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